The Numerical Simulation of Friction Constrained Motions (II): Multiple degrees of freedom models

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Abstract- In a previous article (ref. [1]) the authors discussed the application of operator-splitting methods to the time-discretization of those mathematical relations describing the behavior of elasto-dynamical systems with friction, focusing on one-degree of freedom models. The main goal of the present article is to generalize the methodology discussed in ref. [1]; there are no conceptual difficulty at doing so, the main issue being the computation of a vector-valued multiplier modeling the friction forces (or part of them). An iterative method allowing the computation of this multiplier will be discussed and the results of numerical experiments will be presented.

Keywords-friction constrained motions, multiple degrees of freedom elasto-dynamical systems, dynamical multiplier, operator-splitting.

1 Introduction

In a previous article (ref. [1]), the authors discussed the numerical simulation of elasto-dynamical systems with friction, in the particular case of one degree of freedom models. The methodology they advocate in ref. [1] relies on a time-discretization by operator-splitting, combined with an explicit-implicit scheme to treat friction while elasticity is handled via a non-dissipative second order accurate centered scheme. This approach is generalized to higher dimensions, using an equivalent formulation of the problem involving a vector-valued multiplier modeling the friction forces (or part of them).

2 Modeling of friction constrained motions: Splitting of the model

Some remote manipulator system simulators use *finite number of degrees of freedom* models, like the one below to describe *friction constrained motions*:

$$\begin{cases} M\ddot{X} + AX + C(\operatorname{sgn}(\dot{X}) - \gamma(\dot{X})) = f \quad \text{on } (0,T), \\ X(0) = X_0, \ \dot{X}(0) = V_0, \end{cases} \tag{2.1}$$

where in equation (2.1): X is a *displacement* (here $X(t) \in \mathbf{R}^d$), the *mass matrix* M is symmetric and positive definite, the *stiffness matrix* A is symmetric and positive semi-definite, the *friction matrix* C is diagonal, i.e.

$C = diag(c_1, \cdots, c_d)$, with $c_i \geq 0, \forall i = 1, \cdots, d$ and $\sum_{i=1}^d c_i > 0$, $\text{sgn}(V) = \{\text{sgn}(v_i)\}_{i=1}^d$, $\forall\ V = \{v_i\}_{i=1}^d \in \mathbf{R}^d$, $\gamma(V) = \{\gamma_i(v_i)\}_{i=1}^d$, $\forall\ V = \{v_i\}_{i=1}^d \in \mathbf{R}^d$, γ_i being a *nondecreasing Lipschitz continuous* function vanishing at 0 and such that $\lim_{\xi \mapsto \pm\infty} \gamma_i(\xi) = \pm\beta_i$, with $0 < \beta_i < 1$ (typical functions β_i are described in [1], Section 2), f is an *external force*, $X_0, V_0 \in \mathbf{R}^d$. A rigorous equivalent formulation of (2.1) is given by

$$\begin{cases} \dot{X} = V & \text{on } (0, T), \\ M\dot{V} + AX + C(\lambda - \gamma(V)) = f & \text{on } (0, T), \\ C\lambda(t) \cdot V(t) = \sum_{i=1}^d c_i |v_i(t)|,\ \lambda(t) \in \Lambda & \text{a.e. on } (0, T), \\ X(0) = X_0,\ V(0) = V_0, \end{cases} \tag{2.2}$$

with Λ the closed convex non-empty subset of $\mathbf{R}^d$ defined by

$$\Lambda = \{\mu | \mu \in \mathbf{R}^d, |\mu_i| \leq 1, \forall i = 1, \cdots, d\}$$

and $a \cdot b = \sum_{i=1}^d a_i b_i, \forall a = \{a_i\}_{i=1}^d, b = \{b_i\}_{i=1}^d \in \mathbf{R}^d$. The vector-valued function $C(\lambda - \gamma(V))$ models the friction forces present in the system. Suppose that T is finite and let $\tau = T/N$. In order to solve problem (2.2), we advocate the following *Lie's scheme* (where $t^n = n\tau$):

