Extremal even-cycle-free subgraphs of the complete transposition graphs

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Abstract

Given graphs G and H, the generalized Turán number ex(G, H) is the maximum number of edges in an H-free subgraph of G. In this paper, we obtain an asymptotic upper bound on $ex(CT_n, C_{2l})$ for any $n \ge 3$ and $l \ge 2$, where C_{2l} is the cycle of length 2l and CT_n is the complete transposition graph which is defined as the Cayley graph on the symmetric group S_n with respect to the set of all transpositions of S_n .

 ${\bf Key\ words}~$ Turán number, even-cycle-free subgraph, complete transposition graph, Ramsey-type problem

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1 Introduction

Throughout this paper graphs are finite and undirected with no loops or multiple edges. The vertex and edge sets of a graph G are denoted by V(G) and E(G), respectively. The numbers of vertices and edges of G are denoted by v(G) and e(G), respectively. The degree of a vertex $x \in V(G)$ in G is denoted by $d_G(x)$, and the edge joining vertices u and w are denoted as an unordered pair $\{u, w\}$. A cycle with l edges is called an l-cycle or a cycle of length l, where $l \geq 3$. A path with length l is called an l-path, where $l \geq 1$. As usual an l-cycle is denoted by C_l and an l-path by P_l . Two graphs G and H are said to be *isomorphic* if there exists a bijection f from V(G) to V(H) such that $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$.

Let G and H be graphs. We say that G is H-free if there exists no subgraph of G which is isomorphic to H. The generalized Turán number ex(G, H) is the maximum number of edges in an H-free spanning subgraph of G. This invariant proposed by Erdős [10] is a generalization of the well-known Turán number ex(n, H) which gives the maximum number of edges in an H-free graph with n vertices. In the literature there is a huge amount of work

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on Turán numbers and generalized Turán numbers, beginning with Mantel [23] who proved that $ex(n, K_3) = \lfloor n^2/4 \rfloor$ and Turán [26] who determined $ex(n, K_r)$ for any $r \geq 3$, where K_r is the complete graph with r vertices. In [13], Erdős and Simonovits obtained an asymptotic formula for ex(n, H) in terms of the chromatic number of H. But when H is bipartite the situation is considerably more complicated, and we can only deduce that $ex(n, H) = o(n^2)$. Herein and in the rest of this paper asymptotics are taken as $n \to \infty$. In general, it is a challenging problem to determine ex(G, H) when H is a bipartite graph, especially when H is an even cycle. In this regard, two interesting functions that have received much attention are $ex(G, K_{s,t})$ and $ex(Q_n, C_{2l})$, where $K_{s,t}$ is the complete bipartite graph with s and t vertices, respectively, in the biparts of its bipartition, and Q_n is the n-dimensional hypercube. The problem of determining $ex(K_{m,n}, K_{s,t})$, proposed by Zarankiewicz in [28], is the analogue of Turán's original problem (the one of determining $ex(K_n, K_r) = ex(n, K_r)$) for bipartite graphs, and an excellent survey on this problem can be found in [18]. Besides, some related research was dedicated to showing $ex(G, K_{t,t})$, where G is some other certain restricted graph. See e.g. [15, 16].

The study of $ex(Q_n, C_{2l})$ began with a problem raised by Erdős which asks for the maximum number of edges in a C_4 -free spanning subgraph of Q_n . In [10], Erdős conjectured that $\left(\frac{1}{2} + o(1)\right) e(Q_n)$ should be an upper bound for $ex(Q_n, C_4)$, and he also asked whether $o(e(Q_n))$ edges of Q_n would ensure the existence of a cycle C_{2l} for $l \geq 3$. The best known upper bound for $ex(Q_n, C_4)$, obtained by Balogn et al. [3] and improved slightly the bounds of Chung [7] and Wagner [25], is $(0.6068 + o(1))e(Q_n)$. The problem of determining the value of $ex(Q_n, C_{2l})$ when l = 3 or 5 is still open too, and progresses can be found in [1, 2, 3, 7, 8]. For $l \geq 2$, upper bounds for $ex(Q_n, C_{4l})$ and $ex(Q_n, C_{4l+6})$ were obtained by Chung [7] and Füredi and Özkahya [17], respectively, and their results together imply that $ex(Q_n, C_{2l'}) = o(e(Q_n))$ for $l' \ge 6$ or l' = 4. In [9], Conlon proved that $ex(Q_n, H) = o(e(Q_n))$ for any graph H that admits a k-partite representation. This gives a unified approach to the proof that $ex(Q_n, C_{2l}) = o(e(Q_n))$ for all $l \neq 5$ no less than 4. The doubled Johnson graphs J(n;k,k+1), where $1 \le k \le (n-1)/2$, form an interesting family of spanning subgraphs of Q_n , and in particular the doubled odd graph $O_{k+1} := J(2k+1;k,k+1)$ is known to be distance-transitive. Recently, Cao et al. [6] studied $ex(J(n; k, k+1), C_{2l})$ and proved among other things that $ex(O_{k+1}, C_{2l}) = o(e(O_{k+1}))$ for $l \ge 6$.

In this paper, we study the generalized Turán number $ex(CT_n, C_{2l})$ for the complete transposition graph CT_n , where $n \geq 3$ and $l \geq 2$. The complete transposition graphs are an important family of Cayley graphs which share several interesting properties with hypercubes. For example, both CT_n and Q_n are bipartite and arc-transitive, with only integral eigenvalues, and both graphs are popular topologies for interconnection networks [20]. Over the years several aspects of complete transposition graphs such as automorphisms, eigenvalues, connectivity and bisection width have been studied as one can find in, for example, [19, 21, 22, 24, 27]. In general, given a group G with identity element 1 and an inverse-closed subset S of $G \setminus \{1\}$, the Cayley graph Cay(G, S) on G with respect to the connection set S is defined to be the graph with vertex set G such that $x, y \in G$ are adjacent if and only if $yx^{-1} \in S$. The complete transposition graph CT_n is defined as the Cayley graph on the symmetric group S_n whose connection set is the set of all transpositions of S_n . That is,

$$V(CT_n) = \mathbf{S}_n,$$

 $E(CT_n) = \{\{x, y\} : x, y \in S_n \text{ and } y = ux \text{ for some transposition } u \text{ of } S_n\}.$

It follows that CT_n is a connected $\binom{n}{2}$ -regular bipartite graph with

$$v := v(CT_n) = n!$$

vertices and

$$e(CT_n) = \frac{v}{2} \binom{n}{2}$$

edges.

