# Extremal even-cycle-free subgraphs of the complete transposition graphs 

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#### Abstract

Given graphs $G$ and $H$, the generalized Turán number $\operatorname{ex}(G, H)$ is the maximum number of edges in an $H$-free subgraph of $G$. In this paper, we obtain an asymptotic upper bound on $\operatorname{ex}\left(C T_{n}, C_{2 l}\right)$ for any $n \geq 3$ and $l \geq 2$, where $C_{2 l}$ is the cycle of length $2 l$ and $C T_{n}$ is the complete transposition graph which is defined as the Cayley graph on the symmetric group $S_{n}$ with respect to the set of all transpositions of $S_{n}$.


Key words Turán number, even-cycle-free subgraph, complete transposition graph, Ramsey-type problem

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## 1 Introduction

Throughout this paper graphs are finite and undirected with no loops or multiple edges. The vertex and edge sets of a graph $G$ are denoted by $V(G)$ and $E(G)$, respectively. The numbers of vertices and edges of $G$ are denoted by $v(G)$ and $e(G)$, respectively. The degree of a vertex $x \in V(G)$ in $G$ is denoted by $d_{G}(x)$, and the edge joining vertices $u$ and $w$ are denoted as an unordered pair $\{u, w\}$. A cycle with $l$ edges is called an $l$-cycle or a cycle of length $l$, where $l \geq 3$. A path with length $l$ is called an $l$-path, where $l \geq 1$. As usual an $l$-cycle is denoted by $C_{l}$ and an $l$-path by $P_{l}$. Two graphs $G$ and $H$ are said to be isomorphic if there exists a bijection $f$ from $V(G)$ to $V(H)$ such that $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} \in E(H)$.

Let $G$ and $H$ be graphs. We say that $G$ is $H$-free if there exists no subgraph of $G$ which is isomorphic to $H$. The generalized Turán number ex $(G, H)$ is the maximum number of edges in an $H$-free spanning subgraph of $G$. This invariant proposed by Erdős [10] is a generalization of the well-known Turán number ex $(n, H)$ which gives the maximum number of edges in an $H$-free graph with $n$ vertices. In the literature there is a huge amount of work

[^0]on Turán numbers and generalized Turán numbers, beginning with Mantel [23] who proved that ex $\left(n, K_{3}\right)=\left\lfloor n^{2} / 4\right\rfloor$ and Turán [26] who determined ex $\left(n, K_{r}\right)$ for any $r \geq 3$, where $K_{r}$ is the complete graph with $r$ vertices. In [13], Erdős and Simonovits obtained an asymptotic formula for ex $(n, H)$ in terms of the chromatic number of $H$. But when $H$ is bipartite the situation is considerably more complicated, and we can only deduce that ex $(n, H)=o\left(n^{2}\right)$. Herein and in the rest of this paper asymptotics are taken as $n \rightarrow \infty$. In general, it is a challenging problem to determine $\operatorname{ex}(G, H)$ when $H$ is a bipartite graph, especially when $H$ is an even cycle. In this regard, two interesting functions that have received much attention are $\operatorname{ex}\left(G, K_{s, t}\right)$ and $\operatorname{ex}\left(Q_{n}, C_{2 l}\right)$, where $K_{s, t}$ is the complete bipartite graph with $s$ and $t$ vertices, respectively, in the biparts of its bipartition, and $Q_{n}$ is the $n$-dimensional hypercube. The problem of determining ex $\left(K_{m, n}, K_{s, t}\right)$, proposed by Zarankiewicz in [28], is the analogue of Turán's original problem (the one of determining ex $\left.\left(K_{n}, K_{r}\right)=\operatorname{ex}\left(n, K_{r}\right)\right)$ for bipartite graphs, and an excellent survey on this problem can be found in [18]. Besides, some related research was dedicated to showing $\operatorname{ex}\left(G, K_{t, t}\right)$, where $G$ is some other certain restricted graph. See e.g. $[15,16]$.

The study of $\operatorname{ex}\left(Q_{n}, C_{2 l}\right)$ began with a problem raised by Erdős which asks for the maximum number of edges in a $C_{4}$-free spanning subgraph of $Q_{n}$. In [10], Erdős conjectured that $\left(\frac{1}{2}+o(1)\right) e\left(Q_{n}\right)$ should be an upper bound for $\operatorname{ex}\left(Q_{n}, C_{4}\right)$, and he also asked whether $o\left(e\left(Q_{n}\right)\right)$ edges of $Q_{n}$ would ensure the existence of a cycle $C_{2 l}$ for $l \geq 3$. The best known upper bound for ex $\left(Q_{n}, C_{4}\right)$, obtained by Balogn et al. [3] and improved slightly the bounds of Chung [7] and Wagner [25], is $(0.6068+o(1)) e\left(Q_{n}\right)$. The problem of determining the value of ex $\left(Q_{n}, C_{2 l}\right)$ when $l=3$ or 5 is still open too, and progresses can be found in $[1,2,3,7,8]$. For $l \geq 2$, upper bounds for $\operatorname{ex}\left(Q_{n}, C_{4 l}\right)$ and $\operatorname{ex}\left(Q_{n}, C_{4 l+6}\right)$ were obtained by Chung [7] and Füredi and Özkahya [17], respectively, and their results together imply that $\operatorname{ex}\left(Q_{n}, C_{2 l^{\prime}}\right)=o\left(e\left(Q_{n}\right)\right)$ for $l^{\prime} \geq 6$ or $l^{\prime}=4$. In [9], Conlon proved that ex $\left(Q_{n}, H\right)=o\left(e\left(Q_{n}\right)\right)$ for any graph $H$ that admits a $k$-partite representation. This gives a unified approach to the proof that ex $\left(Q_{n}, C_{2 l}\right)=o\left(e\left(Q_{n}\right)\right)$ for all $l \neq 5$ no less than 4 . The doubled Johnson graphs $J(n ; k, k+1)$, where $1 \leq k \leq(n-1) / 2$, form an interesting family of spanning subgraphs of $Q_{n}$, and in particular the doubled odd graph $\widetilde{O}_{k+1}:=J(2 k+1 ; k, k+1)$ is known to be distance-transitive. Recently, Cao et al. [6] studied $\operatorname{ex}\left(J(n ; k, k+1), C_{2 l}\right)$ and proved among other things that ex $\left(\widetilde{O}_{k+1}, C_{2 l}\right)=o\left(e\left(\widetilde{O}_{k+1}\right)\right)$ for $l \geq 6$.

