# FRACTIONAL KIRCHHOFF EQUATION WITH A GENERAL CRITICAL NONLINEARITY 

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Abstract. In this paper, we study the fractional Kirchhoff equation with critical nonlinearity

$$
\left(a+b \int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}} u\right|^{2} d x\right)(-\Delta)^{s} u+u=f(u) \text { in } \mathbb{R}^{N}
$$

where $N>2 s$ and $(-\Delta)^{s}$ is the fractional Laplacian with $0<s<1$. By using a perturbation approach, we prove the existence of solutions to the above problem without the AmbrosettiRabinowitz condition when the parameter $b$ small. What's more, we obtain the asymptotic behavior of solutions as $b \rightarrow 0$.

## 1. Introduction and main result

In this paper, we are concerned with the following fractional Kirchhoff equation

$$
\begin{equation*}
\left(a+b \int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}} u\right|^{2} d x\right)(-\Delta)^{s} u+u=f(u) \text { in } \mathbb{R}^{N}, \tag{1.1}
\end{equation*}
$$

where $N>2 s$ with $0<s<1, a, b$ are positive constants and $(-\Delta)^{s} u$ is the fractional Laplacian which arises in the description of various phenomena in the applied science, such as the phase transition [19], Markov processes [1] and fractional quantum mechanics [15]. When $a=1$ and $b=0$, (1.1) becomes the fractional Schrödinger equations which have been studied by many authors. We refer the readers to $[2,5-7]$ and the references therein for the details. When $s=1$, the problem (1.1) reduces to the well-known Kirchhoff equation

$$
\begin{equation*}
-\left(a+b \int_{\mathbb{R}^{N}}|\nabla u|^{2} d x\right) \Delta u+u=f(u) \text { in } \mathbb{R}^{N}, \tag{1.2}
\end{equation*}
$$

which has been studied in the last decade, see $[9,12,17]$. The equation (1.2) is related to the stationary analogue of the Kirchhoff equation $u_{t t}-\left(a+b \int_{\Omega}|\nabla u|^{2} d x\right) \Delta u=f(x, u)$ on $\Omega \subset \mathbb{R}^{N}$ bounded, which was proposed by Kirchhoff [13] in 1883 as a generalization the classic D'Alembert's wave equation for free vibrations of elastic strings.

Recently, in bounded regular domains of $\mathbb{R}^{N}$, Fiscella and Valdinoci [11] proposed the following fractional stationary Kirchhoff equation

$$
\left\{\begin{array}{l}
M\left(\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}} u\right|^{2}\right)(-\Delta)^{s} u=f(x, u), \text { in } \Omega,  \tag{1.3}\\
u=0 \text { in } \mathbb{R}^{N} \backslash \Omega,
\end{array}\right.
$$

which models nonlocal aspects of the tension arising from nonlocal measurements of the fractional length of the string. Also in bounded domains, Autuori et al. [4] dealt with the existence and the asymptotic behavior of non-negative solutions of a class of fractional stationary Kirchhoff equation. In the whole of $\mathbb{R}^{N}$, Pucci et al. [18] established the existence and multiplicity of nontrivial non-negative entire solutions of a stationary Kirchhoff eigenvalue problem. In the

[^0]subcritical case, by using minimax arguments, Ambrosio et al. [3] obtained the multiplicity results for (1.2) in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$ with $b$ small. Also in the subcritical case, without the (AR)-condition, the authors [20] investigated the existence of radial solutions by using the variational methods combined with a cut-off function technique. More recently, without the (AR)-condition and monotonicity assumptions, in low dimension $(N=2, N=3)$, Z. Liu et al. [16] studied the existence of ground states in the critical case. To the best of our knowledge, there are few papers on the fractional Kirchhoff equations involving the critical growth in $\mathbb{R}^{N}$ with $N>3$, because of the tough difficulties brought by the nonlocal term and the lack of compactness of the Sobolev embedding $H_{r}^{s}\left(\mathbb{R}^{N}\right) \rightarrow L^{2_{s}^{*}}\left(\mathbb{R}^{N}\right)$.

