# AN INVERSE SPECTRAL PROBLEM FOR SECOND-ORDER FUNCTIONAL-DIFFERENTIAL PENCILS WITH TWO DELAYS 

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#### Abstract

We consider a second order functional-differential pencil with two constant delays of the argument and study the inverse problem of recovering its coefficients from the spectra of two boundary value problems with one common boundary condition. The uniqueness theorem is proved and a constructive procedure for solving this inverse problem along with necessary and sufficient conditions for its solvability is obtained. Moreover, we give a survey on the contemporary state of the inverse spectral theory for operators with delay. The pencil under consideration generalizes Sturm-Liouville-type operators with delay, which allows us to illustrate essential results in this direction, including recently solved open questions.


Key words: functional-differential equation, pencil, deviating argument, constant delay, inverse spectral problem

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## 1. Introduction and main results

Recently, there appeared considerable interest in inverse problems of spectral analysis for Sturm-Liouville-type operators with constant delay:

$$
\begin{equation*}
\ell y \equiv-y^{\prime \prime}(x)+q(x) y(x-a)=\lambda y(x), \quad 0<x<\pi, \tag{1}
\end{equation*}
$$

under two-point boundary conditions, see [17], which are often adequate for modelling various real-world processes frequently possessing a nonlocal nature. Here $q(x)$ is a complex-valued function in $L_{2}(a, \pi)$ vanishing on $(0, a)$. In particular, it is well known that specification of the spectra $\left\{\lambda_{n, j}\right\}, \quad j=0,1$, of two boundary value problems for the functional-differential equation (1) with one common boundary condition in zero, say,

$$
\begin{equation*}
y(0)=y^{(j)}(\pi)=0 \tag{2}
\end{equation*}
$$

uniquely determines the potential $q(x)$ as soon as $a \in[\pi / 2, \pi)$. Moreover, the corresponding inverse problem is overdetermined. Thus, in [5], conditions on an arbitrary increasing sequence of natural numbers $\left\{n_{k}\right\}_{k \geq 1}$ were obtained that are necessary and sufficient for the unique determination of $q(x)$ by specifying the corresponding subspectra $\left\{\lambda_{n_{k}, 0}\right\}$ and $\left\{\lambda_{n_{k}, 1}\right\}$.

For a long time, it was an open question whether the uniqueness result for $a \in[\pi / 2, \pi)$ would remain true also for $a \in(0, \pi / 2)$. The positive answer for $a \in[2 \pi / 5, \pi / 2)$ was given independently in [8], and in [10] for the case of the Robin boundary condition in zero:

$$
\begin{equation*}
y^{\prime}(0)-h y(0)=y^{(j)}(\pi)=0 \tag{3}
\end{equation*}
$$

For $a \in[\pi / 3,2 \pi / 5)$, the authors of [9] have shown that the spectra of both problems consisting of (11) and (2) uniquely determine the potential $q(x)$ on $(a, 3 a / 2) \cup(\pi-a / 2, \pi)$. But the strongest uniqueness result under these settings was obtained in [11], where it was proved that $q(x)$ is uniquely determined on the set $(a, 3 a / 2) \cup(\pi-a, 2 a) \cup(\pi-a / 2, \pi)$. Moreover,

[^0]the authors of [11] proved that, for the complete determination of $q(x)$, it is sufficient to additionally specify it on $(3 a / 2, \pi / 2+a / 4)$ as well as its mean value on $(\pi / 2+a / 4, \pi-a)$.

Meanwhile, the recent paper [17] gave a negative answer to the open question formulated above for boundary conditions (2) as soon as $a \in[\pi / 3,2 \pi / 5)$ by constructing a one-parametric infinite family $B$ of different iso-bispectral potentials $q(x)$, i.e. of those for which both boundary value problems possess one and the same pair of spectra. It is also interesting that potentials in $B$ differ on the set $(3 a / 2, \pi-a) \cup(2 a, \pi-a / 2)$ by an arbitrary multiplicative complex constant. Thus, the uniqueness subdomain for $q(x)$ established in [11] is unimprovable.

This appeared quite unexpected taking into account that the paper [16] announced that specification of the spectra of both boundary value problems consisting of (11) and (3) uniquely determines the potential $q(x)$ also for $a \in[\pi / 3,2 \pi / 5)$, see also [13]. Even though, in [16], Robin boundary conditions were imposed also in the point $\pi$, they can be easily reduced to (3).

Moreover, for any fixed $a \in(0, \pi)$, in the papers [2] and [6] for the cases of boundary conditions (2) and (3), respectively, it was established that if the spectra coincide with the ones of the corresponding problems with the zero potential, then $q(x)$ is zero too.

Among numerous studies devoted to inverse spectral problems for functional-differential operators with delay, to the best of our knowledge, 5] remains a sole work dealing with necessary and sufficient conditions of solvability for inverse problems of this class. In particular, from the results in [5] it follows that, unlike the classical case $a=0$, the spectra $\left\{\lambda_{n, 0}\right\}$ and $\left\{\lambda_{n, 1}\right\}$ may have any finite number of common eigenvalues. In Appendix A, we supplement the work [5], by applying the solvability result of the present paper to (1) for $a \in[2 \pi / 5, \pi)$.

There are also works devoted to inverse problems for operators with several delays:

$$
\begin{equation*}
\ell_{m} y \equiv-y^{\prime \prime}(x)+\sum_{\nu=1}^{m} q_{\nu}(x) y\left(x-a_{\nu}\right)=\lambda y(x), \quad 0<x<\pi, \quad m>1, \tag{4}
\end{equation*}
$$

where $q_{\nu}(x)=0$ on $\left(0, a_{\nu}\right)$ for $\nu=\overline{1, n}$, see [18-22]. However, all potentials $q_{\nu}(x)$ cannot be completely determined simultaneously even by specifying arbitrarily many different spectra. This becomes especially obvious when all $a_{\nu}$ are equal, but even for different $a_{\nu}$ there remain subintervals, where the functions $q_{\nu}(x)$ cannot be distinguished (see Appendix B).

An attempt to generalize the results of papers [2, 6] to the operator $\ell_{2}$ was made in [19], where it was claimed that if the spectra of two boundary value problems for one and the same equation (4) for $m=2$ along with the boundary conditions (2) coincide with the analogous two spectra corresponding to the zero potentials, then $q_{1}(x)$ and $q_{2}(x)$ are zeros too. Thereafter, the generalization of this assertion to arbitrary $m>1$ was announced in [20]. But, unfortunately, both papers [19, 20 contain a serious mistake, and these two assertions cannot be true even for different delays (see the counterexample in Appendix B).

Nevertheless, one can completely recover all potentials $q_{\nu}(x)$ by using the spectra of $2 m$ boundary value problems for $m$ different equations specially composed from (4). For example, in [21, 22] for $m=2$ and $a_{1}, a_{2} \in[\pi / 2, \pi)$, it was shown that, for recovering $q_{1}(x)$ and $q_{2}(x)$, it is sufficient to specify the spectra of four boundary value problems for two equations:

$$
-y^{\prime \prime}(x)+q_{1}(x) y\left(x-a_{1}\right)+(-1)^{\nu} q_{2}(x) y\left(x-a_{2}\right)=\lambda y(x), \quad 0<x<\pi, \quad \nu=1,2 .
$$

In the present paper, we show, in particular, that there is no need to use two different equations if one of delayed terms depends on the spectral parameter. Specifically, we consider the functional-differential equation with nonlinear dependence on the spectral parameter $\rho$ :

$$
\begin{equation*}
y^{\prime \prime}(x)+\rho^{2} y(x)=q_{0}(x) y\left(x-a_{0}\right)+2 \rho q_{1}(x) y\left(x-a_{1}\right), \quad 0<x<\pi, \tag{5}
\end{equation*}
$$

which generalizes equation (1). We assume that $a_{0} \in[\pi / 3, \pi), a_{1} \in[\pi / 2, \pi)$ and $a_{0}+a_{1} \geq \pi$. The case $a_{0}+a_{1}<\pi$ requires a separate investigation (see Section 3 for details). For $\nu=0,1$, let $q_{\nu}(x)$ be a complex-valued function in $W_{2}^{\nu}\left[a_{\nu}, \pi\right], q_{\nu}(x)=0$ on $\left(0, a_{\nu}\right)$ and

$$
\begin{equation*}
\int_{a_{1}}^{\pi} q_{1}(x) d x=0 \tag{6}
\end{equation*}
$$

For $j=0,1$, we denote by $\left\{\rho_{n, j}\right\}$ the spectrum of the boundary value problem $\mathcal{L}_{j}:=\mathcal{L}_{j}\left(q_{0}, q_{1}\right)$ that consists of equation (5) along with the boundary conditions (22). We also use one and the same symbol $\left\{\varkappa_{n}\right\}$ for denoting different sequences in $l_{2}$, and put $\mathbb{Z}_{0}:=\mathbb{Z} \backslash\{0\}$ and $\mathbb{Z}_{1}:=\mathbb{Z}$. Under our assumptions, we prove first the following theorem giving asymptotics of the spectra.

Theorem 1. For $j=0,1$ and $n \in \mathbb{Z}_{j}$, the following asymptotics holds:

$$
\begin{equation*}
\rho_{n, j}=\rho_{n, j}^{0}+\frac{\omega}{\pi n} \cos \rho_{n, j}^{0} a_{0}+\frac{\alpha_{j}}{\pi n} \sin \rho_{n, j}^{0} a_{1}+\frac{\varkappa_{n}}{n}, \quad \alpha_{0}, \alpha_{1}, \omega \in \mathbb{C}, \tag{7}
\end{equation*}
$$

where $\rho_{n, j}^{0}=n-j / 2$. Moreover,

$$
\begin{equation*}
\alpha_{j}=\alpha+(-1)^{j} \beta, \quad \alpha=\frac{q_{1}\left(a_{1}\right)}{2}, \quad \beta=\frac{q_{1}(\pi)}{2}, \quad \omega=\frac{1}{2} \int_{a_{0}}^{\pi} q_{0}(x) d x . \tag{8}
\end{equation*}
$$

For $\alpha=0$, this theorem was announced in the conference papers [23, 24], which became the first works dealing with inverse problems for functional-differential pencils in any form.

Consider the following inverse problem.
Inverse Problem 1. Given the spectra $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}, j=0,1$; find $q_{0}(x)$ and $q_{1}(x)$.
We note that for $a_{0}=a_{1}=0$ this inverse problem was studied in [25]. The following theorem gives uniqueness of its solution under our present settings.

Theorem 2. Let both spectra $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}, j=0,1$, be specified. Then the function $q_{0}(x)$ is uniquely determined a.e. on the union of intervals $I_{1}:=\left(a_{0}, 3 a_{0} / 2\right) \cup\left(\pi-a_{0}, 2 a_{0}\right) \cup\left(\pi-a_{0} / 2, \pi\right)$, while the function $q_{1}(x)$ is uniquely determined on the entire segment $\left[a_{1}, \pi\right]$.

In particular, both functions $q_{0}(x)$ and $q_{1}(x)$ are completely determined when $a_{0} \geq 2 \pi / 5$. However, for $a_{0} \in[\pi / 3,2 \pi / 5)$, the complete uniqueness does not take place. For illustrating this, one can use the same one-parametric family of functions $B=\left\{q_{0, \gamma}(x)\right\}_{\gamma \in \mathbb{C}}$ constructed in [17], for which also the problems $\mathcal{L}_{j}\left(q_{0, \gamma}, q_{1}\right), j=0,1$, will possess one and the same pair of spectra for all $\gamma \in \mathbb{C}$. Since, as was already mentioned above, for different values of $\gamma$ the functions $q_{0, \gamma}(x)$ differ precisely on the set $\left(a_{0}, \pi\right) \backslash \operatorname{cl}\left(I_{1}\right)$ except their common zeros, the uniqueness subdomain $I_{1}$ cannot be refined.

However, as in [11], the function $q_{0}(x)$ would be determined uniquely if some a priori information on it were additionally specified. Namely, the following theorem holds.

Theorem 3. Under the hypothesis of Theorem 2, the specification of the function $q_{0}(x)$ on the subinterval ( $3 a_{0} / 2, \pi / 2+a_{0} / 4$ ) along with the value

$$
\omega_{0}:=\int_{\frac{\pi}{2}+\frac{a_{0}}{4}}^{\pi-a_{0}} q_{0}(x) d x .
$$

determines it also on $\left(\pi / 2+a_{0} / 4, \pi-a_{0}\right) \cup\left(2 a_{0}, \pi-a_{0} / 2\right)$. So $q_{0}(x)$ is determined completely.

As in [5], one can show that, in the case $a_{0} \geq 2 \pi / 5$, Inverse Problem 1 is overdetermined, and also describe subspectra, whose specification would uniquely determine the functions $q_{0}(x)$ and $q_{1}(x)$. Some results of this type can be found in [23, 24]. But here we restrict ourself to dealing with the full spectra, which surprisingly does not prevent us from obtaining necessary and sufficient conditions for the solvability of Inverse Problem 1 for $a_{0} \geq 2 \pi / 5$. Besides asymptotics (7), these conditions include some restrictions on the growth of certain entire functions, which makes the inverse problem consistent even in spite of its overdetermination. Specifically, the following theorem holds.