In this paper, we study the generalized Turán number ex $\left(C T_{n}, C_{2 l}\right)$ for the complete transposition graph $C T_{n}$, where $n \geq 3$ and $l \geq 2$. The complete transposition graphs are an important family of Cayley graphs which share several interesting properties with hypercubes. For example, both $C T_{n}$ and $Q_{n}$ are bipartite and arc-transitive, with only integral eigenvalues, and both graphs are popular topologies for interconnection networks [20]. Over the years several aspects of complete transposition graphs such as automorphisms, eigenvalues, connectivity and bisection width have been studied as one can find in, for example, [19, 21, 22, 24, 27]. In general, given a group $G$ with identity element 1 and an inverse-closed subset $S$ of $G \backslash\{1\}$, the Cayley graph $\operatorname{Cay}(G, S)$ on $G$ with respect to the connection set $S$ is defined to be the graph with vertex set $G$ such that $x, y \in G$ are adjacent if and only if $y x^{-1} \in S$. The complete transposition graph $C T_{n}$ is defined as the Cayley graph on the
symmetric group $S_{n}$ whose connection set is the set of all transpositions of $S_{n}$. That is,

$$
V\left(C T_{n}\right)=\mathrm{S}_{n},
$$

$$
E\left(C T_{n}\right)=\left\{\{x, y\}: x, y \in \mathrm{~S}_{n} \text { and } y=u x \text { for some transposition } u \text { of } \mathrm{S}_{n}\right\} .
$$

It follows that $C T_{n}$ is a connected $\binom{n}{2}$-regular bipartite graph with

$$
v:=v\left(C T_{n}\right)=n!
$$

vertices and

$$
e\left(C T_{n}\right)=\frac{v}{2}\binom{n}{2}
$$

edges.
The main result in this paper is as follows.
Theorem 1.1. Let $n$ and $l$ be integers with $n \geq 3$ and $l \geq 2$.
(i) If $l \geq 4$ and $l$ is even, then $\operatorname{ex}\left(C T_{n}, C_{2 l}\right)=O\left(n^{-1+\frac{2}{l}}\right) e\left(C T_{n}\right)$.
(ii) If $l \geq 4$ and $l$ is odd, then

$$
\operatorname{ex}\left(C T_{n}, C_{2 l}\right)= \begin{cases}O\left(n^{-\frac{1}{l}}\right) e\left(C T_{n}\right), & \text { if } l=7, \\ O\left(n^{\left.-\frac{1}{8}+\frac{1}{4(l-3)}\right) e\left(C T_{n}\right),}\right. & \text { otherwise. }\end{cases}
$$

(iii) If $l=3$, then $\operatorname{ex}\left(C T_{n}, C_{2 l}\right) \leq(\sqrt{2}-1+o(1)) e\left(C T_{n}\right)$.
(iv) If $l=2$, then $\operatorname{ex}\left(C T_{n}, C_{2 l}\right) \leq \frac{3}{4} e\left(C T_{n}\right)$.

An immediate consequence of Theorem 1.1 is that $\operatorname{ex}\left(C T_{n}, C_{2 l}\right)=o\left(e\left(C T_{n}\right)\right)$ for $l \geq 4$. This leads to the following Ramsey-type result.

Corollary 1.2. Let $t$ and $l$ be integers with $t \geq 1$ and $l \geq 4$. If $C T_{n}$ is edge-partitioned into $t$ subgraphs, then one of the subgraphs must contain $C_{2 l}$ provided that $n$ is sufficiently large (depending only on $t$ and $l$ ).

In the next section we will prove some basic properties of cycles in the complete transposition graphs. Using these preparations we will prove parts (i)-(ii) and (iii)-(iv) of Theorem 1.1 in Sections 3 and 4, respectively.

## 2 Preliminaries

We assume that $S_{n}$ is the symmetric group on $\{1,2, \ldots, n\}$, where $n \geq 3$. The identity element of $\mathrm{S}_{n}$ is denoted by id. The support of an element $x \in \mathrm{~S}_{n}$ is defined as $\operatorname{supp}(x)=$ $\left\{i \in\{1,2, \ldots, n\} \mid i^{x} \neq i\right\}$.

Definition 2.1. The support of an edge $\{u, z\}$ of $C T_{n}$, denoted by $\operatorname{supp}(\{u, z\})$, is defined to be the support of the transposition $z u^{-1}$. That is, $\operatorname{supp}(\{u, z\})=\operatorname{supp}\left(z u^{-1}\right)$.

Since $C T_{n}$ is a Cayley graph on the symmetric group $\mathrm{S}_{n}$ whose connection set consists of all transpositions, we know that $\operatorname{supp}(\{u, z\})$ is a 2 -subset of $\{1,2, \ldots, n\}$ for any $\{u, z\} \in$ $E\left(C T_{n}\right)$, and $\operatorname{supp}(\{u, z\})=\operatorname{supp}(\{z, u\})$. Note that, for any two incident edges $\{x, u\}$ and $\{x, z\}$ of $C T_{n}$, we have $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})|=0$ or 1 . For any subgraph $H$ of $C T_{n}$, we define

$$
\operatorname{supp}(H):=\bigcup_{\{u, z\} \in E(H)} \operatorname{supp}(\{u, z\})
$$

Let $P=\left(u_{1}, u_{2}, \ldots, u_{t}\right)$ be a path in $C T_{n}$. Setting $w_{i}=u_{i} u_{i-1}^{-1}$ for $i \in\{2,3, \ldots, t\}$, we have $u_{t} u_{1}^{-1}=w_{t} w_{t-1} \cdots w_{3} w_{2}$ and hence $\operatorname{supp}\left(u_{t} u_{1}^{-1}\right) \subseteq \operatorname{supp}(P)$.

Lemma 2.2. Let $g$ and $h$ be distinct transpositions of $\mathrm{S}_{n}$. Then the only 4-cycles in $C T_{n}$ passing through the 2-path ( $g, \mathrm{id}, h$ ) are (id, $g, h g, h, \mathrm{id}$ ) and (id, $g, g h, h, \mathrm{id}$ ). In particular, if $g h=h g$, then these 4-cycles are identical and they are the only 4-cycle in $C T_{n}$ passing through the 2-path ( $g$, id, $h$ ).

Proof. Suppose $g h=h g$. Then $|\operatorname{supp}(g) \cap \operatorname{supp}(h)|=0$. Note that id, $g$ and $h$ are three vertices in $C T_{n}$. Let $w$ be a common neighbor of the vertices $g$ and $h$ in $C T_{n}$. Then there exist transpositions $x, y$ such that $x g=y h=w$, implying that $g h=x y$. Since the supports of $g$ and $h$ are disjoint, the equation $g h=x y$ holds if and only if $g=x$ and $h=y$, or $g=y$ and $h=x$. Therefore, $w$ is either the vertex id or the vertex $g h$. Thus, there exists a unique 4 -cycle in $C T_{n}$ passing through $g$, id and $h$, which is (id, $g, h g=g h, h$, id).

Suppose $g h \neq h g$. Then $|\operatorname{supp}(g) \cap \operatorname{supp}(h)|=1$. Without loss of generality we may assume $g=(1,2)$, and $h=(1,3)$. Let $w$ be a common neighbor of the vertices $g$ and $h$ in $C T_{n}$. Then there exist transpositions $x, y$ such that $x g=y h=w$, implying that $x y=$ $g h=(1,2)(1,3)=(1,2,3)$. If we decompose $(1,2,3)$ into the product of two transpositions of $S_{n}$, then the supports of these two transpositions must lie in $\{1,2,3\}$ and contain exactly one common letter. Therefore, the only ways to decompose $(1,2,3)$ into the product of two transpositions of $\mathrm{S}_{n}$ are $(1,2,3)=(1,3)(2,3)=(2,3)(1,2)=(1,2)(1,3)$. Hence, we have $x=(1,3)$ and $y=(2,3)$, or $x=(2,3)$ and $y=(1,2)$, or $x=(1,2)$ and $y=(1,3)$, yielding $w \in\{\mathrm{id},(1,3,2),(1,2,3)\}$. Therefore, there are exactly two 4 -cycles in $C T_{n}$ passing through $g$, id and $h$, namely (id, $g, h g, h, \mathrm{id}$ ) and (id, $g, g h, h$, id).