Motivated by the works above, we investigate the existence of the positive solutions of (1.1) in $\mathbb{R}^{N}(N>2 s)$ with the critical growth. Precisely, $f$ satisfies the following conditions:
$\left(f_{1}\right) f \in C^{1}\left(\mathbb{R}^{+}, \mathbb{R}\right), \lim _{t \rightarrow 0} f(t) / t=0$ and $f(t) \equiv 0$ for $t \leq 0$,
( $f_{2}$ ) $\lim _{t \rightarrow \infty} f(t) / t^{2 *}-1=1$, where $2_{s}^{*}=\frac{2 N}{N-2 s}$,
$\left(f_{3}\right)$ there exist $D>0$ and $p<2_{s}^{*}$ such that $f(t) \geq t^{2_{s}^{*}-1}+D t^{p-1}$ for $t \geq 0$.
Our main result can read as
Theorem 1.1. Suppose that $f$ satisfies $\left(f_{1}\right)-\left(f_{3}\right)$ and $\max \left\{2,2_{s}^{*}-2\right\}<p<2_{s}^{*}$, then for $b$ small, (1.1) admits a nontrivial positive radial solution $u_{b}$. What's more, along a subsequence, $u_{b}$ converges to $u$ in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$ as $b \rightarrow 0$, where $u$ is a radial ground state to the limit problem

$$
\begin{equation*}
a(-\Delta)^{s} u+u=f(u), \quad u \in H^{s}\left(\mathbb{R}^{N}\right) \tag{1.4}
\end{equation*}
$$

Because of the presence of the Kirchhoff term, in high dimension $N>4 s$, for the energy functional $I_{b}(u)$ (see section 2), one has $I_{b}(t u) \rightarrow+\infty$ as $t \rightarrow+\infty$ for each $u \neq 0$. That means Mountain pass geometry may not holds and Mountain pass theorem may not be appropriate. To overcome this difficulty, we use the variational method combined with the perturbation approach $[21,22]$ to get a special bounded (PS)-sequence. On the other hand, because of the presence of the Kirchhoff term, for the bounded (PS)-sequence $\left\{u_{n}\right\}$, even $u_{n} \rightarrow u_{0}$ weakly, it doesn't hold in general that $u_{0}$ is the critical point of the energy functional, which brings us more tough to get the compactness. We use the properties of the special (PS)-sequence and some results of the limit problem (1.4) to recover the compactness. Moreover, we obtain the asymptotic behavior of the solutions of (1.1) as $b \rightarrow 0$.

The paper is organized as follows. Some preliminaries are presented in Section 2. In Section 3, we construct the min-max level. In Section 4, we complete the proof of Theorem 1.1.

## 2. Preliminaries and functional Setting

2.1. Fractional order Sobolev spaces. The fractional Laplacian $(-\Delta)^{s}$ with $s \in(0,1)$ of a function $\phi: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is defined by $\mathcal{F}\left((-\Delta)^{s} \phi\right)(\xi)=|\xi|^{2 s} \mathcal{F}(\phi)(\xi)$, where $\mathcal{F}$ is the Fourier transform. If $\phi$ is smooth enough, it can be computed by the following singular integral

$$
(-\Delta)^{s} \phi(x)=c_{s} \text { P.V. } \int_{\mathbb{R}^{N}} \frac{\phi(x)-\phi(y)}{|x-y|^{N+2 s}} \mathrm{~d} y, \quad x \in \mathbb{R}^{N}
$$

where $c_{s}$ is a normalization constant and P.V. stands the principal value. For any $s \in(0,1)$, we consider the fractional order Sobolev space

$$
H^{s}\left(\mathbb{R}^{N}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{N}\right): \int_{\mathbb{R}^{N}}|\xi|^{2 s}|\hat{u}|^{2} \mathrm{~d} \xi<\infty\right\}
$$

endowed with the norm $\|u\|=\left(\int_{\mathbb{R}^{N}}\left(1+a|\xi|^{2 s}\right)|\hat{u}|^{2} \mathrm{~d} \xi\right)^{1 / 2} . H_{r}^{s}\left(\mathbb{R}^{N}\right)$ denotes the space of radial functions in $H^{s}\left(\mathbb{R}^{N}\right)$, i.e. $H_{r}^{s}\left(\mathbb{R}^{N}\right)=\left\{u \in H^{s}\left(\mathbb{R}^{N}\right): u(x)=u(|x|)\right\}$. The homogeneous Sobolev
space $\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ is defined by

$$
\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)=\left\{u \in L^{2_{s}^{*}}\left(\mathbb{R}^{N}\right):|\xi|^{s} \hat{u} \in L^{2}\left(\mathbb{R}^{N}\right)\right\}
$$

which is the completion of $C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ under the norm $\|u\|_{\mathcal{D}^{s, 2}}^{2}=\left\|(-\Delta)^{s / 2} u\right\|_{2}^{2}=\int_{\mathbb{R}^{N}}|\xi|^{2 s}|\hat{u}|^{2} \mathrm{~d} \xi$.
For the further introduction on the fractional order Sobolev space, we refer to [10]. Now, we introduce the following Sobolev embedding theorems.

