# STRONG COMPARISON PRINCIPLE FOR A P-LAPLACE EQUATION INVOLVING SINGULARITY AND ITS APPLICATIONS 

R.DHANYA ${ }^{1 *}$, M.S. INDULEKHA ${ }^{2}$ AND RITABRATA JANA ${ }^{3}$


#### Abstract

We prove a strong comparison principle for radially decreasing solutions $u, v \in$ $C_{0}^{1, \alpha}\left(\overline{B_{R}}\right)$ of the singular equations $-\Delta_{p} u-\frac{\lambda}{u^{o}}=f(x)$ and $-\Delta_{p} v-\frac{\lambda}{v^{o}}=g(x)$ in $B_{R}$, where $1<p \leq 2, \delta \in(0,1)$ and $\lambda>0$. We assume that $f$ and $g$ are continuous radial functions with $0 \leq f \leq g$ and $f \not \equiv g$ in $B_{R}$. Also, a counterexample is provided where the strong comparison principle is violated when $p>2$. In addition, we prove a three solution theorem for $p$-Laplace equation as an application of strong comparison principle. This is illustrated with an example.


Key words: Singular term, Strong Comparison Principle, Three solution theorem Mathematics Subject Classification (2020): 35J92, 35J75, 35J66

## 1. Introduction

Strong comparison principle for p-Laplacian is an inevitable tool in the analysis of partial differential equations. It is useful in establishing existence and uniqueness results, a-priori estimates, symmetry results, etc. We consider the following p-Laplace equations for $p \in$ $(1, \infty)$

$$
\left.\begin{array}{rl}
-\Delta_{p} u-\frac{\lambda}{u^{\delta}} & =f(x) \text { in } B_{R}  \tag{1.1}\\
-\Delta_{p} v-\frac{\lambda}{v^{0}} & =g(x) \text { in } B_{R} \\
u=v & =0 \text { on } \partial B_{R}
\end{array}\right\}
$$

where $B_{R} \subset \mathbb{R}^{n}$ is an open ball of radius $R$ centred at origin, $\delta \in(0,1)$ and $\lambda>0$. The functions $f$ and $g$ belong to $C\left(B_{R}\right)$ and are radial such that $0 \leq f \leq g$ and $f \not \equiv g$ in $B_{R}$. We

[^0]Acknowledgments: $R$ Dhanya was supported by DST/INSPIRE/04/2015/003221 and R Jana was supported by Prime Minister's Research Fellowship when this work was carried out. We thank anonymous referee for their valuable comments which helped in improving the presentation of the paper.
assume that $u$ and $v$ belong to $C_{0}^{1, \alpha}\left(\overline{B_{R}}\right)$ for some $\alpha \in(0,1)$. Clearly, the solutions $u$ and $v$ of (1.1) are positive in $B_{R}$. Given that $f \leq g$, by weak comparison principle we observe that $u \leq v$. The strong comparison principle (SCP) for (1.1) reads as

$$
\begin{equation*}
0<u<v \text { in } B_{R} \text { and } \frac{\partial v}{\partial \nu}<\frac{\partial u}{\partial \nu}<0 \text { on } \partial B_{R} \tag{1.2}
\end{equation*}
$$

where $\nu$ denotes the outward normal vector on $\partial B_{R}$. The main goal of this article is to investigate to what extend the strong comparison principle (1.2) is valid for the p-Laplace equation with a singular nonlinearity as in (1.1).

In the literature, standard methods of strong comparison principle were developed for equations

$$
\left.\begin{array}{rl}
-\Delta_{p} u-b(x, u) & =f(x) \text { in } \Omega  \tag{1.3}\\
-\Delta_{p} v-b(x, v) & =g(x) \text { in } \Omega \\
u=v & =0 \text { on } \partial \Omega
\end{array}\right\}
$$

where $b(x, \cdot)$ is an increasing function for each $x$. If $u \leq v$ and $f(x) \leq g(x)$, then we have $f^{*} \leq g^{*}$, where $f^{*}:=b(x, u)+f(x)$ and $g^{*}:=b(x, v)+g(x)$. Now the comparison principle of [10] is applicable for $-\Delta_{p} u=f^{*}$ and $-\Delta_{p} v=g^{*}$ and yields $u(x)<v(x)$ for all $x \in \Omega$. On the other hand, the above technique is no longer applicable for (1.1) as the function $b(x, \cdot)$ is decreasing for each $x$.