It is well known that any permutation in $S_{n}$ can be expressed as a product of transpositions, and for each $g \in \mathrm{~S}_{n}$ the map $\hat{g}: h \mapsto h g, h \in \mathrm{~S}_{n}$ defines an automorphism of $C T_{n}$. Hence Lemma 2.2 implies the following result.

Corollary 2.3. Let $(u, x, z)$ be a 2-path in $C T_{n}$. If $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})|=0$, then there is a unique 4-cycle in $C T_{n}$ containing ( $u, x, z$ ), namely $\left(x, z, z x^{-1} u, u, x\right)$; and if $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})|=1$, then there are exactly two 4 -cycles in $C T_{n}$ containing $(u, x, z)$, namely $\left(x, z, z x^{-1} u, u, x\right)$ and $\left(x, z, u x^{-1} z, u, x\right)$.

Denote by $n\left(C_{4}\right)$ the number of 4-cycles in $C T_{n}$. Lemma 2.2 and Corollary 2.3 together imply the following result.

Corollary 2.4. The following hold.
(i) The length of a shortest cycle in $C T_{n}$ is 4 .
(ii) For any edge $\{u, z\}$ of $C T_{n}$, there are exactly $\frac{1}{2}(n-2)(n+1)$ cycles of length 4 in $C T_{n}$ containing $\{u, z\}$.
(iii) $n\left(C_{4}\right)=\frac{1}{8}(n-2)(n+1) e\left(C T_{n}\right)$.

Proof. Since $C T_{n}$ is bipartite, it does not contain any 3 -cycle. On the other hand, 4 -cycles exist in $C T_{n}$ by Lemma 2.2. So any shortest cycle in $C T_{n}$ has length 4 as stated in (i).

For any edge $\{u, z\}$ of $C T_{n}$, there are exactly $\binom{n-2}{2} 2$-paths $(u, z, w)$ such that $\mid \operatorname{supp}(\{u, z\}) \cap$ $\operatorname{supp}(\{z, w\}) \mid=0$, and there are exactly $n-2$ 2-paths $(u, z, w)$ such that $\mid \operatorname{supp}(\{u, z\}) \cap$ $\operatorname{supp}(\{z, w\}) \mid=1$. Hence, by Corollary 2.3 , the number of 4 -cycles containing any given edge of $C T_{n}$ is equal to $\binom{n-2}{2}+2(n-2)=\frac{1}{2}(n-2)(n+1)$ as claimed in (ii). We obtain (iii) from (ii) immediately.

Let

$$
\mathcal{F}_{0}=\left\{\text { all transpositions of } S_{n}\right\}
$$

and

$$
\mathcal{F}_{i}=\left\{x \in \mathcal{F}_{0} \mid i \in \operatorname{supp}(x)\right\}
$$

for each $i \in\{1,2, \ldots, n\}$. Clearly, in $\mathrm{S}_{n}$ any pair of transpositions with joint supports are contained in one of $\mathcal{F}_{1}, \mathcal{F}_{2}, \ldots, \mathcal{F}_{n}$. In addition, $\mathcal{F}_{1} \cup \cdots \cup \mathcal{F}_{n}$ contains all transpositions of $\mathrm{S}_{n}$ and each transposition of $\mathrm{S}_{n}$ appears exactly three times in $\mathcal{F}_{0} \cup \mathcal{F}_{1} \cup \cdots \cup \mathcal{F}_{n}$.

The following auxiliary graphs will play an important role in our proof of Theorem 1.1.
Definition 2.5. Let $G$ be a spanning subgraph of $C T_{n}$. For each $i \in\{0,1,2, \ldots, n\}$ and each $x \in \mathrm{~S}_{n}$, define $G_{x}^{i}$ to be the graph with vertex set $V\left(G_{x}^{i}\right)=\left\{y x \in \mathrm{~S}_{n} \mid y \in \mathcal{F}_{i}\right\}$ such that for $u, z \in V\left(G_{x}^{i}\right), u$ and $z$ are adjacent if and only if $|\operatorname{supp}(\{x, u\}) \cap \operatorname{supp}(\{x, z\})|=\delta_{i}$ and there exists a vertex $w$ with $w \neq x$ such that $(u, w, z)$ is a 2-path in $G$, where $\delta_{0}=0$ and $\delta_{i}=1$ for $i \in\{1,2, \ldots, n\}$.

By the definition of $G_{x}^{i}$, it is clear that

$$
\begin{equation*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} v\left(G_{x}^{i}\right)=3 v \cdot\binom{n}{2} \tag{1}
\end{equation*}
$$

where as before $v=n!$ is the number of vertices of $C T_{n}$. Since $\left|V\left(G_{x}^{i}\right) \cap V\left(G_{x}^{j}\right)\right|=1$ and $\left|E\left(G_{x}^{0}\right) \cap E\left(G_{x}^{i}\right)\right|=0$ for any $i, j \in\{1,2, \ldots, n\}$ with $i \neq j$, we have $\left|E\left(G_{x}^{i}\right) \cap E\left(G_{x}^{j}\right)\right|=0$ for any $i, j \in\{0,1, \ldots, n\}$ with $i \neq j$. Hence

$$
\begin{equation*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(G_{x}^{i}\right) \geq \sum_{w \in V(G)}\binom{d_{G}(w)}{2} \tag{2}
\end{equation*}
$$

where the right-hand side gives the number of 2-paths in $G$.
Lemma 2.6. Let $G$ be a spanning subgraph of $C T_{n}$. Let $l$ be an integer with $l \geq 3$. If there exists an l-cycle in $G_{x}^{i}$ for some $i \in\{0,1, \ldots, n\}$ and $x \in V\left(C T_{n}\right)$, then there exists a 2l-cycle in $G$.

Proof. Assume that $C=\left(u_{1}, u_{2}, \ldots, u_{l}, u_{l+1}=u_{1}\right)$ is an $l$-cycle in $G_{x}^{i}$. By the definition of $G_{x}^{i}$, for each $j \in\{1,2, \ldots, l\}$ there exists $w_{j} \in V\left(C T_{n}\right)$ such that $\left(u_{j}, w_{j}, u_{j+1}\right)$ is a 2 path in $G$. By Corollary 2.3, we have $w_{j}=u_{j} x^{-1} u_{j+1}=u_{j+1} x^{-1} u_{j}$ if $i=0$, and $w_{j} \in$ $\left\{u_{j} x^{-1} u_{j+1}, u_{j+1} x^{-1} u_{j}\right\}$ if $i \in\{1,2, \ldots, n\}$.

