# Spacial inhomogeneity and nonlinear tunneling for the forced KdV equation 

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#### Abstract

A variable-coefficient forced Korteweg-de Vries equation with spacial inhomogeneity is investigated in this paper. Under constraints, this equation is transformed into its bilinear form, and multi-soliton solutions are derived. Effects of spacial inhomogeneity for soliton velocity, width and background are discussed. Nonlinear tunneling for this equation is presented, where the soliton amplitude can be amplified or compressed. Our results might be useful for the relevant problems in fluids and plasmas.


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## I. Introduction

In this paper, we will investigate the following variable-coefficient forced KdV equation 1101 with the aid of symbolic computation [11 13],

$$
\begin{equation*}
u_{t}+a(t) u u_{x}+b(t) u_{x x x}+c(t) u+d(t) u_{x}=f(x, t) \tag{1}
\end{equation*}
$$

where $u$ is a function of the scaled spacial coordinate $x$ and temporal coordinate $t, a(t), b(t), c(t)$ and $d(t)$ are the analytic functions of $t$, and $f(x, t)$ is the analytic function of $x$ and $t$. These variable coefficients respectively represent the nonlinear, dispersive, line-damping, dissipative and external-force effects, which are caused by the inhomogeneities of media and boundaries [1-10]. Here, we assume that the spacial inhomogeneity is linear and take the following form

$$
\begin{equation*}
f(x, t)=f_{1}(t) x+f_{2}(t) \tag{2}
\end{equation*}
$$

When coefficients are taken with different cases, Eq. (1) has been seen to describe nonlinear waves in a fluid-filled tube [1-3], weakly nonlinear waves in the water of variable depth [4, 5], trapped quasi-one-dimensional Bose-Einstein condensates [6], internal gravity waves in lakes with changing cross sections [7], the formation of a trailing shelf behind a slowly-varying solitary wave [8], dynamics of a circular rod composed of a general compressible hyperelastic material with the variable cross-sections and material density [9], and atmospheric and oceanic dynamical systems 10].

If the the spacial inhomogeneity is ignored, i.e., $f_{1}(t)=0$, Eq. (1) has been transformed into its several KdV-typed ones with simpler forms [14-17], and has also been solved directly via the bilinear method [18]. The effects of the dispersive, line-damping, dissipative, and externalforce terms on the solitonic velocity, amplitude and background have been discussed [18] with the characteristic-line method [19, 20]; Besides, Wronskian form are derived based on the given bilinear expression [21].

However, since the spacial inhomogeneity in external-force term brings into more difficulties in solving, to our knowledge, the multi-soliton solutions for Eq. (1) in the explicit bilinear forms have not been constructed directly, and the effects of spacial inhomogeneity on solitonic propagation and interaction have not been discussed.

In addition, nonlinear tunneling for the nonautonomous nonlinear Schrödinger equations has attracted attention in recent years [22-25]. The concept of the nonlinear tunneling effect comes from the wave equations steming from the nonlinear dispersion relation, which has shown that the soliton can pass lossless through the barrier/well under special conditions which depend on the ratio between the amplitude of the soliton and the height of the barrier/well $[22-25]$. In this paper, we will apply such concept to Eq. (11), a KdV-typed equation. In section II, a dependent variable transformation and two constraints will be proposed, Eq. (1) will be transformed into its bilinear form, and the multi-soliton solutions in the explicit forms will be constructed. In section III, we will show that different from Ref. [18], the nonlinear coefficient can also affect the
soliton width and amplitude for the existence of spacial inhomogeneity in the forced term. In section IV, we will discuss nonlinear barrier/well of Eq. (1). Finally, Section V will present the conclusions.

## II. Soliton solutions

Through the dependent variable transformation

$$
\begin{equation*}
u=\alpha(t)(\log \Phi)_{x x}+\beta(t)+\gamma(t) x \tag{3}
\end{equation*}
$$

and the coefficient constraints,

$$
\begin{align*}
& b(t)=\frac{\rho a(t)}{6} e^{\int[a(t) \gamma(t)-c(t)] d t}  \tag{4}\\
& f_{1}(t)=a(t) \gamma(t)^{2}+c(t) \gamma(t)+\gamma^{\prime}(t) \tag{5}
\end{align*}
$$

where

$$
\begin{align*}
& \alpha(t)=2 \rho e^{\int[a(t) \gamma(t)-c(t)] d t}  \tag{6}\\
& \beta(t)=e^{\int[-a(t) \gamma(t)-c(t)] d t}\left\{\delta+\int e^{\int[a(t) \gamma(t)+c(t)] d t}\left[f_{2}(t)-d(t) \gamma(t)\right] d t\right\} \tag{7}
\end{align*}
$$

