An Indefinite Convection-Diffusion Operator With Real Spectrum

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1 Introduction

For $0 < \varepsilon < 2$, we consider the operator

$$(Hf)(\theta) := \varepsilon \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{\partial f}{\partial \theta}$$
(1)

initially defined on all C^2 periodic functions $f \in L^2(-\pi, \pi)$; the exact domain is given by taking the closure of the operator defined on the above functions. That such a closure exists will be shown later. In a recent paper [1] Benilov, O'Brien and Sazonov showed that the equation

$$\frac{\partial f}{\partial t} = Hf \tag{2}$$

approximates the evolution of a liquid film inside a rotating horizontal cylinder.

We shall show that the eigenvalue problem

$$-iHf = \lambda f \tag{3}$$

has only real eigenvalues, which were shown to exist by Davies in [3]. In the same paper, he showed that the spectrum of -iH is equal to the set of its eigenvalues, so this implies that the spectrum is real. This was conjectured in [1], and Chugunova and Pelinovsky proved in [2] that all but finitely many eigenvalues are real, and gave numerical evidence that all eigenvalues are real. Our approach is to show that it is sufficient to consider H acting on the Hardy space $H^2(-\pi, \pi)$ and analytically continue any solution of (3) to the unit disc, where the corresponding ODE (8) is now self-adjoint on [0, 1] with regular singularities at the end points. In order to do this we make use of a bound on the Fourier coefficients proved by Davies in [3].

As in [3], by expanding $f \in L^2(-\pi, \pi)$ in the form

$$f(\theta) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbf{Z}} v_n \mathrm{e}^{in\theta},$$

one may rewrite the eigenvalue problem in the form $Av = \lambda v$, where A = -iH is given by

$$(Av)_n = \frac{\varepsilon}{2}n(n-1)v_{n-1} - \frac{\varepsilon}{2}n(n+1)v_{n+1} + nv_n.$$

Here we have identified $l^2(\mathbf{Z})$ and $L^2(-\pi,\pi)$ using the Fourier transform $\mathcal{F}: l^2(\mathbf{Z}) \to L^2(-\pi,\pi)$. We have

$$A = -i\mathcal{F}^{-1}H\mathcal{F}$$

and define $\text{Dom}(H) = \mathcal{F}(\text{Dom}(A)).$

The (unbounded) tridiagonal matrix A is of the form

$$A = \left(\begin{array}{rrr} A_{-} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & A_{+} \end{array}\right)$$

where A_{-} acts in $l^{2}(\mathbf{Z}_{-})$, the central 0 acts in **C** and A_{+} acts in $l^{2}(\mathbf{Z}_{+})$. We assume that A_{+} has its natural maximal domain

$$\mathcal{D} = \{ v \in l^2(\mathbf{Z}_+) : A_+ v \in l^2(\mathbf{Z}_+) \}$$

Davies has shown in [3] that A_+ is closed and that \mathcal{D} is the closure under the graph norm of A_+ of the subspace consisting of those $v \in l^2(\mathbf{Z}_+)$ that have finite support. Let τ be the natural identification between $l^2(\mathbf{Z}_+)$ and $l^2(\mathbf{Z}_-)$; then $\text{Dom}(A) = \tau(\mathcal{D}) \oplus \mathbf{C} \oplus \mathcal{D}$ and τ induces a unitary equivalence between A_+ and A_- . Therefore, in order to prove that all eigenvalues of A are real, we only need to prove that all eigenvalues of A_+ are real. The Fourier transform identifies $l^2(\mathbf{Z}_+)$ with $\{f \in H^2(-\pi,\pi) : \int_{-\pi}^{\pi} f(\theta) d\theta = 0\}$.

Let H_0 be the restriction of H to $\mathcal{C}^2_{\text{per}}([-\pi,\pi])$, which is clearly a subspace of Dom (H). We now show that H is the closure of H_0 .

Proposition 1.1 Where H and H_0 are as above, H is the closure of H_0 .

Proof It follows from Davies' result on the domain of A_+ that the trigonometric polynomials are dense in Dom (H) with respect to the graph norm. Since the trigonometric polynomials are contained in $C^2_{per}([-\pi,\pi])$, this space is also dense in Dom (H), which is closed in graph norm since Dom (A) is.