We claim that $w_{1}, w_{2}, \ldots, w_{l}$ are pairwise distinct. Suppose to the contrary that $w_{j}=$ $w_{s}$ with $j<s$. Since there are at most two cycles of length 4 containing $\left(x, u_{j}, w_{j}\right)$, we have $s=j+1$. If $i=0$, then $\left|\operatorname{supp}\left(\left\{x, u_{j}\right\}\right) \cap \operatorname{supp}\left(\left\{x, u_{j+1}\right\}\right)\right|=0$, which implies that $\left(x u_{j}^{-1}\right)\left(u_{j+1} x^{-1}\right)=\left(u_{j+1} x^{-1}\right)\left(x u_{j}^{-1}\right)=u_{j+1} u_{j}^{-1}$ and $w_{j}^{-1} w_{j+1}=u_{j+1}^{-1} x u_{j}^{-1} u_{j+1} x^{-1} u_{j+2}=$ $u_{j}^{-1} u_{j+2} \neq \mathrm{id}$, a contradiction. If $i \in\{1,2, \ldots, n\}$, then

$$
\left\{u_{j} x^{-1} u_{j+1}, u_{j+1} x^{-1} u_{j}\right\} \cap\left\{u_{j+1} x^{-1} u_{j+2}, u_{j+2} x^{-1} u_{j+1}\right\} \neq \emptyset .
$$

Assume that $u_{j}=\left(i, t_{0}\right) x, u_{j+1}=\left(i, t_{1}\right) x$ and $u_{j+2}=\left(i, t_{2}\right) x$, where $t_{0}, t_{1}, t_{2}$ are distinct elements of $\{1,2, \ldots, n\} \backslash\{i\}$. Then

$$
\left\{\left(i, t_{0}, t_{1}\right) x,\left(i, t_{1}, t_{0}\right) x\right\} \cap\left\{\left(i, t_{1}, t_{2}\right) x,\left(i, t_{2}, t_{1}\right) x\right\} \neq \emptyset
$$

which is impossible.
Since $C T_{n}$ is a bipartite graph, we have $\left\{u_{1}, u_{2}, \ldots, u_{l}\right\} \cap\left\{w_{1}, w_{2}, \ldots, w_{l}\right\}=\emptyset$. Since $w_{1}, w_{2}, \ldots, w_{l}$ are pairwise distinct and the 2 -path $\left(u_{j}, w_{j}, u_{j+1}\right)$ is in $G$ for $j \in\{1,2, \ldots, l\}$, it follows that $\left(u_{1}, w_{1}, u_{2}, w_{2}, \ldots, u_{l}, w_{l}, u_{l+1}=u_{1}\right)$ is a $2 l$-cycle in $G$.

## 3 Proof of the main result when $l \geq 4$

We prove parts (i) and (ii) of Theorem 1.1 in this section.

## $3.14 k$-cycle-free subgraphs of $C T_{n}$

Proof of Theorem 1.1 (i). Suppose $G$ is a $C_{4 k}$-free spanning subgraph of $C T_{n}$ with maximum number of edges, where $k \geq 2$. Then $d_{G}(w) \geq 1$ for each $w \in V(G)$. Since $G$ is $C_{4 k}$-free, by Lemma 2.6, $G_{x}^{i}$ is $C_{2 k}$-free for any $x \in V\left(C T_{n}\right)$ and $i \in\{0,1, \ldots, n\}$. Thus from the main theorem in [4] by Bondy and Simonovits it follows that $G_{x}^{i}$ has at most $c_{k}\left(v\left(G_{x}^{i}\right)\right)^{1+\frac{1}{k}}$ edges, where $c_{k}$ is a positive constant relying on $k$ only. Therefore, we have

$$
\begin{equation*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(G_{x}^{i}\right) \leq c_{k} v \cdot\left(\binom{n}{2}^{1+\frac{1}{k}}+n(n-1)^{1+\frac{1}{k}}\right) \leq c_{k}^{\prime} v \cdot\binom{n}{2}^{1+\frac{1}{k}} . \tag{3}
\end{equation*}
$$

On the other hand, by (2) and the Cauchy-Schwarz inequality, we have

$$
\begin{align*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(G_{x}^{i}\right) & \geq \sum_{w \in V(G)}\binom{d_{G}(w)}{2} \\
& =\frac{1}{2} \sum_{w \in V(G)} d_{G}(w)^{2}-\frac{1}{2} \sum_{w \in V(G)} d_{G}(w) \\
& \geq \frac{1}{2 v}\left(\sum_{w \in V(G)} d_{G}(w)\right)^{2}-\frac{1}{2} \sum_{w \in V(G)} d_{G}(w) . \tag{4}
\end{align*}
$$

Since $\sum_{w \in V(G)} d_{G}(w)=2 e(G)$, it follows from (3) and (4) that

$$
\frac{2 e(G)^{2}}{v}-e(G) \leq c_{k}^{\prime} v \cdot\binom{n}{2}^{1+\frac{1}{k}},
$$

or equivalently,

$$
e(G)^{2} \leq \frac{1}{2} c_{k}^{\prime} v^{2} \cdot\binom{n}{2}^{1+\frac{1}{k}}+\frac{e(G) v}{2}
$$

Set $\pi=e(G) / e\left(C T_{n}\right)$. Observe that $0<\pi<1$. Since $e\left(C T_{n}\right)=\frac{v}{2}\binom{n}{2}$, the inequality above yields

$$
\pi^{2} \leq 2 c_{k}^{\prime} \cdot\binom{n}{2}^{-1+\frac{1}{k}}+\pi \cdot\binom{n}{2}^{-1}
$$

So there exists a constant $c$ depending on $k$ such that

$$
\pi \leq c n^{-1+\frac{1}{k}}
$$

Therefore, we have

$$
\operatorname{ex}\left(C T_{n}, C_{4 k}\right)=e(G)=\pi e\left(C T_{n}\right) \leq c n^{-1+\frac{1}{k}} e\left(C T_{n}\right)
$$

as desired in part (i) of Theorem 1.1.

## $3.2(4 k+2)$-cycle-free subgraphs of $C T_{n}$

In this subsection we assume that $G$ is a $C_{4 k+2}$-free spanning subgraph of $C T_{n}$ and $a$ and $b$ are integers with $a, b \geq 2$ such that $4 a+4 b=4 k+4$, where $k \geq 2$. Note that a cycle of length $4 a$ in $G$ can not intersect a cycle of length $4 b$ in $G$ at a single edge, for otherwise their union would contain a cycle of length $4 k+2$. In what follows we will give an upper bound as well as a lower bound on the number of $4 a$-cycles in $G$. These bounds will be used in the proof of part (ii) of Theorem 1.1 at the end of this subsection.