Lemma 2.1 (see $[8,10,14])$. For any $s \in(0,1), H^{s}\left(\mathbb{R}^{N}\right)$ is continuously embedded into $L^{q}\left(\mathbb{R}^{N}\right)$ for $q \in\left[2,2_{s}^{*}\right]$ and compactly embedded into $L_{\text {loc }}^{q}\left(\mathbb{R}^{N}\right)$ for $q \in\left[1,2_{s}^{*}\right)$. $H_{r}^{s}\left(\mathbb{R}^{N}\right)$ is compactly embedded into $L^{q}\left(\mathbb{R}^{N}\right)$ for $q \in\left(2,2_{s}^{*}\right)$ and $\mathcal{D}^{s, 2}\left(\mathbb{R}^{N}\right)$ is continuously embedded into $L^{2_{s}^{*}}\left(\mathbb{R}^{N}\right)$, i.e., there exists $S_{s}>0$ such that $S_{s}\left(\int_{\mathbb{R}^{N}}|u|^{2_{s}^{*}} \mathrm{~d} x\right)^{2 / 2_{s}^{*}} \leq \int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}} u\right|^{2} \mathrm{~d} x$.
2.2. The variational setting. We define the energy functional $I_{b}: H^{s}\left(\mathbb{R}^{N}\right) \rightarrow \mathbb{R}$ by

$$
I_{b}(u)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(a\left|(-\Delta)^{\frac{s}{2}} u\right|^{2}+u^{2}\right) \mathrm{d} x+\frac{b}{4}\left(\int_{\mathbb{R}^{N}}\left|(-\Delta)^{\frac{s}{2}} u\right|^{2}\right)^{2} \mathrm{~d} x-\int_{\mathbb{R}^{N}} F(u) \mathrm{d} x
$$

with $F(t)=\int_{0}^{t} f(\zeta) \mathrm{d} \zeta$. It is standard to show that $I_{b}$ is of class $C^{1}$.
Definition 2.2. We call $u \in H^{s}\left(\mathbb{R}^{N}\right)$ a weak solution of (1.1) if for any $\phi \in H^{s}\left(\mathbb{R}^{N}\right)$,

$$
\left(a+b\|u\|_{\mathcal{D}^{s, 2}}^{2}\right) \int_{\mathbb{R}^{N}}(-\Delta)^{\frac{s}{2}} u(-\Delta)^{\frac{s}{2}} \phi \mathrm{~d} x+\int_{\mathbb{R}^{N}} u \phi \mathrm{~d} x=\int_{\mathbb{R}^{N}} f(u) \phi \mathrm{d} x .
$$

Obviously, the critical points of $I_{b}$ are the weak solutions of (1.1).
Similar to the proof of Brezis-Lieb Lemma in [21], we can give the following lemma.
Lemma 2.3. For $s \in(0,1)$, assume $\left(f_{1}\right)-\left(f_{2}\right)$ hold. Let $\left\{u_{n}\right\} \subset H^{s}\left(\mathbb{R}^{N}\right)$ such that $u_{n} \rightarrow u$ weakly in $H^{s}\left(\mathbb{R}^{N}\right)$ and a.e. in $\mathbb{R}^{N}$ as $n \rightarrow \infty$, then $\int_{\mathbb{R}^{N}} F\left(u_{n}\right) \rightarrow \int_{\mathbb{R}^{N}} F\left(u_{n}-u\right)+\int_{\mathbb{R}^{N}} F(u)$.

