FRACTIONAL KIRCHHOFF EQUATION WITH A GENERAL CRITICAL NONLINEARITY

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ABSTRACT. In this paper, we study the fractional Kirchhoff equation with critical nonlinearity

$$\left(a+b\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2dx\right)(-\Delta)^su+u=f(u) \text{ in } \mathbb{R}^N,$$

where N > 2s and $(-\Delta)^s$ is the fractional Laplacian with 0 < s < 1. By using a perturbation approach, we prove the existence of solutions to the above problem without the Ambrosetti-Rabinowitz condition when the parameter b small. What's more, we obtain the asymptotic behavior of solutions as $b \to 0$.

1. INTRODUCTION AND MAIN RESULT

In this paper, we are concerned with the following fractional Kirchhoff equation

(1.1)
$$\left(a+b\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 dx\right) (-\Delta)^s u + u = f(u) \text{ in } \mathbb{R}^N$$

where N > 2s with 0 < s < 1, a, b are positive constants and $(-\Delta)^s u$ is the fractional Laplacian which arises in the description of various phenomena in the applied science, such as the phase transition [19], Markov processes [1] and fractional quantum mechanics [15]. When a = 1 and b = 0, (1.1) becomes the fractional Schrödinger equations which have been studied by many authors. We refer the readers to [2, 5–7] and the references therein for the details. When s = 1, the problem (1.1) reduces to the well-known Kirchhoff equation

(1.2)
$$-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+u=f(u) \text{ in } \mathbb{R}^N,$$

which has been studied in the last decade, see [9, 12, 17]. The equation (1.2) is related to the stationary analogue of the Kirchhoff equation $u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u)$ on $\Omega \subset \mathbb{R}^N$ bounded, which was proposed by Kirchhoff [13] in 1883 as a generalization the classic D'Alembert's wave equation for free vibrations of elastic strings.

Recently, in bounded regular domains of \mathbb{R}^N , Fiscella and Valdinoci [11] proposed the following fractional stationary Kirchhoff equation

(1.3)
$$\begin{cases} M\left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2\right)(-\Delta)^s u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases}$$

which models nonlocal aspects of the tension arising from nonlocal measurements of the fractional length of the string. Also in bounded domains, Autuori et al. [4] dealt with the existence and the asymptotic behavior of non-negative solutions of a class of fractional stationary Kirchhoff equation. In the whole of \mathbb{R}^N , Pucci et al. [18] established the existence and multiplicity of nontrivial non-negative entire solutions of a stationary Kirchhoff eigenvalue problem. In the

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subcritical case, by using minimax arguments, Ambrosio et al. [3] obtained the multiplicity results for (1.2) in $H^s_r(\mathbb{R}^N)$ with b small. Also in the subcritical case, without the (AR)-condition, the authors [20] investigated the existence of radial solutions by using the variational methods combined with a cut-off function technique. More recently, without the (AR)-condition and monotonicity assumptions, in low dimension (N = 2, N = 3), Z. Liu et al. [16] studied the existence of ground states in the critical case. To the best of our knowledge, there are few papers on the fractional Kirchhoff equations involving the critical growth in \mathbb{R}^N with N > 3, because of the tough difficulties brought by the nonlocal term and the lack of compactness of the Sobolev embedding $H^s_r(\mathbb{R}^N) \to L^{2^*_s}(\mathbb{R}^N)$.

Motivated by the works above, we investigate the existence of the positive solutions of (1.1) in $\mathbb{R}^{N}(N > 2s)$ with the critical growth. Precisely, f satisfies the following conditions:

- $\begin{array}{ll} (f_1) \ f \in C^1(\mathbb{R}^+,\mathbb{R}), \lim_{t \to 0} f(t)/t = 0 \ \text{and} \ f(t) \equiv 0 \ \text{for} \ t \leq 0, \\ (f_2) \ \lim_{t \to \infty} f(t)/t^{2^*_s 1} = 1, \ \text{where} \ 2^*_s = \frac{2N}{N 2s}, \\ (f_3) \ \text{there exist} \ D > 0 \ \text{and} \ p < 2^*_s \ \text{such that} \ f(t) \geq t^{2^*_s 1} + Dt^{p-1} \ \text{for} \ t \geq 0. \end{array}$