Lemma 3.1. For any $2 l$-cycle $C$ in $C T_{n}$, where $l \geq 2$, we have $|\operatorname{supp}(C)| \leq 2 l$.
Proof. Let $C=\left(u_{0}, u_{1}, u_{2}, \ldots, u_{2 l}=u_{0}\right)$ be a $2 l$-cycle in $C T_{n}$. Set $w_{i}=u_{i} u_{i-1}^{-1}$ for $i \in$ $\{1,2, \ldots, 2 l\}$. Then $\operatorname{supp}\left(\left\{u_{i-1}, u_{i}\right\}\right)=\operatorname{supp}\left(w_{i}\right)$ and $\operatorname{supp}(C)=\cup_{i=1}^{2 l} \operatorname{supp}\left(w_{i}\right)$. Observe that $w_{2 l} w_{2 l-1} \cdots w_{2} w_{1}=$ id. So for any $x \in \operatorname{supp}(C)$ there exist distinct $i, j \in\{1,2, \ldots, 2 l\}$ such that $x \in \operatorname{supp}\left(w_{i}\right) \cap \operatorname{supp}\left(w_{j}\right)$. Hence $|\operatorname{supp}(C)| \leq 2 l$.

Lemma 3.2. Let $C$ and $C^{\prime}$ be cycles of lengths $4 a$ and $4 b$ in $G$, respectively. If $C$ and $C^{\prime}$ have at least one common edge, then $\left|\operatorname{supp}(C) \cap \operatorname{supp}\left(C^{\prime}\right)\right| \geq 3$.

Proof. Suppose $\left\{u_{1}, u_{2}\right\}$ is a common edge of $C$ and $C^{\prime}$. Since $G$ is a ( $4 a+4 b-2$ )-cycle-free subgraph of $C T_{n}$, there exists a vertex $u_{3}$ of $G$ such that $u_{3} \in\left(V(C) \cap V\left(C^{\prime}\right)\right) \backslash\left\{u_{1}, u_{2}\right\}$. Since $\operatorname{supp}\left(u_{3} u_{1}^{-1}\right) \subseteq \operatorname{supp}(C) \cap \operatorname{supp}\left(C^{\prime}\right)$ and $u_{3} \neq u_{2}$, we have

$$
\operatorname{supp}\left(\left\{u_{1}, u_{2}\right\}\right) \neq \operatorname{supp}\left(u_{3} u_{1}^{-1}\right)
$$

This together with $\operatorname{supp}\left(\left\{u_{1}, u_{2}\right\}\right) \subseteq \operatorname{supp}(C) \cap \operatorname{supp}\left(C^{\prime}\right)$ implies that $\left|\operatorname{supp}(C) \cap \operatorname{supp}\left(C^{\prime}\right)\right| \geq$ 3.

For any graphs $H$ and $L$, define $N(H, L)$ to be the number of subgraphs of $H$ which are isomorphic to $L$.

Lemma 3.3. We have

$$
N\left(G, C_{4 a}\right)=O\left(n^{4 a-3}\right) e(G)+O\left(v n^{4 a-1+\frac{1}{b}}\right)
$$

Moreover, if $a=b$, then $N\left(G, C_{4 a}\right)=O\left(n^{4 a-3}\right) e(G)$.
Proof. Denote by $\mathcal{C}$ the set of cycles of length $4 a$ in $G$ and $\mathcal{C}_{e}$ the set of cycles in $\mathcal{C}$ containing a given edge $e$. Note that $|\mathcal{C}|=N\left(G, C_{4 a}\right)$. Let $E=\cup_{C \in \mathbb{C}} E(C)$. Let $E_{1}$ be the set of edges in $E$ that are contained in a cycle of length $4 b$ in $G$, and let $E_{2}:=E \backslash E_{1}$. Then $E=E_{1} \cup E_{2}$ and

$$
\begin{equation*}
4 a N\left(G, C_{4 a}\right)=\sum_{e_{1} \in E_{1}}\left|\mathcal{C}_{e_{1}}\right|+\sum_{e_{2} \in E_{2}}\left|\mathcal{C}_{e_{2}}\right| . \tag{5}
\end{equation*}
$$

Assume that $e=\left\{u_{1}, u_{4 a}\right\}$. Observe that for any $4 a$-cycle $\left(u_{1}, u_{2}, \ldots, u_{4 a}, u_{1}\right)$, there is a unique sequence $\left(A_{1}, A_{2}, \ldots, A_{4 a-1}\right)$ of length $4 a-1$ such that $A_{i}=\operatorname{supp}\left(\left\{u_{i}, u_{i+1}\right\}\right)$ for any $i \in\{1,2, \ldots, 4 a-1\}$. For each $B \in\left\{\operatorname{supp}\left(C^{*}\right) \mid C^{*} \in \mathcal{C}_{e}\right\}$, there are $\binom{|B|}{2}^{4 a-1}$ sequences $\left(A_{1}, A_{2}, \ldots, A_{4 a-1}\right)$ of length $4 a-1$ such that $A_{i} \subseteq B$ and $\left|A_{i}\right|=2$ for each $i \in\{1,2, \ldots, 4 a-1\}$, and hence there are at most $\binom{|B|}{2}^{4 a-1} 4 a$-cycles $C$ containing $e$ such that $\operatorname{supp}(C)=B$.

For each $e_{1} \in E_{1}\left(\right.$ if $\left.E_{1} \neq \emptyset\right)$, let $C^{\prime}$ be a fixed $4 b$-cycle with $e_{1} \in E\left(C^{\prime}\right)$. For any $4 a$-cycle $C^{*} \in \mathcal{C}_{e_{1}}$, we have $\operatorname{supp}\left(e_{1}\right) \subseteq \operatorname{supp}\left(C^{*}\right)$ and $\left|\operatorname{supp}\left(C^{*}\right) \cap \operatorname{supp}\left(C^{\prime}\right)\right| \geq 3$ by Lemma 3.2. Hence, by Lemma 3.1, we have

$$
\left|\left\{\operatorname{supp}\left(C^{*}\right) \mid C^{*} \in \mathcal{C}_{e_{1}}\right\}\right| \leq \sum_{i=1}^{4 a-2}\binom{\left|\operatorname{supp}\left(C^{\prime}\right)\right|-2}{i} \sum_{j=0}^{4 a-2-i}\binom{n-\left|\operatorname{supp}\left(C^{\prime}\right)\right|}{j}
$$

which implies

$$
\begin{align*}
\left|\mathcal{C}_{e_{1}}\right| & \leq \sum_{i=1}^{4 a-2}\binom{\left|\operatorname{supp}\left(C^{\prime}\right)\right|-2}{i} \sum_{j=0}^{4 a-2-i}\binom{n-\left|\operatorname{supp}\left(C^{\prime}\right)\right|}{j}\binom{i+j+2}{2}^{4 a-1} \\
& =O\left(n^{4 a-3}\right) \tag{6}
\end{align*}
$$

For each $e_{2} \in E_{2}\left(\right.$ if $\left.E_{2} \neq \emptyset\right)$, by Lemma 3.1 again, we have

$$
\left|\left\{\operatorname{supp}\left(C^{*}\right) \mid C^{*} \in \mathcal{C}_{e_{2}}\right\}\right| \leq \sum_{i=0}^{4 a-2}\binom{n-2}{i}
$$

which implies

$$
\begin{align*}
\left|\mathcal{C}_{e_{2}}\right| & \leq \sum_{i=0}^{4 a-2}\binom{n-2}{i}\binom{i+2}{2}^{4 a-1} \\
& =O\left(n^{4 a-2}\right) . \tag{7}
\end{align*}
$$