Giacomoni et. al. in [7] derived a strong comparison principle for quasilinear elliptic equation with singular non-linearity. Here the authors proved that $u<v$ in $\Omega$ for the same set of equations (1.1), but with a stronger assumption $0 \leq f<g$ in $\Omega$. In contrast to this, we no longer assume $f<g$ and hence the result obtained is stronger. In [12], the SCP is shown for PDE of the type $-\Delta_{p} u-\frac{\lambda}{u^{\delta}}+\sigma u^{p-1}=f(x)$ with similar assumptions as in [7]. It is noteworthy to mention that in both these articles authors have used the fact that $g-f$ attains a positive minimum in any compact subset of $\Omega$. Our main focus here is to investigate the validity of SCP relaxing this condition. In section 2 , we state the main result as Theorem 1.1, where we prove that the SCP is valid in $B_{R}$ if $1<p \leq 2$ and $0 \leq f \leq g$. In addition to this, we provide a counterexample for the SCP when $p>2$.

Theorem 1.1. Let $1<p \leq 2, \lambda>0$ and $f, g$ be continuous radial functions in $B_{R}$ such that $0 \leq f \leq g$ in $B_{R}$ and $f \not \equiv g$ in $B_{R}$. Assume that $u, v \in C^{1, \alpha}\left(\overline{B_{R}}\right)$, are radially decreasing solutions of $-\Delta_{p} u-\frac{\lambda}{u^{\delta}}=f(x)$ and $-\Delta_{p} v-\frac{\lambda}{v^{\delta}}=g(x), u=v=0$ on $\partial B_{R}$. Then $0<u<v$ in $B_{R}$ and $\frac{\partial v}{\partial \nu}<\frac{\partial u}{\partial \nu}<0$ on $\partial B_{R}$.

If $f$ and $g$ are $L^{\infty}$ functions in $B_{R}$, then by the regularity results in [7], the solutions $u$ and $v$ belong to $C^{1, \alpha}\left(\overline{B_{R}}\right)$. If we assume that $f, g$ are radial and radially decreasing, the solutions are expected to be radially decreasing by a recent work of [6].

The existence of multiple solutions of elliptic problems is another interesting area of research. In the third section of this paper, we shall see how an SCP is helpful in obtaining a third solution when two pairs of ordered sub and super-solutions are known(see also the example given at the end of section (3). In this regard, we consider the following elliptic problem in a bounded open set $\Omega$ in $\mathbb{R}^{N}$ :

$$
\begin{equation*}
-\Delta_{p} u=\lambda\left(\frac{1}{u^{\delta}}+G(u)\right) \text { in } \Omega ; u>0 \text { in } \Omega, \quad u=0 \text { on } \partial \Omega . \tag{1.4}
\end{equation*}
$$

We assume that $0<\delta<1$ and the function $G: \mathbb{R} \rightarrow[0, \infty)$ is monotonically increasing in $\mathbb{R}^{+}$with $G(0)=0$. We define the solution operator $A_{G}$ in definition 3.3, section 3 and prove the three solution theorem.

Theorem 1.2. (Three solution theorem) Suppose there exists two pairs of ordered sub and supersolutions $\left(\psi_{1}, \phi_{1}\right)$ and $\left(\psi_{2}, \phi_{2}\right)$ of (1.4) with the property $\psi_{1} \leq \psi_{2} \leq \phi_{1}, \psi_{1} \leq \phi_{2} \leq \phi_{1}$ and $\psi_{2} \not \leq \phi_{2}$. Additionally assume that $\psi_{2}, \phi_{2}$ are not solutions of (1.4) and $A_{G}\left(\phi_{2}\right)<\phi_{2}$ and $A_{G}\left(\psi_{2}\right)>\psi_{2}$. Then there exists at least three solutions $u_{i}, i=1,2,3$ for (1.4) where $u_{1} \in\left[\psi_{1}, \phi_{2}\right], u_{2} \in\left[\psi_{2}, \phi_{1}\right]$ and $u_{3} \in\left[\psi_{1}, \phi_{1}\right] \backslash\left(\left[\psi_{1}, \phi_{2}\right] \cup\left[\psi_{2}, \phi_{1}\right]\right)$.