When $b=0$, problem (1.1) becomes the limit problem (1.4) which plays a crucial role in our paper. The energy functional of (1.4) is defined as

$$
L(u)=\frac{1}{2} \int_{\mathbb{R}^{N}}\left(a\left|(-\Delta)^{\frac{s}{2}} u\right|^{2}+u^{2}\right) \mathrm{d} x-\int_{\mathbb{R}^{N}} F(u) \mathrm{d} x, \quad u \in H^{s}\left(\mathbb{R}^{N}\right) .
$$

With the same assumptions on $f$ in Theorem 1.1, it is not difficult to check that $L(u)$ satisfies the Mountain pass geometry. The Mountain pass value denoted by $c$ is defined by

$$
c=\inf _{\gamma \in \Gamma_{L}} \max _{t \in[0,1]} L(\gamma(t))>0,
$$

where $\Gamma_{L}=\left\{\gamma \in C\left([0,1], H^{s}\left(\mathbb{R}^{N}\right)\right), \gamma(0)=0, L(\gamma(1))<0\right\}$. In the following, we present some results of the ground states of (1.4) and the proof is similar as that in [22].

Proposition 2.4. Suppose $f$ satisfies $\left(f_{1}\right)-\left(f_{3}\right)$ and $\max \left\{2,2_{s}^{*}-2\right\}<p<2_{s}^{*}$. Let $S_{r}$ be the set of positive radial ground states of (1.4), then
(i) $S_{r}$ is not empty and $S_{r}$ is compact in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$,
(ii) $c<\frac{s}{N}\left(a S_{s}\right)^{\frac{N}{2 s}}$ and $c$ agrees with the least energy level denoted by $E$, that is, there exists $\gamma \in \Gamma_{L}$ such that $u \in \gamma(t)$ and $\max _{[0,1]} L(\gamma(t))=E$, where $u \in S_{r}$,
(iii) $u \in S_{r}$ satisfies the Pohozăev identity

$$
\begin{equation*}
\frac{N-2 s}{2} \int_{\mathbb{R}^{N}} a\left|(-\Delta)^{s / 2} u\right|^{2} \mathrm{~d} x+\frac{N}{2} \int_{\mathbb{R}^{N}} u^{2} \mathrm{~d} x=N \int_{\mathbb{R}^{N}} F(u) \mathrm{d} x . \tag{2.1}
\end{equation*}
$$

## 3. The minimax level

In order to get a bounded (PS)-sequence by the local deformation argument, a different min-max level is needed. Take $U \in S_{r}$ be arbitrary but fixed. By the definition of $\hat{U}=$ $\mathcal{F}(U)$, for $U_{\tau}(x)=U\left(\frac{x}{\tau}\right), \tau>0$, we have $\hat{U}_{\tau}(\cdot)=\tau^{N} \hat{U}(\tau \cdot)$. Thus $\int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} U_{\tau}\right|^{2} \mathrm{~d} x=$ $\tau^{N-2 s} \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} U\right|^{2} \mathrm{~d} x$. From the Pohozǎev identity (2.1), we obtain

$$
L\left(U_{\tau}\right)=\left(\frac{a \tau^{N-2 s}}{2}-\frac{N-2 s}{2 N} \tau^{N}\right) \int_{\mathbb{R}^{N}}\left|(-\Delta)^{s / 2} U\right|^{2}
$$

So, there exists $\tau_{0}>1$ such that $L\left(U_{\tau}\right)<-2$ for $\tau \geq \tau_{0}$. Set $D_{b} \equiv \max _{\tau \in\left[0, \tau_{0}\right]} I_{b}\left(U_{\tau}\right)$. Noting that $I_{b}\left(U_{\tau}\right)=L\left(U_{\tau}\right)+\frac{b}{4}\left\|U_{\tau}\right\|_{\mathcal{D}^{s, 2}}^{4}$ and $\max _{\tau \in\left[0, \tau_{0}\right]} L\left(U_{\tau}\right)=E$, we have $D_{b} \rightarrow E$ as $b \rightarrow 0^{+}$.
Lemma 3.1. There exist $b_{1}>0$ and $\mathcal{C}_{0}>0$, such that for any $0<b<b_{1}$ there hold

$$
I_{b}\left(U_{\tau_{0}}\right)<-2, \quad\left\|U_{\tau}\right\| \leq \mathcal{C}_{0}, \quad \forall \tau \in\left(0, \tau_{0}\right], \quad\|u\| \leq \mathcal{C}_{0}, \quad \forall u \in S_{r}
$$