Our main result can read as

Theorem 1.1. Suppose that f satisfies $(f_1) - (f_3)$ and $\max\{2, 2_s^* - 2\} , then for b$ small, (1.1) admits a nontrivial positive radial solution u_b . What's more, along a subsequence, u_b converges to u in $H^s_r(\mathbb{R}^N)$ as $b \to 0$, where u is a radial ground state to the limit problem

(1.4)
$$a(-\Delta)^s u + u = f(u), \quad u \in H^s(\mathbb{R}^N).$$

Because of the presence of the Kirchhoff term, in high dimension N > 4s, for the energy functional $I_b(u)$ (see section 2), one has $I_b(tu) \to +\infty$ as $t \to +\infty$ for each $u \neq 0$. That means Mountain pass geometry may not holds and Mountain pass theorem may not be appropriate. To overcome this difficulty, we use the variational method combined with the perturbation approach [21, 22] to get a special bounded (PS)-sequence. On the other hand, because of the presence of the Kirchhoff term, for the bounded (PS)-sequence $\{u_n\}$, even $u_n \to u_0$ weakly, it doesn't hold in general that u_0 is the critical point of the energy functional, which brings us more tough to get the compactness. We use the properties of the special (PS)-sequence and some results of the limit problem (1.4) to recover the compactness. Moreover, we obtain the asymptotic behavior of the solutions of (1.1) as $b \to 0$.

The paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we construct the min-max level. In Section 4, we complete the proof of Theorem 1.1.

2. Preliminaries and functional setting

2.1. Fractional order Sobolev spaces. The fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$ of a function $\phi : \mathbb{R}^N \to \mathbb{R}$ is defined by $\mathcal{F}((-\Delta)^s \phi)(\xi) = |\xi|^{2s} \mathcal{F}(\phi)(\xi)$, where \mathcal{F} is the Fourier transform. If ϕ is smooth enough, it can be computed by the following singular integral

$$(-\Delta)^s \phi(x) = c_s \operatorname{P.V.} \int_{\mathbb{R}^N} \frac{\phi(x) - \phi(y)}{|x - y|^{N + 2s}} \, \mathrm{d}y, \ x \in \mathbb{R}^N,$$

where c_s is a normalization constant and P.V. stands the principal value. For any $s \in (0, 1)$, we consider the fractional order Sobolev space

$$H^{s}(\mathbb{R}^{N}) = \left\{ u \in L^{2}(\mathbb{R}^{N}) : \int_{\mathbb{R}^{N}} |\xi|^{2s} |\hat{u}|^{2} \,\mathrm{d}\xi < \infty \right\},\$$

endowed with the norm $||u|| = \left(\int_{\mathbb{R}^N} (1+a|\xi|^{2s})|\hat{u}|^2 d\xi\right)^{1/2}$. $H_r^s(\mathbb{R}^N)$ denotes the space of radial functions in $H^s(\mathbb{R}^N)$, i.e. $H_r^s(\mathbb{R}^N) = \{u \in H^s(\mathbb{R}^N) : u(x) = u(|x|)\}$. The homogeneous Sobolev

space $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is defined by

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \{ u \in L^{2^*_s}(\mathbb{R}^N) : |\xi|^s \hat{u} \in L^2(\mathbb{R}^N) \},\$$

which is the completion of $C_0^{\infty}(\mathbb{R}^N)$ under the norm $\|u\|_{\mathcal{D}^{s,2}}^2 = \|(-\Delta)^{s/2}u\|_2^2 = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi.$

For the further introduction on the fractional order Sobolev space, we refer to [10]. Now, we introduce the following Sobolev embedding theorems.