The main result in this paper is as follows.

Theorem 1.1. Let n and l be integers with $n \ge 3$ and $l \ge 2$.

- (i) If $l \ge 4$ and l is even, then $\exp(CT_n, C_{2l}) = O(n^{-1+\frac{2}{l}})e(CT_n)$.
- (ii) If $l \ge 4$ and l is odd, then

$$ex(CT_n, C_{2l}) = \begin{cases} O(n^{-\frac{1}{l}})e(CT_n), & \text{if } l = 7, \\ O(n^{-\frac{1}{8} + \frac{1}{4(l-3)}})e(CT_n), & \text{otherwise.} \end{cases}$$

- (iii) If l = 3, then $ex(CT_n, C_{2l}) \le (\sqrt{2} 1 + o(1))e(CT_n)$.
- (iv) If l = 2, then $ex(CT_n, C_{2l}) \le \frac{3}{4}e(CT_n)$.

An immediate consequence of Theorem 1.1 is that $ex(CT_n, C_{2l}) = o(e(CT_n))$ for $l \ge 4$. This leads to the following Ramsey-type result.

Corollary 1.2. Let t and l be integers with $t \ge 1$ and $l \ge 4$. If CT_n is edge-partitioned into t subgraphs, then one of the subgraphs must contain C_{2l} provided that n is sufficiently large (depending only on t and l).

In the next section we will prove some basic properties of cycles in the complete transposition graphs. Using these preparations we will prove parts (i)-(ii) and (iii)-(iv) of Theorem 1.1 in Sections 3 and 4, respectively.

2 Preliminaries

We assume that S_n is the symmetric group on $\{1, 2, ..., n\}$, where $n \ge 3$. The identity element of S_n is denoted by id. The *support* of an element $x \in S_n$ is defined as $supp(x) = \{i \in \{1, 2, ..., n\} \mid i^x \neq i\}$.

Definition 2.1. The support of an edge $\{u, z\}$ of CT_n , denoted by $\operatorname{supp}(\{u, z\})$, is defined to be the support of the transposition zu^{-1} . That is, $\operatorname{supp}(\{u, z\}) = \operatorname{supp}(zu^{-1})$.

Since CT_n is a Cayley graph on the symmetric group S_n whose connection set consists of all transpositions, we know that $\operatorname{supp}(\{u, z\})$ is a 2-subset of $\{1, 2, \ldots, n\}$ for any $\{u, z\} \in E(CT_n)$, and $\operatorname{supp}(\{u, z\}) = \operatorname{supp}(\{z, u\})$. Note that, for any two incident edges $\{x, u\}$ and $\{x, z\}$ of CT_n , we have $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})| = 0$ or 1. For any subgraph H of CT_n , we define

$$\operatorname{supp}(H) := \bigcup_{\{u,z\} \in E(H)} \operatorname{supp}(\{u,z\}).$$

Let $P = (u_1, u_2, ..., u_t)$ be a path in CT_n . Setting $w_i = u_i u_{i-1}^{-1}$ for $i \in \{2, 3, ..., t\}$, we have $u_t u_1^{-1} = w_t w_{t-1} \cdots w_3 w_2$ and hence $\text{supp}(u_t u_1^{-1}) \subseteq \text{supp}(P)$.

Lemma 2.2. Let g and h be distinct transpositions of S_n . Then the only 4-cycles in CT_n passing through the 2-path (g, id, h) are $(\mathrm{id}, g, hg, h, \mathrm{id})$ and $(\mathrm{id}, g, gh, h, \mathrm{id})$. In particular, if gh = hg, then these 4-cycles are identical and they are the only 4-cycle in CT_n passing through the 2-path (g, id, h) .

Proof. Suppose gh = hg. Then $|\operatorname{supp}(g) \cap \operatorname{supp}(h)| = 0$. Note that id, g and h are three vertices in CT_n . Let w be a common neighbor of the vertices g and h in CT_n . Then there exist transpositions x, y such that xg = yh = w, implying that gh = xy. Since the supports of g and h are disjoint, the equation gh = xy holds if and only if g = x and h = y, or g = y and h = x. Therefore, w is either the vertex id or the vertex gh. Thus, there exists a unique 4-cycle in CT_n passing through g, id and h, which is (id, g, hg = gh, h, id).

Suppose $gh \neq hg$. Then $|\operatorname{supp}(g) \cap \operatorname{supp}(h)| = 1$. Without loss of generality we may assume g = (1,2), and h = (1,3). Let w be a common neighbor of the vertices g and hin CT_n . Then there exist transpositions x, y such that xg = yh = w, implying that xy =gh = (1,2)(1,3) = (1,2,3). If we decompose (1,2,3) into the product of two transpositions of S_n , then the supports of these two transpositions must lie in $\{1,2,3\}$ and contain exactly one common letter. Therefore, the only ways to decompose (1,2,3) into the product of two transpositions of S_n are (1,2,3) = (1,3)(2,3) = (2,3)(1,2) = (1,2)(1,3). Hence, we have x = (1,3) and y = (2,3), or x = (2,3) and y = (1,2), or x = (1,2) and y = (1,3), yielding $w \in \{\operatorname{id}, (1,3,2), (1,2,3)\}$. Therefore, there are exactly two 4-cycles in CT_n passing through g, id and h, namely (id, g, hg, h, id) and (id, g, gh, h, id).

It is well known that any permutation in S_n can be expressed as a product of transpositions, and for each $g \in S_n$ the map $\hat{g} : h \mapsto hg$, $h \in S_n$ defines an automorphism of CT_n . Hence Lemma 2.2 implies the following result.