Proof. Since $S_{r}$ is compact, it is easy to verify that there exists $\mathcal{C}_{0}>0$ such that the second and third part of the assertion hold. It follows from $I_{b}\left(U_{\tau_{0}}\right) \leq L\left(U_{\tau_{0}}\right)+\frac{b}{4} \mathcal{C}_{0}^{4}$ and $L\left(U_{\tau_{0}}\right)<-2$ that the first part holds for any $0<b<b_{1}$, where $b_{1}>0$ small. The proof is completed.
Now, for any $b \in\left(0, b_{1}\right)$, we define a min-max value $C_{b}:=\inf _{\gamma \in \Upsilon_{b}} \max _{\tau \in\left[0, \tau_{0}\right]} I_{b}(\gamma(\tau))$, where

$$
\Upsilon_{b}=\left\{\gamma \in C\left(\left[0, \tau_{0}\right], H_{r}^{s}\left(\mathbb{R}^{N}\right)\right): \gamma(0)=0, \gamma\left(\tau_{0}\right)=U_{\tau_{0}},\|\gamma(\tau)\| \leq \mathcal{C}_{0}+1, \tau \in\left[0, \tau_{0}\right]\right\} .
$$

Proposition 3.2. $\lim _{b \rightarrow 0^{+}} C_{b}=E$.
Proof. For $\tau>0$, by $\left\|U_{\tau}\right\|^{2}=a \tau^{N-2 s}\|U\|_{\mathcal{D}^{s}, 2}^{2}+\tau^{N}\|U\|_{2}^{2}$, we can define $U_{0} \equiv 0$. So $U_{\tau} \in \Upsilon_{b}$. Moreover, $\lim \sup _{b \rightarrow 0^{+}} C_{b} \leq \lim _{b \rightarrow 0^{+}} D_{b}=E=c$. On the other hand, for any $\gamma \in \Upsilon_{b}$, it follows from $L\left(U_{\tau_{0}}\right)<-2$ that $\tilde{\gamma}(\cdot)=\gamma\left(\tau_{0} \cdot\right) \in \Gamma_{L}$. Thus, from the definition of $c$ and $C_{b}$, we obtain $C_{b} \geq E$ for any $b \in\left(0, b_{1}\right)$. The proof is completed.

## 4. The Proof of Theorem 1.1

For $\alpha, d>0$, we define

$$
I_{b}^{\alpha}:=\left\{u \in H_{r}^{s}\left(\mathbb{R}^{N}\right): I_{b}(u) \leq \alpha\right\}
$$

and

$$
S^{d}:=\left\{u \in H_{r}^{s}\left(\mathbb{R}^{N}\right): \inf _{v \in S_{r}}\|u-v\| \leq d\right\} .
$$

Proposition 4.1. Let $\left\{b_{n}\right\}_{n=1}^{\infty}$ be such that $\lim _{n \rightarrow \infty} b_{n}=0$ and $\left\{u_{b_{n}}\right\} \subset S^{d}$ with

$$
\lim _{n \rightarrow \infty} I_{b_{n}}\left(u_{b_{n}}\right) \leq E \text { and } \lim _{n \rightarrow \infty} I_{b_{n}}^{\prime}\left(u_{b_{n}}\right)=0
$$

Then for $d$ small, there is $u_{0} \in S_{r}$, up to a subsequence, such that $u_{b_{n}} \rightarrow u_{0}$ strongly in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$.
Proof. For convenience, we write $u_{b_{n}}$ for $u_{b}$. Since $u_{b} \in S^{d}$, there exists $\tilde{u}_{b} \in S_{r}$ such that $\left\|u_{b}-\tilde{u}_{b}\right\| \leq d$. Let $v_{b}=u_{b}-\tilde{u}_{b}$. By the fact that $S_{r}$ is compact and $\left\|v_{b}\right\| \leq d$, up to a subsequence, there exist $\tilde{u}_{0} \in S_{r}$ and $v_{0} \in H^{s}\left(\mathbb{R}^{N}\right)$, such that $\tilde{u}_{b} \rightarrow \tilde{u}_{0}$ strongly in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$ and $v_{b} \rightarrow v_{0}$ weakly in $H^{s}\left(\mathbb{R}^{N}\right)$. Denoting $u_{0}=\tilde{u}_{0}+v_{0}$, then $u_{0} \in S^{d}$ and $u_{b} \rightarrow u_{0}$ weakly in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$. Next, we show $u_{b} \rightarrow u_{0}$ strongly in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$. Since $\lim _{n \rightarrow \infty} I_{b}^{\prime}\left(u_{b}\right)=0$, then for any $\phi \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$,

$$
I_{b}^{\prime}\left(u_{b}\right) \phi=L^{\prime}\left(u_{b}\right) \phi+b\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2} \int_{\mathbb{R}^{N}}(-\Delta)^{s / 2} u_{b}(-\Delta)^{s / 2} \phi
$$