Lemma 2.1 (see [8,10,14]). For any $s \in (0,1)$, $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2,2_s^*]$ and compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for $q \in [1,2_s^*)$. $H^s_r(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2,2_s^*)$ and $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is continuously embedded into $L^{2_s^*}(\mathbb{R}^N)$, i.e., there exists $S_s > 0$ such that $S_s \left(\int_{\mathbb{R}^N} |u|^{2_s^*} dx \right)^{2/2_s^*} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 dx$.

2.2. The variational setting. We define the energy functional $I_b: H^s(\mathbb{R}^N) \to \mathbb{R}$ by

$$I_b(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(a |(-\Delta)^{\frac{s}{2}} u|^2 + u^2 \right) \, \mathrm{d}x + \frac{b}{4} \left(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 \right)^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x$$

with $F(t) = \int_0^t f(\zeta) d\zeta$. It is standard to show that I_b is of class C^1 .

Definition 2.2. We call $u \in H^{s}(\mathbb{R}^{N})$ a weak solution of (1.1) if for any $\phi \in H^{s}(\mathbb{R}^{N})$,

$$\left(a+b\|u\|_{\mathcal{D}^{s,2}}^2\right)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi\,\mathrm{d}x+\int_{\mathbb{R}^N}u\phi\,\mathrm{d}x=\int_{\mathbb{R}^N}f(u)\phi\,\mathrm{d}x.$$

Obviously, the critical points of I_b are the weak solutions of (1.1).

Similar to the proof of Brezis-Lieb Lemma in [21], we can give the following lemma.

Lemma 2.3. For $s \in (0,1)$, assume $(f_1) - (f_2)$ hold. Let $\{u_n\} \subset H^s(\mathbb{R}^N)$ such that $u_n \to u$ weakly in $H^s(\mathbb{R}^N)$ and a.e. in \mathbb{R}^N as $n \to \infty$, then $\int_{\mathbb{R}^N} F(u_n) \to \int_{\mathbb{R}^N} F(u_n-u) + \int_{\mathbb{R}^N} F(u)$.

When b = 0, problem (1.1) becomes the limit problem (1.4) which plays a crucial role in our paper. The energy functional of (1.4) is defined as

$$L(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(a |(-\Delta)^{\frac{s}{2}} u|^2 + u^2 \right) \, \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x, \ u \in H^s(\mathbb{R}^N).$$

With the same assumptions on f in Theorem 1.1, it is not difficult to check that L(u) satisfies the Mountain pass geometry. The Mountain pass value denoted by c is defined by

$$c = \inf_{\gamma \in \Gamma_L} \max_{t \in [0,1]} L(\gamma(t)) > 0,$$

where $\Gamma_L = \{\gamma \in C([0,1], H^s(\mathbb{R}^N)), \gamma(0) = 0, L(\gamma(1)) < 0\}$. In the following, we present some results of the ground states of (1.4) and the proof is similar as that in [22].

Proposition 2.4. Suppose f satisfies $(f_1) - (f_3)$ and $\max\{2, 2_s^* - 2\} . Let <math>S_r$ be the set of positive radial ground states of (1.4), then

(i) S_r is not empty and S_r is compact in $H^s_r(\mathbb{R}^N)$,

(ii) $c < \frac{s}{N}(aS_s)^{\frac{N}{2s}}$ and c agrees with the least energy level denoted by E, that is, there exists $\gamma \in \Gamma_L$ such that $u \in \gamma(t)$ and $\max_{[0,1]} L(\gamma(t)) = E$, where $u \in S_r$,

 $(iii)u \in S_r$ satisfies the Pohozăev identity

(2.1)
$$\frac{N-2s}{2} \int_{\mathbb{R}^N} a |(-\Delta)^{s/2} u|^2 \, \mathrm{d}x + \frac{N}{2} \int_{\mathbb{R}^N} u^2 \, \mathrm{d}x = N \int_{\mathbb{R}^N} F(u) \, \mathrm{d}x.$$