Theorem 4. Let $a_{0} \geq 2 \pi / 5$. Then for any sequences of complex numbers $\left\{\rho_{n, 0}\right\}_{|n| \in \mathbb{N}}$ and $\left\{\rho_{n, 1}\right\}_{n \in \mathbb{Z}}$ to be the spectra of some boundary value problems $\mathcal{L}_{0}\left(q_{0}, q_{1}\right)$ and $\mathcal{L}_{1}\left(q_{0}, q_{1}\right)$, respectively, it is necessary and sufficient to satisfy the following two conditions:
(i) For $j=0,1$, the sequence $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}$ has the form (7);
(ii) For $j, \nu=0,1$, the exponential type of the function $g_{j, \nu}(\rho)$ does not exceed $\pi-a_{\nu}$, where

$$
\begin{gather*}
g_{j, \nu}(\rho)=\theta_{j}(\rho)+(-1)^{j+\nu} \theta_{j}(-\rho),  \tag{9}\\
\theta_{0}(\rho)=\rho^{2} \Delta_{0}(\rho)-\rho \sin \rho \pi+\omega \cos \rho\left(\pi-a_{0}\right)-\alpha_{0} \sin \rho\left(\pi-a_{1}\right),  \tag{10}\\
\theta_{1}(\rho)=\rho \Delta_{1}(\rho)-\rho \cos \rho \pi-\omega \sin \rho\left(\pi-a_{0}\right)-\alpha_{1} \cos \rho\left(\pi-a_{1}\right), \tag{11}
\end{gather*}
$$

while the functions $\Delta_{j}(\rho)$ are determined by the formula

$$
\begin{equation*}
\Delta_{j}(\rho)=\pi^{1-j} \prod_{n \in \mathbb{Z}_{j}} \frac{\rho_{n, j}-\rho}{\rho_{n, j}^{0}} \exp \left(\frac{\rho}{\rho_{n, j}^{0}}\right), \quad j=0,1 \tag{12}
\end{equation*}
$$

For $j=0,1$, the function $\Delta_{j}(\rho)$ determined by (12) is the characteristic function of the problem $\mathcal{L}_{j}$ (see the next section). The proof of Theorem 4 is constructive and gives an algorithm for solving Inverse Problem 1 (Algorithm 1 in Section 6). For proving Theorem 4, we obtain and study a transformation operator associated with equation (1), which allows reducing the inverse problem to the so-called main vectorial integral equation. For applications of the transformation operator approach to other classes of nonlocal operators, see the survey [26].

The paper is organized as follows. In the next section, we construct a transformation operator for the sine-type solution of equation (5), and study the characteristic functions of the problems $\mathcal{L}_{j}$. Therein, we also give the proof of Theorem 1. In Section 3, we derive and study the main equation of Inverse Problem 1. In Section 4, we prove the uniqueness theorems (Theorems 2 and 3). In Section 5, we obtain important representations for the functions determined by (12) with arbitrary complex zeros of the form (7). In Section 6, we prove Theorem 4 and obtain an algorithm for solving the inverse problem. In Appendix A, we provide an ana$\log$ of Theorem 4 for the Sturm-Liouville-type operator $\ell$ with the delay $a \in[2 \pi / 5, \pi)$, i.e. when $q_{1}(x) \equiv 0$. In Appendix B, we give a counterexample, showing that specification of any spectra does not uniquely determine all potentials in equation (4) even for $m=2$ and $a_{1} \neq a_{2}$.

## 2. Transformation operator and characteristic functions

Let $y=S(x, \rho)$ be the sine-type solution of equation (5), i.e. the solution satisfying the initial conditions $S(0, \rho)=0$ and $S^{\prime}(0, \rho)=1$. By virtue of its uniqueness, eigenvalues of the problem $\mathcal{L}_{j}, \quad j=0,1$, coincide with zeros of the entire function

$$
\begin{equation*}
\Delta_{j}(\rho):=S^{(j)}(\pi, \rho), \tag{13}
\end{equation*}
$$

which is called characteristic function of $\mathcal{L}_{j}$. We introduce the designations

$$
Q_{\nu}(x):=\int_{a_{\nu}}^{x} q_{\nu}(t) d t, \quad \nu=0,1, \quad c_{0}(x):=\cos x, \quad c_{1}(x):=\sin x,
$$

where the latter two ones are aimed to be used only occasionally for convenience.
The following lemma gives the transformation operator that connects the solutions $\rho^{-1} \sin \rho x$ and $\cos \rho x$ of the simplest equation (5), possessing zero coefficients, with the solution $S(x, \rho)$.

Lemma 1. The following representation holds:

$$
\begin{equation*}
S(x, \rho)=\frac{\sin \rho x}{\rho}-Q_{1}(x) \frac{\cos \rho\left(x-a_{1}\right)}{\rho}+\sum_{\nu=0}^{1} \int_{a_{\nu}}^{x} K_{\nu}(x, t) \frac{c_{1-\nu}(\rho(x-t))}{\rho} d t, \quad 0 \leq x \leq \pi \tag{14}
\end{equation*}
$$

where $K_{\nu}(x, t)=0$ in the exterior of the triangle $a_{\nu} \leq t \leq x \leq \pi$ as $\nu=0,1$, and

$$
\begin{equation*}
K_{1}(x, t)=\frac{1}{2}\left(q_{1}\left(x-\frac{t-a_{1}}{2}\right)+q_{1}\left(\frac{t+a_{1}}{2}\right)\right), \quad a_{1} \leq t \leq x \leq \pi, \tag{15}
\end{equation*}
$$

while the kernel $K_{0}(x, t)$ satisfies the following integral equation:

$$
\begin{equation*}
K_{0}(x, t)=\frac{1}{2} \int_{\frac{t+a_{0}}{2}}^{x-\frac{t-a_{0}}{2}} q_{0}(\tau) d \tau+\frac{A(x, t)}{2}, \quad a_{0} \leq t \leq x \leq \pi \tag{16}
\end{equation*}
$$

where

$$
A(x, t)=\left\{\begin{array}{l}
0, \quad a_{0} \leq t \leq \min \left\{x, 2 a_{0}\right\}, x \leq \pi,  \tag{17}\\
\int_{t}^{x} q_{0}(\tau) d \tau \int_{a_{0}}^{t-a_{0}} K_{0}\left(\tau-a_{0}, \eta\right) d \eta-\int_{\frac{t}{2}+a_{0}}^{x} q_{0}(\tau) d \tau \int_{a_{0}}^{2(\tau-x)+t-a_{0}} K_{0}\left(\tau-a_{0}, \eta\right) d \eta \\
\quad+\int_{a_{0}}^{t} q_{0}(\tau) d \tau \int_{0}^{2 \tau-t-a_{0}} K_{0}\left(\tau-a_{0}, \eta\right) d \eta, \quad 2 a_{0}<t \leq x \leq \pi
\end{array}\right.
$$

In particular, the following relations hold:

$$
\begin{equation*}
K_{0}(x, x)=0, \quad K_{0}\left(x, a_{0}\right)=\frac{Q_{0}(x)}{2}, \quad K_{1}\left(x, a_{1}\right)=\frac{q_{1}(x)}{2}+\alpha, \quad A\left(\pi, 2 a_{0}\right)=A(\pi, \pi)=0 . \tag{18}
\end{equation*}
$$

Remark 1. Since $\pi \leq 3 a_{0}$, the integration variable $\eta$ in (17) never exceeds $2 a_{0}$. Thus, the function $K_{0}\left(\tau-a_{0}, \eta\right)$ under the integrals in (17) is determined by the formula

$$
\begin{equation*}
K_{0}\left(\tau-a_{0}, \eta\right)=\frac{1}{2} \int_{\frac{\eta+a_{0}}{2}}^{\tau-\frac{\eta+a_{0}}{2}} q_{0}(\zeta) d \zeta, \quad a_{0} \leq \eta \leq \min \left\{\tau-a_{0}, 2 a_{0}\right\}, \quad \tau \leq \pi \tag{19}
\end{equation*}
$$

Proof of Lemma 1. Clearly, the function $S(x, \rho)$ obeys the following integral equation:

$$
\begin{equation*}
S(x, \rho)=\frac{\sin \rho x}{\rho}+\sum_{\nu=0}^{1}(2 \rho)^{\nu} \int_{a_{\nu}}^{x} \frac{\sin \rho(x-t)}{\rho} q_{\nu}(t) S\left(t-a_{\nu}, \rho\right) d t, \quad 0 \leq x \leq \pi . \tag{20}
\end{equation*}
$$

Since $a_{\nu}+a_{1} \geq \pi$ for $\nu=0,1$, substituting (14) into (20), we arrive at the relation

$$
\begin{equation*}
\int_{a_{0}}^{x} K_{0}(x, t) \frac{\sin \rho(x-t)}{\rho} d t+\int_{a_{1}}^{x} K_{1}(x, t) \frac{\cos \rho(x-t)}{\rho} d t-Q_{1}(x) \frac{\cos \rho\left(x-a_{1}\right)}{\rho}=\sum_{j=0}^{2} \mathcal{A}_{j}(x, \rho), \tag{21}
\end{equation*}
$$

where

$$
\begin{gathered}
\mathcal{A}_{\nu}(x, \rho)=2^{\nu} \rho^{\nu-1} \int_{a_{\nu}}^{x} \sin \rho(x-t) q_{\nu}(t) d t \int_{0}^{t-a_{\nu}} \cos \rho \tau d \tau, \quad \nu=0,1, \\
\mathcal{A}_{2}(x, \rho)=\int_{2 a_{0}}^{x} \frac{\sin \rho(x-t)}{\rho} q_{0}(t) d t \int_{a_{0}}^{t-a_{0}} K_{0}\left(t-a_{0}, \tau\right) d \tau \int_{0}^{t-\tau-a_{0}} \cos \rho \xi d \xi
\end{gathered}
$$

Using the formula $2 \sin \rho(x-t) \cos \rho \tau=\sin \rho(x-t+\tau)+\sin \rho(x-t-\tau)$ and changing the integration variables along with its order, we get

$$
\begin{gather*}
\mathcal{A}_{\nu}(x, \rho)=(2 \rho)^{\nu-1} \int_{a_{\nu}}^{x} \sin \rho(x-t) d t \int_{\frac{t+a_{\nu}}{2}}^{x-\frac{t-a_{\nu}}{2}} q_{\nu}(\tau) d \tau, \quad \nu=0,1,  \tag{22}\\
\mathcal{A}_{2}(x, \rho)=\int_{2 a_{0}}^{x} A(x, t) \frac{\sin \rho(x-t)}{2 \rho} d t \tag{23}
\end{gather*}
$$

where $A(x, t)$ is determined by formula (17). Moreover, integration by parts gives

$$
\begin{equation*}
\mathcal{A}_{1}(x, \rho)=-Q_{1}(x) \frac{\cos \rho\left(x-a_{1}\right)}{\rho}+\int_{a_{1}}^{x}\left(q_{1}\left(x-\frac{t-a_{1}}{2}\right)+q_{1}\left(\frac{t+a_{1}}{2}\right)\right) \frac{\cos \rho(x-t)}{2 \rho} d t . \tag{24}
\end{equation*}
$$

Substituting (22) for $\nu=0$ as well as (23) and (24) into (21), we arrive at (14)-(17). Relations (18) are obvious.

The next lemma gives fundamental representations for the characteristic functions.
Lemma 2. The following representations hold:

$$
\begin{align*}
& \Delta_{0}(\rho)=\frac{\sin \rho \pi}{\rho}-\omega \frac{\cos \rho\left(\pi-a_{0}\right)}{\rho^{2}}+\alpha_{0} \frac{\sin \rho\left(\pi-a_{1}\right)}{\rho^{2}}+\sum_{\nu=0}^{1} \int_{0}^{\pi-a_{\nu}} w_{0, \nu}(x) \frac{c_{\nu}(\rho x)}{\rho^{2}} d x  \tag{25}\\
& \Delta_{1}(\rho)=\cos \rho \pi+\omega \frac{\sin \rho\left(\pi-a_{0}\right)}{\rho}+\alpha_{1} \frac{\cos \rho\left(\pi-a_{1}\right)}{\rho}+\sum_{\nu=0}^{1} \int_{0}^{\pi-a_{\nu}} w_{1, \nu}(x) \frac{c_{1-\nu}(\rho x)}{\rho} d x \tag{26}
\end{align*}
$$

where $w_{j, \nu}(x) \in L_{2}\left(0, \pi-a_{\nu}\right), \quad j, \nu=0,1$, and

$$
\begin{equation*}
\int_{0}^{\pi-a_{0}} w_{0,0}(x) d x=\omega, \quad \int_{0}^{\pi-a_{1}} x w_{0,1}(x) d x=\alpha_{0}\left(a_{1}-\pi\right), \quad \int_{0}^{\pi-a_{1}} w_{1,1}(x) d x=-\alpha_{1} . \tag{27}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
w_{0, \nu}(x)=(-1)^{\nu+1} K_{\nu, 2}(\pi, \pi-x), \quad w_{1, \nu}(x)=P_{\nu}(\pi, \pi-x), \quad \nu=0,1, \tag{28}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{\nu, 1}(x, t):=\frac{\partial}{\partial x} K_{\nu}(x, t), \quad K_{\nu, 2}(x, t):=\frac{\partial}{\partial t} K_{\nu}(x, t), \quad P_{\nu}(x, t):=K_{\nu, 1}(x, t)+K_{\nu, 2}(x, t) . \tag{29}
\end{equation*}
$$

Proof. Integrating by parts in (14) with account of (18), and recalling (6), (8) along with (13), we get (25). Differentiating (14) with respect to $x$ and then using integration by parts, we obtain (26). Finally, although relations (27) can be established by direct calculations, we accept them just as a simple corollary from entireness of the functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$.