Corollary 2.3. Let (u, x, z) be a 2-path in CT_n . If $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})| = 0$, then there is a unique 4-cycle in CT_n containing (u, x, z), namely $(x, z, zx^{-1}u, u, x)$; and if $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})| = 1$, then there are exactly two 4-cycles in CT_n containing (u, x, z), namely $(x, z, zx^{-1}u, u, x)$ and $(x, z, ux^{-1}z, u, x)$.

Denote by $n(C_4)$ the number of 4-cycles in CT_n . Lemma 2.2 and Corollary 2.3 together imply the following result.

Corollary 2.4. The following hold.

- (i) The length of a shortest cycle in CT_n is 4.
- (ii) For any edge $\{u, z\}$ of CT_n , there are exactly $\frac{1}{2}(n-2)(n+1)$ cycles of length 4 in CT_n containing $\{u, z\}$.
- (iii) $n(C_4) = \frac{1}{8}(n-2)(n+1)e(CT_n)$.

Proof. Since CT_n is bipartite, it does not contain any 3-cycle. On the other hand, 4-cycles exist in CT_n by Lemma 2.2. So any shortest cycle in CT_n has length 4 as stated in (i).

For any edge $\{u, z\}$ of CT_n , there are exactly $\binom{n-2}{2}$ 2-paths (u, z, w) such that $|\operatorname{supp}(\{u, z\}) \cap \operatorname{supp}(\{z, w\})| = 0$, and there are exactly n - 2 2-paths (u, z, w) such that $|\operatorname{supp}(\{u, z\}) \cap \operatorname{supp}(\{z, w\})| = 1$. Hence, by Corollary 2.3, the number of 4-cycles containing any given edge of CT_n is equal to $\binom{n-2}{2} + 2(n-2) = \frac{1}{2}(n-2)(n+1)$ as claimed in (ii). We obtain (iii) from (ii) immediately.

Let

 $\mathcal{F}_0 = \{ \text{all transpositions of } S_n \}$

and

$$\mathcal{F}_i = \{ x \in \mathcal{F}_0 \mid i \in \operatorname{supp}(x) \}$$

for each $i \in \{1, 2, ..., n\}$. Clearly, in S_n any pair of transpositions with joint supports are contained in one of $\mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_n$. In addition, $\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$ contains all transpositions of S_n and each transposition of S_n appears exactly three times in $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n$.

The following auxiliary graphs will play an important role in our proof of Theorem 1.1.

Definition 2.5. Let G be a spanning subgraph of CT_n . For each $i \in \{0, 1, 2, ..., n\}$ and each $x \in S_n$, define G_x^i to be the graph with vertex set $V(G_x^i) = \{yx \in S_n \mid y \in \mathcal{F}_i\}$ such that for $u, z \in V(G_x^i)$, u and z are adjacent if and only if $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})| = \delta_i$ and there exists a vertex w with $w \neq x$ such that (u, w, z) is a 2-path in G, where $\delta_0 = 0$ and $\delta_i = 1$ for $i \in \{1, 2, ..., n\}$.

By the definition of G_x^i , it is clear that

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n v(G_x^i) = 3v \cdot \binom{n}{2},\tag{1}$$

where as before v = n! is the number of vertices of CT_n . Since $|V(G_x^i) \cap V(G_x^j)| = 1$ and $|E(G_x^0) \cap E(G_x^i)| = 0$ for any $i, j \in \{1, 2, ..., n\}$ with $i \neq j$, we have $|E(G_x^i) \cap E(G_x^j)| = 0$ for any $i, j \in \{0, 1, ..., n\}$ with $i \neq j$. Hence

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n e(G_x^i) \ge \sum_{w \in V(G)} \binom{d_G(w)}{2},\tag{2}$$

where the right-hand side gives the number of 2-paths in G.

Lemma 2.6. Let G be a spanning subgraph of CT_n . Let l be an integer with $l \geq 3$. If there exists an l-cycle in G_x^i for some $i \in \{0, 1, ..., n\}$ and $x \in V(CT_n)$, then there exists a 2l-cycle in G.

Proof. Assume that $C = (u_1, u_2, \ldots, u_l, u_{l+1} = u_1)$ is an *l*-cycle in G_x^i . By the definition of G_x^i , for each $j \in \{1, 2, \ldots, l\}$ there exists $w_j \in V(CT_n)$ such that (u_j, w_j, u_{j+1}) is a 2-path in G. By Corollary 2.3, we have $w_j = u_j x^{-1} u_{j+1} = u_{j+1} x^{-1} u_j$ if i = 0, and $w_j \in \{u_j x^{-1} u_{j+1}, u_{j+1} x^{-1} u_j\}$ if $i \in \{1, 2, \ldots, n\}$.

We claim that w_1, w_2, \ldots, w_l are pairwise distinct. Suppose to the contrary that $w_j = w_s$ with j < s. Since there are at most two cycles of length 4 containing (x, u_j, w_j) , we have s = j + 1. If i = 0, then $|\operatorname{supp}(\{x, u_j\}) \cap \operatorname{supp}(\{x, u_{j+1}\})| = 0$, which implies that $(xu_j^{-1})(u_{j+1}x^{-1}) = (u_{j+1}x^{-1})(xu_j^{-1}) = u_{j+1}u_j^{-1}$ and $w_j^{-1}w_{j+1} = u_{j+1}^{-1}xu_j^{-1}u_{j+1}x^{-1}u_{j+2} = u_j^{-1}u_{j+2} \neq id$, a contradiction. If $i \in \{1, 2, \ldots, n\}$, then

$$\{u_j x^{-1} u_{j+1}, u_{j+1} x^{-1} u_j\} \cap \{u_{j+1} x^{-1} u_{j+2}, u_{j+2} x^{-1} u_{j+1}\} \neq \emptyset.$$

Assume that $u_j = (i, t_0)x$, $u_{j+1} = (i, t_1)x$ and $u_{j+2} = (i, t_2)x$, where t_0, t_1, t_2 are distinct elements of $\{1, 2, ..., n\} \setminus \{i\}$. Then

$$\{(i,t_0,t_1)x,(i,t_1,t_0)x\} \cap \{(i,t_1,t_2)x,(i,t_2,t_1)x\} \neq \emptyset,$$

which is impossible.