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3. The minimax level

In order to get a bounded (PS)-sequence by the local deformation argument, a different min-max level is needed. Take $U \in S_r$ be arbitrary but fixed. By the definition of $\hat{U} = \mathcal{F}(U)$, for $U_{\tau}(x) = U(\frac{x}{\tau}), \tau > 0$, we have $\hat{U}_{\tau}(\cdot) = \tau^N \hat{U}(\tau \cdot)$. Thus $\int_{\mathbb{R}^N} |(-\Delta)^{s/2} U_{\tau}|^2 dx = \tau^{N-2s} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} U|^2 dx$. From the Pohozǎev identity (2.1), we obtain

$$L(U_{\tau}) = \left(\frac{a\tau^{N-2s}}{2} - \frac{N-2s}{2N}\tau^{N}\right) \int_{\mathbb{R}^{N}} |(-\Delta)^{s/2}U|^{2}$$

So, there exists $\tau_0 > 1$ such that $L(U_{\tau}) < -2$ for $\tau \ge \tau_0$. Set $D_b \equiv \max_{\tau \in [0,\tau_0]} I_b(U_{\tau})$. Noting that $I_b(U_{\tau}) = L(U_{\tau}) + \frac{b}{4} ||U_{\tau}||^4_{\mathcal{D}^{s,2}}$ and $\max_{\tau \in [0,\tau_0]} L(U_{\tau}) = E$, we have $D_b \to E$ as $b \to 0^+$.

Lemma 3.1. There exist $b_1 > 0$ and $C_0 > 0$, such that for any $0 < b < b_1$ there hold

$$I_b(U_{\tau_0}) < -2, \qquad ||U_\tau|| \le C_0, \ \forall \tau \in (0, \tau_0], \qquad ||u|| \le C_0, \ \forall u \in S_r.$$

Proof. Since S_r is compact, it is easy to verify that there exists $C_0 > 0$ such that the second and third part of the assertion hold. It follows from $I_b(U_{\tau_0}) \leq L(U_{\tau_0}) + \frac{b}{4}C_0^4$ and $L(U_{\tau_0}) < -2$ that the first part holds for any $0 < b < b_1$, where $b_1 > 0$ small. The proof is completed.

Now, for any $b \in (0, b_1)$, we define a min-max value $C_b := \inf_{\gamma \in \Upsilon_b} \max_{\tau \in [0, \tau_0]} I_b(\gamma(\tau))$, where

$$\Upsilon_b = \left\{ \gamma \in C([0, \tau_0], H^s_r(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(\tau_0) = U_{\tau_0}, \|\gamma(\tau)\| \le \mathcal{C}_0 + 1, \tau \in [0, \tau_0] \right\}$$

Proposition 3.2. $\lim_{b \to 0^+} C_b = E.$

Proof. For $\tau > 0$, by $||U_{\tau}||^2 = a\tau^{N-2s}||U||_{\mathcal{D}^{s,2}}^2 + \tau^N||U||_2^2$, we can define $U_0 \equiv 0$. So $U_{\tau} \in \Upsilon_b$. Moreover, $\limsup_{b\to 0^+} C_b \leq \lim_{b\to 0^+} D_b = E = c$. On the other hand, for any $\gamma \in \Upsilon_b$, it follows from $L(U_{\tau_0}) < -2$ that $\tilde{\gamma}(\cdot) = \gamma(\tau_0 \cdot) \in \Gamma_L$. Thus, from the definition of c and C_b , we obtain $C_b \geq E$ for any $b \in (0, b_1)$. The proof is completed.

4. The Proof of Theorem 1.1

For $\alpha, d > 0$, we define

$$I_b^{\alpha} := \{ u \in H_r^s(\mathbb{R}^N) : I_b(u) \le \alpha \}$$

and

$$S^d := \left\{ u \in H^s_r(\mathbb{R}^N) : \inf_{v \in S_r} \|u - v\| \le d \right\}.$$

Proposition 4.1. Let $\{b_n\}_{n=1}^{\infty}$ be such that $\lim_{n\to\infty} b_n = 0$ and $\{u_{b_n}\} \subset S^d$ with

$$\lim_{n \to \infty} I_{b_n}(u_{b_n}) \le E \text{ and } \lim_{n \to \infty} I'_{b_n}(u_{b_n}) = 0.$$

Then for d small, there is $u_0 \in S_r$, up to a subsequence, such that $u_{b_n} \to u_0$ strongly in $H^s_r(\mathbb{R}^N)$.

