Fractional variational problems with the Riesz-Caputo derivative

Ricardo Almeida*

Abstract

In this paper we investigate optimality conditions for fractional variational problems, with a Lagrangian depending on the Riesz-Caputo derivative. First we prove a generalized Euler-Lagrange equation for the case when the interval of integration of the functional is different from the interval of the fractional derivative. Next we consider integral dynamic constraints on the problem, for several different cases. Finally, we determine optimality conditions for functionals depending not only on the admissible functions, but on time also, and we present a necessary condition for a pair function–time to be an optimal solution to the problem.

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1 Introduction

Fractional variational calculus deals with problems of minimizing functionals that involves some fractional derivatives and/or fractional integrals. During the last few decades, fractional calculus has called the attention of many researchers in different fields, not only on pure and applied mathematics, but also physics, chemistry, mechanics, economics, electrical engineering, viscoelastic, robotics, etc. In fact, it is currently one of the most interdisciplinary fields of mathematics. Just to mention some recent works on fractional variational calculus, see e.g. [2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18, 20]. Most of the work done so far deals with Riemann-Liouville or Caputo fractional derivatives, and a few with other types (modified Riemann-Liouville, Riesz, Riesz-Caputo, symmetric fractional derivative, etc.). The study of variational optimal conditions, for functionals containing a Riesz-Caputo fractional derivative type, was carried out in [1]. The basic variational problem was considered, with a fractional Euler-Lagrange equation. Other problems, e.g. with free-end points, the problem of Lagrange and multiple integrals are studied as well. We mention also [11], where a Noether's type theorem is presented, within the Riesz-Caputo fractional derivative context. The purpose of this paper is to present solutions to other fundamental problems, and show how they can be obtained using standard techniques of variational calculus. The paper is organized in the following way. In section 2 we review some necessary definitions on fractional calculus, and results that will be needed later; for more on this subject we refer to [17, 19, 21]. Section 3 deals with necessary optimality conditions for the fundamental problem of fractional variational calculus. The problem is defined via the Riesz-Caputo fractional derivative. We extend known results (see [1]) to the case when the interval of integration of the functional is a subset of the domain of the admissible functions. In section 4 we consider the optimization problem in the presence of an integral constraint. The constraint is given by a functional of the same type as the cost functional, i.e., it depends on the Riesz-Caputo fractional derivative as well. The case where the interval of integration is not the whole interval is also presented. In the last section we study a more general case, when we are not only interested in finding the optimal admissible function, but also the time where the minimum is obtained.

2 Preliminaries

There exist numerous definitions of fractional integrals and fractional derivatives. This paper deals with the Riesz and Riesz-Caputo fractional derivatives. Throughout the work, $y : [a, b] \to \mathbb{R}$ is a function of class C^1 and $\alpha \in (0, 1)$.

*ricardo.almeida@ua.pt, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

The left and right Riemann-Liouville fractional integrals of order α are defined respectively by

$$ {}_aI_x^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} y(t)dt \text{ and } {}_xI_b^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} y(t)dt. $$

The Riesz fractional integral ${}_a^RI_b^\alpha y$ is given by

$$ {}_a^RI_b^\alpha y(x) = \frac{1}{2}({}_aI_x^\alpha y(x) + {}_xI_b^\alpha y(x)). $$

The left and right Riemann-Liouville fractional derivatives of order α are defined respectively by

$$ {}_aD_x^\alpha y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{-\alpha} y(t)dt \text{ and } {}_xD_b^\alpha y(x) = \frac{-1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (t-x)^{-\alpha} y(t)dt. $$

The Riesz fractional derivative ${}_a^RD_b^\alpha y$ is given by

$$ {}_a^RD_b^\alpha y(x) = \frac{1}{2}({}_aD_x^\alpha y(x) - {}_xD_b^\alpha y(x)). $$

The left and right Caputo fractional derivatives of order α are defined respectively by

$$ {}_a^CD_x^\alpha y(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x (x-t)^{-\alpha} \frac{d}{dx} y(t)dt \text{ and } {}_xD_b^\alpha y(x) = \frac{-1}{\Gamma(1-\alpha)} \int_x^b (t-x)^{-\alpha} \frac{d}{dx} y(t)dt. $$