2

Since CT_n is a bipartite graph, we have $\{u_1, u_2, \ldots, u_l\} \cap \{w_1, w_2, \ldots, w_l\} = \emptyset$. Since w_1, w_2, \ldots, w_l are pairwise distinct and the 2-path (u_j, w_j, u_{j+1}) is in G for $j \in \{1, 2, \ldots, l\}$, it follows that $(u_1, w_1, u_2, w_2, \ldots, u_l, w_l, u_{l+1} = u_1)$ is a 2*l*-cycle in G. \Box

3 Proof of the main result when $l \ge 4$

We prove parts (i) and (ii) of Theorem 1.1 in this section.

3.1 4*k*-cycle-free subgraphs of CT_n

Proof of Theorem 1.1 (i). Suppose G is a C_{4k} -free spanning subgraph of CT_n with maximum number of edges, where $k \ge 2$. Then $d_G(w) \ge 1$ for each $w \in V(G)$. Since G is C_{4k} -free, by Lemma 2.6, G_x^i is C_{2k} -free for any $x \in V(CT_n)$ and $i \in \{0, 1, \ldots, n\}$. Thus from the main theorem in [4] by Bondy and Simonovits it follows that G_x^i has at most $c_k(v(G_x^i))^{1+\frac{1}{k}}$ edges, where c_k is a positive constant relying on k only. Therefore, we have

$$\sum_{v \in V(CT_n)} \sum_{i=0}^n e(G_x^i) \le c_k v \cdot \left(\binom{n}{2}^{1+\frac{1}{k}} + n(n-1)^{1+\frac{1}{k}} \right) \le c'_k v \cdot \binom{n}{2}^{1+\frac{1}{k}}.$$
 (3)

On the other hand, by (2) and the Cauchy-Schwarz inequality, we have

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n e(G_x^i) \ge \sum_{w \in V(G)} \binom{d_G(w)}{2}$$

= $\frac{1}{2} \sum_{w \in V(G)} d_G(w)^2 - \frac{1}{2} \sum_{w \in V(G)} d_G(w)$
 $\ge \frac{1}{2v} \left(\sum_{w \in V(G)} d_G(w)\right)^2 - \frac{1}{2} \sum_{w \in V(G)} d_G(w).$ (4)

Since $\sum_{w \in V(G)} d_G(w) = 2e(G)$, it follows from (3) and (4) that

$$\frac{2e(G)^2}{v} - e(G) \le c'_k v \cdot \binom{n}{2}^{1 + \frac{1}{k}},$$

or equivalently,

$$e(G)^2 \le \frac{1}{2}c'_k v^2 \cdot {\binom{n}{2}}^{1+\frac{1}{k}} + \frac{e(G)v}{2}.$$

Set $\pi = e(G)/e(CT_n)$. Observe that $0 < \pi < 1$. Since $e(CT_n) = \frac{v}{2} {n \choose 2}$, the inequality above yields

$$\pi^2 \le 2c'_k \cdot \binom{n}{2}^{-1+\frac{1}{k}} + \pi \cdot \binom{n}{2}^{-1}.$$

So there exists a constant c depending on k such that

$$\pi \le cn^{-1+\frac{1}{k}}$$

Therefore, we have

$$ex(CT_n, C_{4k}) = e(G) = \pi e(CT_n) \le cn^{-1 + \frac{1}{k}} e(CT_n),$$

as desired in part (i) of Theorem 1.1.

3.2 (4k+2)-cycle-free subgraphs of CT_n

In this subsection we assume that G is a C_{4k+2} -free spanning subgraph of CT_n and a and b are integers with $a, b \ge 2$ such that 4a + 4b = 4k + 4, where $k \ge 2$. Note that a cycle of length 4a in G can not intersect a cycle of length 4b in G at a single edge, for otherwise their union would contain a cycle of length 4k + 2. In what follows we will give an upper bound as well as a lower bound on the number of 4a-cycles in G. These bounds will be used in the proof of part (ii) of Theorem 1.1 at the end of this subsection.

Lemma 3.1. For any 2*l*-cycle C in CT_n , where $l \ge 2$, we have $|\operatorname{supp}(C)| \le 2l$.

Proof. Let $C = (u_0, u_1, u_2, \ldots, u_{2l} = u_0)$ be a 2*l*-cycle in CT_n . Set $w_i = u_i u_{i-1}^{-1}$ for $i \in \{1, 2, \ldots, 2l\}$. Then $\operatorname{supp}(\{u_{i-1}, u_i\}) = \operatorname{supp}(w_i)$ and $\operatorname{supp}(C) = \bigcup_{i=1}^{2l} \operatorname{supp}(w_i)$. Observe that $w_{2l}w_{2l-1}\cdots w_2w_1 = \operatorname{id}$. So for any $x \in \operatorname{supp}(C)$ there exist distinct $i, j \in \{1, 2, \ldots, 2l\}$ such that $x \in \operatorname{supp}(w_i) \cap \operatorname{supp}(w_j)$. Hence $|\operatorname{supp}(C)| \leq 2l$.

Lemma 3.2. Let C and C' be cycles of lengths 4a and 4b in G, respectively. If C and C' have at least one common edge, then $|\operatorname{supp}(C) \cap \operatorname{supp}(C')| \geq 3$.

Proof. Suppose $\{u_1, u_2\}$ is a common edge of C and C'. Since G is a (4a + 4b - 2)-cycle-free subgraph of CT_n , there exists a vertex u_3 of G such that $u_3 \in (V(C) \cap V(C')) \setminus \{u_1, u_2\}$. Since $\operatorname{supp}(u_3u_1^{-1}) \subseteq \operatorname{supp}(C) \cap \operatorname{supp}(C')$ and $u_3 \neq u_2$, we have

$$supp(\{u_1, u_2\}) \neq supp(u_3 u_1^{-1}).$$

This together with $\operatorname{supp}(\{u_1, u_2\}) \subseteq \operatorname{supp}(C) \cap \operatorname{supp}(C')$ implies that $|\operatorname{supp}(C) \cap \operatorname{supp}(C')| \ge 3$.

For any graphs H and L, define N(H, L) to be the number of subgraphs of H which are isomorphic to L.