It follows from Lemma 2.1 and $u_{b} \in S^{d}$ that $L^{\prime}\left(u_{0}\right)=0$ as $b \rightarrow 0$. Obviously $u_{0} \not \equiv 0$ by $u_{0} \in S^{d}$ with $d$ small. Thus $L\left(u_{0}\right) \geq E$. Meanwhile, from Lemma 2.3, $I_{b}\left(u_{b}\right)=L\left(u_{b}\right)+\frac{b}{4}\left\|u_{b}\right\|_{\mathcal{D}^{s}, 2}^{4}=$
$L\left(u_{0}\right)+L\left(u_{b}-u_{0}\right)+o(1)$. Together with $\lim _{n \rightarrow \infty} I_{b_{n}}\left(u_{b_{n}}\right) \leq E$, we obtain $L\left(u_{b}-u_{0}\right) \leq o(1)$. Thus, by $\left(f_{1}\right)-\left(f_{2}\right)$ and Sobolev embedding theorem, there exists constant $c_{1}>0$ such that $\left\|u_{b}-u_{0}\right\|^{2} \leq c_{1}\left\|u_{b}-u_{0}\right\|^{2_{s}^{*}}$. If $\left\|u_{b}-u_{0}\right\| \nrightarrow 0$ as $b \rightarrow 0$, there exists constant $c_{2}>0$ such that $\left\|u_{b}-u_{0}\right\| \geq c_{2}$ for $b$ small. On the other hand, from $\tilde{u}_{0} \in S_{r}$ and $u_{0} \in S^{d}$, we get $\left\|\tilde{u}_{0}-u_{0}\right\| \leq d$. Then $\left\|u_{b}-u_{0}\right\| \leq\left\|u_{b}-\tilde{u}_{b}\right\|+\left\|\tilde{u}_{b}-\tilde{u}_{0}\right\|+\left\|\tilde{u}_{0}-u_{0}\right\| \leq 2 d+o(1)$, which is a contradiction for $d$ small. The proof is completed.
Remark 4.2. By Proposition 4.1, for small $d \in(0,1)$, there exist $\omega>0, b_{0}>0$ such that

$$
\begin{equation*}
\left\|I_{b}^{\prime}(u)\right\| \geq \omega \text { for } u \in I_{b}^{D_{b}} \bigcap\left(S^{d} \backslash S^{\frac{d}{2}}\right) \text { and } b \in\left(0, b_{0}\right) \tag{4.1}
\end{equation*}
$$

Thus, we have the following proposition.
Proposition 4.3. There exists $\alpha>0$ such that for small $b>0$ and $\gamma(\tau)=U(\dot{\bar{\tau}}), \tau \in\left(0, \tau_{0}\right]$,

$$
I_{b}(\gamma(\tau)) \geq C_{b}-\alpha \text { implies that } \gamma(\tau) \in S^{\frac{d}{2}}
$$

Proof. By the Pohozǎev identity $(2.1), I_{b}(\gamma(\tau))=\left(\frac{a \tau^{N-2 s}}{2}-\frac{N-2 s}{2 N} \tau^{N}\right)\|U\|_{\mathcal{D}^{s, 2}}^{2}+\frac{b}{4} \tau^{2 N-4 s}\|U\|_{\mathcal{D}^{s, 2}}^{4}$. Then $\lim _{b \rightarrow 0^{+}} \max _{\tau \in\left[0, \tau_{0}\right]} I_{b}(\gamma(\tau))=\max _{\tau \in\left[0, \tau_{0}\right]}\left(\frac{a \tau^{N-2 s}}{2}-\frac{N-2 s}{2 N} \tau^{N}\right)\|U\|_{\mathcal{D}^{s, 2}}^{2}=E$. On the other hand, $\lim _{b \rightarrow 0^{+}} C_{b}=E$. The conclusion follows.