The Riesz-Caputo fractional derivative ${}_a^{RC}D_b^\alpha y$ is given by

$$ {}_a^{RC}D_b^\alpha y(x) = \frac{1}{2}({}_a^CD_x^\alpha y(x) - {}_x^CD_b^\alpha y(x)). $$

In the discussion to follow, we need a fractional integration by parts formula. For the Caputo derivative, we have (cf. [1])

$$ \int_a^b w(x) \cdot {}_a^CD_x^\alpha z(x)dx = \int_a^b {}_xD_b^\alpha w(x) \cdot z(x)dx + {}_xI_b^{1-\alpha} w(x) \cdot z(x)\Big|_{x=a}^{x=b}, $$

and

$$ \int_a^b w(x) \cdot {}_x^CD_b^\alpha z(x)dx = \int_a^b {}_aD_x^\alpha w(x) \cdot z(x)dx - {}_aI_x^{1-\alpha} w(x) \cdot z(x)\Big|_{x=a}^{x=b}. $$

Thus, in the case of the Riesz-Caputo fractional derivative, one has

$$ \int_a^b w(x) \cdot {}_a^{RC}D_b^\alpha z(x)dx = -\int_a^b {}_a^RD_b^\alpha w(x) \cdot z(x)dx + {}_a^RI_b^{1-\alpha} w(x) \cdot z(x)\Big|_{x=a}^{x=b}. $$

3 The Euler-Lagrange equation

The fundamental fractional variational problem is stated in the following way.

(P1) Among all C^1 functions $y : [a,b] \to \mathbb{R}$, with fixed values on $x = a$ and $x = b$, say

$$ y(a) = y_a \text{ and } y(b) = y_b, \quad y_a, y_b \in \mathbb{R}, $$

find the ones for which the functional

$$ J(y) = \int_a^b L\left(x, y(x), {}_a^{RC}D_b^\alpha y(x)\right) dx $$

attains a minimum value.

We are assuming, here and from now on, that the Lagrange function $L : [a,b] \times \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous first and second partial derivatives with respect to all its arguments, and ${}_a^{RC}D_b^\alpha y$ exists and is continuous on the closed interval $[a,b]$.

The possible extremizers for problem (P1) can be obtained by solving a fractional differential equation, the so called fractional Euler-Lagrange equation.

Theorem 1. *[1] Let y be a solution to problem (P1). Then, y is a solution of the fractional Euler-Lagrange equation*

$$\frac{\partial L}{\partial y} - {}_a^R D_b^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} = 0 \tag{1}$$

for all $x \in [a, b]$.

We remark that, when $\alpha = 1$, equation (1) is the (standard) Euler-Lagrange equation: if y is a minimizer of

$$J(y) = \int_a^b L(x, y(x), y'(x))dx,$$

restricted to the boundary conditions

$$y(a) = y_a \text{ and } y(b) = y_b,$$

then y is a solution of the differential equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'} = 0.$$

Definition 2. *A function y that is a solution of equation* (1) *is called an extremal for J.*

We first extend this result to functionals where the interval of integration is $[A, B] \subset [a, b]$. The idea comes from [5, 6], where similar problems with a Lagrangian function depending on the Riemann-Liouville [6] or the Caputo [5] fractional derivatives are considered.

The new problem (P2) deals with finding optimality conditions for functionals of type

$$J(y) = \int_A^B L(x, y(x), {}_a^{RC} D_b^\alpha y(x))dx,$$

with the boundary conditions

$$y(a) = y_a \text{ and } y(b) = y_b, \quad y_a, y_b \in \mathbb{R}.$$

Theorem 3. *Let y be a solution of problem (P2). Then, y satisfies the following equations:*

$$\begin{cases} \dfrac{\partial L}{\partial y} - {}_A^R D_B^\alpha \dfrac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} = 0, & \text{for all } x \in [A, B], \\ {}_x D_B^\alpha \dfrac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} - {}_x D_A^\alpha \dfrac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} = 0, & \text{for all } x \in [a, A], \\ {}_B D_x^\alpha \dfrac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} - {}_A D_x^\alpha \dfrac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} = 0, & \text{for all } x \in [B, b]. \end{cases}$$