Now we are in position to give the proof of Theorem 1.
Proof of Theorem 1. By the standard approach involving Rouché's theorem (see, e.g., [27]), using representation (25), one can show that the function $\Delta_{0}(\lambda)$ has infinitely many zeros of the form $\rho_{n, 0}=n+\varepsilon_{n, 0}$, where $|n| \in \mathbb{N}$, while $\varepsilon_{n, 0} \rightarrow 0$ as $|n| \rightarrow \infty$. Substituting this representation into (25), we obtain

$$
\begin{equation*}
\sin \rho_{n, 0} \pi=\frac{\omega}{n} \cos \left(n+\varepsilon_{n, 0}\right)\left(\pi-a_{0}\right)-\frac{\alpha_{0}}{n} \sin \left(n+\varepsilon_{n, 0}\right)\left(\pi-a_{1}\right)+\frac{\varkappa_{n}}{n} . \tag{30}
\end{equation*}
$$

Since $\sin \rho_{n, 0} \pi=\sin \left(n+\varepsilon_{n, 0}\right) \pi=(-1)^{n} \varepsilon_{n, 0} \pi+O\left(\varepsilon_{n, 0}^{3}\right)$ as $|n| \rightarrow \infty$, we refine $\varepsilon_{n, 0}=O\left(n^{-1}\right)$ for $|n| \rightarrow \infty$. Hence, we have the asymptotic formulae

$$
\begin{gathered}
\sin \rho_{n, 0} \pi=(-1)^{n} \varepsilon_{n, 0} \pi+O\left(\frac{1}{n^{3}}\right) \\
\cos \left(n+\varepsilon_{n, 0}\right)\left(\pi-a_{0}\right)=(-1)^{n} \cos n a_{0}+O\left(\frac{1}{n}\right), \\
\sin \left(n+\varepsilon_{n, 0}\right)\left(\pi-a_{1}\right)=(-1)^{n+1} \sin n a_{1}+O\left(\frac{1}{n}\right)
\end{gathered}
$$

as soon as $|n| \rightarrow \infty$. Substituting them into (30), we arrive at

$$
\varepsilon_{n, 0}=\frac{\omega}{\pi n} \cos n a_{0}+\frac{\alpha_{0}}{\pi n} \sin n a_{1}+\frac{\varkappa_{n}}{n},
$$

which implies (7) for $j=0$.
Analogously, applying Rouchés theorem to representation (26), we get $\rho_{n, 1}=\rho_{n, 1}^{0}+\varepsilon_{n, 1}$, where $n \in \mathbb{Z}$, while $\varepsilon_{n, 1} \rightarrow 0$ as $|n| \rightarrow \infty$. Substituting this into (26), we obtain

$$
\begin{equation*}
\cos \rho_{n, 1} \pi=-\frac{\omega}{n} \sin \left(\rho_{n, 1}^{0}+\varepsilon_{n, 1}\right)\left(\pi-a_{0}\right)-\frac{\alpha_{1}}{n} \cos \left(\rho_{n, 1}^{0}+\varepsilon_{n, 1}\right)\left(\pi-a_{1}\right)+\frac{\varkappa_{n}}{n} . \tag{31}
\end{equation*}
$$

Since $\cos \rho_{n, 1} \pi=\cos \left(\rho_{n, 1}^{0}+\varepsilon_{n, 1}\right) \pi=(-1)^{n} \varepsilon_{n, 1} \pi+O\left(\varepsilon_{n, 1}^{3}\right)$ as $|n| \rightarrow \infty$, we refine $\varepsilon_{n, 1}=O\left(n^{-1}\right)$ for $|n| \rightarrow \infty$. Then substituting the asymptotic formulae

$$
\begin{aligned}
& \sin \left(\rho_{n, 1}^{0}+\varepsilon_{n, 1}\right)\left(\pi-a_{0}\right)=(-1)^{n+1} \cos \rho_{n, 1}^{0} a_{0}+O\left(\frac{1}{n}\right), \\
& \cos \left(\rho_{n, 1}^{0}+\varepsilon_{n, 1}\right)\left(\pi-a_{1}\right)=(-1)^{n+1} \sin \rho_{n, 1}^{0} a_{1}+O\left(\frac{1}{n}\right),
\end{aligned}
$$

for $|n| \rightarrow \infty$ into (31), we arrive at

$$
\varepsilon_{n, 1}=\frac{\omega}{\pi n} \cos \rho_{n, 1}^{0} a_{0}+\frac{\alpha_{1}}{\pi n} \sin \rho_{n, 1}^{0} a_{1}+\frac{\varkappa_{n}}{n},
$$

which implies (7) for $j=1$.

Finally, we obtain formulae for recovering the characteristic functions from their zeros.
Lemma 3. For any $a_{0}, a_{1} \in[0,2 \pi]$, each function $\Delta_{0}(\rho)$ and $\Delta_{1}(\rho)$ of the form described in (25)-(27) is determined by its zeros uniquely. Moreover, representation (12) holds.

Proof. By virtue of Hadamard's factorization theorem (see, e.g., [28]), we get

$$
\begin{equation*}
\Delta_{j}(\rho)=C_{j} \rho^{s_{j}} \exp \left(b_{j} \rho\right) \prod_{\rho_{n, j} \neq 0}\left(1-\frac{\rho}{\rho_{n, j}}\right) \exp \left(\frac{\rho}{\rho_{n, j}}\right), \quad j=0,1, \tag{32}
\end{equation*}
$$

where $C_{j}$ and $b_{j}$ are some constants, while $s_{j}$ is the multiplicity of the null zero $\rho_{n, j}=0$. In particular, we have

$$
\begin{equation*}
\rho^{j-1} c_{1-j}(\rho \pi)=\pi^{1-j} \prod_{n \in \mathbb{Z}_{j}}\left(1-\frac{\rho}{\rho_{n, j}^{0}}\right) \exp \left(\frac{\rho}{\rho_{n, j}^{0}}\right), \quad j=0,1 . \tag{33}
\end{equation*}
$$

Dividing (32) by (33), we obtain

$$
\begin{align*}
\frac{\rho^{1-j} \Delta_{j}(\rho)}{c_{1-j}(\rho \pi)}=\frac{C_{j}}{\pi^{1-j}} \exp & \left(\left(b_{j}+\sum_{\rho_{n, j} \neq 0}\left(\frac{1}{\rho_{n, j}}-\frac{1}{\rho_{n, j}^{0}}\right)-\sum_{\rho_{n, j}=0} \frac{1}{\rho_{n, j}^{0}}\right) \rho\right) \\
& \times \prod_{\rho_{n, j}=0}\left(\frac{1}{\rho}-\frac{1}{\rho_{n, j}^{0}}\right)^{-1} \prod_{\rho_{n, j} \neq 0} \frac{\rho_{n, j}^{0}}{\rho_{n, j}} \prod_{\rho_{n, j} \neq 0} \frac{\rho_{n, j}-\rho}{\rho_{n, j}^{0}-\rho}, \quad j=0,1 . \tag{34}
\end{align*}
$$

On the other hand, (25) and (26) imply $\rho^{1-j}\left(c_{1-j}(\rho \pi)\right)^{-1} \Delta_{j}(\rho) \rightarrow 1$ for $j=0,1$ as $\rho^{2} \rightarrow-\infty$, which along with (34) gives

$$
C_{j}=\pi^{1-j}(-1)^{s_{j}} \prod_{\rho_{n, j}=0} \frac{1}{\rho_{n, j}^{0}} \prod_{\rho_{n, j} \neq 0} \frac{\rho_{n, j}}{\rho_{n, j}^{0}}, \quad b_{j}=\sum_{\rho_{n, j}=0} \frac{1}{\rho_{n, j}^{0}}+\sum_{\rho_{n, j} \neq 0}\left(\frac{1}{\rho_{n, j}^{0}}-\frac{1}{\rho_{n, j}}\right) .
$$

Substituting this into (32), we arrive at (12).

## 3. Main equation of the inverse problem

The relations in (28) can be considered as a system of equations with respect to the functions $q_{0}(x)$ and $p(x):=q_{1}^{\prime}(x)$, which we refer to as main (vectorial) equation of Inverse Problems 1. By virtue of our standing assumption $a_{0}+a_{1} \geq \pi$, for each $\nu \in\{0,1\}$, the functions $K_{\nu, 2}(x, t)$ and $P_{\nu}(x, t)$ depend only on $q_{\nu}(x)$, while for $\nu=1$ they depend even only on $p(x)$. Hence, the main equation can be splitted into two independent subsystems for $\nu=0$ and $\nu=1$ :

$$
\begin{gather*}
w_{0,0}(x)=-K_{0,2}\left(\pi, \pi-x ; q_{0}\right), \quad w_{1,0}(x)=P_{0}\left(\pi, \pi-x ; q_{0}\right)  \tag{35}\\
w_{0,1}(x)=K_{1,2}(\pi, \pi-x ; p), \quad w_{1,1}(x)=P_{1}(\pi, \pi-x ; p) \tag{36}
\end{gather*}
$$

respectively. Here and below, in order to emphasize dependence of a certain function $F\left(x_{1}, x_{2}\right)$ on some function $f(x)$, sometimes we write $F\left(x_{1}, x_{2} ; f\right)$.

According to (15)-(17) and (29), the subsystem (35) is nonlinear when $a_{0} \in[\pi / 3, \pi / 2)$, while the subsystem (36) is always linear because $a_{1} \geq \pi / 2$.

Consider first the linear subsystem (36). By virtue of (15) and (29), we get

$$
K_{1,1}(x, t)=\frac{1}{2} p\left(x-\frac{t-a_{1}}{2}\right), \quad K_{1,2}(x, t)=\frac{1}{4}\left(p\left(\frac{t+a_{1}}{2}\right)-p\left(x-\frac{t-a_{1}}{2}\right)\right)
$$

$$
P_{1}(x, t)=\frac{1}{4}\left(p\left(\frac{t+a_{1}}{2}\right)+p\left(x-\frac{t-a_{1}}{2}\right)\right), \quad a_{1}<t<x<\pi .
$$

Thus, the subsystem (36) is equivalent to the system

$$
w_{j, 1}(x)=\frac{1}{4}\left(p\left(\frac{\pi+a_{1}-x}{2}\right)-(-1)^{j} p\left(\frac{\pi+a_{1}+x}{2}\right)\right), \quad 0<x<\pi-a_{1}, \quad j=0,1 .
$$

Solving this linear system, we get

$$
p\left(\frac{\pi+a_{1}-x}{2}\right)=2\left(w_{0,1}+w_{1,1}\right)(x), \quad p\left(\frac{\pi+a_{1}+x}{2}\right)=2\left(w_{1,1}-w_{0,1}\right)(x), \quad x \in\left(0, \pi-a_{1}\right),
$$

or, after changing the variable, we have

$$
p(x)=2 \begin{cases}\left(w_{1,1}+w_{0,1}\right)\left(\pi+a_{1}-2 x\right), & a_{1}<x<\frac{a_{1}+\pi}{2}  \tag{37}\\ \left(w_{1,1}-w_{0,1}\right)\left(2 x-\pi-a_{1}\right), & \frac{a_{1}+\pi}{2}<x<\pi\end{cases}
$$

Thus, we arrive at the following theorem.
Theorem 5. Let $a_{1} \in[\pi / 2, \pi)$. Then for any functions $w_{0,1}(x), w_{1,1}(x) \in L_{2}\left(0, \pi-a_{1}\right)$ the linear subsystem (36) has a unique solution $p(x) \in L_{2}\left(a_{1}, \pi\right)$, which can be constructed by formula (37). Moreover,

$$
\begin{equation*}
\int_{a_{1}}^{\pi} p(x) d x=2 \int_{0}^{\pi-a_{1}} w_{1,1}(x) d x, \quad \int_{a_{1}}^{\pi} x p(x) d x=\frac{\pi+a_{1}}{2} \int_{a_{1}}^{\pi} p(x) d x-\int_{0}^{\pi-a_{1}} x w_{0,1}(x) d x \tag{38}
\end{equation*}
$$

Proof. It remains to prove (38). Changing the integration variable, one can easily obtain the following relations for any integrable function $f(x)$ :

$$
\begin{equation*}
\int_{a}^{\frac{\pi+a}{2}} f(\pi+a-2 x) d x=\frac{1}{2} \int_{0}^{\pi-a} f(x) d x, \int_{\frac{\pi+a}{2}}^{\pi} f(2 x-\pi-a) d x=\frac{1}{2} \int_{0}^{\pi-a} f(x) d x \tag{39}
\end{equation*}
$$

which along with (37) give the first relation in (38). Analogously, using (37) and the relations

$$
\int_{a}^{\frac{\pi+a}{2}} x f(\pi+a-2 x) d x=\int_{0}^{\pi-a} \frac{\pi+a-x}{4} f(x) d x, \int_{\frac{\pi+a}{2}}^{\pi} x f(2 x-\pi-a) d x=\int_{0}^{\pi-a} \frac{\pi+a+x}{4} f(x) d x
$$

one can obtain the second identity in (38).
Further, differentiating (16) and (17), and taking (29) into account, we get

$$
K_{0, l}(x, t)=\frac{1}{2}\left\{\begin{align*}
q_{0}\left(x-\frac{t-a_{0}}{2}\right)+\frac{\partial}{\partial x} A(x, t), & l=1  \tag{40}\\
-\frac{1}{2}\left(q_{0}\left(\frac{t+a_{0}}{2}\right)+q_{0}\left(x-\frac{t-a_{0}}{2}\right)\right)+\frac{\partial}{\partial t} A(x, t), & l=2
\end{align*}\right.
$$

where

$$
\frac{\partial}{\partial x} A(x, t)=\left\{\begin{array}{l}
0, \quad a_{0} \leq t \leq \min \left\{x, 2 a_{0}\right\}, \quad x \leq \pi  \tag{41}\\
2 \int_{x-\frac{t}{2}+a_{0}}^{x} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2(\tau-x)+t-a_{0}\right) d \tau, \quad 2 a_{0}<t \leq x \leq \pi
\end{array}\right.
$$

and

$$
\frac{\partial}{\partial t} A(x, t)=\left\{\begin{array}{l}
0, \quad a_{0} \leq t \leq \min \left\{x, 2 a_{0}\right\}, \quad x \leq \pi  \tag{42}\\
\int_{t}^{x} q_{0}(\tau) K_{0}\left(\tau-a_{0}, t-a_{0}\right) d \tau \\
-\int_{x-\frac{t}{2}+a_{0}}^{x} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2(\tau-x)+t-a_{0}\right) d \tau \\
\quad-\int_{\frac{t}{2}+a_{0}}^{t} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2 \tau-t-a_{0}\right) d \tau, \quad 2 a_{0}<t \leq x \leq \pi
\end{array}\right.
$$