Thanks to (4.1) and Proposition 4.3, we can prove the following proposition, which assures the existence of a bounded (PS)-sequence for $I_{b}$. The proof is similar as that in [21,22]. We omit the details here.

Proposition 4.4. For $b>0$ small, there exist $\left\{u_{n}\right\}_{n} \subset I_{b}^{D_{b}} \cap S^{d}$ such that $I_{b}^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$.

## The completion of Proof of Theorem 1.1

Proof. It follows from Proposition 4.4 that there exists $b_{0}>0$ such that for $b \in\left(0, b_{0}\right)$, there exists $\left\{u_{n}\right\} \in I_{b}^{D_{b}} \cap S^{d}$ with $I_{b}^{\prime}\left(u_{n}\right) \rightarrow 0$ as $n \rightarrow \infty$. Thus, there exists $u_{b} \in H_{r}^{s}\left(\mathbb{R}^{N}\right)$, up to a subsequence, such that $u_{n} \rightarrow u_{b}$ weakly in $H_{r}^{s}\left(\mathbb{R}^{N}\right), u_{n} \rightarrow u_{b}$ strongly in $L^{p}\left(\mathbb{R}^{N}\right), p \in\left(2,2_{s}^{*}\right)$ and $u_{n} \rightarrow u_{b}$ a.e in $\mathbb{R}^{N}$. Next, we claim that $I_{b}^{\prime}\left(u_{b}\right)=0$ for $b$ small. Set $f(t)=g(t)+t^{2_{s}^{*}-1}$. By Lemma 2.1, we have $\int_{\mathbb{R}^{N}} g\left(u_{n}\right) \varphi=\int_{\mathbb{R}^{N}} g\left(u_{b}\right) \varphi+o_{n}(1)$ for any $\varphi \in C_{0}^{\infty}\left(\mathbb{R}^{N}\right)$ and $\int_{\mathbb{R}^{N}} g\left(u_{n}\right) u_{n}=\int_{\mathbb{R}^{N}} g\left(u_{b}\right) u_{b}+o_{n}(1)$. Let $v_{n}=u_{n}-u_{b}$ and $\left\|v_{n}\right\|_{\mathcal{D}^{s, 2}}^{2} \rightarrow A \geq 0$, then $\left\|u_{n}\right\|_{\mathcal{D}^{s, 2}}^{2}=\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+A+o_{n}(1)$. From $I_{b}^{\prime}\left(u_{n}\right) \rightarrow 0$, we have

$$
\begin{equation*}
\left(a+b\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+b A\right)\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+\left\|u_{b}\right\|_{2}^{2}=\int_{\mathbb{R}^{N}} g\left(u_{b}\right) u_{b}+\left\|u_{b}\right\|_{2_{s}^{*}}^{2_{s}^{*}} . \tag{4.2}
\end{equation*}
$$

The corresponding Pohozǎev identity is

$$
\begin{equation*}
\frac{N-2 s}{2}\left(a+b\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+b A\right)\left\|u_{b}\right\|_{\mathcal{D}^{s}, 2}^{2}+\frac{N}{2}\left\|u_{b}\right\|_{2}^{2}=N \int_{\mathbb{R}^{N}} G\left(u_{b}\right)+\frac{N}{2_{s}^{*}}\left\|u_{b}\right\|_{2_{s}^{s}}^{2_{s}^{*}} . \tag{4.3}
\end{equation*}
$$

It follows from $I_{b}^{\prime}\left(u_{n}\right) u_{n} \rightarrow 0$ and Brezis-Lieb Lemma that

$$
\left(a+b\left\|u_{b}\right\|_{\mathcal{D}^{s}, 2}^{2}+b A\right)\left(\left\|u_{b}\right\|_{\mathcal{D}^{s}, 2}^{2}+A\right)+\left(\left\|u_{b}\right\|_{2}^{2}+\left\|v_{n}\right\|_{2}^{2}\right)=\int_{\mathbb{R}^{N}} g\left(u_{b}\right) u_{b}+\left\|u_{b}\right\|_{2_{s}^{*}}^{2_{s}^{*}}+\left\|v_{n}\right\|_{2_{s}^{*}}^{2_{s}^{*}}+o_{n}(1)
$$