2 Reality Of The Eigenvalues

If $A_+v = \lambda v$, then v is a solution of the recurrence relation

$$\frac{\varepsilon}{2}(n+1)(n+2)v_{n+2} + (\lambda - n - 1)v_{n+1} - \frac{\varepsilon}{2}n(n+1)v_n = 0$$
(4)

satisfying the initial condition $\varepsilon v_2 = (1-\lambda)v_1$. We shall study the generating function, $\sum_{k=1}^{\infty} v_k z^k$ of (v_k) .

Lemma 2.1 Let $v \in l^2(\mathbf{Z}_+)$ be such that $A_+v = \lambda v$. Then the function $u(z) := \sum_{k=1}^{\infty} v_k z^k$, defined for |z| < 1, satisfies the differential equation

$$u'' - 2\frac{z + 1/\varepsilon}{(1-z)(1+z)}u' + \frac{2\lambda/\varepsilon}{z(1-z)(1+z)}u = 0.$$
 (5)

Proof The constant term in

$$z(1-z)(1+z)u'' - 2(z+1/\varepsilon)zu' + (2\lambda/\varepsilon)u$$

is clearly 0, and the coefficient of z is

$$2v_2 - 2v_1/\varepsilon + 2\lambda v_1/\varepsilon = 2\left(v_2 - \frac{(1-\lambda)}{\varepsilon}v_1\right) = 0.$$

The coefficient of z^n is

$$n(n+1)v_{n+1} - (n-1)(n-2)v_{n-1} - \frac{2}{\varepsilon}nv_n - 2(n-1)v_{n-1} + \frac{2\lambda}{\varepsilon}v_n$$

= $\frac{2}{\varepsilon} \left[\frac{\varepsilon}{2}n(n+1)v_{n+1} + (\lambda - n)v_n - \frac{\varepsilon}{2}n(n-1)v_{n-1}\right] = 0$

for $n \geq 2$.

From here on we assume that $\lambda \in \mathbf{C}$ is an eigenvalue of the operator A_+ and v is a corresponding non-zero eigenvector. In [3], Davies proved that there exist constants b, m such that

$$||v||_{\infty,c} \le b|\lambda|^m ||v||_2,$$
 (6)

where $c = 1 + 1/\varepsilon$ and $||w||_{\infty,c} := \sup\{|w_n|n^c : 1 \le n < \infty\}$. Since $\varepsilon > 0$, this implies that $v \in l^1(\mathbf{Z}_+)$, and hence that $\sum_{k=1}^{\infty} v_k z^k$ is absolutely convergent for $|z| \le 1$. The equation (5) can be written in the form of Heun's equation

$$u'' + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z-a}\right)u' + \frac{\alpha\beta z - \mu}{z(z-1)(z-a)}u = 0$$
(7)

with $\alpha = 1$, $\beta = 0$, $\gamma = 0$, $\delta = 1 + 1/\varepsilon$, $\epsilon = 1 - 1/\varepsilon$, a = -1 and $\mu = 2\lambda/\varepsilon$. This is a Fuchsian equation with four regular singular points, at 0, 1, -1 and ∞ , with $\{0, 1 - \gamma\}$, $\{0, 1 - \delta\}$, $\{0, 1 - \epsilon\}$ and $\{\alpha, \beta\}$ as the roots of the corresponding indicial equations (for the Frobenius series at each regular singular point). For background information on Heun's equation, see [4].

Lemma 2.2 Suppose that $0 < \varepsilon < 2$, $1/\varepsilon \notin \mathbf{Z}$, $\lambda \in \mathbf{C}$ is an eigenvalue of A_+ and $v \in l^2(\mathbf{Z}_+)$ is a corresponding non-zero eigenvector. Then there exists a solution u of (7) which is analytic in an open set containing [0, 1] and such that $u(z) = \sum_{k=1}^{\infty} v_k z^k$ for all z such that $|z| \leq 1$.

