# An Indefinite Convection-Diffusion Operator With Real Spectrum 

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## 1 Introduction

For $0<\varepsilon<2$, we consider the operator

$$
\begin{equation*}
(H f)(\theta):=\varepsilon \frac{\partial}{\partial \theta}\left(\sin (\theta) \frac{\partial f}{\partial \theta}\right)+\frac{\partial f}{\partial \theta} \tag{1}
\end{equation*}
$$

initially defined on all $\mathcal{C}^{2}$ periodic functions $f \in L^{2}(-\pi, \pi)$; the exact domain is given by taking the closure of the operator defined on the above functions. That such a closure exists will be shown later. In a recent paper [1] Benilov, O'Brien and Sazonov showed that the equation

$$
\begin{equation*}
\frac{\partial f}{\partial t}=H f \tag{2}
\end{equation*}
$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder.

We shall show that the eigenvalue problem

$$
\begin{equation*}
-i H f=\lambda f \tag{3}
\end{equation*}
$$

has only real eigenvalues, which were shown to exist by Davies in [3]. In the same paper, he showed that the spectrum of $-i H$ is equal to the set of its eigenvalues, so this implies that the spectrum is real. This was conjectured in [1], and Chugunova and Pelinovsky proved in [2] that all but finitely many eigenvalues are real, and gave numerical evidence that all eigenvalues are real. Our approach is to show that it is sufficient to consider $H$ acting on the Hardy space $H^{2}(-\pi, \pi)$ and analytically continue any solution of (3) to the unit disc, where the corresponding ODE (8) is now self-adjoint on $[0,1]$
with regular singularities at the end points. In order to do this we make use of a bound on the Fourier coefficients proved by Davies in [3].

As in [3], by expanding $f \in L^{2}(-\pi, \pi)$ in the form

$$
f(\theta)=\frac{1}{\sqrt{2 \pi}} \sum_{n \in \mathbf{Z}} v_{n} \mathrm{e}^{i n \theta},
$$

one may rewrite the eigenvalue problem in the form $A v=\lambda v$, where $A=-i H$ is given by

$$
(A v)_{n}=\frac{\varepsilon}{2} n(n-1) v_{n-1}-\frac{\varepsilon}{2} n(n+1) v_{n+1}+n v_{n} .
$$

Here we have identified $l^{2}(\mathbf{Z})$ and $L^{2}(-\pi, \pi)$ using the Fourier transform $\mathcal{F}: l^{2}(\mathbf{Z}) \rightarrow L^{2}(-\pi, \pi)$. We have

$$
A=-i \mathcal{F}^{-1} H \mathcal{F}
$$

and define $\operatorname{Dom}(H)=\mathcal{F}(\operatorname{Dom}(A))$.
The (unbounded) tridiagonal matrix $A$ is of the form

$$
A=\left(\begin{array}{ccc}
A_{-} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & A_{+}
\end{array}\right)
$$

where $A_{-}$acts in $l^{2}\left(\mathbf{Z}_{-}\right)$, the central 0 acts in $\mathbf{C}$ and $A_{+}$acts in $l^{2}\left(\mathbf{Z}_{+}\right)$. We assume that $A_{+}$has its natural maximal domain

$$
\mathcal{D}=\left\{v \in l^{2}\left(\mathbf{Z}_{+}\right): A_{+} v \in l^{2}\left(\mathbf{Z}_{+}\right)\right\} .
$$

Davies has shown in [3] that $A_{+}$is closed and that $\mathcal{D}$ is the closure under the graph norm of $A_{+}$of the subspace consisting of those $v \in l^{2}\left(\mathbf{Z}_{+}\right)$that have finite support. Let $\tau$ be the natural identification between $l^{2}\left(\mathbf{Z}_{+}\right)$and $l^{2}\left(\mathbf{Z}_{-}\right)$; then $\operatorname{Dom}(A)=\tau(\mathcal{D}) \oplus \mathbf{C} \oplus \mathcal{D}$ and $\tau$ induces a unitary equivalence between $A_{+}$and $A_{-}$. Therefore, in order to prove that all eigenvalues of $A$ are real, we only need to prove that all eigenvalues of $A_{+}$are real. The Fourier transform identifies $l^{2}\left(\mathbf{Z}_{+}\right)$with $\left\{f \in H^{2}(-\pi, \pi): \int_{-\pi}^{\pi} f(\theta) \mathrm{d} \theta=0\right\}$.

Let $H_{0}$ be the restriction of $H$ to $\mathcal{C}_{\text {per }}^{2}([-\pi, \pi])$, which is clearly a subspace of $\operatorname{Dom}(H)$. We now show that $H$ is the closure of $H_{0}$.

Proposition 1.1 Where $H$ and $H_{0}$ are as above, $H$ is the closure of $H_{0}$.
Proof It follows from Davies' result on the domain of $A_{+}$that the trigonometric polynomials are dense in $\operatorname{Dom}(H)$ with respect to the graph norm. Since the trigonometric polynomials are contained in $\mathcal{C}_{\text {per }}^{2}([-\pi, \pi])$, this space is also dense in $\operatorname{Dom}(H)$, which is closed in graph norm since $\operatorname{Dom}(A)$ is.