$$X^0 = X_0, \quad V^0 = V_0; \tag{2.3}$$

for $n = 1, \cdots, N, X^n$ and V^n being known, solve

$$\begin{cases} M\dot{V} + C(\lambda - \gamma(V)) = f & \text{on } \ (t^n, t^{n+1}), \\ C\lambda(t) \cdot V(t) = \sum_{i=1}^d c_i |v_i(t)|, & \lambda(t) \in \Lambda \text{ a.e. on } (t^n, t^{n+1}), \\ \dot{X} = 0 & \text{on } \ (t^n, t^{n+1}), \\ V(t^n) = V^n, \quad X(t^n) = X^n, \end{cases} \tag{2.4}$$

and set

$$V^{n+1/2} = V(t^{n+1}), \quad X^{n+1/2} = X^n, \tag{2.5}$$

next solve

$$\begin{cases} M\dot{V} + AX = 0 & \text{on } \ (t^n, t^{n+1}), \\ \dot{X} = V & \text{on } \ (t^n, t^{n+1}), \\ V(t^n) = V^{n+1/2}, & X(t^n) = X^{n+1/2}, \end{cases} \tag{2.6}$$

and set

$$V^{n+1} = V(t^{n+1}), \quad X^{n+1} = X(t^{n+1}). \tag{2.7}$$

Problem (2.6)(the elastic step), is *equivalent* to

$$\begin{cases} M\ddot{X} + AX = 0 \quad \text{on } \ (t^n, t^{n+1}), \\ X(t^n) = X^{n+1/2}, \quad \dot{X}(t^n) = V^{n+1/2}, \end{cases} \tag{2.8}$$

while (2.7) reads as

$$X^{n+1} = X(t^{n+1}), \quad V^{n+1} = \dot{X}(t^{n+1}). \tag{2.9}$$

Problems (2.6), (2.8) is a standard one whose numerical solution is a well-documented topic. On the other hand, solving problem (2.4) (the friction step) is a more critical issue which is the main study of this article and is addressed in the following section.

3 Time-discretization of problem (2.4)

Problem (2.4) is a special case of

$$\begin{cases} M\dot{W} + C(\lambda - \gamma(W)) = f & \text{on } (t_0, t_f), \\ C\lambda(t) \cdot W(t) = \sum_{i=1}^d c_i |w_i(t)|, & \lambda(t) \in \Lambda \text{ a.e. on } (t_0, t_f), \\ W(t_0) = W_0. \end{cases} \tag{3.10}$$

In order to time-discretize (3.10), we advocate the following implicit-explicit scheme (with $\tau_f = (t_f - t_0)/P$):

$$W^0 = W_0; \tag{3.11}$$

for $p = 1, \cdots, P, W^{p-1}$ being known solve the following system of equations

$$\begin{cases} M\frac{W^p - W^{p-1}}{\tau_f} + C\ \lambda^p = C\gamma(W^{p-1}) + f^p, \\ C\ \lambda^p \cdot W^p = \sum_{i=1}^d c_i |w_i^p|, \quad \lambda^p \in \Lambda, \end{cases} \tag{3.12}$$

where $f^p = f(t_0 + p\tau_f)$ (or an approximation of it). Using *compactness* arguments we can show that

$$\lim_{\tau_f \to 0} \max_{1 \le p \le P} \|W^p - W(t_0 + p\tau_f)\| = 0,$$

and weak-* convergence to λ in $L^\infty(t_0, t_f; \mathbf{R}^d)$, for the sequence $\{\{\lambda^p\}_{p=1}^P\}_p$, where $\{W, \lambda\}$ is the unique solution of system (3.10). The iterative solution of system such as (3.12) will be briefly discussed in the following section.

4 Iterative Solution of System (3.12)

Let $b^p = MW^{p-1} + \tau_f C\gamma(W^{p-1}) + \tau_f f^p$, then drop the superscript p in problem (3.12). It takes then the following form:

$$\begin{cases} MW + \tau_f C\ \lambda = b, \\ C\ \lambda \cdot W = \sum_{i=1}^d c_i |w_i|, \quad \lambda \in \Lambda. \end{cases} \tag{4.13}$$