Lemma 3.3. We have

$$N(G, C_{4a}) = O(n^{4a-3})e(G) + O(vn^{4a-1+\frac{1}{b}}).$$

Moreover, if a = b, then $N(G, C_{4a}) = O(n^{4a-3})e(G)$.

Proof. Denote by C the set of cycles of length 4a in G and C_e the set of cycles in C containing a given edge e. Note that $|C| = N(G, C_{4a})$. Let $E = \bigcup_{C \in C} E(C)$. Let E_1 be the set of edges in E that are contained in a cycle of length 4b in G, and let $E_2 := E \setminus E_1$. Then $E = E_1 \cup E_2$ and

$$4aN(G, C_{4a}) = \sum_{e_1 \in E_1} |\mathcal{C}_{e_1}| + \sum_{e_2 \in E_2} |\mathcal{C}_{e_2}|.$$
(5)

Assume that $e = \{u_1, u_{4a}\}$. Observe that for any 4*a*-cycle $(u_1, u_2, \ldots, u_{4a}, u_1)$, there is a unique sequence $(A_1, A_2, \ldots, A_{4a-1})$ of length 4a - 1 such that $A_i = \operatorname{supp}(\{u_i, u_{i+1}\})$ for any $i \in \{1, 2, \ldots, 4a - 1\}$. For each $B \in \{\operatorname{supp}(C^*) \mid C^* \in \mathbb{C}_e\}$, there are $\binom{|B|}{2}^{4a-1}$ sequences $(A_1, A_2, \ldots, A_{4a-1})$ of length 4a - 1 such that $A_i \subseteq B$ and $|A_i| = 2$ for each $i \in \{1, 2, \ldots, 4a - 1\}$, and hence there are at most $\binom{|B|}{2}^{4a-1}$ 4*a*-cycles *C* containing *e* such that $\operatorname{supp}(C) = B$.

For each $e_1 \in E_1$ (if $E_1 \neq \emptyset$), let C' be a fixed 4b-cycle with $e_1 \in E(C')$. For any 4a-cycle $C^* \in \mathcal{C}_{e_1}$, we have $\operatorname{supp}(e_1) \subseteq \operatorname{supp}(C^*)$ and $|\operatorname{supp}(C^*) \cap \operatorname{supp}(C')| \geq 3$ by Lemma 3.2. Hence, by Lemma 3.1, we have

$$|\{\operatorname{supp}(C^*) \mid C^* \in \mathcal{C}_{e_1}\}| \le \sum_{i=1}^{4a-2} \binom{|\operatorname{supp}(C')| - 2}{i} \sum_{j=0}^{4a-2-i} \binom{n - |\operatorname{supp}(C')|}{j},$$

which implies

$$\mathcal{C}_{e_1} \leq \sum_{i=1}^{4a-2} \binom{|\operatorname{supp}(C')| - 2}{i} \sum_{j=0}^{4a-2-i} \binom{n - |\operatorname{supp}(C')|}{j} \binom{i+j+2}{2}^{4a-1} = O(n^{4a-3}).$$
(6)

For each $e_2 \in E_2$ (if $E_2 \neq \emptyset$), by Lemma 3.1 again, we have

$$|\{\operatorname{supp}(C^*) \mid C^* \in \mathfrak{C}_{e_2}\}| \le \sum_{i=0}^{4a-2} \binom{n-2}{i},$$

which implies

$$\begin{aligned} |\mathfrak{C}_{e_2}| &\leq \sum_{i=0}^{4a-2} \binom{n-2}{i} \binom{i+2}{2}^{4a-1} \\ &= O(n^{4a-2}). \end{aligned}$$
(7)

Note that $|E_1| \leq e(G)$. Note also that $|E_2| \leq e(CT_n, C_{4b})$ as the subgraph induced by E_2 is C_{4b} -free. Using part (i) of Theorem 1.1 (which has been proved already), we have $|E_2| \leq cn^{-1+1/b}e(CT_n)$ for some positive constant c. Combining (5), (6) and (7), we obtain

$$N(G, C_{4a}) \le \frac{1}{4a} \left(\sum_{e \in E_1} O(n^{4a-3}) + \sum_{e \in E_2} O(n^{4a-2}) \right)$$
$$\le O(n^{4a-3})e(G) + O(vn^{4a-1+\frac{1}{b}}).$$

In particular, if a = b, then $|E_2| = 0$ and hence

$$N(G, C_{4a}) \le \frac{1}{4a} \sum_{e \in E_1} O(n^{4a-3}) \le O(n^{4a-3})e(G)$$

This completes the proof.

Proposition 3.4. (Erdős and Simonovits [13]) Let L be a bipartite graph, where there exist vertices x and y such that $L \setminus \{x, y\}$ is a tree. Then there exist constants c_1 , $c_2 > 0$ such that if H is a graph containing more than $c_1v(H)^{\frac{3}{2}}$ edges, then

$$N(H,L) \ge c_2 \frac{e(H)^{e(L)}}{v(H)^{2e(L)-v(L)}}.$$

With the help of this proposition and the auxiliary graphs G_x^i as defined in Definition 2.5, we now prove a lower bound on $N(G, C_{4a})$.

Lemma 3.5. We have

$$N(G, C_{4a}) \ge cv \frac{d^{4a}}{n^{4a}} - O(vn^{2a})$$

for some positive constant c depending on a, where d = 2e(G)/v.