## 2. Strong Comparison Principle

In this section we prove the main result, Theorem 1.1:
Proof. Given that $f \leq g$, using the test function $(u-v)^{+}$in the weak formulation of the problem we can find that $u(x) \leq v(x) \forall x \in B_{R}$. Now, for any $0<r<R$, define $U_{r}:=$ $B_{R} \backslash B_{r}$. Since $u$ and $v$ are radially decreasing in $B_{R} \backslash\{0\}$ we have $\frac{d u}{d r}<0$ and $\frac{d v}{d r}<0$. Next we write $w=v-u$ and following the idea of [7] the system of equations in (1.1) can be re-written as

$$
\begin{align*}
-\operatorname{div}(A(x) \nabla w)-\lambda B(x) w & =g-f \geq 0 \text { in } U_{r}  \tag{2.5}\\
w & \geq 0 \text { on } \partial U_{r}
\end{align*}
$$

for a matrix $A(x)=\left[a_{i j}(x)\right]$ and a scalar function $B(x)$. Here,

$$
\begin{aligned}
& a_{i j}(x)=\int_{0}^{1}|(1-t) \nabla u(x)+t \nabla v(x)|^{p-2}\left[\delta_{i j}+(p-2) \frac{\left((1-t) u_{x_{i}}+t v_{x_{i}}\right)\left((1-t) u_{x_{j}}+t v_{x_{j}}\right)}{|(1-t) \nabla u(x)+t \nabla v(x)|^{2}}\right] d t \\
& \text { and } B(x)=-\delta \int_{0}^{1} \frac{d t}{((1-t) u(x)+t v(x))^{\delta+1}} .
\end{aligned}
$$

Using the assumptions on $u$ and $v$, we note that $A(x)=\left[a_{i j}(x)\right]$ is uniformly elliptic in $U_{r}$ for every $r>0$. We now fix an $r_{0}>0$ such that $f-g \not \equiv 0$ in $U_{r_{0}}$, which is possible as $f, g$ are assumed to be continuous in $B_{R}$. Now applying the strong maximum principle Theorem 2.5.2 of [13] we conclude that $w>0$ in $U_{r}$ for all $r<r_{0}$. In fact this implies that $w(x)>0$ for all $x \neq 0$.

In the next step, by exploiting the ideas in section 3 of [4] we will show that $w$ is strictly positive in $B_{r_{0}}$ as well . Using the radial symmetry of solutions, the problem (1.1) can be reduced to a system of ODEs:

$$
\begin{array}{r}
u_{1}^{\prime}=\alpha\left(r, u_{2}\right), u_{1}\left(r_{1}\right)=u_{1,0} \\
u_{2}^{\prime}=-\frac{N-1}{r} u_{2}+\beta_{f}\left(r, u_{1}\right), u_{2}\left(r_{1}\right)=u_{2,0} \tag{2.6}
\end{array}
$$

where $r_{1} \in(0, R), u_{1}(r)=u(r), u_{2}(r)=\left|u^{\prime}(r)\right|^{p-2} u^{\prime}(r)$. We denote by $\beta_{f}(r, y)$ the function $-\left(\frac{\lambda}{y^{\delta}}+f(r)\right)$ and $\alpha(r, y):(0, R) \times \mathbb{R} \rightarrow \mathbb{R}$ is given by

$$
\alpha(r, y)=\left\{\begin{align*}
y^{\frac{1}{p-1}} & \text { if } y \geq 0  \tag{2.7}\\
|y|^{\frac{1}{p-1}} & \text { if } y<0
\end{align*}\right.
$$

Clearly $u_{1}(R)=u_{2}(0)=0$. Analogously we can write

$$
v_{1}^{\prime}=\alpha\left(r, v_{2}\right), v_{1}\left(r_{1}\right)=v_{1,0}
$$

$$
\begin{equation*}
v_{2}^{\prime}=-\frac{N-1}{r} v_{2}+\beta_{g}\left(r, v_{1}\right), v_{2}\left(r_{1}\right)=v_{2,0} \tag{2.8}
\end{equation*}
$$