By virtue of (29) and (40), we get

$$
\begin{equation*}
P_{0}(x, t)=\frac{1}{4}\left(q_{0}\left(x-\frac{t-a_{0}}{2}\right)-q_{0}\left(\frac{t+a_{0}}{2}\right)\right)+B(x, t), \tag{43}
\end{equation*}
$$

where

$$
B(x, t)=\frac{1}{2}\left(\frac{\partial}{\partial x} A(x, t)+\frac{\partial}{\partial t} A(x, t)\right) .
$$

Then, summing up (41) and (42) and dividing by 2 , we arrive at

$$
B(x, t)=\frac{1}{2}\left\{\begin{array}{l}
0, \quad a_{0} \leq t \leq \min \left\{x, 2 a_{0}\right\}, \quad x \leq \pi  \tag{44}\\
\int_{t}^{x} q_{0}(\tau) K_{0}\left(\tau-a_{0}, t-a_{0}\right) d \tau \\
\quad+\int_{x-\frac{t}{2}+a_{0}}^{x} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2(\tau-x)+t-a_{0}\right) d \tau \\
\quad-\int_{\frac{t}{2}+a_{0}}^{t} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2 \tau-t-a_{0}\right) d \tau, \quad 2 a_{0}<t \leq x \leq \pi
\end{array}\right.
$$

According to formula (40) for $l=2$ along with (43), the subsystem (35) takes the form

$$
\begin{equation*}
w_{j, 0}(x)=\frac{1}{4}\left(q_{0}\left(\frac{\pi+x+a_{0}}{2}\right)+(-1)^{j} q_{0}\left(\frac{\pi-x+a_{0}}{2}\right)\right)+u_{j}(x), \quad 0 \leq x \leq \pi-a_{0}, \quad j=0,1, \tag{45}
\end{equation*}
$$

where

$$
\begin{equation*}
u_{0}(x)=-\left.\frac{1}{2} \frac{\partial}{\partial t} A(\pi, t)\right|_{t=\pi-x}=\frac{1}{2} \frac{d}{d x} A(\pi, \pi-x), \quad u_{1}(x)=B(\pi, \pi-x) \tag{46}
\end{equation*}
$$

Thus, by virtue of (42) and (44), we get

$$
u_{j}(x)=\frac{1}{2}\left\{\begin{array}{l}
(-1)^{j+1} \int_{\pi-x}^{\pi} q_{0}(\tau) K_{0}\left(\tau-a_{0}, \pi-x-a_{0}\right) d \tau  \tag{47}\\
+\int_{\pi_{0}}^{\frac{\pi+x}{2}+a_{0}} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2 \tau-x-\pi-a_{0}\right) d \tau \\
+(-1)^{j} \int_{\substack{\pi-x}}^{\pi-x+a_{0}} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2 \tau+x-\pi-a_{0}\right) d \tau, \quad 0 \leq x<\pi-2 a_{0}, \\
0, \quad \pi-2 a_{0} \leq x \leq \pi-a_{0},
\end{array}\right.
$$

Transforming (45) and taking (47) into account, we obtain

$$
2\left(w_{0,0}+(-1)^{j} w_{1,0}\right)(x)=q_{0}\left(\frac{\pi+a_{0}+(-1)^{j} x}{2}\right)+\left\{\begin{array}{l}
2\left(u_{0}+(-1)^{j} u_{1}\right)(x), \quad 0 \leq x<\pi-2 a_{0}, \\
0, \quad \pi-2 a_{0} \leq x \leq \pi-a_{0}
\end{array}\right.
$$

for $j=0,1$. After changing the variables, we get

$$
\begin{align*}
& 2\left(w_{0,0}-w_{1,0}\right)\left(\pi+a_{0}-2 x\right)=\left\{\begin{array}{l}
q_{0}(x), \quad a_{0} \leq x \leq \frac{3 a_{0}}{2}, \\
q_{0}(x)+2 v(x), \quad \frac{3 a_{0}}{2}<x \leq \frac{a_{0}+\pi}{2},
\end{array}\right.  \tag{48}\\
& 2\left(w_{0,0}+w_{1,0}\right)\left(2 x-\pi-a_{0}\right)=\left\{\begin{array}{l}
q_{0}(x)+2 v(x), \quad \frac{a_{0}+\pi}{2} \leq x<\pi-\frac{a_{0}}{2}, \\
q_{0}(x), \quad \pi-\frac{a_{0}}{2} \leq x \leq \pi,
\end{array}\right. \tag{49}
\end{align*}
$$

where (note that $\left.u_{1}(0)=0\right)$

$$
v(x)= \begin{cases}\left(u_{0}-u_{1}\right)\left(\pi+a_{0}-2 x\right), & \frac{3 a_{0}}{2}<x \leq \frac{a_{0}+\pi}{2}  \tag{50}\\ \left(u_{0}+u_{1}\right)\left(2 x-\pi-a_{0}\right), & \frac{a_{0}+\pi}{2}<x<\pi-\frac{a_{0}}{2}\end{cases}
$$

Obviously, formulae (48) and (49) immediately give the solution $q_{0}(x)$ of the subsystem (35) on $I_{2}:=\left[a_{0}, 3 a_{0} / 2\right] \cup\left[\pi-a_{0} / 2, \pi\right]$. For $a_{0} \geq \pi / 2$, we have $\left[a_{0}, \pi\right] \subset I_{2}$ and, hence, the function $q_{0}(x)$ is completely obtained. For $a_{0}<\pi / 2$, the subsystem (35) becomes nonlinear, and its solvability on $I_{3}:=\left(3 a_{0} / 2, \pi-a_{0} / 2\right)$ is conditioned by the following lemma.

Lemma 4. Let $v(x)$ be determined on the interval $I_{3}$ by formula (50) with $u_{0}(x)$ and $u_{1}(x)$ constructed in $(47)$. Then $\left.v\right|_{D}$ does not depend on $\left.q_{0}\right|_{I_{3}}$ if and only if $D \subset\left[\pi-a_{0}, 2 a_{0}\right]$.

Here and below $\left.f\right|_{\mathcal{S}}$ denotes the restriction of the function $f$ to the set $\mathcal{S}$.
Proof. Substituting (47) into (50), we obtain the formulae

$$
\begin{align*}
v(x)= & \int_{x+\frac{a_{0}}{2}}^{2 x-a_{0}} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2(\tau-x)\right) d \tau \\
& \quad-\int_{2 x-a_{0}}^{\pi} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2\left(x-a_{0}\right)\right) d \tau, \quad \frac{3 a_{0}}{2}<x \leq \frac{a_{0}+\pi}{2}, \tag{51}
\end{align*}
$$

and

$$
\begin{equation*}
v(x)=\int_{x+\frac{a_{0}}{2}}^{\pi} q_{0}(\tau) K_{0}\left(\tau-a_{0}, 2(\tau-x)\right) d \tau, \quad \frac{a_{0}+\pi}{2}<x<\pi-\frac{a_{0}}{2} . \tag{52}
\end{equation*}
$$

where, according to Remark 1, the function $K_{0}\left(\tau-a_{0}, \cdot\right)$ is determined by (19), i.e.

$$
K_{0}\left(\tau-a_{0}, 2(\tau-x)\right)=-K_{0}\left(\tau-a_{0}, 2\left(x-a_{0}\right)\right)=\frac{1}{2} \int_{\tau-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}} q_{0}(\zeta) d \zeta
$$

Substituting this into (51) and (52), we arrive at

$$
\begin{array}{r}
v(x)=\frac{1}{2} \int_{x+\frac{a_{0}}{2}}^{2 x-a_{0}} q_{0}(\tau) d \tau \int_{\tau-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}} q_{0}(\zeta) d \zeta-\frac{1}{2} \int_{2 x-a_{0}}^{\pi} q_{0}(\tau) d \tau \int_{x-\frac{a_{0}}{2}}^{\tau-x+\frac{a_{0}}{2}} q_{0}(\zeta) d \zeta \\
=\frac{1}{2} \int_{x+\frac{a_{0}}{2}}^{\pi} q_{0}(\tau) d \tau \int_{\tau-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}} q_{0}(\zeta) d \zeta, \quad \frac{3 a_{0}}{2}<x<\pi-\frac{a_{0}}{2} . \tag{53}
\end{array}
$$

Thus, the function $v(x)$ does not depend on $\left.q_{0}\right|_{\left[\pi-a_{0}, 2 a_{0}\right]}$. Moreover, according to the first representation in (53), it depends on $\left.q_{0}\right|_{\left(3 a_{0} / 2, \pi-a_{0}\right)}$ if and only if

$$
x-\frac{a_{0}}{2}>\frac{3 a_{0}}{2} \quad \text { or } \quad \pi-x+\frac{a_{0}}{2}>\frac{3 a_{0}}{2},
$$

i.e. $x \in I_{3} \backslash\left[\pi-a_{0}, 2 a_{0}\right]$. Analogously, $v(x)$ depends on $\left.q_{0}\right|_{\left(2 a_{0}, \pi-a_{0} / 2\right)}$ if and only if $x<\pi-a_{0}$. Hence, $v(x)$ is independent of $\left.q_{0}\right|_{I_{3}}$ if and only if $x \in\left[\pi-a_{0}, 2 a_{0}\right]$.

Lemma 4 along with formulae (48) and (49) guaranties solvability of the subsystem (35) on the set $I_{1}=\left(a_{0}, 3 a_{0} / 2\right) \cup\left(\pi-a_{0}, 2 a_{0}\right) \cup\left(\pi-a_{0} / 2, \pi\right)$. The following corollary gives a condition of the solvability on the entire interval $\left(a_{0}, \pi\right)$

Corollary 1. $\left.v\right|_{I_{3}}$ does not depend on $\left.q_{0}\right|_{I_{3}}$ if and only if $a_{0} \geq 2 \pi / 5$.
Proof. According to Lemma 4, $\left.v\right|_{I_{3}}$ does not depend on $\left.q_{0}\right|_{I_{3}}$ if and only if $I_{3} \subset\left[\pi-a_{0}, 2 a_{0}\right]$, which, in turn, is equivalent to $a_{0} \geq 2 \pi / 5$.

According to Corollary 1, formulae (48), (49) and (53) give the representation

$$
\begin{align*}
v(x)=2 \int_{x+\frac{a_{0}}{2}}^{\pi}\left(w_{0,0}+w_{1,0}\right)\left(2 \tau-\pi-a_{0}\right) d \tau & \int_{\tau-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}}\left(w_{0,0}-w_{1,0}\right)\left(\pi+a_{0}-2 \zeta\right) d \zeta \\
& =\frac{1}{2} \int_{2 x-\pi}^{\pi-a_{0}}\left(w_{0,0}+w_{1,0}\right)(\tau) d \tau  \tag{54}\\
\int_{\pi+2 a_{0}-2 x}^{2 x-a_{0}-\tau} & \left(w_{0,0}-w_{1,0}\right)(\zeta) d \zeta, \quad \frac{3 a_{0}}{2}<x<\frac{a_{0}+\pi}{2},
\end{align*}
$$

as soon as $a_{0} \in[2 \pi / 5, \pi / 2)$. Thus, we arrive at the following theorem.
Theorem 6. Let $a_{0} \in[2 \pi / 5, \pi)$. Then for any functions $w_{0,0}(x), w_{1,0}(x) \in L_{2}\left(0, \pi-a_{0}\right)$ subsystem (35) has a unique solution $q_{0}(x) \in L_{2}\left(a_{0}, \pi\right)$, which can be constructed by the formula

$$
q_{0}(x)=2\left\{\begin{array}{lc}
\left(w_{0,0}-w_{1,0}\right)\left(\pi+a_{0}-2 x\right), & a_{0}<x<\frac{3 a_{0}}{2}  \tag{55}\\
\left(w_{0,0}-w_{1,0}\right)\left(\pi+a_{0}-2 x\right)-v(x), & \frac{3 a_{0}}{2}<x<\frac{a_{0}+\pi}{2}, \\
\left(w_{0,0}+w_{1,0}\right)\left(2 x-\pi-a_{0}\right)-v(x), & \frac{a_{0}+\pi}{2}<x<\pi-\frac{a_{0}}{2} \\
\left(w_{0,0}+w_{1,0}\right)\left(2 x-\pi-a_{0}\right), & \pi-\frac{a_{0}}{2}<x<\pi
\end{array}\right.
$$

where the function $v(x)$ is determined by formula (54). Moreover,

$$
\begin{equation*}
\int_{a_{0}}^{\pi} q_{0}(x) d x=2 \int_{0}^{\pi-a_{0}} w_{0,0}(x) d x \tag{56}
\end{equation*}
$$

Proof. It remains to prove (56). Indeed, integrating (55) and using (39), we get

$$
\int_{a_{0}}^{\pi} q_{0}(x) d x=2 \int_{0}^{\pi-a_{0}} w_{0,0}(x) d x-2 \int_{\frac{3 a_{0}}{2}}^{\pi-\frac{a_{0}}{2}} v(x) d x
$$

where, according to (50), we have

$$
\int_{\frac{3 a_{0}}{2}}^{\pi-\frac{a_{0}}{2}} v(x) d x=\int_{\frac{3 a_{0}}{2}}^{\frac{\pi+a_{0}}{2}}\left(u_{0}-u_{1}\right)\left(\pi+a_{0}-2 x\right) d x+\int_{\frac{\pi+a_{0}}{2}}^{\pi-\frac{a_{0}}{2}}\left(u_{0}+u_{1}\right)\left(2 x-\pi-a_{0}\right) d x=\int_{0}^{\pi-2 a_{0}} u_{0}(x) d x
$$

Finally, using (46) and the last two equalities in (18), we calculate

$$
\int_{0}^{\pi-2 a_{0}} u_{0}(x) d x=\frac{A\left(\pi, 2 a_{0}\right)-A(\pi, \pi)}{2}=0
$$

which finishes the proof.