Proof. For convenience, we write u_{b_n} for u_b . Since $u_b \in S^d$, there exists $\tilde{u}_b \in S_r$ such that $||u_b - \tilde{u}_b|| \leq d$. Let $v_b = u_b - \tilde{u}_b$. By the fact that S_r is compact and $||v_b|| \leq d$, up to a subsequence, there exist $\tilde{u}_0 \in S_r$ and $v_0 \in H^s(\mathbb{R}^N)$, such that $\tilde{u}_b \to \tilde{u}_0$ strongly in $H^s_r(\mathbb{R}^N)$ and $v_b \to v_0$ weakly in $H^s(\mathbb{R}^N)$. Denoting $u_0 = \tilde{u}_0 + v_0$, then $u_0 \in S^d$ and $u_b \to u_0$ weakly in $H^s_r(\mathbb{R}^N)$. Next, we show $u_b \to u_0$ strongly in $H^s_r(\mathbb{R}^N)$. Since $\lim_{n\to\infty} I'_b(u_b) = 0$, then for any $\phi \in C^\infty_0(\mathbb{R}^N)$,

$$I'_{b}(u_{b})\phi = L'(u_{b})\phi + b\|u_{b}\|_{\mathcal{D}^{s,2}}^{2} \int_{\mathbb{R}^{N}} (-\Delta)^{s/2} u_{b}(-\Delta)^{s/2} \phi$$

It follows from Lemma 2.1 and $u_b \in S^d$ that $L'(u_0) = 0$ as $b \to 0$. Obviously $u_0 \not\equiv 0$ by $u_0 \in S^d$ with d small. Thus $L(u_0) \geq E$. Meanwhile, from Lemma 2.3, $I_b(u_b) = L(u_b) + \frac{b}{4} ||u_b||^4_{\mathcal{D}^{s,2}} =$

 $L(u_0) + L(u_b - u_0) + o(1)$. Together with $\lim_{n\to\infty} I_{b_n}(u_{b_n}) \leq E$, we obtain $L(u_b - u_0) \leq o(1)$. Thus, by $(f_1) - (f_2)$ and Sobolev embedding theorem, there exists constant $c_1 > 0$ such that $||u_b - u_0||^2 \leq c_1 ||u_b - u_0||^{2^*_s}$. If $||u_b - u_0|| \neq 0$ as $b \to 0$, there exists constant $c_2 > 0$ such that $||u_b - u_0|| \geq c_2$ for b small. On the other hand, from $\tilde{u}_0 \in S_r$ and $u_0 \in S^d$, we get $||\tilde{u}_0 - u_0|| \leq d$. Then $||u_b - u_0|| \leq ||u_b - \tilde{u}_b|| + ||\tilde{u}_b - \tilde{u}_0|| + ||\tilde{u}_0 - u_0|| \leq 2d + o(1)$, which is a contradiction for d small. The proof is completed.

Remark 4.2. By Proposition 4.1, for small $d \in (0,1)$, there exist $\omega > 0, b_0 > 0$ such that

(4.1)
$$||I'_b(u)|| \ge \omega \text{ for } u \in I_b^{D_b} \bigcap (S^d \setminus S^{\frac{d}{2}}) \text{ and } b \in (0, b_0).$$

Thus, we have the following proposition.