Proof. By Lemma 2.6, we have

$$N(G, C_{4a}) \ge \sum_{x \in V(CT_n)} \sum_{i=0}^n N(G_x^i, C_{2a}).$$
(8)

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Setting $L = C_{2a}$ in Proposition 3.4, there exist two positive constants c_1 and c_2 such that

$$N(G_x^i, C_{2a}) \ge c_2 \left(\frac{e(G_x^i)^{2a}}{v(G_x^i)^{2a}} - \frac{(c_1 v(G_x^i)^{3/2})^{2a}}{v(G_x^i)^{2a}} \right)$$

Combining this with (8), we obtain

$$N(G, C_{4a}) \ge \sum_{x \in V(CT_n)} \sum_{i=0}^n c_2 \left(\frac{e(G_x^i)^{2a}}{v(G_x^i)^{2a}} - \frac{(c_1 v(G_x^i)^{3/2})^{2a}}{v(G_x^i)^{2a}} \right)$$
$$\ge \sum_{x \in V(CT_n)} \left(\frac{c_2 e(G_x^0)^{2a}}{\binom{n}{2}^{2a}} + \sum_{i=1}^n \frac{c_2 e(G_x^i)^{2a}}{(n-1)^{2a}} \right) - \sum_{x \in V(CT_n)} \sum_{i=0}^n c_1^{2a} v(G_x^i)^a.$$

By Hölder's inequality, we then have

$$\begin{split} N(G, C_{4a}) &\geq c_2 \sum_{x \in V(CT_n)} \left(\frac{e(G_x^0)^{2a}}{\binom{n}{2}^{2a}} + \frac{\left(\sum_{i=1}^n e(G_x^i)\right)^{2a}}{n^{2a-1}(n-1)^{2a}} \right) - O(vn^{2a}) \\ &\geq \frac{c_2}{n^{4a}} \sum_{x \in V(CT_n)} \left(e(G_x^0)^{2a} + \left(\sum_{i=1}^n e(G_x^i)\right)^{2a} \right) - O(vn^{2a}) \\ &\geq \frac{c_a}{n^{4a}} \sum_{x \in V(CT_n)} \left(\sum_{i=0}^n e(G_x^i)\right)^{2a} - O(vn^{2a}) \\ &\geq \frac{c_a v}{n^{4a}} \left(\sum_{x \in V(CT_n)} \sum_{i=0}^n \frac{e(G_x^i)}{v}\right)^{2a} - O(vn^{2a}) \\ &\geq \frac{c_a v}{n^{4a}} \left(\sum_{w \in V(G)} \frac{\binom{d_G(w)}{2}}{v}\right)^{2a} - O(vn^{2a}), \end{split}$$

where c_a is a positive constant depending on a and inequality (2) is used in the last step. Setting d = 2e(G)/v and applying Hölder's inequality again, we obtain

$$N(G, C_{4a}) \ge \frac{c_a v}{n^{4a}} \left(\sum_{w \in V(G)} \frac{d_G(w)}{v} \right)^{2a} - O(vn^{2a})$$
$$= \frac{c_a v}{n^{4a}} \left(\frac{d}{2} \right)^{2a} - O(vn^{2a})$$
$$\ge cv \frac{d^{4a}}{n^{4a}} - O(vn^{2a})$$

for some positive constant c depending on a. This completes the proof.

Proof of Theorem 1.1 (ii). Suppose G is a C_{4k+2} -free spanning subgraph of CT_n with maximum number of edges. Then $ex(CT_n, C_{2l}) = e(G)$, where l = 2k + 1. Set d = 2e(G)/v.

Combining Lemma 3.3 and Lemma 3.5, we have

$$cv\frac{d^{4a}}{n^{4a}} \le O(n^{4a-3})e(G) + O(vn^{4a-1+\frac{1}{b}}) + O(vn^{2a}),$$
$$d^{4a} \le O(n^{8a-3})d + O(n^{8a-1+\frac{1}{b}}) + O(n^{6a}).$$

Hence $d = \max\left\{O(n^{2-\frac{1}{4a-1}}), O(n^{2-\frac{1-\frac{1}{b}}{4a}})\right\}$. This bound is minimized when a = 2 and b = k - 1, and this choice of (a, b) yields $d = O(n^{2-\frac{1}{8}+\frac{1}{8(k-1)}})$. Since e(G) = vd/2 and $e(CT_n) = \frac{v}{2}\binom{n}{2}$, it follows that

$$e(G) = O(vn^{2-\frac{1}{8}+\frac{1}{8(k-1)}})$$

= $O(n^{-\frac{1}{8}+\frac{1}{8(k-1)}})e(CT_n)$
= $O(n^{-\frac{1}{8}+\frac{1}{4(l-3)}})e(CT_n).$ (9)

Consider the case when a = b = (k + 1)/2 with k odd. By Lemmas 3.3 and 3.5, we have

$$d^{4a} \le O(n^{8a-3})d + O(n^{6a}),$$

which yields

$$e(G) = O(n^{2 - \frac{1}{4a - 1}})$$

= $O(vn^{2 - \frac{1}{2k + 1}})$
= $O(n^{-\frac{1}{2k + 1}})e(CT_n)$
= $O(n^{-\frac{1}{l}})e(CT_n).$ (10)

Observe that when k is odd we have $n^{-\frac{1}{2k+1}} \leq n^{-\frac{1}{8} + \frac{1}{8(k-1)}}$ if and only if 0 < k < 4.9. So (10) is a better bound than (9) when k = 3. Therefore, $e(G) = O(n^{-\frac{1}{l}})e(CT_n)$ when l = 7. This competes the proof.

So far we have completed the proof of Theorem 1.1 (i) and (ii). These results imply that $ex(CT_n, C_{2l}) = o(e(CT_n))$ for any fixed positive integer $l \ge 4$. Thus, for any $t \ge 1$ and $l \ge 4$, there exists a positive integer n(t, l) such that for any n > n(t, l) and any edge-coloring of CT_n with t colors, CT_n contains a monochromatic copy of C_{2l} , as claimed in Corollary 1.2.

Remark. The theta graph $\Theta_{i,j,k}$ is the graph with i + j + k - 1 vertices which consists of three internally vertex-disjoint paths between the same pair of vertices with lengths i, j and k, respectively. As a by-product of the proof of Theorem 1.1 (i) and (ii), we obtain that

$$ex(CT_n, \Theta_{4a-1,1,4b-1}) = o(e(CT_n))$$

for any $a, b \geq 2$.



Figure 1: Possibilities for $G \cap H$ when $H \in \mathfrak{C}_4$.