$\Phi$ is a function of $x$ and $t, \rho$ and $\delta$ are constants, and ' denotes the derivative with respect to $t$, Eq. (1) can be transformed into the following bilinear form,

$$
\begin{equation*}
\left\{D_{x} D_{t}+b(t) D_{x}^{4}+[d(t)+a(t) \beta(t)+a(t) \gamma(t) x] D_{x}^{2}+a(t) \gamma(t) \frac{\partial}{\partial x}\right\} \Phi \cdot \Phi=0 \tag{8}
\end{equation*}
$$

where $D_{x}^{m} D_{t}^{n}$ is the bilinear derivative operator [26, 27] defined by

$$
\begin{equation*}
\left.D_{x}^{m} D_{t}^{n} a \cdot b \equiv\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x^{\prime}}\right)^{m}\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t^{\prime}}\right)^{n} a(x, t) b\left(x^{\prime}, t^{\prime}\right)\right|_{x^{\prime}=x, t^{\prime}=t} \tag{9}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial}{\partial x} \Phi \cdot \Phi=2 \Phi \Phi_{x} \tag{10}
\end{equation*}
$$

Note that the independence of $f_{1}(t)$ is transformed to that of $\gamma(t)$ through constraint (5).
We expand $\Phi$ in the power series of a parameter $\epsilon$ as

$$
\begin{equation*}
\Phi=1+\epsilon \Phi_{1}+\epsilon^{2} \Phi_{2}+\cdots \tag{11}
\end{equation*}
$$

Substituting Expansion (11) into Eq. (8) and collecting the coefficients of the same power of $\epsilon$, through the standard process of the Hirota bilinear method, we can derive the $N$-soliton-like solutions for Eq. (1), which can be denoted as

$$
\begin{equation*}
u=\alpha(t) \frac{\partial^{2}}{\partial x^{2}}\left\{\log \left[\sum_{\mu=0,1} \exp \left(\sum_{j=1}^{N} \mu_{j} \xi_{j}+\sum_{1 \leq j<l}^{N} \mu_{j} \mu_{l} A_{j l}\right)\right]\right\}+\beta(t)+\gamma(t) x \tag{12}
\end{equation*}
$$

with

$$
\begin{align*}
& \xi_{j}=k_{j}(t) x+\omega_{j}(t)+\xi_{j}^{0},  \tag{13}\\
& k_{j}(t)=e^{-\int a(t) \gamma(t) d t} k_{j},  \tag{14}\\
& \omega_{j}(t)=-\int k_{j}^{3}(t) b(t) d t-\int k_{j}(t)[d(t)+a(t) \beta(t)] d t,  \tag{15}\\
& e^{A_{j l}}=\frac{\left(k_{j}-k_{l}\right)^{2}}{\left(k_{j}+k_{l}\right)^{2}}, \tag{16}
\end{align*}
$$

where $k_{j}$ and $\xi_{j}^{0}(j=1,2, \cdots, N)$ are arbitrary real constants, $\sum_{\mu=0,1}$ is a summation over all possible combinations of $\mu_{1}=0,1, \mu_{2}=0,1, \cdots, \mu_{N}=0,1$, and $\sum_{1 \leq j<l}^{N}$ means a summation over all possible pairs $(j, l)$ chosen from the set $(1,2, \ldots, N)$, with the condition that $1 \leq j<$ $l$ [27].

Specially, one soliton solution can be expressed as

$$
\begin{equation*}
\Phi=1+\exp \left[k_{1}(t) x+\omega_{1}(t)+\xi_{10}\right] \tag{17}
\end{equation*}
$$

and two soliton solution can be expressed as

$$
\begin{align*}
\Phi & =1+\exp \left[k_{1}(t) x+\omega_{1}(t)+\xi_{10}\right]+\exp \left[k_{2}(t) x+\omega_{2}(t)+\xi_{20}\right] \\
& +\exp \left[k_{1}(t) x+\omega_{1}(t)+\xi_{10}+k_{2}(t) x+\omega_{2}(t)+\xi_{20}+A_{12}\right] . \tag{18}
\end{align*}
$$

## III. Spacial inhomogeneity

The coefficients $a(t), b(t), c(t), d(t)$ and $f_{2}(t)$ have the similar influences on the soliton velocity, amplitude and background, which have been discussed in Refs. [18, 21]. Thus, we will mainly discuss the influence of the spacial inhomogeneity in the forced term.

As shown in Fig. [1, the soliton width broadens, amplitude increases, and position of soliton raises. In expression (14), $a(t)$ and $\gamma(t)$ occur simultaneously, so nonlinear coefficient $a(t)$ can affect the soliton width and amplitude.

(a)

(b)

Fig. 1. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=\delta=1, d(t)=a(t)=1, c(t)=\gamma(t)=0.1, \xi_{10}=-10 ;(b)$ Profile of Fig. 1 (a) at $t=0, t=4, t=6$.

Fig. 2 presents a case that the soliton velocity, amplitude and background are periodic. Fig. 3 corresponds to the periodic two soliton solution.

(a)

(b)

Fig. 2. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=1, d(t)=f_{2}(t)=\delta=0, \xi_{10}=-10, a(t)=[2+\sin (t)]^{-1}$, $e^{\int[-a(t) \gamma(t)] d t}=2+\sin (t), e^{\int-c(t) d t}=1 ;(b)$ Profile of Fig. $2($ a) at $t=-4, t=0, t=5$.