Proposition 4.3. There exists $\alpha > 0$ such that for small b > 0 and $\gamma(\tau) = U(\frac{\cdot}{\tau}), \tau \in (0, \tau_0]$,

 $I_b(\gamma(\tau)) \ge C_b - \alpha$ implies that $\gamma(\tau) \in S^{\frac{d}{2}}$,

Proof. By the Pohozăev identity (2.1), $I_b(\gamma(\tau)) = \left(\frac{a\tau^{N-2s}}{2} - \frac{N-2s}{2N}\tau^N\right) \|U\|_{\mathcal{D}^{s,2}}^2 + \frac{b}{4}\tau^{2N-4s}\|U\|_{\mathcal{D}^{s,2}}^4$. Then $\lim_{b\to 0^+} \max_{\tau\in[0,\tau_0]} I_b(\gamma(\tau)) = \max_{\tau\in[0,\tau_0]} \left(\frac{a\tau^{N-2s}}{2} - \frac{N-2s}{2N}\tau^N\right) \|U\|_{\mathcal{D}^{s,2}}^2 = E$. On the other hand, $\lim_{b\to 0^+} C_b = E$. The conclusion follows.

Thanks to (4.1) and Proposition 4.3, we can prove the following proposition, which assures the existence of a bounded (PS)-sequence for I_b . The proof is similar as that in [21, 22]. We omit the details here.

Proposition 4.4. For b > 0 small, there exist $\{u_n\}_n \subset I_b^{D_b} \cap S^d$ such that $I'_b(u_n) \to 0$ as $n \to \infty$.

The completion of Proof of Theorem 1.1

Proof. It follows from Proposition 4.4 that there exists $b_0 > 0$ such that for $b \in (0, b_0)$, there exists $\{u_n\} \in I_b^{D_b} \cap S^d$ with $I_b'(u_n) \to 0$ as $n \to \infty$. Thus, there exists $u_b \in H_r^s(\mathbb{R}^N)$, up to a subsequence, such that $u_n \to u_b$ weakly in $H_r^s(\mathbb{R}^N)$, $u_n \to u_b$ strongly in $L^p(\mathbb{R}^N)$, $p \in (2, 2_s^*)$ and $u_n \to u_b$ a.e in \mathbb{R}^N . Next, we claim that $I_b'(u_b) = 0$ for b small. Set $f(t) = g(t) + t^{2_s^* - 1}$. By Lemma 2.1, we have $\int_{\mathbb{R}^N} g(u_n)\varphi = \int_{\mathbb{R}^N} g(u_b)\varphi + o_n(1)$ for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} g(u_n)u_n = \int_{\mathbb{R}^N} g(u_b)u_b + o_n(1)$. Let $v_n = u_n - u_b$ and $\|v_n\|_{\mathcal{D}^{s,2}}^2 \to A \ge 0$, then $\|u_n\|_{\mathcal{D}^{s,2}}^2 = \|u_b\|_{\mathcal{D}^{s,2}}^2 + A + o_n(1)$. From $I_b'(u_n) \to 0$, we have

(4.2)
$$(a+b\|u_b\|_{\mathcal{D}^{s,2}}^2+bA) \|u_b\|_{\mathcal{D}^{s,2}}^2 + \|u_b\|_2^2 = \int_{\mathbb{R}^N} g(u_b)u_b + \|u_b\|_{2_s^*}^{2_s^*}.$$

The corresponding Pohozǎev identity is

(4.3)
$$\frac{N-2s}{2} \left(a+b\|u_b\|_{\mathcal{D}^{s,2}}^2+bA\right) \|u_b\|_{\mathcal{D}^{s,2}}^2 + \frac{N}{2}\|u_b\|_2^2 = N \int_{\mathbb{R}^N} G(u_b) + \frac{N}{2_s^*} \|u_b\|_{2_s^*}^{2_s^*}.$$

It follows from $I'_b(u_n)u_n \to 0$ and Brezis-Lieb Lemma that

$$(a+b\|u_b\|_{\mathcal{D}^{s,2}}^2+bA) (\|u_b\|_{\mathcal{D}^{s,2}}^2+A) + (\|u_b\|_2^2+\|v_n\|_2^2) = \int_{\mathbb{R}^N} g(u_b)u_b + \|u_b\|_{2_s^*}^{2_s^*} + \|v_n\|_{2_s^*}^{2_s^*} + o_n(1).$$