Proof. To obtain the necessary conditions, we first consider variation functions of the type $y+\epsilon\eta$, where $\eta : [a, b] \to \mathbb{R}$ is a function of class C^1 such that $\eta(a) = \eta(b) = 0$. For convenience, we also assume that $\eta(A) = \eta(B) = 0$. Let $j(\epsilon) = J(y + \epsilon\eta)$. Since $j'(0) = 0$, integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_A^B \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}{}_a^{RC} D_b^\alpha\right]\eta\, dx \\ &= \int_A^B \frac{\partial L}{\partial y}\eta\, dx + \frac{1}{2}\left[\int_a^B \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}{}_a^C D_x^\alpha \eta\, dx - \int_a^A \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}{}_a^C D_x^\alpha \eta\, dx\right] \\ &\quad - \frac{1}{2}\left[\int_A^b \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}{}_x^C D_b^\alpha \eta\, dx - \int_B^b \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}{}_x^C D_b^\alpha \eta\, dx\right] \\ &= \int_A^B \frac{\partial L}{\partial y}\eta\, dx + \frac{1}{2}\left[\int_a^B {}_x D_B^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\eta\, dx - \int_a^A {}_x D_A^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\eta\, dx\right] \\ &\quad - \frac{1}{2}\left[\int_A^b {}_A D_x^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\eta\, dx - \int_B^b {}_B D_x^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\eta\, dx\right] \\ &= \int_A^B \left[\frac{\partial L}{\partial y} - {}_A^R D_B^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\right]\eta\, dx + \frac{1}{2}\int_a^A \left[{}_x D_B^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} - {}_x D_A^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\right]\eta\, dx \\ &\quad - \frac{1}{2}\int_B^b \left[{}_B D_x^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y} - {}_A D_x^\alpha \frac{\partial L}{\partial {}_a^{RC} D_b^\alpha y}\right]\eta\, dx. \end{aligned}$$

For appropriate choices of η, we prove the necessary conditions. $\square$

Obviously, in the case $A = a$ and $B = b$, Theorem 3 reduces to Theorem 1. Other cases could be deduced from Theorem 3, namely when $A = a$ and $B \neq b$, or $A \neq a$ and $B = b$.

4 The fractional isoperimetric problem

The original isoperimetric problem is addressed in the following way: among all closed plane curves, without self-intersecting, such that the total length has a given value, find the ones for which the enclosed area is the greatest. Nowadays, isoperimetric problems are the ones that involve some integral constraint on the dynamics, and have become one of the classical problems of the calculus of variations. The fractional isoperimetric problem is stated as follows.

(P3) Find a function $y : [a, b] \to \mathbb{R}$ of class C^1, such that it minimizes the functional

$$J(y) = \int_a^b L(x, y(x), {}_a^{RC}D_b^\alpha y(x))dx$$

when restricted to the boundary conditions

$$y(a) = y_a \text{ and } y(b) = y_b, \quad y_a, y_b \in \mathbb{R},$$

and to an integral constraint

$$I(y) = \int_a^b g(x, y(x), {}_a^{RC}D_b^\alpha y(x))dx = l,$$

where l is a fixed real. As before, we assume that $g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is a function with continuous first and second partial derivatives with respect to all its arguments.

Theorem 4. *Let y be a solution to problem (P3). If y is not an extremal for I, then there exists a constant λ such that y satisfies the equation*

$$\frac{\partial F}{\partial y} - {}_a^R D_b^\alpha \frac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} = 0$$

for all $x \in [a, b]$, where $F = L - \lambda g$.