Proof Define $u(z) = \sum_{k=1}^{\infty} v_k z^k$ on $\{z \in \mathbf{C} : |z| < 1\}$. Let u_1 be the solution of (7) with exponent 0 about 1 and u_2 be the solution with exponent $-1/\varepsilon$ about 1. Let U be the intersection of the open discs of unit radius about 0 and 1. The space of solutions of (7) in U is two-dimensional, and u, u_1 and u_2 lie in this space. Hence there exist constants a, b such that $u = au_1 + bu_2$ in U. Since $v \in l^1(\mathbf{Z}_+), u(z)$ converges to a finite limit as $z \to 1$ in U. Also $u_1(1)$ is finite, but $u_2(z) \to \infty$ as $z \to 1$ in U. Therefore we must have b = 0and $u = au_1$ in U. Let W be the union of the open discs of unit radius about 0 and 1. We now extend u to all of W by $u(z) = au_1(z)$ on the open disc of unit radius about 1. Now u is an analytic solution of (7) on W, which is an open set containing [0, 1], such that $u(z) = \sum_{k=1}^{\infty} v_k z^k$ for all z such that $|z| \leq 1$.

Theorem 2.3 Suppose that $0 < \varepsilon < 2$, $1/\varepsilon \notin \mathbb{Z}$ and $\lambda \in \mathbb{C}$ is an eigenvalue of A_+ . Then $\lambda \in \mathbb{R}$.

Proof Let $v \in l^2(\mathbf{Z}_+)$ be a non-zero eigenvector corresponding to λ . Let u be as in Lemma 2.2 and put $\mu = 2\lambda/\varepsilon$. Following [5], the equation (7) can be written as

$$-(pu')' + qu = \mu wu \tag{8}$$

on the complex plane cut along $[1,\infty)$ and $(-\infty,0]$, where

$$\begin{array}{rcl} p(z) &=& z^{\gamma}(1-z)^{\delta}(z-a)^{\epsilon} &=& (1-z)^{1+1/\varepsilon}(z+1)^{1-1/\varepsilon}\\ q(z) &=& \alpha\beta z^{\gamma}(1-z)^{\delta-1}(z-a)^{\epsilon-1} &=& 0\\ w(z) &=& z^{\gamma-1}(1-z)^{\delta-1}(z-a)^{\epsilon-1} &=& z^{-1}(1-z)^{1/\varepsilon}(z+1)^{-1/\varepsilon}. \end{array}$$

We now restrict u to [0, 1]. It is clear that $u \in \mathcal{C}^{\infty}([0, 1])$ and

$$-(pu')' = \mu w u$$

on (0, 1). Note that p > 0 on [0, 1) and w > 0 on (0, 1) with $p(x), w(x) \to 0$ as $x \to 1$ from below. Since u has a zero of order 1 at 0 and w has a pole of order 1 at 0, $wu \in \mathcal{C}([0, 1])$. Therefore $|u|^2 w \in L^1(0, 1)$. Now

$$\begin{split} \mu \int_{0}^{1} |u(x)|^{2} w(x) \mathrm{d}x &= \lim_{n \to \infty} \left\{ -\int_{1/n}^{1-1/n} (pu')'(x) \overline{u(x)} \mathrm{d}x \right\} \\ &= \lim_{n \to \infty} \left\{ -\left[p(x)u'(x) \overline{u(x)} \right]_{1/n}^{1-1/n} + \int_{1/n}^{1-1/n} p(x)u'(x) \overline{u'(x)} \mathrm{d}x \right\} \\ &= \lim_{n \to \infty} \left\{ \left[p(x) \left\{ u(x) \overline{u'(x)} - u'(x) \overline{u(x)} \right\} \right]_{1/n}^{1-1/n} - \int_{1/n}^{1-1/n} \overline{(pu')'(x)} u(x) \mathrm{d}x \right\} \\ &= -\int_{0}^{1} \overline{(pu')'(x)} u(x) \mathrm{d}x \\ &= \overline{\mu} \int_{0}^{1} |u(x)|^{2} w(x) \mathrm{d}x. \end{split}$$

Since u is a non-zero solution of (8) and w > 0 a.e. we have $\mu = \overline{\mu}$ and hence $\mu \in \mathbf{R}$. Since $\lambda = \frac{\varepsilon}{2}\mu$, we also have $\lambda \in \mathbf{R}$.

References

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