Note that $\left|E_{1}\right| \leq e(G)$. Note also that $\left|E_{2}\right| \leq \operatorname{ex}\left(C T_{n}, C_{4 b}\right)$ as the subgraph induced by $E_{2}$ is $C_{4 b}$-free. Using part (i) of Theorem 1.1 (which has been proved already), we have $\left|E_{2}\right| \leq c n^{-1+1 / b} e\left(C T_{n}\right)$ for some positive constant $c$. Combining (5), (6) and (7), we obtain

$$
\begin{aligned}
N\left(G, C_{4 a}\right) & \leq \frac{1}{4 a}\left(\sum_{e \in E_{1}} O\left(n^{4 a-3}\right)+\sum_{e \in E_{2}} O\left(n^{4 a-2}\right)\right) \\
& \leq O\left(n^{4 a-3}\right) e(G)+O\left(v n^{4 a-1+\frac{1}{b}}\right) .
\end{aligned}
$$

In particular, if $a=b$, then $\left|E_{2}\right|=0$ and hence

$$
N\left(G, C_{4 a}\right) \leq \frac{1}{4 a} \sum_{e \in E_{1}} O\left(n^{4 a-3}\right) \leq O\left(n^{4 a-3}\right) e(G) .
$$

This completes the proof.

Proposition 3.4. (Erdős and Simonovits [13]) Let L be a bipartite graph, where there exist vertices $x$ and $y$ such that $L \backslash\{x, y\}$ is a tree. Then there exist constants $c_{1}, c_{2}>0$ such that if $H$ is a graph containing more than $c_{1} v(H)^{\frac{3}{2}}$ edges, then

$$
N(H, L) \geq c_{2} \frac{e(H)^{e(L)}}{v(H)^{2 e(L)-v(L)}} .
$$

With the help of this proposition and the auxiliary graphs $G_{x}^{i}$ as defined in Definition 2.5, we now prove a lower bound on $N\left(G, C_{4 a}\right)$.

Lemma 3.5. We have

$$
N\left(G, C_{4 a}\right) \geq c v \frac{d^{4 a}}{n^{4 a}}-O\left(v n^{2 a}\right)
$$

for some positive constant $c$ depending on $a$, where $d=2 e(G) / v$.
Proof. By Lemma 2.6, we have

$$
\begin{equation*}
N\left(G, C_{4 a}\right) \geq \sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} N\left(G_{x}^{i}, C_{2 a}\right) . \tag{8}
\end{equation*}
$$

Setting $L=C_{2 a}$ in Proposition 3.4, there exist two positive constants $c_{1}$ and $c_{2}$ such that

$$
N\left(G_{x}^{i}, C_{2 a}\right) \geq c_{2}\left(\frac{e\left(G_{x}^{i}\right)^{2 a}}{v\left(G_{x}^{i}\right)^{2 a}}-\frac{\left(c_{1} v\left(G_{x}^{i}\right)^{3 / 2}\right)^{2 a}}{v\left(G_{x}^{i}\right)^{2 a}}\right) .
$$

Combining this with (8), we obtain

$$
\begin{aligned}
N\left(G, C_{4 a}\right) & \geq \sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} c_{2}\left(\frac{e\left(G_{x}^{i}\right)^{2 a}}{v\left(G_{x}^{i}\right)^{2 a}}-\frac{\left(c_{1} v\left(G_{x}^{i}\right)^{3 / 2}\right)^{2 a}}{v\left(G_{x}^{i}\right)^{2 a}}\right) \\
& \geq \sum_{x \in V\left(C T_{n}\right)}\left(\frac{c_{2} e\left(G_{x}^{0}\right)^{2 a}}{\binom{n}{2}^{2 a}}+\sum_{i=1}^{n} \frac{c_{2} e\left(G_{x}^{i}\right)^{2 a}}{(n-1)^{2 a}}\right)-\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} c_{1}^{2 a} v\left(G_{x}^{i}\right)^{a} .
\end{aligned}
$$

By Hölder's inequality, we then have

$$
\begin{aligned}
N\left(G, C_{4 a}\right) & \geq c_{2} \sum_{x \in V\left(C T_{n}\right)}\left(\frac{e\left(G_{x}^{0}\right)^{2 a}}{\binom{n}{2}^{2 a}}+\frac{\left(\sum_{i=1}^{n} e\left(G_{x}^{i}\right)\right)^{2 a}}{n^{2 a-1}(n-1)^{2 a}}\right)-O\left(v n^{2 a}\right) \\
& \geq \frac{c_{2}}{n^{4 a}} \sum_{x \in V\left(C T_{n}\right)}\left(e\left(G_{x}^{0}\right)^{2 a}+\left(\sum_{i=1}^{n} e\left(G_{x}^{i}\right)\right)^{2 a}\right)-O\left(v n^{2 a}\right) \\
& \geq \frac{c_{a}}{n^{4 a}} \sum_{x \in V\left(C T_{n}\right)}\left(\sum_{i=0}^{n} e\left(G_{x}^{i}\right)\right)^{2 a}-O\left(v n^{2 a}\right) \\
& \geq \frac{c_{a} v}{n^{4 a}}\left(\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} \frac{e\left(G_{x}^{i}\right)}{v}\right)^{2 a}-O\left(v n^{2 a}\right) \\
& \geq \frac{c_{a} v}{n^{4 a}}\left(\sum_{w \in V(G)} \frac{\left(d_{G}(w)\right.}{v}\right)^{2 a}-O\left(v n^{2 a}\right),
\end{aligned}
$$

where $c_{a}$ is a positive constant depending on $a$ and inequality (2) is used in the last step. Setting $d=2 e(G) / v$ and applying Hölder's inequality again, we obtain

$$
\begin{aligned}
N\left(G, C_{4 a}\right) & \geq \frac{c_{a} v}{n^{4 a}}\binom{\sum_{w \in V(G)} \frac{d_{G}(w)}{v}}{2}^{2 a}-O\left(v n^{2 a}\right) \\
& =\frac{c_{a} v}{n^{4 a}}\binom{d}{2}^{2 a}-O\left(v n^{2 a}\right) \\
& \geq c v \frac{d^{4 a}}{n^{4 a}}-O\left(v n^{2 a}\right)
\end{aligned}
$$

for some positive constant $c$ depending on $a$. This completes the proof.

Proof of Theorem 1.1 (ii). Suppose $G$ is a $C_{4 k+2}$-free spanning subgraph of $C T_{n}$ with maximum number of edges. Then $\operatorname{ex}\left(C T_{n}, C_{2 l}\right)=e(G)$, where $l=2 k+1$. Set $d=2 e(G) / v$.