Proof. Consider variations of y of the form

$$y + \epsilon_1\eta_1 + \epsilon_2\eta_2,$$

where η_i is a function of class C^1, and $\eta_i(a) = \eta_i(b) = 0$, for each $i \in \{1, 2\}$. Define two functions j and i by

$$j(\epsilon_1, \epsilon_2) = J(y + \epsilon_1\eta_1 + \epsilon_2\eta_2) \text{ and } i(\epsilon_1, \epsilon_2) = I(y + \epsilon_1\eta_1 + \epsilon_2\eta_2) - l.$$

Since y is not an extremal for I, there exists a function η_2 for which

$$\left.\frac{\partial i}{\partial \epsilon_2}\right|_{(0,0)} \neq 0.$$

Also, since $i(0, 0) = 0$, by the implicit function theorem, there exists a function $\epsilon_2(\cdot)$ satisfying the relation $i(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. In other words, there exists a subfamily of variation functions satisfying the integral constraint. Moreover, observe that j has a minimum at zero subject to the constraint $i(\cdot, \cdot) = 0$, and we just proved that $\nabla i(0, 0) \neq (0, 0)$. Then, by the Lagrange multiplier rule, there exists a constant λ such that

$$\nabla(j(0, 0) - \lambda i(0, 0)) = (0, 0).$$

Differentiating j and i with respect to ϵ_1, at $(\epsilon_1, \epsilon_2) = (0, 0)$, we prove the theorem. $\square$

We now shall present a more general result.

Theorem 5. *Let y be solution to problem (P3). Then there exist two constants λ_0 and λ, not both zero, such that y satisfies the equation*

$$\frac{\partial K}{\partial y} - {}_a^R D_b^\alpha \frac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} = 0,$$

for all $x \in [a, b]$, where $K = \lambda_0 L - \lambda g$.

Proof. It follows the same pattern as in the proof of Theorem 4, and using the abnormal Lagrange multiplier rule. □

The isoperimetric problem for functionals where the interval of integration is $[A,B] \subset [a,b]$ can be solved in a similar way. We sate problem (P4) as:

$$\text{minimize } J(y) = \int_A^B L\left(x, y(x), {}_a^{RC}D_b^\alpha y(x)\right) dx,$$

when restricted to the constraints

$$y(a) = y_a \text{ and } y(b) = y_b,$$

and subject to an integral constraint

$$I(y) = \int_A^B g(x, y(x), {}_a^{RC}D_b^\alpha y(x))dx = l.$$

Similarly, we say that y is an extremal for I if

$$\frac{\partial L}{\partial y}(x) - {}_A^C D_B^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y} = 0$$

for all $x \in [A,B]$. Using the same techniques as the ones presented in Theorems 3 and 4, the following two results can be proven.

Theorem 6. *If y is a solution to problem (P4), and if y is not an extremal for I, then there exists a constant λ such that*

$$\begin{cases} \dfrac{\partial F}{\partial y} - {}_A^R D_B^\alpha \dfrac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [A,B], \\ {}_x D_B^\alpha \dfrac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} - {}_x D_A^\alpha \dfrac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [a,A], \\ {}_B D_x^\alpha \dfrac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} - {}_A D_x^\alpha \dfrac{\partial F}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [B,b], \end{cases}$$

with $F = L - \lambda g$.

Theorem 7. *If y is a solution to problem (P4), then there exist two constants λ_0 and λ, not both zero, such that*

$$\begin{cases} \dfrac{\partial K}{\partial y} - {}_A^R D_B^\alpha \dfrac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [A,B], \\ {}_x D_B^\alpha \dfrac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} - {}_x D_A^\alpha \dfrac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [a,A], \\ {}_B D_x^\alpha \dfrac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} - {}_A D_x^\alpha \dfrac{\partial K}{\partial {}_a^{RC}D_b^\alpha y} = 0 & \text{for all } x \in [B,b], \end{cases}$$

with $K = \lambda_0 L - \lambda g$.

5 Optimal time problem

In this section, we are interested not only in finding an optimal admissible function for the variational problem, but also the optimal time T. We state the problem in the following way: consider the functional

$$J(y,T) = \int_a^T L(x, y(x), {}_a^{RC}D_b^\alpha y(x))dx,$$

where

$$(y,T) \in \{C^1[a,b] \times [a,b] \,|\, y(a) = y_a\}.$$

Problem (P5) is the following one: find a pair (y,T) for which J attains a minimum value.