If $d = 1$ computing the *closed form solution* of problem (4.13) is easy as shown in ref. [1]. On the other hand, if $d \ge 2$, then we must rely on *iterative techniques*. A simple one is provided by the following algorithm

$$\lambda^0 \ \text{ given in } \ \Lambda; \tag{4.14}$$

for $k \ge 0$, λ^k being known, solve

$$MW^k = b - \tau_f C\lambda^k \tag{4.15}$$

and update λ^k via

$$\lambda^{k+1} = P_\Lambda(\lambda^k + \rho CW^k). \tag{4.16}$$

In (4.16), the *projection* operator $P_\Lambda : \mathbf{R}^d \to \Lambda$ is defined by

$$P_\Lambda(\mu) = \{\min(1, \max(-1, \mu_i))\}_{i=1}^d, \ \forall \mu = \{\mu_i\}_{i=1}^d \in \mathbf{R}^d. \tag{4.17}$$

The set Λ being closed, convex (and non-empty), operator P_Λ is a *contraction*. Concerning the convergence of algorithm (4.14)-(4.16), we then have the following

Theorem 4.1 *Suppose that*

$$0 < \rho < \frac{2}{\tau_f \beta_d}, \tag{4.18}$$

where β_d is the largest eigenvalue of matrix $M^{-1}C^2$; we have then, $\forall \lambda^0 \in \Lambda$,

$$\lim_{k \to +\infty} \{W^k, \lambda^k\} = \{W, \lambda\}, \tag{4.19}$$

where $\{W, \lambda\}$ is the solution of system (4.13).

An estimate of the speed of convergence of (4.14)-(4.16) will be given in a forthcoming publication (ref. [2]).

5 Numerical experiments

We will describe in this section the numerical results obtained when applying the methodology of the previous sections to a *2-degree of freedom* model problem (2.1), (2.2). We take $T = 4$ and

- the mass matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, the stiffness matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, the friction matrix $C = I$,
- $\gamma = \{\gamma_i\}_{i=1}^2$ with $\beta_i = \frac{1}{3}$ and $\varepsilon_i = 10^{-1}, i = 1, 2.$ (see [1], Section 2),
- the forcing term $f = \{f_i\}_{i=1}^2$, where

$$f_1(t) = \begin{cases} 2(t - \frac{t^2}{2}) - 1 - \gamma_1(1-t) & \text{if } 0 \le t \le 1, \\ 1 + (t - \frac{3}{2}) - \gamma_1(0) & \text{if } 1 \le t \le 2, \\ 3t^3 - 23t^2 + 70t - \frac{238}{3} - \gamma_1(4(t-3)(t-2)) & \text{if } 2 \le t \le 3, \\ \frac{t^3}{3} - 3t^2 + 6t - \frac{17}{6} - \gamma_1(0) & \text{if } 3 \le t \le 4, \end{cases}$$

and

$$f_2(t) = \begin{cases} \frac{t^2}{2} - 2t - \gamma_2(0) & \text{if } 0 \le t \le 1, \\ \frac{1}{2} - t - \gamma_2(0) & \text{if } 1 \le t \le 2, \\ -2t^3 + 16t^2 - 36t + \frac{163}{6} - \gamma_2(1 - (t-3)^2) & \text{if } 2 \le t \le 3, \\ \frac{-2}{3}t^3 + 6t^2 - 20t + \frac{175}{6} - \gamma_2(1 - (t-3)^2) & \text{if } 3 \le t \le 4. \end{cases}$$

For the above data, the solution of problem (2.1) is given by

$$v_1(t) = \begin{cases} 1 - t & \text{if } 0 \le t < 1, \\ 0 & \text{if } 1 \le t < 2, \\ 4(t-3)(t-2) & \text{if } 2 \le t < 3, \\ 0 & \text{if } 3 \le t \le 4, \end{cases}$$

$$v_2(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ 1 - (t-3)^2 & \text{if } 2 \le t \le 4, \end{cases}$$

and

$$x_1(t) = \begin{cases} t - \frac{t^2}{2} & \text{if } 0 \le t < 1, \\ \frac{1}{2} & \text{if } 1 \le t < 2, \\ \frac{1}{2} + 4[\frac{1}{3}(t^3 - 8) - \frac{5}{2}(t^2 - 4) + 6(t-2)] & \text{if } 2 \le t < 3, \\ \frac{-1}{6} & \text{if } 3 \le t \le 4, \end{cases}$$

$$x_2(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ (t-2) - \frac{1}{3}((t-3)^3 + 1) & \text{if } 2 \le t \le 4, \end{cases}$$

while the corresponding function λ is given by

$$\lambda_1(t) = \begin{cases} 1 & \text{if } 0 < t < 1, \\ t - \frac{3}{2} & \text{if } 1 < t < 2, \\ -1 & \text{if } 2 < t < 3, \\ \frac{-1}{2} & \text{if } 3 < t < 4, \end{cases}$$

and

$$\lambda_2(t) = \begin{cases} 1 - t & \text{if } 0 < t < 2, \\ 1 & \text{if } 2 < t < 4. \end{cases}$$