4 Proof of the main result when l = 2, 3

In this section, \mathcal{C}_4 denotes the set of 4-cycles in CT_n , and for each $e \in E(CT_n)$, $(\mathcal{C}_4)_e$ denotes the set of 4-cycles in CT_n containing e. Suppose G is a 2*l*-cycle-free spanning subgraph of CT_n with maximum number of edges. For any subgraphs H and L of CT_n , let $G \cap H$ be the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

Note that for any 4-cycle $H \in C_4$, $G \cap H$ is isomorphic to one of the six graphs in Figure 1. Denote by $\chi_0, \chi_1, \chi_2^1, \chi_2^2, \chi_3, \chi_4$ the ratio of the number of 4-cycles H with $G \cap H$ isomorphic to the graphs (1)–(6) in Figure 1 to the total number of 4-cycles in CT_n , respectively. Of course we have

$$\chi_0 + \chi_1 + \chi_2^1 + \chi_2^2 + \chi_3 + \chi_4 = 1.$$
(11)

By double counting the cardinality of $\{(e, H) \mid H \in \mathcal{C}_4, e \in E(G \cap H)\}$, we obtain

$$\sum_{H \in \mathfrak{C}_4} e(G \cap H) = \sum_{e \in E(G)} |(\mathfrak{C}_4)_e|,$$

which by Corollary 2.4 (ii) implies

$$\left(\chi_1 + 2(\chi_2^1 + \chi_2^2) + 3\chi_3 + 4\chi_4\right) \cdot n(C_4) = e(G) \cdot \frac{1}{2}(n-2)(n+1)$$

where as before $n(C_4)$ is the number of 4-cycles in CT_n . Set $\pi = e(G)/e(CT_n)$. By Corollary 2.4 (iii), we have

$$\chi_1 + 2(\chi_2^1 + \chi_2^2) + 3\chi_3 + 4\chi_4 = 4\pi.$$
(12)

Proof of Theorem 1.1 (iv). Suppose G is a C_4 -free spanning subgraph of CT_n with maximum number of edges. Then $d_G(w) \ge 1$ for any $w \in V(G)$ and $\chi_4 = 0$ as G is C_4 -free. Hence, by (11) and (12), we have

$$\pi = \frac{1}{4} \left(\chi_1 + 2(\chi_2^1 + \chi_2^2) + 3\chi_3 \right) \le \frac{3}{4} \left(\chi_0 + \chi_1 + \chi_2^1 + \chi_2^2 + \chi_3 \right) = \frac{3}{4}$$

Thus $ex(CT_n, C_4) = e(G) = \pi e(CT_n) \leq \frac{3}{4}e(CT_n)$ as desired in part (iv) of Theorem 1.1. \Box

Proof of Theorem 1.1 (iii). Suppose G is a C_6 -free spanning subgraph of CT_n with maximum number of edges. For each $i \in \{0, 1, 2, ..., n\}$ and each $x \in V(CT_n)$, let H_x^i be the subgraph

of G_x^i (see Definition 2.5) induced by the subset $\{u \in V(G_x^i) \mid \{u, x\} \notin E(G)\}$ of $V(G_x^i)$. Then

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n v(H_x^i) = 3 \sum_{x \in V(CT_n)} \left(\binom{n}{2} - d_G(x) \right).$$

Since $|E(H_x^i) \cap E(H_x^j)| = 0$ for distinct $i, j \in \{0, 1, \dots, n\}$, we have

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n e(H_x^i) + (4\chi_4 + 2\chi_3) \cdot n(C_4) \ge \sum_{w \in V(G)} \binom{d_G(w)}{2}.$$
 (13)

We claim that for any $e \in E(CT_n)$ there are at most two 4-cycles H in C_4 containing esuch that $(H \cap G) - e$ is isomorphic to the graph (5) in Figure 1. Suppose to the contrary that there exist three such 4-cycles in C_4 , say, C_1, C_2 and C_3 . Suppose $e = \{u, z\}$. Since G is C_6 free, we have $(V(C_i) \setminus \{u, z\}) \cap (V(C_j) \setminus \{u, z\}) \neq \emptyset$ for any distinct $i, j \in \{1, 2, 3\}$. Setting $C_1 = (u, z, x_1, x_2, u)$ and $C_2 = (u, z, x_1, x_3, u)$. If $V(C_3) = \{u, z, x_1, x_4\}$, then there are three 4-cycles containing the 2-path (u, z, x_1) , which contradicts Corollary 2.3. If $V(C_3) =$ $\{u, z, x_2, x_3\}$, then there exists a triangle in G, a contradiction. This proves our claim. By double counting the number of pairs (e, H) with $e \in E(CT_n)$ and $H \in C_4$ such that $(G \cap H) - e$ is isomorphic to the graph (5) in Figure 1, we obtain $2e(CT_n) \ge (\chi_3 + 4\chi_4) \cdot n(C_4)$. This together with Corollary 2.4 (iii) implies $\chi_3 + 4\chi_4 \le 2e(CT_n)/n(C_4) = 16/(n-2)(n+1)$. Therefore,

$$2\chi_3 + 4\chi_4 = o(1). \tag{14}$$

Since G is C_6 -free and H_x^i is a subgraph of G_x^i , by Lemma 2.6, H_x^i contains no 3-cycles for any $x \in V(CT_n)$ and $i \in \{0, 1, ..., n\}$. So by Mantel's theorem [23] we have $e(H_x^0) \leq (\binom{n}{2} - d_G(x))^2 / 4$ and $e(H_x^i) \leq (n-1)^2 / 4$ for $i \in \{1, 2, ..., n\}$. Since $|E(H_x^i) \cap E(H_x^j)| = 0$ for distinct $i, j \in \{0, 1, ..., n\}$, we have

$$\sum_{i=0}^{n} e(H_x^i) \le \frac{1}{4} \left(\binom{n}{2} - d_G(x) \right)^2 + \frac{1}{4} n(n-1)^2$$
$$= \frac{1}{4} \left(\binom{n}{2}^2 + n(n-1)^2 - 2\binom{n}{2} d_G(x) + d_G(x)^2 \right).$$

Since $\sum_{x \in V(G)} d_G(x) = 2e(G)$, it follows that

$$\sum_{x \in V(CT_n)} \sum_{i=0}^n e(H_x^i) \le \frac{v}{4} {\binom{n}{2}}^2 - {\binom{n}{2}} e(G) + \frac{vn(n-1)^2}{4} + \frac{1}{4} \sum_{x \in V(CT_n)} d_G(x)^2.$$
(15)