where $v_{1}(R)=v_{2}(0)=0$.
Suppose $u\left(r^{\prime}\right)=v\left(r^{\prime}\right)$ for some $r^{\prime}<r_{0}$ (where $r_{0}$ is as in the first part of the proof). As $w \geq 0$ in $B_{R}$, its minimum is attained at $r^{\prime}$ and hence $\frac{d w}{d r}\left(r^{\prime}\right)=0$. Taking $r_{1}=r^{\prime}$ in the systems of ODE, $u_{1,0}=v_{1,0}$ and $u_{2,0}=v_{2,0}$. For the function $b(x, u)=\lambda u^{-\delta}$ we have $0 \leq-\frac{\partial b}{\partial u} \in L_{l o c}^{\infty}((-R, R) \times(0, \infty))$, and hence by using Lemma 3.2 of [4], we obtain $v_{1}(r) \leq u_{1}(r) \forall r \in\left[r_{1}, R\right)$ which contradicts the fact that $w>0$ in $U_{r_{0}}$. Therefore, $0<u<v$ in $B_{R}$. Finally we note that since $w>0$ in $B_{R}$, we can apply Theorem 2.7.1 of Pucci and Serrin[13] to conclude that $\frac{\partial v}{\partial \nu}<\frac{\partial u}{\partial \nu}<0$.

From a careful observation of the above proof we note that the hypothesis of the Theorem 1.1 can be modified as in the next theorem and still the strong comparison principle holds.

Theorem 2.1. Let $1<p \leq 2$ and $u$, $v$ be positive radially decreasing solutions of (1.1). Also assume that $f, g$ are continuous radial functions in $B_{R}$ such that $f \leq g$ and $f \not \equiv g$. Then $u(x)<v(x)$ for all $x \in B_{R}$.

When $p>2$, under the given assumptions of the above theorem we can show that $u<v$ in $B_{R} \backslash\{0\}$. On the other hand, when $1<p \leq 2$, our Theorem 1.1 uses the smoothness of the map $t \rightarrow t^{\frac{1}{p-1}}$ along with the Muller Kamke theorem [14] to prove $u(0)<v(0)$. In the next example we prove that the above result(Theorem 2.1) need not be true when $p>2$.
Counter example to Theorem 2.1 when $p>2$ : For $0<\theta<\infty$, define $u_{\theta}(x):=1-r^{\theta}$ and $f_{\theta}(x):=((p-1)(\theta-1)-1+N) \theta^{(p-1)} r^{(p-1)(\theta-1)-1}-\lambda\left(1-r^{\theta}\right)^{-\delta}$, where $r=|x|$. Clearly,

$$
\begin{array}{r}
-\Delta_{p} u_{\theta}-\lambda u_{\theta}^{-\delta}=f_{\theta} \text { in } B_{1} \\
u_{\theta}=0 \text { on } \partial B_{1} \tag{2.9}
\end{array}
$$

Also, $u_{\theta}>0$ in $B_{1}$ and $f_{\theta} \in C\left(B_{1}\right)$ for all $\theta \in(0, \infty)$. We observe that $u_{\theta}(0)=1$ for all $\theta$ and $u_{\theta_{1}}(x)<u_{\theta_{2}}(x)$ for all $x$ in $B_{1} \backslash\{0\}$ when $0<\theta_{1}<\theta_{2}<\infty$. We claim that, we can choose $\theta_{1}, \theta_{2}$ and $\lambda>0$ appropriately so that $f_{\theta_{1}}(x) \leq f_{\theta_{2}}(x)$ in $B_{1}$ and thus the strong comparison principle (Theorem 2.1) is violated. To this end, it is enough to prove that $\partial_{\theta} f_{\theta} \geq 0$. Now,

$$
\begin{aligned}
\partial_{\theta} f_{\theta}(r)= & (p-1)(p(\theta-1)+N) \theta^{p-2} r^{(p-1)(\theta-1)-1}+ \\
& (p-1)(p(\theta-1)+N-\theta) \theta^{p-1} r^{(p-1)(\theta-1)-1} \ln (r)+ \\
& \left(-\lambda \delta\left(1-r^{\theta}\right)^{-\delta-1} r^{\theta} \ln (r)\right)
\end{aligned}
$$