Together with (4.2), we have

(4.4)
$$(a+b||u_b||_{\mathcal{D}^{s,2}}^2+bA)A+||v_n||_2^2=||v_n||_{2_s^*}^{2_s^*}+o_n(1).$$

It follows from Lemma 2.1 that $A \leq \frac{1}{a} \left(\frac{A}{S_s}\right)^{\frac{2^*}{2}} + o(1)$. If A = 0, we have done. If A > 0, then $A \geq a^{\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}$. By the Pohozǎev identity (4.3) and (4.4),

$$I_{b}(u_{n}) = \left(\frac{1}{2} - \frac{1}{2_{s}^{*}}\right) a(\|u_{b}\|_{\mathcal{D}^{s,2}}^{2} + A) + \left(\frac{1}{4} - \frac{1}{2_{s}^{*}}\right) b(\|u_{b}\|_{\mathcal{D}^{s,2}}^{2} + A)^{2} + \left(\frac{1}{2} - \frac{1}{2_{s}^{*}}\right) \|v_{n}\|_{2}^{2} + o(1)$$
$$\geq \left(\frac{1}{2} - \frac{1}{2_{s}^{*}}\right) aA + b\left(\frac{1}{4} - \frac{1}{2_{s}^{*}}\right) (\|u_{b}\|_{\mathcal{D}^{s,2}}^{2} + A)^{2} + o(1).$$

On the other hand, from $\{u_n\} \in S^d$, for d small, there exist $\tilde{u}_n \in S_r$ and $\tilde{v}_n \in H^s(\mathbb{R}^N)$ such that $u_n = \tilde{u}_n + \tilde{v}_n$ with $\|\tilde{v}_n\| \leq d$. Thus $\|u_n\|_{\mathcal{D}^{s,2}}^2 \leq \|\tilde{v}_n\|_{\mathcal{D}^{s,2}}^2 + \|\tilde{u}_n\|_{\mathcal{D}^{s,2}}^2 \leq 1 + \sup_{v \in S_r} \|v\|_{\mathcal{D}^{s,2}}^2 \triangleq B$ which implies that $\|u_b\|_{\mathcal{D}^{s,2}}^2 + A \leq 2B$, where B is independent of b, n and d. So

$$I_b(u_n) \ge \left(\frac{1}{2} - \frac{1}{2_s^*}\right) aA - 4b \left|\frac{1}{4} - \frac{1}{2_s^*}\right| B^2 + o(1).$$

Meanwhile, from $\limsup_{n\to\infty} I_b(u_n) \leq D_b$, we get

$$\left(\frac{1}{2} - \frac{1}{2_s^*}\right) aA \le D_b + b \left|\frac{1}{4} - \frac{1}{2_s^*}\right| B^2.$$

Together with $A \ge a^{\frac{N-2s}{2s}} S_s^{\frac{N}{2s}}$, we have $\frac{s}{N} (aS_s)^{\frac{N}{2s}} \le D_b + b \left| \frac{1}{4} - \frac{1}{2_s^*} \right| B^2 \to E$, as $b \to 0$, which is a contradiction with $E < \frac{s}{N} (aS_s)^{\frac{N}{2s}}$. So, the claim is true. Since $u_n \in S^d$, then for d small, $u_b \neq 0$. Thus, for b and d small, there exists $u_b \in H_r^s(\mathbb{R}^N)$ which is a nontrivial solution of (1.1). In the following, we investigate the asymptotic behavior of u_b as $b \to 0$. Noting that $D_b \to E$ as $b \to 0$, the similar proof as that in Proposition 4.1, we obtain that there exist $u \neq 0$ such that $u_b \to u$ strongly in $H_r^s(\mathbb{R}^N)$ with L'(u) = 0 and L(u) = E. The proof is finished. \Box

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