One the other hand, by (13) and (14), we have

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$$\sum_{x \in V(CT_n)} \sum_{i=0}^n e(H_x^i) \ge \sum_{w \in V(G)} {\binom{d_G(w)}{2}} - o(n(C_4)), \qquad (16)$$
$$\ge \frac{1}{2} \sum_{w \in V(G)} d_G(w)^2 - e(G) - o(n(C_4)).$$

Combining (15) with (16), we have

$$\frac{v}{4} \binom{n}{2}^2 - \binom{n}{2} e(G) + \frac{vn(n-1)^2}{4} \ge \frac{1}{4} \sum_{w \in V(G)} d_G(w)^2 - e(G) - o(n(C_4))$$
$$\ge \frac{1}{4v} \left(\sum_{w \in V(G)} d_G(w)\right)^2 - e(G) - o(n(C_4))$$
$$= \frac{e(G)^2}{v} - e(G) - o(n(C_4)).$$

Dividing both sides by $\frac{v}{4} {\binom{n}{2}}^2$, we then obtain

$$1 - \frac{2e(G)}{e(CT_n)} + \frac{4}{n} - \frac{e(G)^2}{e(CT_n)^2} + \frac{2e(G)}{e(CT_n)\binom{n}{2}} + o(1) \ge 0.$$

Recall that $\pi = e(G)/e(CT_n)$. Since $0 < \pi < 1$, $\frac{4}{n} = o(1)$ and $2\pi/\binom{n}{2} = o(1)$, we have $1 - 2\pi - \pi^2 + o(1) \ge 0$, which implies $\pi \le \sqrt{2} - 1 + o(1)$. Therefore, we have $e(G) = \pi e(CT_n) \le (\sqrt{2} - 1 + o(1))e(CT_n)$ as desired in part (iii) of Theorem 1.1.

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References

- N. Alon, A. Krech and T. Szabó, Turán's theorem in hypercube, SIAM J. Discrete Math. 21 (2007) 66–72.
- [2] N. Alon, R. Radoičić, B. Sudakov and J. Vondrák, A Ramsey-type result for the hypercube, J. Graph Theory 53 (2006) 196–208.
- [3] J. Balogh, P. Hu, B. Lidický, H. Liu, Upper bounds on the size of 4- and 6-cycle-free subgraphs of the hypercube, *European J. Combin.* 35 (2014) 75–85.
- [4] J. A. Bondy and M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16(2) (1974) 97–105.
- [5] A. E. Brouwer, A. M. Cohen and A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, Heidelberg, New York, 1989.
- [6] M. Cao, B. Lv and K. Wang, On even-cycle-free subgraphs of the doubled Johnson graphs, arXiv:1907.02725.
- [7] F. Chung, Subgraphs of a hypercube containing no small even cycles, J. Graph Theory 16 (1992) 273–286.
- [8] M. Conder, Hexagon-free subgraphs of hypercubes, J. Graph Theory 17 (1993) 477–479.
- [9] D. Conlon, An extremal theorem in the hypercube, *Electron. J. Combin.* 17 (2010) #R111.

- [10] P. Erdős, On some problems in graph theory, combinatorial analysis and combinatorial number theory, *Graph Theory and Combinatorics (Cambridge, 1983)*, 1–17, Academic Press, London, 1984.
- [11] P. Erdős, Some of my favourite unsolved problems, A Tribute to Paul Erdős, 467–478, Cambridge Univ. Press, Cambridge, 1990.
- [12] P. Erdős and M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966) 51–57.
- [13] P. Erdős and M. Simonovits, Cube-supersaturated graphs and related problems, Progress in Graph Theory (Waterloo, Ont., 1982), 203–218, Academic Press, Toronto, ON, 1984.
- [14] P. Erdős and A. H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946) 1087–1091.
- [15] J. Fox and J. Pach, Separator theorems and Turán-type results for planar intersection graph, Adv. Math. 219(3) (2008) 1070–1080.
- [16] J. Fox, J. Pach, A. Sheffer, A. Suk, and J. Zahl, A semi-algebraic version of Zarankiewiczs problem, J. Eur. Math. Soc. 19(6) (2017) 1785–1810.
- [17] Z. Füredi and L. Ozkahya, On even-cycle-free subgraphs of the hypercube, J. Combin. Theory Ser. A 118 (2011) 1816–1819.
- [18] Z. Füredi and M. Simonovits, The history of degenerate (bipartite) extremal graph problems, *Erdős Centennial*, Springer, Berlin Heidelberg, (2013) 169–264.
- [19] A. Ganesan, Automorphism group of the complete transposition graph, J. Alg. Combin. 42 (2015) 793–801.
- [20] M. C. Heydemann, Cayley graphs and interconnection networks, in: G. Hahn and G. Sabidussi eds., *Graph Symmetry*, 167–224, Kluwer Academic Publishing, Dordrecht, 1997.
- [21] J. S. Jwo, Properties of star graph, bubble-sort graph, prefix-reversal graph and complete-transposition graph, J. Inf. Sci. Eng. 12 (1996) 603–617.
- [22] K. Kalpakis and Y. Yesha, On the bisection width of the transposition network, Networks 29 (1997) 69–76.
- [23] W. Mantel, Problem 28, Wiskundige Opgaven 10 (1907) 60–61.
- [24] L. Stacho and I. Vrt'o, Bisection width of transposition graphs, Discrete Appl. Math. 84 (1998) 211–235.
- [25] A. Thomason and P. Wagner, Bounding the size of square-free subgraphs of the hypercube, *Discrete Math.* 309 (2009) 1730–1735.

- [26] P. Turán, On an extremal problem in graph theory, Mat. Fiz. Lapok 48 (1941) 436–452 (in Hungarian).
- [27] G. L. Wang, H. Z. Shi, F. F. Hou and Y. L. Bai, Some conditional vertex connectivities of complete-transposition graphs, *Inf. Sci.* 295 (2015) 536–543.
- [28] K. Zarankiewicz, Problem of P101, Colloq. Math. 2 (1951) 301.