Define $l_{p}(\theta):=\frac{1}{\theta}+\frac{1}{p(\theta-1)+N-\theta}>0$ for $1<\theta<\infty$. If $r \in\left[e^{-l_{p}(\theta)}, 1\right]$, the first two summands of $\partial_{\theta} f_{\theta}$ give non-negative sum and since $\lambda>0$ the third summand is also positive. This gives $\partial_{\theta} f_{\theta}(r) \geq 0$ when $e^{-l_{p}(\theta)} \leq r \leq 1$. Next when $r \in\left(0, e^{-l_{p}(\theta)}\right)$, we first choose $\theta \geq \frac{p}{p-2}$ so that we get
$(p-1)(p(\theta-1)+N-\theta) \theta^{p-1} r^{(p-1)(\theta-1)-1-\theta}-\lambda \delta\left(1-r^{\theta}\right)^{-\delta-1} \leq(p-1)(p(\theta-1)+N-\theta) \theta^{p-1}-\lambda \delta$.
Now we choose $\lambda$ large enough, for instance $\lambda \delta \geq(p-1)(p(\theta-1)+N-\theta) \theta^{p-1}$ so that the sum of last two terms in $\partial_{\theta} f_{\theta}(r)$ is positive. The first term of $\partial_{\theta} f_{\theta}(r)$ is always positive and thus $\partial_{\theta} f_{\theta}(r) \geq 0$ for $r \in\left(0, e^{-l_{p}(\theta)}\right)$ as well. Thus we conclude that the strong comparison principle does not hold true if we choose $\frac{p}{p-2} \leq \theta_{1}<\theta_{2}<\infty$ and $\lambda$ large enough.

## 3. Three Solution Theorem

In this section we consider the following quasilinear BVP with singular nonlinearity:

$$
\begin{equation*}
-\Delta_{p} u=\lambda\left(\frac{1}{u^{\delta}}+G(u)\right) \text { in } \Omega ; u>0 \text { in } \Omega, \quad u=0 \text { on } \partial \Omega \tag{3.10}
\end{equation*}
$$

$\Omega$ is a bounded open subset of $\mathbb{R}^{N}, N \geq 1$ with smooth boundary $\partial \Omega$ and $0<\delta<1$. The function $G: \mathbb{R} \rightarrow[0, \infty)$ is monotonically increasing in $\mathbb{R}^{+}$with $G(0)=0$. We prove the existence of three solutions of (3.10) whenever there exists two pairs of ordered sub and super solutions. We use a technique similar to that in [5], where the authors have proved this result for the linear case $p=2$. We remark here that all the results in this section can be concluded for $-\Delta_{p} u=\lambda\left(\frac{c}{u^{\delta}}+G(u)\right)$ where $c$ is any positive constant.

Definition 3.1. A function $u \in C^{1, \alpha}(\bar{\Omega})$ is said to be a sub-solution(super solution) of (3.10) if $u>0$ in $\Omega, u=0$ on $\partial \Omega$ and

$$
\int_{\Omega}|\nabla u|^{p-2} \nabla u \nabla \phi \leq(\geq) \lambda \int_{\Omega}\left(\frac{1}{u^{\delta}}+G(u)\right) \phi
$$

holds for all non-negative test functions $\phi \in C_{c}^{\infty}(\Omega)$. If a function $u$ is both sub solution and super solution, then it is called a solution of (3.10).

Definition 3.2. Given $\lambda>0$ and $0<\delta<1$, we define $\xi_{\lambda}$ as the unique positive solution of $-\Delta_{p} \xi_{\lambda}=\lambda \xi_{\lambda}^{-\delta}$ in $\Omega ;\left.\xi_{\lambda}\right|_{\partial \Omega}=0$. By [7], we know that there exists positive constants $l, L$ for which $l d(x) \leq \xi_{\lambda} \leq L d(x)$, where $d(x)=d(x, \partial \Omega)$.

Definition 3.3. For a given $\lambda>0$, we define the map $A_{G}: C_{0}(\bar{\Omega}) \rightarrow C_{0}^{1, \alpha}(\bar{\Omega})$ as $A_{G}(u)=w$ iff $w$ is a weak solution of $-\Delta_{p} w-\frac{\lambda}{w^{\delta}}=\lambda G(u)$ in $\Omega ; w>0$ in $\Omega, w=0$ on $\partial \Omega$.

Lemma 3.4. The map $A_{G}$ is well defined, monotone operator from $C_{0}(\bar{\Omega})$ to $C_{0}^{1, \alpha}(\bar{\Omega})$.
Proof. For a given $u$, existence and uniqueness of a non-negative weak solution $w=$ $A_{G}(u) \in W_{0}^{1, p}(\Omega)$ can be proved by the minimization of a suitable energy functional as discussed in Lemma 3.1 of [7] or by following the idea of proof of Theorem 3.2 in [8]. Again using the results in Appendix B of [7], it can be shown that $w \in C^{1, \alpha}(\bar{\Omega})$. Now the monotonicity of the map $A_{G}$ easily follows as $G$ is assumed to be a monotonically increasing function.