(a)

(b)

Fig. 3. Two solitons given by Expression (18) with parameters: $k_{1}=2, k_{2}=1, \rho=1, d(t)=f_{2}(t)=\delta=0, \xi_{10}=-10$, $a(t)=[2+\sin (t)]^{-1}, e^{\int[-a(t) \gamma(t)] d t}=2+\sin (t), e^{\int-c(t) d t}=1 ;(b)$ Profile of Fig. 3 (a) at $t=-4, t=0, t=5$.

## IV. Nonlinear tunneling

Nonlinear tunneling has been discussed for the nonlinear Schrödinger equation [22-25]. Hereby, we will investigate the nonlinear tunneling for the KdV equation.

Fig. 4 shows the one soliton through well with $e^{\int-c(t) d t}=1-0.9 \operatorname{sech}(t)$, while Fig. 5 shows the one soliton through barrier with $e^{\int-c(t) d t}=1+0.9 \operatorname{sech}(t)$. In Fig. 6, the soliton passes through multiple well or barrier with $e^{\int-c(t) d t}=1+h_{1} \operatorname{sech}\left(t+t_{1}\right)+h_{2} \operatorname{sech}\left(t+t_{2}\right)$. Thereinto, $h_{1}$ and $h_{2}$ denote the height of the barrier/well, $t_{1}$ and $t_{2}$ denote the position, and $\left|t_{1}-t_{2}\right|$ denotes the separation distance of the barrier/well.

(a)

(b)

Fig. 4. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=\delta=1, d(t)=a(t)=1, \xi_{10}=-10, f_{2}(t)=0$, $e^{\int[-a(t) \gamma(t)] d t}=1, e^{\int-c(t) d t}=1-0.9 \operatorname{sech}(t) ;(b)$ Profile of Fig. 4 (a) at $t=-3, t=0, t=3$.


Fig. 5. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=\delta=1, d(t)=a(t)=1, \xi_{10}=-10, f_{2}(t)=0$, $e^{\int[-a(t) \gamma(t)] d t}=1, e^{\int-c(t) d t}=1+0.9 \operatorname{sech}(t) ;(b)$ Profile of Fig. 5 (a) at $t=-3, t=0, t=2$.

(a)

(b)

Fig. 6. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=\delta=1, d(t)=a(t)=1, \xi_{10}=-10, f_{2}(t)=0$, $e^{\int[-a(t) \gamma(t)] d t}=1 ;(a) e^{\int-c(t) d t}=1-0.9 \operatorname{sech}(t-2)-0.9 \operatorname{sech}(t+2) ;(b) e^{\int-c(t) d t}=1+0.9 \operatorname{sech}(t-2)+0.9 \operatorname{sech}(t+2)$.

Fig. 7 presents the one soliton through well with periodic background and characteristic line, and Fig. 8 corresponds to the two soliton cases of Fig. 7 .

(a)

(b)

Fig. 7. One soliton given by Expression (17) with parameters: $k_{1}=2, \rho=1, d(t)=f_{2}(t)=\delta=0, \xi_{10}=-10, a(t)=\{[2+\sin (t)][1-$ $0.9 \operatorname{sech}(t)]\}^{-1}, e^{\int[-a(t) \gamma(t)] d t}=2+\sin (t), e^{\int-c(t) d t}=1-0.9 \operatorname{sech}(t) ;(b)$ Profile of Fig. 7 (a) at $t=-4, t=0, t=5$.


Fig. 8. Two solitons given by Expression (18) with parameters: $k_{1}=2, k_{2}=1, \rho=1, d(t)=f_{2}(t)=\delta=0, \xi_{10}=-10$, $a(t)=\{[2+\sin (t)][1-0.9 \operatorname{sech}(t)]\}^{-1}, e^{\int[-a(t) \gamma(t)] d t}=2+\sin (t), e^{\int-c(t) d t}=1-0.9 \operatorname{sech}(t) ;(b)$ Profile of Fig. 8 (a) at $t=-4, t=0$, $t=5$.

## V. Conclusions

In this paper, Eq. (1), a variable-coefficient model with spacial inhomogeneity in fluids [1 10], is investigated with symbolic computation. Under coefficient constraints (4) and (5), Eq. (11) is transformed into its bilinear form, and the multi-soliton solutions are constructed. The function $\gamma(t)$ corresponds to spacial inhomogeneity, and the nonlinear coefficient $a(t)$ can also affect the soliton width and amplitude for the existence of $\gamma(t)$, as shown in Figs. 17 3,

Nonlinear tunneling for Eq. (1) is a special state of amplitude, so it can be regarded as a kind of variable coefficient effects. With $e^{\int-c(t) d t}$ taken as $1+\sum h_{n} \operatorname{sech}\left(t+t_{n}\right)$, nonlinear tunneling in Figs. 4 - 6 is illustrated, where $h_{n}$ denotes the height of the barrier/well, $t_{n}$ denotes the position, and $\left|t_{n}-t_{l}\right|$ denotes the separation distance of the barrier/well. Finally, Figs. 7 and 8 display the combination of nonlinear tunneling and variable coefficient effects.

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