## 4. Proof of the uniqueness theorems

Let us begin with the following assertion.
Lemma 5. Let $a_{0}, a_{1} \in(0, \pi)$. Then specification of any pair of sequences $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}$, $j=0,1$, of the form (7) uniquely determines the values $\alpha_{0}, \alpha_{1}$ and $\omega$.

Proof. According to (7), we arrive at the asymptotic formulae

$$
\begin{equation*}
\omega \cos n a_{0}=\frac{\gamma_{n, 0}+\gamma_{-n, 0}}{2}+o(1), \quad \alpha_{j} \sin \rho_{n, j}^{0} a_{1}=\frac{\gamma_{n, j}-\gamma_{j-n, j}}{2}+o(1), \quad|n| \rightarrow \infty \tag{57}
\end{equation*}
$$

where we denoted

$$
\begin{equation*}
\gamma_{n, j}:=\pi n\left(\rho_{n, j}-\rho_{n, j}^{0}\right), \quad n \in \mathbb{Z}_{j}, \quad j=0,1 \tag{58}
\end{equation*}
$$

By virtue of Lemma 3.3 in [12], the sequences $\left\{\cos n a_{0}\right\}_{n \geq 1}$ and $\left\{\sin \rho_{n, j}^{0} a_{1}\right\}_{n \geq 1}, j=0,1$, do not converge, i.e. each of them has at least two different partial limits. Choose increasing sequences of natural numbers $\left\{m_{k, l}\right\}, l=\overline{1,3}$, so that

$$
\begin{equation*}
r_{1}:=\lim _{k \rightarrow \infty} \cos m_{k, 1} a_{0} \neq 0, \quad r_{l}:=\lim _{k \rightarrow \infty} \sin \rho_{m_{k, l}, l-2}^{0} a_{1} \neq 0, \quad l=2,3 \tag{59}
\end{equation*}
$$

According to (57) and (59), we calculate the values $\omega, \alpha_{0}$ and $\alpha_{1}$ by the formulae

$$
\begin{equation*}
\omega=\lim _{k \rightarrow \infty} \frac{\gamma_{m_{k, 1}, 0}+\gamma_{-m_{k, 1}, 0}}{2 r_{1}}, \quad \alpha_{j}=\lim _{k \rightarrow \infty} \frac{\gamma_{m_{k, j+2}, j}-\gamma_{j-m_{k, j+2, j}}}{2 r_{j+2}}, \quad j=0,1 \tag{60}
\end{equation*}
$$

which finishes the proof.
Proof of Theorem 2. According to Lemmas 3 and 5, under the hypothesis of the theorem, the functions $\Delta_{j}(\rho), j=0,1$, as well as the numbers $\alpha_{0}, \alpha_{1}$, and $\omega$ are determine uniquely. Then, by virtue of Lemma 2, the functions $w_{j, \nu}(x), j, \nu=0,1$, are uniquely determined too.

Thus, by Theorem 5 and Lemma 4, the function $p(x)=q_{1}^{\prime}(x)$ is uniquely determined a.e. on the interval $\left(a_{1}, \pi\right)$, while $q_{0}(x)$ is so on $I_{1}$. Finally, taking (8) into account, we get

$$
\begin{equation*}
q_{1}(x)=\alpha_{0}+\alpha_{1}+\int_{a_{1}}^{x} p(t) d t, \quad a_{1} \leq x \leq \pi, \tag{61}
\end{equation*}
$$

which finishes the proof.
Proof of Theorem 3. Following the proof of Theorem 2 in [11], we denote

$$
p_{1}:=\left.q_{0}\right|_{\left(a_{0}, \pi-a_{0}\right)}, \quad p_{2}:=\left.q_{0}\right|_{\left(2 a_{0}, \pi\right)}, \quad R_{1}(x, t):=\int_{t-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}} p_{1}(\tau) d \tau, \quad R_{2}(x):=\int_{x}^{\pi} p_{2}(t) d t .
$$

Then, by virtue of (53) and (55) we have the relation

$$
\begin{equation*}
F(x)=q_{0}(x)+\int_{x+\frac{a_{0}}{2}}^{\pi} p_{2}(t) d t \int_{t-x+\frac{a_{0}}{2}}^{x-\frac{a_{0}}{2}} p_{1}(\tau) d \tau, \quad \frac{3 a_{0}}{2}<x<\pi-\frac{a_{0}}{2}, \tag{62}
\end{equation*}
$$

where

$$
F(x)=2 \begin{cases}\left(w_{0,0}-w_{1,0}\right)\left(\pi+a_{0}-2 x\right), & \frac{3 a_{0}}{2}<x<\frac{a_{0}+\pi}{2} \\ \left(w_{0,0}+w_{1,0}\right)\left(2 x-\pi-a_{0}\right), & \frac{a_{0}+\pi}{2}<x<\pi-\frac{a_{0}}{2}\end{cases}
$$

After changing the order of integration, relation (62) on the target intervals takes the forms:

$$
\begin{equation*}
F_{1}(x)=p_{1}(x)+\int_{x+\frac{a_{0}}{2}}^{\pi-\frac{a_{0}}{2}} R_{1}(x, t) p_{2}(t) d t-\int_{\frac{3 a_{0}}{2}}^{\pi-x+\frac{a_{0}}{2}} R_{2}\left(x+t-\frac{a_{0}}{2}\right) p_{1}(t) d t, \quad \frac{3 a_{0}}{2}<x<\pi-a_{0}, \tag{63}
\end{equation*}
$$

where

$$
F_{1}(x):=F(x)+\int_{\pi-x}^{\frac{3 a_{0}}{2}} R_{2}\left(x+t-\frac{a_{0}}{2}\right) p_{1}(t) d t+R_{2}\left(\pi-\frac{a_{0}}{2}\right) \int_{x-\frac{a_{0}}{2}}^{\pi-x} p_{1}(t) d t, \quad \frac{3 a_{0}}{2}<x<\pi-a_{0}
$$

and

$$
\begin{equation*}
F_{2}(x)=p_{2}(x)+R_{2}\left(x+\frac{a_{2}}{2}\right) \int_{\frac{3 a_{0}}{2}}^{x-\frac{a_{0}}{2}} p_{1}(\tau) d \tau, \quad 2 a_{0}<x<\pi-\frac{a_{0}}{2} \tag{64}
\end{equation*}
$$

where

$$
F_{2}(x):=F(x)-\int_{x+\frac{a_{0}}{2}}^{\pi} p_{2}(t) d t \int_{t-x+\frac{a_{0}}{2}}^{\frac{3 a_{0}}{2}} p_{1}(\tau) d \tau, \quad 2 a_{0}<x<\pi-\frac{a_{0}}{2} .
$$

Note that the functions $F_{1}(x), \quad F_{2}(x), \quad R_{1}(x, t)$ and $R_{2}(x)$ involved in (63) and (64), being dependent only on $\left.q\right|_{I_{2}}$, are already known. Substituting (64) into (63) and changing the order of integration, we obtain the integral equation

$$
F_{3}(x)=p_{1}(x)-R_{3}(x, x) \int_{\frac{3 a_{0}}{2}}^{x} p_{1}(t) d t-\int_{x}^{\pi-a_{0}} R_{3}(x, t) p_{1}(t) d t
$$

$$
\begin{equation*}
-\int_{\frac{3 a_{0}}{2}}^{\pi-x+\frac{a_{0}}{2}} R_{2}\left(x+t-\frac{a_{0}}{2}\right) p_{1}(t) d t, \quad \frac{3 a_{0}}{2}<x<\pi-a_{0} \tag{65}
\end{equation*}
$$

where

$$
F_{3}(x)=F_{1}(x)-\int_{x+\frac{a_{0}}{2}}^{\pi-\frac{a_{0}}{2}} R_{1}(x, t) F_{2}(t) d t, \quad R_{3}(x, t)=\int_{t+\frac{a_{0}}{2}}^{\pi-\frac{a_{0}}{2}} R_{1}(x, \tau) R_{2}\left(\tau-\frac{a_{0}}{2}\right) d \tau
$$

Since, according to the hypothesis of the theorem, the value

$$
\omega_{1}:=\omega_{0}+\int_{\frac{3 a_{0}}{2}}^{\frac{\pi}{2}+\frac{a_{0}}{4}} q_{0}(x) d x=\int_{\frac{3 a_{0}}{2}}^{\pi-a_{0}} p_{1}(x) d x
$$

is known, equation (655) takes the form

$$
\begin{equation*}
F_{4}(x)=p_{1}(x)+\int_{x}^{\pi-a} R_{4}(x, t) p_{1}(t) d t-\int_{\frac{3 a_{0}}{2}}^{\pi-x+\frac{a_{0}}{2}} R_{2}\left(x+t-\frac{a_{0}}{2}\right) p_{1}(t) d t, \quad \frac{3 a_{0}}{2}<x<\pi-a_{0} \tag{66}
\end{equation*}
$$

where both functions $F_{4}(x)=F_{3}(x)+\omega_{1} R_{3}(x, x)$ and $R_{4}(x, t)=R_{3}(x, x)-R_{3}(x, t)$ are known too. Taking into account that the function $p_{1}(x)$ on the interval $\left(3 a_{0} / 2, \pi / 2+a_{0} / 4\right)$ is given by the hypothesis, we find it on $\left(\pi / 2+a_{0} / 4, \pi-a_{0}\right)$ by solving there equation (66), in which the second integral becomes known. Finally, substituting the completely found function $p_{1}(x)$ into relation (64), we find $p_{2}(x)$ on $\left(2 a_{0}, \pi-a_{0} / 2\right)$, which finishes the proof.

## 5. Other representations for the infinite products

The results of this section are valid for any $a_{0}, a_{1} \in[0, \pi]$.
In Section 3, we proved, in particular, that any functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$ of the forms described in (25)-(27) have infinitely many zeros obeying (7) for the corresponding $j \in\{0,1\}$. Moreover, these functions are determined by their zeros uniquely by formula (12). However, the functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$ constructed by (12) with arbitrary sequences of complex numbers of the form (17), generally speaking, do not have the forms as in (25)-(27). This fact is connected, in particular, with excessiveness of the input data of Inverse Problem 1 for recovering the functions $w_{j, \nu}(x)$ in (25) and (26). Nevertheless, the following lemma holds.

Lemma 6. Let $j \in\{0,1\}$. Then for any sequence of complex numbers $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}$ obeying (7), the function $\Delta_{j}(\rho)$ constructed by formula (12) has the form

$$
\begin{equation*}
\Delta_{0}(\rho)=\frac{\sin \rho \pi}{\rho}-\omega \frac{\cos \rho\left(\pi-a_{0}\right)}{\rho^{2}}+\alpha_{0} \frac{\sin \rho\left(\pi-a_{1}\right)}{\rho^{2}}+\gamma_{0} \frac{\sin \rho \pi}{\rho^{2}}+\sum_{\nu=0}^{1} \int_{0}^{\pi} w_{0, \nu}(x) \frac{c_{\nu}(\rho x)}{\rho^{2}} d x \tag{67}
\end{equation*}
$$

for $j=0$, and

$$
\begin{equation*}
\Delta_{1}(\rho)=\cos \rho \pi+\omega \frac{\sin \rho\left(\pi-a_{0}\right)}{\rho}+\alpha_{1} \frac{\cos \rho\left(\pi-a_{1}\right)}{\rho}+\gamma_{1} \frac{\cos \rho \pi}{\rho}+\sum_{\nu=0}^{1} \int_{0}^{\pi} w_{1, \nu}(x) \frac{c_{1-\nu}(\rho x)}{\rho} d x \tag{68}
\end{equation*}
$$

for $j=1$. Here $w_{j, \nu}(x) \in L_{2}(0, \pi)$ for $j, \nu=0,1$, and

$$
\begin{equation*}
\int_{0}^{\pi} w_{0,0}(x) d x=\omega, \quad \int_{0}^{\pi} x w_{0,1}(x) d x=\alpha_{0}\left(a_{1}-\pi\right)-\gamma_{0} \pi, \quad \int_{0}^{\pi} w_{1,1}(x) d x=-\alpha_{1}-\gamma_{1} \tag{69}
\end{equation*}
$$

Before proceeding directly to the proof of Lemma 6, we establish the following auxiliary assertion giving some important subtle estimates that will be required for the proof.