Together with (4.2), we have

$$
\begin{equation*}
\left(a+b\left\|u_{b}\right\|_{\mathcal{D}_{s, 2}}^{2}+b A\right) A+\left\|v_{n}\right\|_{2}^{2}=\left\|v_{n}\right\|_{2_{s}^{*}}^{2_{s}^{*}}+o_{n}(1) . \tag{4.4}
\end{equation*}
$$

It follows from Lemma 2.1 that $A \leq \frac{1}{a}\left(\frac{A}{S_{s}}\right)^{\frac{2_{s}^{*}}{2}}+o(1)$. If $A=0$, we have done. If $A>0$, then $A \geq a^{\frac{N-2 s}{2 s}} S_{s}^{\frac{N}{2 s}}$. By the Pohozǎev identity (4.3) and (4.4),

$$
\begin{aligned}
I_{b}\left(u_{n}\right) & =\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right) a\left(\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+A\right)+\left(\frac{1}{4}-\frac{1}{2_{s}^{*}}\right) b\left(\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+A\right)^{2}+\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right)\left\|v_{n}\right\|_{2}^{2}+o(1) \\
& \geq\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right) a A+b\left(\frac{1}{4}-\frac{1}{2_{s}^{*}}\right)\left(\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+A\right)^{2}+o(1)
\end{aligned}
$$

On the other hand, from $\left\{u_{n}\right\} \in S^{d}$, for $d$ small, there exist $\tilde{u}_{n} \in S_{r}$ and $\tilde{v}_{n} \in H^{s}\left(\mathbb{R}^{N}\right)$ such that $u_{n}=\tilde{u}_{n}+\tilde{v}_{n}$ with $\left\|\tilde{v}_{n}\right\| \leq d$. Thus $\left\|u_{n}\right\|_{\mathcal{D}^{s, 2}}^{2} \leq\left\|\tilde{v}_{n}\right\|_{\mathcal{D}^{s, 2}}^{2}+\left\|\tilde{u}_{n}\right\|_{\mathcal{D}^{s, 2}}^{2} \leq 1+\sup _{v \in S_{r}}\|v\|_{\mathcal{D}^{s, 2}}^{2} \triangleq B$ which implies that $\left\|u_{b}\right\|_{\mathcal{D}^{s, 2}}^{2}+A \leq 2 B$, where $B$ is independent of $b, n$ and $d$. So

$$
I_{b}\left(u_{n}\right) \geq\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right) a A-4 b\left|\frac{1}{4}-\frac{1}{2_{s}^{*}}\right| B^{2}+o(1)
$$

Meanwhile, from $\lim \sup _{n \rightarrow \infty} I_{b}\left(u_{n}\right) \leq D_{b}$, we get

$$
\left(\frac{1}{2}-\frac{1}{2_{s}^{*}}\right) a A \leq D_{b}+b\left|\frac{1}{4}-\frac{1}{2_{s}^{*}}\right| B^{2} .
$$

Together with $A \geq a^{\frac{N-2 s}{2 s}} S_{s}^{\frac{N}{2 s}}$, we have $\frac{s}{N}\left(a S_{s}\right)^{\frac{N}{2 s}} \leq D_{b}+b\left|\frac{1}{4}-\frac{1}{2_{s}^{*}}\right| B^{2} \rightarrow E$, as $b \rightarrow 0$, which is a contradiction with $E<\frac{s}{N}\left(a S_{s}\right)^{\frac{N}{2 s}}$. So, the claim is true. Since $u_{n} \in S^{d}$, then for $d$ small, $u_{b} \not \equiv 0$. Thus, for $b$ and $d$ small, there exists $u_{b} \in H_{r}^{s}\left(\mathbb{R}^{N}\right)$ which is a nontrivial solution of (1.1). In the following, we investigate the asymptotic behavior of $u_{b}$ as $b \rightarrow 0$. Noting that $D_{b} \rightarrow E$ as $b \rightarrow 0$, the similar proof as that in Proposition 4.1, we obtain that there exist $u \not \equiv 0$ such that $u_{b} \rightarrow u$ strongly in $H_{r}^{s}\left(\mathbb{R}^{N}\right)$ with $L^{\prime}(u)=0$ and $L(u)=E$. The proof is finished.

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