Theorem 8. *If (y,T) is a solution to problem (P5), then it satisfies the following four conditions:*

1. $L(T, y(T), {}_a^{RC}D_b^\alpha y(T)) = 0$,

2. $\dfrac{\partial L}{\partial y} - {}_a^R D_T^\alpha \dfrac{\partial L}{\partial {}_a^{RC}D_b^\alpha y} = 0$ *for all* $x \in [a, T]$,

3. ${}_a D_x^\alpha \dfrac{\partial L}{\partial {}_a^{RC}D_b^\alpha y} = {}_T D_x^\alpha \dfrac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}$ *for all* $x \in [T, b]$,

4. ${}_a I_x^{1-\alpha} \left.\dfrac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right|_{x=b} = {}_T I_x^{1-\alpha} \left.\dfrac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right|_{x=b}$.

Proof. Let $\eta : [a, b] \to \mathbb{R}$ be a function of class C^1 such that $\eta(a) = 0$, and let $\Delta T \in \mathbb{R}$. Define j as

$$j(\epsilon) = J(y + \epsilon\eta, T + \epsilon\Delta T).$$

Since $j'(0) = 0$, we get

$$\int_a^T \left[\frac{\partial L}{\partial y}\eta + \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_a^{RC}D_b^\alpha \eta\right] dx + \Delta T \cdot L(T, y(T), {}_a^{RC}D_b^\alpha y(T)) = 0.$$

On the other hand, observe that

$$\begin{aligned}
\int_a^T \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_a^{RC}D_b^\alpha \eta\, dx &= \frac{1}{2}\left[\int_a^T \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_a^C D_x^\alpha \eta\, dx - \int_a^T \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_x^C D_b^\alpha \eta\, dx\right] \\
&= \frac{1}{2}\left[\int_a^T \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_a^C D_x^\alpha \eta\, dx - \int_a^b \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_x^C D_b^\alpha \eta\, dx + \int_T^b \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}{}_x^C D_b^\alpha \eta\, dx\right] \\
&= \star
\end{aligned}$$

Integrating by parts each of the last three terms, we get

$$\begin{aligned}
\star = \frac{1}{2}\Bigg[&\int_a^T {}_x D_T^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\, dx + {}_x I_T^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=T} - \int_a^b {}_a D_x^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\, dx + {}_a I_x^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=b} \\
&+ \int_T^b {}_T D_x^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\, dx - {}_T I_x^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=b} + {}_T I_x^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=T}\Bigg].
\end{aligned}$$

Some of the previous terms vanish (cf. [17, pag 46]):

$${}_x I_T^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=T} = 0 \quad \text{and} \quad {}_T I_x^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\eta\right|_{x=T} = 0.$$

Thus, we have proven the relation

$$\begin{aligned}
0 = &\int_a^T \left[\frac{\partial L}{\partial y} - {}_a^R D_T^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right]\eta\, dx + \Delta T \cdot L(T, y(T), {}_a^{RC}D_b^\alpha y(T)) \\
&+ \frac{1}{2}\int_T^b \left[{}_T D_x^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y} - {}_a D_x^\alpha \frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right]\eta\, dx + \frac{1}{2}\left[{}_a I_x^{1-\alpha}\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y} - {}_T I_x^{1-\alpha}\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right]\eta\Bigg|_{x=b}.
\end{aligned}$$

If we fix $\eta \equiv 0$, by the arbitrariness of ΔT, we obtain equation *1* of the theorem. If η is free on $[a, T)$ and zero on $[T, b]$, we obtain equation *2*. To prove the two remaining conditions, choose first η free on $]T, b[$ and zero on $x = b$, and then η such that $\eta(b) \neq 0$. $\square$

The transversality condition obtained in [1] can be seen as a particular case of Theorem 8. Let $T = b$ (time is fixed), then $\Delta T = 0$. Following the proof, equation *1* of Theorem 8 is no longer a necessary condition. Equation *2* becomes the fractional Euler-Lagrange equation as in Theorem 1. Equation *3* is obviously satisfied. About equation *4*, it reads as (cf. [17, pag 46])

$${}_a I_x^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right|_{x=b} = 0,$$

which is equivalent to the one obtained in [1]:

$${}_a^R I_b^{1-\alpha} \left.\frac{\partial L}{\partial {}_a^{RC}D_b^\alpha y}\right|_{x=b} = 0.$$

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