Proposition 1. Put $h_{n, j}:=\beta_{0} \cos \rho_{n, j}^{0} a_{0}+\beta_{1} \sin \rho_{n, j}^{0} a_{1}+\varkappa_{n}, \quad|n| \in \mathbb{N}, \quad \beta_{j} \in \mathbb{C}, \quad j=0,1$. Then

$$
a_{n, j}:=\sum_{k \neq 0, n} \frac{h_{k, j}}{k(n-k)}=O\left(\frac{1}{n}\right), \quad|n| \rightarrow \infty .
$$

Proof. We have

$$
a_{n, j}=\frac{1}{n} \lim _{N \rightarrow \infty} \sum_{\substack{k \neq 0, n \\ k=-N}}^{N} h_{k, j}\left(\frac{1}{k}+\frac{1}{n-k}\right)=\frac{1}{n} \lim _{N \rightarrow \infty}\left(\sum_{\substack{k \neq 0, n \\ k=-N}}^{N} \frac{h_{k, j}}{k}-\sum_{\substack{k \neq-n, 0 \\ k=-N-n}}^{N-n} \frac{h_{n+k, j}}{k}\right) .
$$

One can easily calculate

$$
a_{n, j}=\frac{1}{n} \sum_{\substack{k \neq|n| \\ k=1}}^{\infty} \frac{h_{k, n, j}}{k}-\frac{h_{-n, j}+h_{2 n, j}}{n^{2}}, \quad h_{k, n, j}=h_{k, j}-h_{n+k, j}+h_{n-k, j}-h_{-k, j} .
$$

Denote $\varkappa_{k, n}:=\varkappa_{k}-\varkappa_{n+k}+\varkappa_{n-k}-\varkappa_{-k}$. It remains to note that, since

$$
h_{k, n, j}=4 \beta_{0} \sin \frac{n a_{0}}{2} \cos \frac{(n-j) a_{0}}{2} \sin k a_{0}+4 \beta_{1} \sin \frac{n a_{1}}{2} \sin \frac{(n-j) a_{1}}{2} \sin k a_{1}+\varkappa_{k, n}
$$

and the series $\sum_{k=1}^{\infty} \frac{\sin k a}{k}$ converges for any $a$, the series $\sum_{k=1}^{\infty} \frac{h_{k, n, j}}{k}, j=0,1$, are convergent too, and their sums are uniformly bounded with respect to $n$.

Proof of Lemma 6. Fix $j \in\{0,1\}$. Let us show first that $\left\{\theta_{j}\left(\rho_{n, j}^{0}\right)\right\}_{n \in \mathbb{Z}_{j}} \in l_{2}$, where $\theta_{0}(\rho)$ and $\theta_{1}(\rho)$ are determined by (10) and (11), respectively. It is easy to see that

$$
\begin{equation*}
\theta_{j}\left(\rho_{n, j}^{0}\right)=\left(\rho_{n, j}^{0}\right)^{2-j} \Delta_{j}\left(\rho_{n, j}^{0}\right)+(-1)^{n} \omega_{n, j}, \quad \omega_{n, j}=\omega \cos \rho_{n, j}^{0} a_{0}+\alpha_{j} \sin \rho_{n, j}^{0} a_{1} \tag{70}
\end{equation*}
$$

By virtue of (12) and (33), we obtain

$$
\rho^{2-j} \Delta_{j}(\rho)=\rho\left(\rho_{n, j}-\rho\right) \frac{c_{1-j}(\rho \pi)}{\rho_{n, j}^{0}-\rho} \prod_{k \in \mathbb{Z}_{j} \backslash\{n\}} \frac{\rho_{k, j}-\rho}{\rho_{k, j}^{0}-\rho}
$$

Thus, having put

$$
\begin{equation*}
\varepsilon_{n, j}:=\rho_{n, j}-\rho_{n, j}^{0}=\frac{\omega_{n, j}}{\pi n}+\frac{\varkappa_{n}}{n}, \tag{71}
\end{equation*}
$$

we get

$$
\left(\rho_{n, j}^{0}\right)^{2-j} \Delta_{j}\left(\rho_{n, j}^{0}\right)=(-1)^{n+1} \pi \rho_{n, j}^{0} \varepsilon_{n, j} b_{n, j}, \quad b_{n, j}=\prod_{k \in \mathbb{Z}_{j} \backslash\{n\}}\left(1+\frac{\varepsilon_{k, j}}{k-n}\right), \quad n \in \mathbb{Z}_{j} .
$$

Since $\pi \rho_{n, j}^{0} \varepsilon_{n, j}=\omega_{n, j}+\varkappa_{n}$, we get $\theta_{j}\left(\rho_{n, j}^{0}\right)=(-1)^{n}\left(1-b_{n, j}\right) \omega_{n, j}+\varkappa_{n}$. Thus, we need to prove that $\left\{1-b_{n, j}\right\} \in l_{2}$. For this purpose, we choose $N \in \mathbb{N}$ so that $\left|\varepsilon_{n, j}\right| \leq 1 / 2$ as soon as $|n| \geq N$, and represent $b_{n, j}$ in the form $\left.b_{n, j}=b_{n, j}^{(1)}\right)_{n, j}^{(2)}$, where

$$
b_{n, j}^{(1)}=\prod_{\substack{k \neq n \\ 1-j \leq|k|<N}}\left(1+\frac{\varepsilon_{k, j}}{k-n}\right)=1+O\left(\frac{1}{n}\right), \quad|n| \rightarrow \infty, \quad b_{n, j}^{(2)}=\prod_{\substack{k \neq n \\|k| \geq N}}\left(1+\frac{\varepsilon_{k, j}}{k-n}\right) .
$$

Our choice of $N$ allows one to represent

$$
b_{n, j}^{(2)}=\exp \left(\sum_{\substack{k \neq n \\|k| \geq N}} \ln \left(1+\frac{\varepsilon_{k, j}}{k-n}\right)\right)=\exp \left(\sum_{\substack{k \neq n \\|k| \geq N}} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu+1}\left(\frac{\varepsilon_{k, j}}{k-n}\right)^{\nu+1}\right) .
$$

Therefore, we have

$$
\left|b_{n, j}^{(2)}-1\right| \leq \sum_{\nu=1}^{\infty} \frac{\left(\Omega_{n, j}^{(1)}+2 \Omega_{n, j}^{(2)}\right)^{\nu}}{\nu!}, \quad \Omega_{n, j}^{(1)}=\left|\sum_{\substack{k \neq n \\|k| \geq N}} \frac{\varepsilon_{k, j}}{k-n}\right|, \quad \Omega_{n, j}^{(2)}=\sum_{\substack{k \neq n \\|k| \geq N}} \frac{\left|\varepsilon_{k, j}\right|^{2}}{(k-n)^{2}} .
$$

By virtue of (70) and (71) along with Proposition 1, we have $\Omega_{n, j}^{(1)}=O\left(n^{-1}\right), \quad|n| \rightarrow \infty$. Let us show that $\left\{\Omega_{n, j}^{(2)}\right\} \in l_{2}$. Indeed, using the generalized Minkovskii inequality, we get the estimate

$$
\begin{aligned}
\sqrt{\sum_{|n| \in \mathbb{N}}\left(\Omega_{n, j}^{(2)}\right)^{2}} & \leq C \sqrt{\sum_{|n| \in \mathbb{N}}\left(\sum_{k \neq 0, n} \frac{1}{k^{2}(k-n)^{2}}\right)^{2}} \\
& \leq C \sum_{|k| \in \mathbb{N}} \sqrt{\sum_{n \neq 0, k}\left(\frac{1}{k^{2}(k-n)^{2}}\right)^{2}}<C \sum_{|k| \in \mathbb{N}} \frac{1}{k^{2}} \sqrt{\sum_{|n| \in \mathbb{N}} \frac{1}{n^{4}}}<\infty .
\end{aligned}
$$

Thus, we arrive at $\left\{\theta_{j}\left(\rho_{n, j}^{0}\right)\right\}_{n \in \mathbb{Z}_{j}} \in l_{2}$. Further, since the systems of vector-functions

$$
\left\{\left[\cos \rho_{n, j}^{0} x, \sin \rho_{n, j}^{0} x\right]\right\}_{n \in \mathbb{Z}},
$$

where $\rho_{0,0}^{0}=0$, is an almost normalized orthogonal basis in $\left(L_{2}(0, \pi)\right)^{2}$, there exist unique functions $w_{j, \nu}(x) \in L_{2}(0, \pi), \quad \nu=0,1$, such that

$$
\theta_{j}\left(\rho_{n, j}^{0}\right)=\int_{0}^{\pi} w_{j, j}(x) \cos \rho_{n, j}^{0} x d x+\int_{0}^{\pi} w_{j, 1-j}(x) \sin \rho_{n, j}^{0} x d x, \quad n \in \mathbb{Z}
$$

Consider the function

$$
\tilde{\theta}_{j}(\rho)=\int_{0}^{\pi} w_{j, j}(x) \cos \rho x d x+\int_{0}^{\pi} w_{j, 1-j}(x) \sin \rho x d x
$$

According to (10) and (11), it remains to prove that

$$
\begin{equation*}
\theta_{j}(\rho)-\tilde{\theta}_{j}(\rho)=\gamma_{j} c_{1-j}(\rho \pi), \quad \gamma_{j} \equiv \text { const. } \tag{72}
\end{equation*}
$$

For this purpose, we consider the entire function

$$
\begin{equation*}
\Theta_{j}(\rho):=\frac{\theta_{j}(\rho)-\tilde{\theta}_{j}(\rho)}{c_{1-j}(\rho \pi)}=\Theta_{j, 1}(\rho)+\Theta_{j, 2}(\rho), \tag{73}
\end{equation*}
$$

where

$$
\Theta_{j, 1}(\rho)=\rho\left(\frac{\rho^{1-j} \Delta_{j}(\rho)}{c_{1-j}(\rho \pi)}-1\right), \quad \Theta_{j, 2}(\rho)=\frac{(-1)^{j} \omega c_{j}\left(\rho\left(\pi-a_{0}\right)\right)-\alpha_{j} c_{1-j}\left(\rho\left(\pi-a_{1}\right)\right)-\tilde{\theta}_{j}(\rho)}{c_{1-j}(\rho \pi)} .
$$

Clearly, $\Theta_{j, 2}(\rho)=O(1)$ as soon as $\rho \in G_{\delta}^{j}:=\left\{\rho:\left|\rho-\rho_{n, j}^{0}\right| \geq \delta, n \in \mathbb{Z}\right\}$ for a fixed $\delta>0$ and $\rho \rightarrow \infty$, and $\Theta_{j, 2}(\rho)=o(1)$ for $|\operatorname{Im} \rho| \rightarrow \infty$. Further, dividing (12)) by (33), we get

$$
\Theta_{j, 1}(\rho)=\rho\left(F_{j, 1}(\rho) F_{j, 2}(\rho)-1\right)
$$

where

$$
F_{j, 1}(\rho)=\prod_{1-j \leq|n|<N}\left(1+\frac{\varepsilon_{n, j}}{\rho_{n, j}^{0}-\rho}\right), \quad F_{j, 2}(\rho)=\prod_{|n| \geq N}\left(1+\frac{\varepsilon_{n, j}}{\rho_{n, j}^{0}-\rho}\right),
$$

while $N$ is chosen so that $\left|\varepsilon_{n, j}\right| \leq \delta / 2$ as soon as $|n| \geq N$. Hence, one can represent

$$
F_{j, 2}(\rho)=\exp \left(\sum_{|n| \geq N} \ln \left(1+\frac{\varepsilon_{n, j}}{\rho_{n, j}^{0}-\rho}\right)\right)=\exp \left(\sum_{|n| \geq N} \sum_{\nu=0}^{\infty} \frac{(-1)^{\nu}}{\nu+1}\left(\frac{\varepsilon_{n, j}}{\rho_{n, j}^{0}-\rho}\right)^{\nu+1}\right), \quad \rho \in G_{\delta}^{j} .
$$

Then the following estimates hold:

$$
\left|F_{j, 2}(\rho)-1\right| \leq \sum_{\nu=1}^{\infty} \frac{1}{\nu!}\left(2 \sum_{|n| \geq N} \frac{\left|\varepsilon_{n, 0}\right|}{\left|\rho_{n, j}^{0}-\rho\right|}\right)^{\nu}<C \sqrt{\sum_{n \in \mathbb{Z}} \frac{1}{\left|\rho_{n, j}^{0}-\rho\right|^{2}}}, \quad \rho \in G_{\delta}^{j},
$$

where the right-hand side, as a function of $\rho$, has the period 1 . Thus, we arrive at the estimates

$$
F_{j, \nu}(\rho)=1+O\left(\rho^{\nu-2}\right), \quad \rho \in G_{\delta}^{j}, \quad \rho \rightarrow \infty, \quad \nu=1,2, \quad F_{j, 2}(\rho)=1+o(1), \quad|\operatorname{Im} \rho| \rightarrow \infty
$$

which imply $\Theta_{j, 1}(\rho)=O(\rho)$ for $\rho \rightarrow \infty$ in $G_{\delta}^{j}$, and $\Theta_{j, 1}(\rho)=o(\rho)$ for $|\operatorname{Im} \rho| \rightarrow \infty$. Hence, we get $\gamma_{j}:=\Theta_{j}(\rho) \equiv$ const, which along with (73) implies (721), i.e. (67) and (68)) are proven.

Finally, note that relations (69) follow from entireness of the functions $\Delta_{j}(\rho), j=0,1$.

## 6. Solution of the inverse problem

In this section, besides our initial assumptions on $a_{0}$ and $a_{1}$, we also assume $a_{0} \geq 5 \pi / 2$. The preliminary work fulfilled in Sections 3 and 5 allows us to give the proof of Theorem 4.

Proof of Theorem 4. By necessity, the asymptotics (7) is already established in Theorem 1. Let us prove (ii). According to Lemma 3, the functions $\Delta_{0}(\rho)$ and $\Delta_{1}(\rho)$ determined by formula (12) are the characteristic functions, which, by virtue of Lemma 2 , have representations (25) and (26), respectively. Hence, according to (10) and (11), we have the representations

$$
\theta_{j}(\rho)=\int_{0}^{\pi-a_{j}} w_{j, j}(x) \cos \rho x d x+\int_{0}^{\pi-a_{1}-j} w_{j, 1-j}(x) \sin \rho x d x, \quad j=0,1
$$

which along with (9) give

$$
\begin{equation*}
g_{j, j}(\rho)=2 \int_{0}^{\pi-a_{j}} w_{j, j}(x) \cos \rho x d x, \quad g_{j, 1-j}(\rho)=2 \int_{0}^{\pi-a_{1-j}} w_{j, 1-j}(x) \sin \rho x d x, \quad j=0,1, \tag{74}
\end{equation*}
$$

which, in turn, implies (ii) and finishes the proof of the necessity.