Combining Lemma 3.3 and Lemma 3.5, we have

$$
\begin{aligned}
c v \frac{d^{4 a}}{n^{4 a}} & \leq O\left(n^{4 a-3}\right) e(G)+O\left(v n^{4 a-1+\frac{1}{b}}\right)+O\left(v n^{2 a}\right), \\
d^{4 a} & \leq O\left(n^{8 a-3}\right) d+O\left(n^{8 a-1+\frac{1}{b}}\right)+O\left(n^{6 a}\right) .
\end{aligned}
$$

Hence $d=\max \left\{O\left(n^{2-\frac{1}{4 a-1}}\right), O\left(n^{2-\frac{1-\frac{1}{b}}{4 a}}\right)\right\}$. This bound is minimized when $a=2$ and $b=k-1$, and this choice of $(a, b)$ yields $d=O\left(n^{2-\frac{1}{8}+\frac{1}{8(k-1)}}\right)$. Since $e(G)=v d / 2$ and $e\left(C T_{n}\right)=\frac{v}{2}\binom{n}{2}$, it follows that

$$
\begin{align*}
e(G) & =O\left(v n^{2-\frac{1}{8}+\frac{1}{8(k-1)}}\right) \\
& =O\left(n^{-\frac{1}{8}+\frac{1}{8(k-1)}}\right) e\left(C T_{n}\right) \\
& =O\left(n^{-\frac{1}{8}+\frac{1}{4(l-3)}}\right) e\left(C T_{n}\right) . \tag{9}
\end{align*}
$$

Consider the case when $a=b=(k+1) / 2$ with $k$ odd. By Lemmas 3.3 and 3.5, we have

$$
d^{4 a} \leq O\left(n^{8 a-3}\right) d+O\left(n^{6 a}\right),
$$

which yields

$$
\begin{align*}
e(G) & =O\left(n^{2-\frac{1}{4 a-1}}\right) \\
& =O\left(v n^{2-\frac{1}{2 k+1}}\right) \\
& =O\left(n^{-\frac{1}{2 k+1}}\right) e\left(C T_{n}\right) \\
& =O\left(n^{-\frac{1}{l}}\right) e\left(C T_{n}\right) . \tag{10}
\end{align*}
$$

Observe that when $k$ is odd we have $n^{-\frac{1}{2 k+1}} \leq n^{-\frac{1}{8}+\frac{1}{8(k-1)}}$ if and only if $0<k<4.9$. So (10) is a better bound than (9) when $k=3$. Therefore, $e(G)=O\left(n^{-\frac{1}{l}}\right) e\left(C T_{n}\right)$ when $l=7$. This competes the proof.

So far we have completed the proof of Theorem 1.1 (i) and (ii). These results imply that $\operatorname{ex}\left(C T_{n}, C_{2 l}\right)=o\left(e\left(C T_{n}\right)\right)$ for any fixed positive integer $l \geq 4$. Thus, for any $t \geq 1$ and $l \geq 4$, there exists a positive integer $n(t, l)$ such that for any $n>n(t, l)$ and any edge-coloring of $C T_{n}$ with $t$ colors, $C T_{n}$ contains a monochromatic copy of $C_{2 l}$, as claimed in Corollary 1.2.

Remark. The theta graph $\Theta_{i, j, k}$ is the graph with $i+j+k-1$ vertices which consists of three internally vertex-disjoint paths between the same pair of vertices with lengths $i, j$ and $k$, respectively. As a by-product of the proof of Theorem 1.1 (i) and (ii), we obtain that

$$
\operatorname{ex}\left(C T_{n}, \Theta_{4 a-1,1,4 b-1}\right)=o\left(e\left(C T_{n}\right)\right)
$$

for any $a, b \geq 2$.


Figure 1: Possibilities for $G \cap H$ when $H \in \mathcal{C}_{4}$.

## 4 Proof of the main result when $l=2,3$

In this section, $\mathcal{C}_{4}$ denotes the set of 4 -cycles in $C T_{n}$, and for each $e \in E\left(C T_{n}\right),\left(\complement_{4}\right)_{e}$ denotes the set of 4 -cycles in $C T_{n}$ containing $e$. Suppose $G$ is a $2 l$-cycle-free spanning subgraph of $C T_{n}$ with maximum number of edges. For any subgraphs $H$ and $L$ of $C T_{n}$, let $G \cap H$ be the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$.

Note that for any 4-cycle $H \in \mathcal{C}_{4}, G \cap H$ is isomorphic to one of the six graphs in Figure 1. Denote by $\chi_{0}, \chi_{1}, \chi_{2}^{1}, \chi_{2}^{2}, \chi_{3}, \chi_{4}$ the ratio of the number of 4-cycles $H$ with $G \cap H$ isomorphic to the graphs (1)-(6) in Figure 1 to the total number of 4 -cycles in $C T_{n}$, respectively. Of course we have

$$
\begin{equation*}
\chi_{0}+\chi_{1}+\chi_{2}^{1}+\chi_{2}^{2}+\chi_{3}+\chi_{4}=1 \tag{11}
\end{equation*}
$$

By double counting the cardinality of $\left\{(e, H) \mid H \in \mathcal{C}_{4}, e \in E(G \cap H)\right\}$, we obtain

$$
\sum_{H \in \mathcal{C}_{4}} e(G \cap H)=\sum_{e \in E(G)}\left|\left(\mathfrak{C}_{4}\right)_{e}\right|
$$

which by Corollary 2.4 (ii) implies

$$
\left(\chi_{1}+2\left(\chi_{2}^{1}+\chi_{2}^{2}\right)+3 \chi_{3}+4 \chi_{4}\right) \cdot n\left(C_{4}\right)=e(G) \cdot \frac{1}{2}(n-2)(n+1)
$$

where as before $n\left(C_{4}\right)$ is the number of 4 -cycles in $C T_{n}$. Set $\pi=e(G) / e\left(C T_{n}\right)$. By Corollary 2.4 (iii), we have

$$
\begin{equation*}
\chi_{1}+2\left(\chi_{2}^{1}+\chi_{2}^{2}\right)+3 \chi_{3}+4 \chi_{4}=4 \pi \tag{12}
\end{equation*}
$$

Proof of Theorem 1.1 (iv). Suppose $G$ is a $C_{4}$-free spanning subgraph of $C T_{n}$ with maximum number of edges. Then $d_{G}(w) \geq 1$ for any $w \in V(G)$ and $\chi_{4}=0$ as $G$ is $C_{4}$-free. Hence, by (11) and (12), we have

$$
\pi=\frac{1}{4}\left(\chi_{1}+2\left(\chi_{2}^{1}+\chi_{2}^{2}\right)+3 \chi_{3}\right) \leq \frac{3}{4}\left(\chi_{0}+\chi_{1}+\chi_{2}^{1}+\chi_{2}^{2}+\chi_{3}\right)=\frac{3}{4}
$$

Thus $\operatorname{ex}\left(C T_{n}, C_{4}\right)=e(G)=\pi e\left(C T_{n}\right) \leq \frac{3}{4} e\left(C T_{n}\right)$ as desired in part (iv) of Theorem 1.1.