## 2 Reality Of The Eigenvalues

If $A_{+} v=\lambda v$, then $v$ is a solution of the recurrence relation

$$
\begin{equation*}
\frac{\varepsilon}{2}(n+1)(n+2) v_{n+2}+(\lambda-n-1) v_{n+1}-\frac{\varepsilon}{2} n(n+1) v_{n}=0 \tag{4}
\end{equation*}
$$

satisfying the initial condition $\varepsilon v_{2}=(1-\lambda) v_{1}$. We shall study the generating function, $\sum_{k=1}^{\infty} v_{k} z^{k}$ of $\left(v_{k}\right)$.

Lemma 2.1 Let $v \in l^{2}\left(\mathbf{Z}_{+}\right)$be such that $A_{+} v=\lambda v$. Then the function $u(z):=\sum_{k=1}^{\infty} v_{k} z^{k}$, defined for $|z|<1$, satisfies the differential equation

$$
\begin{equation*}
u^{\prime \prime}-2 \frac{z+1 / \varepsilon}{(1-z)(1+z)} u^{\prime}+\frac{2 \lambda / \varepsilon}{z(1-z)(1+z)} u=0 \tag{5}
\end{equation*}
$$

Proof The constant term in

$$
z(1-z)(1+z) u^{\prime \prime}-2(z+1 / \varepsilon) z u^{\prime}+(2 \lambda / \varepsilon) u
$$

is clearly 0 , and the coefficient of $z$ is

$$
2 v_{2}-2 v_{1} / \varepsilon+2 \lambda v_{1} / \varepsilon=2\left(v_{2}-\frac{(1-\lambda)}{\varepsilon} v_{1}\right)=0
$$

The coefficient of $z^{n}$ is

$$
\begin{array}{r}
n(n+1) v_{n+1}-(n-1)(n-2) v_{n-1}-\frac{2}{\varepsilon} n v_{n}-2(n-1) v_{n-1}+\frac{2 \lambda}{\varepsilon} v_{n} \\
=\frac{2}{\varepsilon}\left[\frac{\varepsilon}{2} n(n+1) v_{n+1}+(\lambda-n) v_{n}-\frac{\varepsilon}{2} n(n-1) v_{n-1}\right]=0
\end{array}
$$

for $n \geq 2$.
From here on we assume that $\lambda \in \mathbf{C}$ is an eigenvalue of the operator $A_{+}$ and $v$ is a corresponding non-zero eigenvector. In [3], Davies proved that there exist constants $b, m$ such that

$$
\begin{equation*}
\|v\|_{\infty, c} \leq b|\lambda|^{m}\|v\|_{2} \tag{6}
\end{equation*}
$$

where $c=1+1 / \varepsilon$ and $\|w\|_{\infty, c}:=\sup \left\{\left|w_{n}\right| n^{c}: 1 \leq n<\infty\right\}$. Since $\varepsilon>0$, this implies that $v \in l^{1}\left(\mathbf{Z}_{+}\right)$, and hence that $\sum_{k=1}^{\infty} v_{k} z^{k}$ is absolutely convergent for $|z| \leq 1$. The equation (5) can be written in the form of Heun's equation

$$
\begin{equation*}
u^{\prime \prime}+\left(\frac{\gamma}{z}+\frac{\delta}{z-1}+\frac{\epsilon}{z-a}\right) u^{\prime}+\frac{\alpha \beta z-\mu}{z(z-1)(z-a)} u=0 \tag{7}
\end{equation*}
$$

with $\alpha=1, \beta=0, \gamma=0, \delta=1+1 / \varepsilon, \epsilon=1-1 / \varepsilon, a=-1$ and $\mu=2 \lambda / \varepsilon$. This is a Fuchsian equation with four regular singular points, at $0,1,-1$ and $\infty$, with $\{0,1-\gamma\},\{0,1-\delta\},\{0,1-\epsilon\}$ and $\{\alpha, \beta\}$ as the roots of the corresponding indicial equations (for the Frobenius series at each regular singular point). For background information on Heun's equation, see [4].

Lemma 2.2 Suppose that $0<\varepsilon<2,1 / \varepsilon \notin \mathbf{Z}, \lambda \in \mathbf{C}$ is an eigenvalue of $A_{+}$ and $v \in l^{2}\left(\mathbf{Z}_{+}\right)$is a corresponding non-zero eigenvector. Then there exists a solution $u$ of (7) which is analytic in an open set containing $[0,1]$ and such that $u(z)=\sum_{k=1}^{\infty} v_{k} z^{k}$ for all $z$ such that $|z| \leq 1$.