For the sufficiency, we assume that some complex sequences $\left\{\rho_{n, 0}\right\}_{|n| \in \mathbb{N}}$ and $\left\{\rho_{n, 1}\right\}_{n \in \mathbb{Z}}$ obeying (i) and (ii) are given. Find the values $\alpha_{0}, \alpha_{1}$ and $\omega$ as in the proof of Lemma 5 . Then, by formula (12), construct the functions $\Delta_{0}(\rho)$ and $\Delta_{1}(\rho)$, which, according to Lemma 6 , have representations (67) and (68), respectively, with some numbers $\gamma_{0}$ and $\gamma_{1}$ and some functions $w_{\nu, j}(x) \in L_{2}(0, \pi), j, \nu=0,1$, obeying (69). Using (9)-(11) and (67), (68), we calculate

$$
\begin{aligned}
& g_{0,0}(\rho)=2 \int_{0}^{\pi} w_{0,0}(x) \cos \rho x d x, \quad g_{0,1}(\rho)=2 \gamma_{0} \sin \rho \pi+2 \int_{0}^{\pi} w_{0,1}(x) \sin \rho x d x \\
& g_{1,0}(\rho)=2 \int_{0}^{\pi} w_{1,0}(x) \sin \rho x d x, \quad g_{1,1}(\rho)=2 \gamma_{1} \cos \rho \pi+2 \int_{0}^{\pi} w_{1,1}(x) \cos \rho x d x
\end{aligned}
$$

Thus, condition (ii) implies $\gamma_{0}=\gamma_{1}=0$ and hence, by virtue of the Paley-Wiener theorem, $w_{j, \nu}(x)=0$ a.e. on $\left(\pi-a_{\nu}, \pi\right)$ for $j, \nu=0,1$, which along with (69) gives (27). Therefore, the functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$ has the forms (25) and (26), respectively. By virtue of Theorems 5 and 6 , the subsystems (35) and (36) with these $w_{j, \nu}(x)$ have unique solutions $q_{0}(x) \in L_{2}\left(a_{0}, \pi\right)$ and $p(x) \in L_{2}\left(a_{1}, \pi\right)$, satisfying (56) and (38), respectively. Construct the function $q_{1}(x) \in W_{2}^{1}\left[a_{1}, \pi\right]$ by the formula

$$
\begin{equation*}
q_{1}(x)=\frac{1}{\pi-a_{1}} \int_{a_{1}}^{\pi} d t \int_{t}^{\pi} p(\tau) d \tau-\int_{x}^{\pi} p(t) d t \tag{75}
\end{equation*}
$$

which, obviously, obeys (6). Thus, we constructed the boundary value problems $\mathcal{L}_{0}\left(q_{0}, q_{1}\right)$ and $\mathcal{L}_{1}\left(q_{0}, q_{1}\right)$. Let $\tilde{\Delta}_{0}(\lambda)$ and $\tilde{\Delta}_{1}(\lambda)$ be their characteristic functions, respectively. According to Lemma 2, they have the representations

$$
\begin{align*}
& \tilde{\Delta}_{0}(\rho)=\frac{\sin \rho \pi}{\rho}-\tilde{\omega} \frac{\cos \rho\left(\pi-a_{0}\right)}{\rho^{2}}+\tilde{\alpha}_{0} \frac{\sin \rho\left(\pi-a_{1}\right)}{\rho^{2}}+\sum_{\nu=0}^{1} \int_{0}^{\pi-a_{\nu}} \tilde{w}_{0, \nu}(x) \frac{c_{\nu}(\rho x)}{\rho^{2}} d x  \tag{76}\\
& \tilde{\Delta}_{1}(\rho)=\cos \rho \pi+\tilde{\omega} \frac{\sin \rho\left(\pi-a_{0}\right)}{\rho}+\tilde{\alpha}_{1} \frac{\cos \rho\left(\pi-a_{1}\right)}{\rho}+\sum_{\nu=0}^{1} \int_{0}^{\pi-a_{\nu}} \tilde{w}_{1, \nu}(x) \frac{c_{1-\nu}(\rho x)}{\rho} d x \tag{77}
\end{align*}
$$

where

$$
\begin{gather*}
\tilde{\omega}=\frac{1}{2} \int_{a_{0}}^{\pi} q_{0}(x) d x, \quad \tilde{\alpha}_{j}=\tilde{\alpha}+(-1)^{j} \tilde{\beta}, \quad j=0,1, \quad \tilde{\alpha}=\frac{q_{1}\left(a_{1}\right)}{2}, \quad \tilde{\beta}=\frac{q_{1}(\pi)}{2},  \tag{78}\\
\tilde{w}_{0,0}(x)=-K_{0,2}\left(\pi, \pi-x ; q_{0}\right), \quad \tilde{w}_{1,0}(x)=P_{0}\left(\pi, \pi-x ; q_{0}\right)  \tag{79}\\
\tilde{w}_{0,1}(x)=K_{1,2}(\pi, \pi-x ; p), \quad \tilde{w}_{1,1}(x)=P_{1}(\pi, \pi-x ; p) \tag{80}
\end{gather*}
$$

Comparing (79) and (80) with (35) and (36), respectively, we arrive at

$$
\begin{equation*}
\tilde{w}_{j, \nu}(x)=w_{j, \nu}(x), \quad j, \nu=0,1 \tag{81}
\end{equation*}
$$

Successively using the first equality in (78), identity (56) and the first equality in (27), we get

$$
\begin{equation*}
\tilde{\omega}=\frac{1}{2} \int_{a_{0}}^{\pi} q_{0}(x) d x=\int_{0}^{\pi-a_{0}} w_{0,0}(x) d x=\omega \tag{82}
\end{equation*}
$$

Further, using the first equality in (38) along with the third one in (27), we obtain

$$
\begin{equation*}
\int_{a_{1}}^{\pi} p(x) d x=2 \int_{0}^{\pi-a_{1}} w_{1,1}(x) d x=-2 \alpha_{1} \tag{83}
\end{equation*}
$$

while the second equalities in (27) and in (38) along with (83) give

$$
\begin{equation*}
\int_{a_{1}}^{\pi} x p(x) d x=\frac{\pi+a_{1}}{2} \int_{a_{1}}^{\pi} p(x) d x+\left(\pi-a_{1}\right) \alpha_{0}=\left(\pi-a_{1}\right) \alpha_{0}-\left(\pi+a_{1}\right) \alpha_{1} \tag{84}
\end{equation*}
$$

On the other hand, successively using (78), (75) and (83), we get

$$
\begin{equation*}
\tilde{\alpha}_{1}=\frac{q_{1}\left(a_{1}\right)-q_{1}(\pi)}{2}=-\frac{1}{2} \int_{a_{1}}^{\pi} p(x) d x=\alpha_{1}, \tag{85}
\end{equation*}
$$

while the second and the last equalities in (78) along with (75), (83) and (84) imply

$$
\begin{equation*}
\left(\pi-a_{1}\right)\left(\tilde{\alpha}_{0}-\tilde{\alpha}_{1}\right)=\int_{a_{1}}^{\pi} d t \int_{t}^{\pi} p(\tau) d \tau=\int_{a_{1}}^{\pi}\left(x-a_{1}\right) p(x) d x=\left(\pi-a_{1}\right)\left(\alpha_{0}-\alpha_{1}\right) . \tag{86}
\end{equation*}
$$

By virtue of (85) and (86), we have $\tilde{\alpha}_{j}=\alpha_{j}, j=0,1$, which along with (255), (261), (766), (77), (81) and (82) gives $\tilde{\Delta}_{j}(\rho) \equiv \Delta_{j}(\rho), \quad j=0,1$. Hence, each given sequence $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}$ is the spectrum of the corresponding problem $\mathcal{L}_{j}\left(q_{0}, q_{1}\right), \quad j=0,1$.

The Paley-Wiener theorem implies the following corollary from Theorem 4.
Corollary 2. Arbitrary complex sequences $\left\{\rho_{n, 0}\right\}_{|n| \in \mathbb{N}}$ and $\left\{\rho_{n, 1}\right\}_{n \in \mathbb{Z}}$ are the spectra of some boundary value problems $\mathcal{L}_{0}\left(q_{0}, q_{1}\right)$ and $\mathcal{L}_{1}\left(q_{0}, q_{1}\right)$, respectively, if and only if their convergence exponents are equal to 1 , and there exist some $\alpha_{1}, \alpha_{2}, \omega \in \mathbb{C}$ such that the functions $g_{j, \nu}(\rho), \quad j, \nu=0,1$, determined by formulae (9)-(12) satisfy the following conditions:

$$
\begin{equation*}
g_{j, \nu}(x) \in L_{2}(-\infty, \infty), \quad\left|g_{j, \nu}(\rho)\right| \leq C \exp \left(\left(\pi-a_{\nu}\right)|\rho|\right), \quad g_{j, \nu}(-\rho)=(-1)^{j+\nu} g_{j, \nu}(\rho) . \tag{87}
\end{equation*}
$$

Proof. In addition to the proof of Theorem 4, it is sufficient to note that, by virtue of the Paley-Wiener theorem, conditions (87) are equivalent to the representations (74) with some functions $w_{j, \nu}(x) \in L_{2}\left(0, \pi-a_{\nu}\right), \quad j, \nu=0,1$.

The proof of Theorem 4 gives the following algorithm for solving Inverse Problem 1.
Algorithm 1. Let the spectra $\left\{\rho_{n, j}\right\}_{n \in \mathbb{Z}_{j}}$ of some problems $\mathcal{L}_{j}\left(q_{0}, q_{1}\right), j=0,1$, be given.
(i) Calculate $\alpha_{0}, \alpha_{1}$ and $\omega$ by the formulae (58) and (60), in which the sequences $\left\{m_{k, l}\right\}$, $l=\overline{1,3}$, are chosen so that (59) is fulfilled;
(ii) Construct the functions $w_{j, \nu}(x) \in L_{2}\left(0, \pi-a_{\nu}\right), \quad j, \nu=0,1$, in representations (255) and (261) by inverting the corresponding Fourier transforms:

$$
\left[\begin{array}{c}
w_{j, j}(x) \\
w_{j, 1-j}(x)
\end{array}\right]=\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \theta_{j}(n)\left[\begin{array}{c}
\cos n x \\
\sin n x
\end{array}\right], \quad j=0,1,
$$

where the functions $\theta_{0}(\rho)$ and $\theta_{1}(\rho)$ are determined by formulae (10) and (11), respectively, with $\Delta_{0}(\rho)$ and $\Delta_{1}(\rho)$ constructed by (12);
(iii) Find the functions $q_{0}(x) \in L\left(a_{0}, \pi\right)$ and $p(x) \in L\left(a_{1}, \pi\right)$ by formulae (54) and (55) and by formula (37), respectively, with $w_{j, \nu}(x), j, \nu=0,1$, constructed on step (ii);
(iv) Finally, construct the function $q_{1}(x) \in W_{2}^{1}\left[a_{1}, \pi\right]$ by formula (61) or by formula (75).

Remark 2. As in [5], step (ii) of Algorithm 1 can be refined by changing to recovering the functions $w_{j, \nu}(x), j, \nu=0,1$, from certain subspectra depending on the values $a_{0}$ and $a_{1}$.

## Appendix A

Here we obtain an analog of Theorem 4 for the boundary value problems $\mathcal{L}_{j}(q), j=0,1$, that consist of (1) and (21):

$$
-y^{\prime \prime}(x)+q(x) y(x-a)=\lambda y(x), \quad 0<x<\pi, \quad y(0)=y^{(j)}(\pi)=0
$$

where $q(x) \in L_{2}(0, \pi)$ is a complex-valued function, $q(x)=0$ on $(0, a)$, while $a \in[2 \pi / 5, \pi)$. Let $\left\{\lambda_{n, j}\right\}_{n \geq 1}$ be the spectrum of $\mathcal{L}_{j}(q)$. Consider the following inverse problem

Inverse Problem A. Given $\left\{\lambda_{n, 0}\right\}_{n \geq 1}$ and $\left\{\lambda_{n, 1}\right\}_{n \geq 1}$, find the potential $q(x)$.
As was mentioned in Introduction with references to [5], Inverse Problem A is overdetermined. In [5] for $a \in[\pi / 2, \pi)$, it was established, in particular, that for unique determination of $q(x)$ by the subspectra $\left\{\lambda_{n_{k}, 0}\right\}_{k \geq 1}$ and $\left\{\lambda_{n_{k}, 1}\right\}_{k \geq 1}$, it is necessary and sufficient that each of the functional systems $\left\{\cos n_{k} x\right\}_{k \geq 1}$ and $\left\{\sin \left(n_{k}-1 / 2\right) x\right\}_{k \geq 1}$ is complete in $L_{2}(0, \pi-a)$. Moreover, the appropriate asymptotics along with Riesz-basisness of these two systems is sufficient for solvability of Inverse Problem A. In particular, solely the asymptotics is a necessary and sufficient condition of the solvability when this Riesz-basisness is patently the case. Analogous results can be obtained also for $a \in[2 \pi / 5, \pi / 2)$. So far, these results remain sole ones dealing with the question of solvability of Inverse Problem A.