To solve problem (2.1), we have used the splitting scheme (2.3)− (2.7). The subproblem (2.4) is solved via scheme (3.11), (3.12), while the subproblem (2.6) is solved via a classical finite difference centered scheme. The following results have been obtained with $\tau = 0.003$. On Figs. 1-6, we have shown the graphs of the approximation of $\dot{X}$, X, λ, respectively. On Figs. 7-9, we have shown the L^2-error, on $\dot{X}$, X, λ, as functions

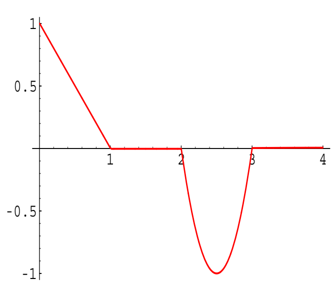

Fig. 1. The computed $v_1(t)$

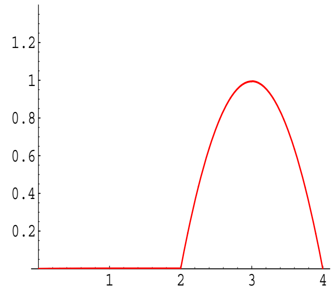

Fig. 2. The computed $v_2(t)$

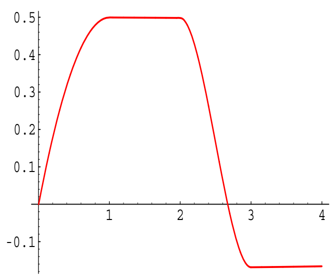

Fig. 3. The computed $x_1(t)$

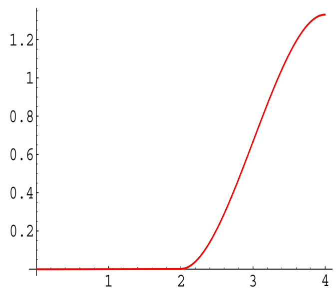

Fig. 4. The computed $x_2(t)$

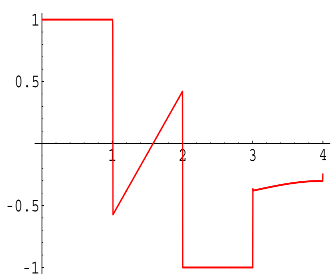

Fig. 5. The computed $\lambda_1(t)$

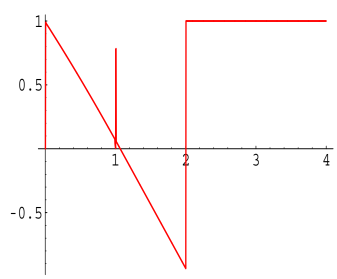

Fig. 6. The computed $\lambda_2(t)$

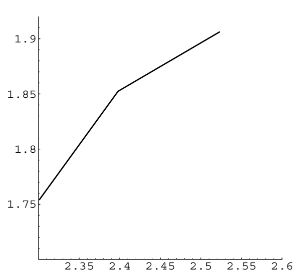

Fig. 7.
L^2-error on $\dot{X}$ versus τ

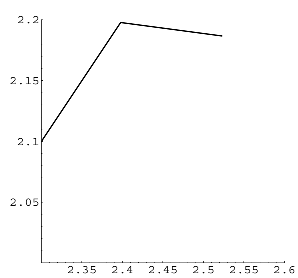

Fig. 8.
L^2-error on X versus τ

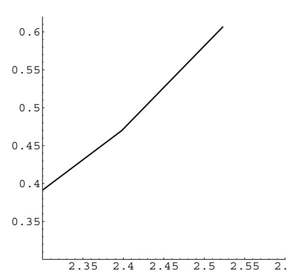

Fig. 9.
L^2-error on λ versus τ

of τ. We clearly have *first order accuracy.* We observe also that the computed discrete multipliers do not exhibit spurious oscillations, as it is the case with other discretization schemes.

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References

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