Proof Define $u(z)=\sum_{k=1}^{\infty} v_{k} z^{k}$ on $\{z \in \mathbf{C}:|z|<1\}$. Let $u_{1}$ be the solution of (7) with exponent 0 about 1 and $u_{2}$ be the solution with exponent $-1 / \varepsilon$ about 1 . Let $U$ be the intersection of the open discs of unit radius about 0 and 1. The space of solutions of (7) in $U$ is two-dimensional, and $u, u_{1}$ and $u_{2}$ lie in this space. Hence there exist constants $a, b$ such that $u=a u_{1}+b u_{2}$ in $U$. Since $v \in l^{1}\left(\mathbf{Z}_{+}\right), u(z)$ converges to a finite limit as $z \rightarrow 1$ in $U$. Also $u_{1}(1)$ is finite, but $u_{2}(z) \rightarrow \infty$ as $z \rightarrow 1$ in $U$. Therefore we must have $b=0$ and $u=a u_{1}$ in $U$. Let $W$ be the union of the open discs of unit radius about 0 and 1 . We now extend $u$ to all of $W$ by $u(z)=a u_{1}(z)$ on the open disc of unit radius about 1 . Now $u$ is an analytic solution of (7) on $W$, which is an open set containing $[0,1]$, such that $u(z)=\sum_{k=1}^{\infty} v_{k} z^{k}$ for all $z$ such that $|z| \leq 1$.

Theorem 2.3 Suppose that $0<\varepsilon<2,1 / \varepsilon \notin \mathbf{Z}$ and $\lambda \in \mathbf{C}$ is an eigenvalue of $A_{+}$. Then $\lambda \in \mathbf{R}$.

Proof Let $v \in l^{2}\left(\mathbf{Z}_{+}\right)$be a non-zero eigenvector corresponding to $\lambda$. Let $u$ be as in Lemma 2.2 and put $\mu=2 \lambda / \varepsilon$. Following [5], the equation (77) can be written as

$$
\begin{equation*}
-\left(p u^{\prime}\right)^{\prime}+q u=\mu w u \tag{8}
\end{equation*}
$$

on the complex plane cut along $[1, \infty)$ and $(-\infty, 0]$, where

$$
\begin{aligned}
& p(z)=z^{\gamma}(1-z)^{\delta}(z-a)^{\epsilon}=(1-z)^{1+1 / \varepsilon}(z+1)^{1-1 / \varepsilon} \\
& q(z)=\alpha \beta z^{\gamma}(1-z)^{\delta-1}(z-a)^{\epsilon-1}=0 \\
& w(z)=z^{\gamma-1}(1-z)^{\delta-1}(z-a)^{\epsilon-1}=z^{-1}(1-z)^{1 / \varepsilon}(z+1)^{-1 / \varepsilon} .
\end{aligned}
$$

We now restrict $u$ to $[0,1]$. It is clear that $u \in \mathcal{C}^{\infty}([0,1])$ and

$$
-\left(p u^{\prime}\right)^{\prime}=\mu w u
$$

on $(0,1)$. Note that $p>0$ on $[0,1)$ and $w>0$ on $(0,1)$ with $p(x), w(x) \rightarrow 0$ as $x \rightarrow 1$ from below. Since $u$ has a zero of order 1 at 0 and $w$ has a pole of order 1 at $0, w u \in \mathcal{C}([0,1])$. Therefore $|u|^{2} w \in L^{1}(0,1)$. Now

$$
\begin{aligned}
\mu \int_{0}^{1}|u(x)|^{2} w(x) \mathrm{d} x & =\lim _{n \rightarrow \infty}\left\{-\int_{1 / n}^{1-1 / n}\left(p u^{\prime}\right)^{\prime}(x) \overline{u(x)} \mathrm{d} x\right\} \\
& =\lim _{n \rightarrow \infty}\left\{-\left[p(x) u^{\prime}(x) \overline{u(x)}\right]_{1 / n}^{1-1 / n}+\int_{1 / n}^{1-1 / n} p(x) u^{\prime}(x) \overline{u^{\prime}(x)} \mathrm{d} x\right\} \\
& =\lim _{n \rightarrow \infty}\left\{\left[p(x)\left\{u(x) \overline{u^{\prime}(x)}-u^{\prime}(x) \overline{u(x)}\right\}\right]_{1 / n}^{1-1 / n}-\int_{1 / n}^{1-1 / n} \overline{\left(p u^{\prime}\right)^{\prime}(x)} u(x) \mathrm{d} x\right\} \\
& =-\int_{0}^{1} \overline{\left(p u^{\prime}\right)^{\prime}(x)} u(x) \mathrm{d} x \\
& =\bar{\mu} \int_{0}^{1}|u(x)|^{2} w(x) \mathrm{d} x
\end{aligned}
$$

Since $u$ is a non-zero solution of (8) and $w>0$ a.e. we have $\mu=\bar{\mu}$ and hence $\mu \in \mathbf{R}$. Since $\lambda=\frac{\varepsilon}{2} \mu$, we also have $\lambda \in \mathbf{R}$.

## References

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