Despite the overdetermination of Inverse Problem A, one can obtain necessary and sufficient conditions for it solvability given the full spectra as a particular case of Theorem 4.

Theorem A. Let $a \in[2 \pi / 5, \pi)$. Then for any sequences of complex numbers $\left\{\lambda_{n, 0}\right\}_{n \geq 1}$ and $\left\{\lambda_{n, 1}\right\}_{n \geq 1}$ to be the spectra of some boundary value problems $\mathcal{L}_{0}(q)$ and $\mathcal{L}_{1}(q)$, respectively, it is necessary and sufficient to satisfy the following two conditions:
(i) For $j=0,1$, the following asymptorics holds:

$$
\lambda_{n, j}=\left(n-\frac{j}{2}+\frac{\omega \cos (n-j / 2) a}{\pi n}+\frac{\varkappa_{n}}{n}\right)^{2}, \quad \omega \in \mathbb{C},
$$

where, as before, one and the same symbol $\left\{\varkappa_{n}\right\}$ denotes different sequences in $l_{2}$;
(ii) The exponential types of the functions $\theta_{0}(\rho)$ and $\theta_{1}(\rho)$ do not exceed $\pi-a$, where

$$
\begin{gather*}
\theta_{0}(\rho)=\rho^{2} \Delta_{0}(\rho)-\rho \sin \rho \pi+\omega \cos \rho(\pi-a), \quad \theta_{1}(\rho)=\rho \Delta_{1}(\rho)-\rho \cos \rho \pi-\omega \sin \rho(\pi-a),  \tag{88}\\
\Delta_{j}(\rho)=\pi^{1-j} \prod_{n=1}^{\infty} \frac{\lambda_{n, j}-\rho^{2}}{(n-j / 2)^{2}}, \quad j=0,1 . \tag{89}
\end{gather*}
$$

As Corollary 2 from Theorem 4, one can obtain the following corollary from Theorem A.
Corollary A. Let $a \in[2 \pi / 5, \pi)$. Then arbitrary sequences of complex numbers $\left\{\lambda_{n, 0}\right\}_{n \geq 1}$ and $\left\{\lambda_{n, 1}\right\}_{n \geq 1}$ are the spectra of some boundary value problems $\mathcal{L}_{0}(q)$ and $\mathcal{L}_{1}(q)$, respectively, if and only if their convergence exponents are equal to $1 / 2$, and the functions $\theta_{j}(\rho), j=0,1$, determined by (88) and (89) satisfy the following conditions:

$$
\theta_{j}(x) \in L_{2}(-\infty, \infty), \quad\left|\theta_{j}(\rho)\right| \leq C \exp ((\pi-a)|\rho|), \quad \theta_{j}(-\rho)=(-1)^{j} \theta_{j}(\rho)
$$

## Appendix B

Let $\pi / 2 \leq a_{1} \leq a_{2}<\pi$ and consider the boundary value problem $\mathcal{B}$ for the equation

$$
\begin{equation*}
-y^{\prime \prime}(x)+q_{1}(x) y\left(x-a_{1}\right)+q_{2}(x) y\left(x-a_{2}\right)=\lambda y(x), \quad 0<x<\pi, \tag{90}
\end{equation*}
$$

where $q_{\nu}(x)=0$ on $\left(0, a_{\nu}\right)$ and $q_{\nu}(x) \in L_{2}\left(a_{\nu}, \pi\right)$, along with the two-point boundary conditions of the general form:

$$
\begin{equation*}
U_{\nu}(y):=h_{0, \nu} y(0)+h_{1, \nu} y^{\prime}(0)+H_{0, \nu} y(\pi)+H_{1, \nu} y^{\prime}(\pi)=0, \quad \nu=1,2 \tag{91}
\end{equation*}
$$

with arbitrary complex coefficients $h_{j, \nu}$ and $H_{j, \nu}$. Let $S(x, \lambda)$ and $C(x, \lambda)$ be solutions of equation (90) under the initial conditions $S(0, \lambda)=C^{\prime}(0, \lambda)=0$ and $S^{\prime}(0, \lambda)=C(0, \lambda)=1$. The spectrum $\operatorname{sp}(\mathcal{B})$ of the problem $\mathcal{B}$ coincides with zeros of its characteristic function

$$
\Delta(\lambda):=\left|\begin{array}{cc}
U_{1}(S) & U_{1}(C) \\
U_{2}(S) & U_{2}(C)
\end{array}\right|=\left|\begin{array}{ll}
h_{1,1}+H_{0,1} \Delta_{0}(\lambda)+H_{1,1} \Delta_{1}(\lambda) & h_{0,1}+H_{0,1} \Theta_{0}(\lambda)+H_{1,1} \Theta_{1}(\lambda) \\
h_{1,2}+H_{0,2} \Delta_{0}(\lambda)+H_{1,2} \Delta_{1}(\lambda) & h_{0,2}+H_{0,2} \Theta_{0}(\lambda)+H_{1,2} \Theta_{1}(\lambda)
\end{array}\right|,
$$

where $\Delta_{j}(\lambda)=S^{(j)}(\pi, \lambda)$ is the characteristic function of the boundary value problem $\mathcal{B}_{j}:=$ $\mathcal{B}_{j}\left(q_{0}, q_{1}\right)$ consisting of equation (90) and the boundary conditions (22), while $\Theta_{j}(\lambda)=C^{(j)}(\pi, \lambda)$ is the characteristic function of the problem for equation (90) under the boundary conditions

$$
y^{\prime}(0)=y^{(j)}(\pi)=0
$$

The following proposition means, actually, that specification of $\operatorname{sp}(\mathcal{B})$ gives no additional information on the potentials $q_{1}(x)$ and $q_{2}(x)$ after specifying the spectra $\operatorname{sp}\left(\mathcal{B}_{0}\right)$ and $\operatorname{sp}\left(\mathcal{B}_{1}\right)$.

Proposition B. Specification of the functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$ along with the coefficients $h_{j, \nu}$ and $H_{j, \nu}$ for $j=0,1$ and $\nu=1,2$ uniquely determines the function $\Delta(\lambda)$.

Proof. It is sufficient to show that specification of the functions $\Delta_{0}(\lambda)$ and $\Delta_{1}(\lambda)$ uniquely determines the functions $\Theta_{0}(\lambda)$ and $\Theta_{1}(\lambda)$. Indeed, analogously to Lemma 2 one can obtain the following representations

$$
\begin{align*}
& \Delta_{0}(\lambda)=\Delta_{0}^{0}(\lambda)+\sum_{\nu=1}^{2} \int_{0}^{\pi-a_{\nu}} w_{0, \nu}(x) \frac{\cos \rho x}{\rho^{2}} d x, \quad \Delta_{0}^{0}(\lambda):=\frac{\sin \rho \pi}{\rho}-\sum_{\nu=1}^{2} \omega_{\nu} \frac{\cos \rho\left(\pi-a_{\nu}\right)}{\rho^{2}},  \tag{92}\\
& \Delta_{1}(\lambda)=\Delta_{1}^{0}(\lambda)+\sum_{\nu=1}^{2} \int_{0}^{\pi-a_{\nu}} w_{1, \nu}(x) \frac{\sin \rho x}{\rho} d x, \quad \Delta_{1}^{0}(\lambda):=\cos \rho \pi+\sum_{\nu=1}^{2} \omega_{\nu} \frac{\sin \rho\left(\pi-a_{\nu}\right)}{\rho}, \tag{93}
\end{align*}
$$

where $\rho^{2}=\lambda$ and

$$
\omega_{\nu}=\frac{1}{2} \int_{a_{\nu}}^{\pi} q_{\nu}(x) d x, \quad w_{j, \nu}(x)=\frac{1}{4}\left(q_{\nu}\left(\frac{\pi+x+a_{\nu}}{2}\right)+(-1)^{j} q_{\nu}\left(\frac{\pi-x+a_{\nu}}{2}\right)\right)
$$

Moreover, in a similar way, one can get also the representations

$$
\Theta_{0}(\lambda)=\Delta_{1}^{0}(\lambda)-\sum_{\nu=1}^{2} \int_{0}^{\pi-a_{\nu}} w_{1, \nu}(x) \frac{\sin \rho x}{\rho} d x, \Theta_{1}(\lambda)=-\lambda \Delta_{0}^{0}(\lambda)+\sum_{\nu=1}^{2} \int_{0}^{\pi-a_{\nu}} w_{0, \nu}(x) \cos \rho x d x
$$

Thus, we arrive at the relations

$$
\Theta_{0}(\lambda)=2 \Delta_{1}^{0}(\lambda)-\Delta_{1}(\lambda), \quad \Theta_{1}(\lambda)=\lambda\left(\Delta_{0}(\lambda)-2 \Delta_{0}^{0}(\lambda)\right),
$$

which finish the proof because $\Delta_{j}^{0}(\lambda)$ is determined by specifying $\Delta_{j}(\lambda)$.
Obviously, no set of spectra determines the functions $q_{1}(x)$ and $q_{2}(x)$ separately if $a_{1}=a_{2}$. The following example shows that they cannot be completely distinguished also when $a_{1} \neq a_{2}$.

Example B. Let

$$
q_{1}(x)=\left\{\begin{array}{rc}
0, & a_{1}<x<\frac{a_{1}+a_{2}}{2},  \tag{94}\\
1, & \frac{a_{1}+a_{2}}{2}<x<\frac{a_{1}+\pi}{2}, \\
-1, & \frac{a_{1}+\pi}{2}<x<\pi-\frac{a_{2}-a_{1}}{2}, \\
0, & \pi-\frac{a_{2}-a_{1}}{2}<x<\pi,
\end{array} \quad q_{2}(x)=\left\{\begin{array}{rc}
-1, & a_{2}<x<\frac{a_{2}+\pi}{2}, \\
1, & \frac{a_{2}+\pi}{2}<x<\pi,
\end{array}\right.\right.
$$

and, hence,

$$
\begin{equation*}
q_{2}(x)=-q_{1}\left(x-\frac{a_{2}-a_{1}}{2}\right), \quad a_{2}<x<\pi . \tag{95}
\end{equation*}
$$

Then the spectra of the problems $\mathcal{B}_{0}\left(q_{0}, q_{1}\right)$ and $\mathcal{B}_{1}\left(q_{0}, q_{1}\right)$ coincide with the ones of $\mathcal{B}_{0}(0,0)$ and $\mathcal{B}_{1}(0,0)$, respectively. Indeed, according to the relations

$$
\mathfrak{L}(\rho):=\Delta_{1}(\lambda)+i \rho \Delta_{0}(\lambda), \quad \Delta_{0}(\lambda)=\frac{\mathfrak{L}(\rho)-\mathfrak{L}(-\rho)}{2 i \rho}, \quad \Delta_{1}(\lambda)=\frac{\mathfrak{L}(\rho)+\mathfrak{L}(-\rho)}{2},
$$

specification of both spectra is equivalent to specification of the function $\mathfrak{L}(\rho)$, which, in turn, is the characteristic function of the Regge-type problem for equation (90) along with the boundary conditions

$$
\begin{equation*}
y(0)=y^{\prime}(\pi)+i \rho y(\pi)=0 . \tag{96}
\end{equation*}
$$

On the other hand, the following representation holds (see, e.g., [19]):

$$
\begin{equation*}
\mathfrak{L}(\rho) \exp (-i \rho \pi)-1=\sum_{\nu=1}^{2} \frac{\omega_{\nu}}{i \rho} \exp \left(-i \rho a_{\nu}\right)-\sum_{\nu=1}^{2} \frac{\exp \left(i \rho a_{\nu}\right)}{2 i \rho} \int_{a_{\nu}}^{\pi} q_{\nu}(x) \exp (-2 i \rho x) d x \tag{97}
\end{equation*}
$$

which also can be easily obtained by using (92) and (93). According to (94), we have

$$
2 \omega_{1}=\int_{\frac{a_{1}+a_{2}}{2}}^{\frac{a_{1}+\pi}{2}} d x-\int_{\frac{a_{1}+\pi}{2}}^{\pi-\frac{a_{2}-a_{1}}{2}} d x=0, \quad 2 \omega_{2}=-\int_{a_{2}}^{\frac{a_{2}+\pi}{2}} d x+\int_{\frac{a_{2}+\pi}{2}}^{\pi} d x=0 .
$$

Moreover, by virtue of (94) and (95), we get

$$
\begin{aligned}
\sum_{\nu=1}^{2} \exp \left(i \rho a_{\nu}\right) \int_{a_{\nu}}^{\pi} q_{\nu}(x) \exp (-2 i \rho x) d x & =\exp \left(i \rho a_{1}\right) \int_{\frac{a_{1}+a_{2}}{2}}^{\pi-\frac{a_{2}-a_{1}}{2}} q_{1}(x) \exp (-2 i \rho x) d x \\
& -\exp \left(i \rho a_{2}\right) \int_{a_{2}}^{\pi} q_{1}\left(x-\frac{a_{2}-a_{1}}{2}\right) \exp (-2 i \rho x) d x=0 .
\end{aligned}
$$

Hence, according to (97), we have $\mathfrak{L}(\rho)=\exp (i \rho \pi)$, which is the characteristic function of the problem (90), (96) with the zero potentials. Thus, specification of the spectra of the problems $\mathcal{B}_{0}\left(q_{1}, q_{2}\right)$ and $\mathcal{B}_{1}\left(q_{1}, q_{2}\right)$ does not uniquely determine the functions $q_{1}(x)$ and $q_{2}(x)$.

Finally, we note that Example B refutes both Theorem 3.1 in [19] and Theorem 3.1 in [20]. Moreover, according to this counterexample along with Proposition B, the functions $q_{1}(x)$ and $q_{2}(x)$ cannot be uniquely determined by specifying any set of the spectra of boundary value problems having the form (90) and (91).

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