Proof of Theorem 1.1 (iii). Suppose $G$ is a $C_{6}$-free spanning subgraph of $C T_{n}$ with maximum number of edges. For each $i \in\{0,1,2, \ldots, n\}$ and each $x \in V\left(C T_{n}\right)$, let $H_{x}^{i}$ be the subgraph
of $G_{x}^{i}$ (see Definition 2.5) induced by the subset $\left\{u \in V\left(G_{x}^{i}\right) \mid\{u, x\} \notin E(G)\right\}$ of $V\left(G_{x}^{i}\right)$. Then

$$
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} v\left(H_{x}^{i}\right)=3 \sum_{x \in V\left(C T_{n}\right)}\left(\binom{n}{2}-d_{G}(x)\right) .
$$

Since $\left|E\left(H_{x}^{i}\right) \cap E\left(H_{x}^{j}\right)\right|=0$ for distinct $i, j \in\{0,1, \ldots, n\}$, we have

$$
\begin{equation*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(H_{x}^{i}\right)+\left(4 \chi_{4}+2 \chi_{3}\right) \cdot n\left(C_{4}\right) \geq \sum_{w \in V(G)}\binom{d_{G}(w)}{2} . \tag{13}
\end{equation*}
$$

We claim that for any $e \in E\left(C T_{n}\right)$ there are at most two 4 -cycles $H$ in $\mathcal{C}_{4}$ containing $e$ such that $(H \cap G)-e$ is isomorphic to the graph (5) in Figure 1. Suppose to the contrary that there exist three such 4-cycles in $\mathcal{C}_{4}$, say, $C_{1}, C_{2}$ and $C_{3}$. Suppose $e=\{u, z\}$. Since $G$ is $C_{6}{ }^{-}$ free, we have $\left(V\left(C_{i}\right) \backslash\{u, z\}\right) \cap\left(V\left(C_{j}\right) \backslash\{u, z\}\right) \neq \emptyset$ for any distinct $i, j \in\{1,2,3\}$. Setting $C_{1}=\left(u, z, x_{1}, x_{2}, u\right)$ and $C_{2}=\left(u, z, x_{1}, x_{3}, u\right)$. If $V\left(C_{3}\right)=\left\{u, z, x_{1}, x_{4}\right\}$, then there are three 4 -cycles containing the 2-path $\left(u, z, x_{1}\right)$, which contradicts Corollary 2.3. If $V\left(C_{3}\right)=$ $\left\{u, z, x_{2}, x_{3}\right\}$, then there exists a triangle in $G$, a contradiction. This proves our claim. By double counting the number of pairs $(e, H)$ with $e \in E\left(C T_{n}\right)$ and $H \in \mathcal{C}_{4}$ such that $(G \cap H)-e$ is isomorphic to the graph (5) in Figure 1, we obtain $2 e\left(C T_{n}\right) \geq\left(\chi_{3}+4 \chi_{4}\right) \cdot n\left(C_{4}\right)$. This together with Corollary 2.4 (iii) implies $\chi_{3}+4 \chi_{4} \leq 2 e\left(C T_{n}\right) / n\left(C_{4}\right)=16 /(n-2)(n+1)$. Therefore,

$$
\begin{equation*}
2 \chi_{3}+4 \chi_{4}=o(1) \tag{14}
\end{equation*}
$$

Since $G$ is $C_{6}$-free and $H_{x}^{i}$ is a subgraph of $G_{x}^{i}$, by Lemma 2.6, $H_{x}^{i}$ contains no 3-cycles for any $x \in V\left(C T_{n}\right)$ and $i \in\{0,1, \ldots, n\}$. So by Mantel's theorem [23] we have $e\left(H_{x}^{0}\right) \leq$ $\left.\binom{n}{2}-d_{G}(x)\right)^{2} / 4$ and $e\left(H_{x}^{i}\right) \leq(n-1)^{2} / 4$ for $i \in\{1,2, \ldots, n\}$. Since $\left|E\left(H_{x}^{i}\right) \cap E\left(H_{x}^{j}\right)\right|=0$ for distinct $i, j \in\{0,1, \ldots, n\}$, we have

$$
\begin{aligned}
\sum_{i=0}^{n} e\left(H_{x}^{i}\right) & \leq \frac{1}{4}\left(\binom{n}{2}-d_{G}(x)\right)^{2}+\frac{1}{4} n(n-1)^{2} \\
& =\frac{1}{4}\left(\binom{n}{2}^{2}+n(n-1)^{2}-2\binom{n}{2} d_{G}(x)+d_{G}(x)^{2}\right) .
\end{aligned}
$$

Since $\sum_{x \in V(G)} d_{G}(x)=2 e(G)$, it follows that

$$
\begin{equation*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(H_{x}^{i}\right) \leq \frac{v}{4}\binom{n}{2}^{2}-\binom{n}{2} e(G)+\frac{v n(n-1)^{2}}{4}+\frac{1}{4} \sum_{x \in V\left(C T_{n}\right)} d_{G}(x)^{2} . \tag{15}
\end{equation*}
$$

One the other hand, by (13) and (14), we have

$$
\begin{align*}
\sum_{x \in V\left(C T_{n}\right)} \sum_{i=0}^{n} e\left(H_{x}^{i}\right) & \geq \sum_{w \in V(G)}\binom{d_{G}(w)}{2}-o\left(n\left(C_{4}\right)\right),  \tag{16}\\
& \geq \frac{1}{2} \sum_{w \in V(G)} d_{G}(w)^{2}-e(G)-o\left(n\left(C_{4}\right)\right) .
\end{align*}
$$

Combining (15) with (16), we have

$$
\begin{aligned}
\frac{v}{4}\binom{n}{2}^{2}-\binom{n}{2} e(G)+\frac{v n(n-1)^{2}}{4} & \geq \frac{1}{4} \sum_{w \in V(G)} d_{G}(w)^{2}-e(G)-o\left(n\left(C_{4}\right)\right) \\
& \geq \frac{1}{4 v}\left(\sum_{w \in V(G)} d_{G}(w)\right)^{2}-e(G)-o\left(n\left(C_{4}\right)\right) \\
& =\frac{e(G)^{2}}{v}-e(G)-o\left(n\left(C_{4}\right)\right)
\end{aligned}
$$

Dividing both sides by $\frac{v}{4}\binom{n}{2}^{2}$, we then obtain

$$
1-\frac{2 e(G)}{e\left(C T_{n}\right)}+\frac{4}{n}-\frac{e(G)^{2}}{e\left(C T_{n}\right)^{2}}+\frac{2 e(G)}{e\left(C T_{n}\right)\binom{n}{2}}+o(1) \geq 0
$$

Recall that $\pi=e(G) / e\left(C T_{n}\right)$. Since $0<\pi<1, \frac{4}{n}=o(1)$ and $2 \pi /\binom{n}{2}=o(1)$, we have $1-2 \pi-\pi^{2}+o(1) \geq 0$, which implies $\pi \leq \sqrt{2}-1+o(1)$. Therefore, we have $e(G)=$ $\pi e\left(C T_{n}\right) \leq(\sqrt{2}-1+o(1)) e\left(C T_{n}\right)$ as desired in part (iii) of Theorem 1.1.

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