We define $e \in C^{1, \alpha}(\bar{\Omega})$ as the unique positive solution of $-\Delta_{p} e=1$ in $\Omega$ with zero Dirichlet boundary condition. $C_{e}(\bar{\Omega})$ is the set of functions in $C_{0}(\bar{\Omega})$ such that $|u| \leq t e(x)$ for some $t>0 . C_{e}(\bar{\Omega})$ is a Banach space equipped with the norm $\|u\|_{e}=\inf \{t>0:|u(x)| \leq$ $t e(x)\}$ (see [5] for more details).

Proposition 3.5. The map $A_{G}: C_{e}(\bar{\Omega}) \longrightarrow C_{e}(\bar{\Omega})$ is completely continuous.
Proof. Recalling the continuous embedding $C_{0}^{1, \alpha}(\bar{\Omega}) \hookrightarrow C_{0}^{1}(\bar{\Omega}) \hookrightarrow C_{e}(\bar{\Omega}) \hookrightarrow C_{0}(\bar{\Omega})$ it is enough to show that $A_{G}: C_{0}(\bar{\Omega}) \longrightarrow C_{0}^{1, \alpha}(\bar{\Omega})$ is continuous. Let $\left\{u_{h}\right\} \subset C_{0}(\bar{\Omega})$ be such that $\left\|u_{h}-u\right\|_{C_{0}(\bar{\Omega})} \rightarrow 0$ as $h \rightarrow 0$. Let $A_{G}(u)=w$ and $A_{G}\left(u_{h}\right)=w_{h}$. Since $G$ is positive, we get $-\Delta_{p} w_{h}-\frac{\lambda}{w_{h}^{\delta}}=\lambda G\left(u_{h}\right) \geq 0$. Using weak comparison principle, we conclude $w_{h} \geq \xi_{\lambda} \geq$
$c d(x)$. Thus, for some positive constants $C$ and $C_{1}$ independent of $h$, we have

$$
w_{h}^{-\delta}+G\left(u_{h}\right) \leq \frac{C}{d(x)^{\delta}} \leq \frac{C_{1}}{\xi_{\lambda}^{\delta}}
$$

Again from the weak comparison principle we have $\xi_{\lambda} \leq w_{h} \leq k \xi_{\lambda}$. Now we can use Theorem B. 1 of [7] and obtain a $C_{0}^{1, \alpha}$ uniform bound for $\left\{w_{h}\right\}$, that is, there exists $M>0$ such that

$$
\sup _{h}\left\|w_{h}\right\|_{C_{0}^{1, \alpha}(\bar{\Omega})} \leq M
$$

By the compact embedding $C_{0}^{1, \alpha}(\bar{\Omega}) \subset \subset C_{0}^{1, \alpha^{\prime}}(\bar{\Omega})$ where $0<\alpha^{\prime}<\alpha$, the sequence $w_{h}$ has a convergent subsequence in $C_{0}^{1, \alpha^{\prime}}(\bar{\Omega})$, namely $\left\{w_{h_{i}}\right\}$. The uniqueness of weak solution of the equation $-\Delta_{p} w-\frac{\lambda}{w^{\beta}}=\lambda G(u)$ would imply that $w_{h_{i}} \rightarrow w$ in $C_{0}^{1, \alpha^{\prime}}(\bar{\Omega})$. Through a standard subsequence argument it can be shown that $w_{h} \rightarrow w$ in $C_{0}^{1, \alpha^{\prime}}(\bar{\Omega})$ and thus the the map $A_{G}: C_{0}(\bar{\Omega}) \rightarrow C_{0}^{1, \alpha^{\prime}}(\bar{\Omega})$ is continuous. Once again using Ascoli Arzela theorem it is easy to prove that the map $A_{G}$ is completely continuous from $C_{e}(\bar{\Omega})$ to itself.

Authors in [8] consider a system of quasilinear equations with a singular non-linearity and prove that the associated operator is completely continuous. Furthermore, they show the existence of its solution using Schauder's fixed point theorem. Our aim is to use a fixed point theorem due to Amann [2] to prove the existence of three solutions to (3.10).
Proof of Theorem [1.2: Existence of two solutions $u_{1} \in\left[\psi_{1}, \phi_{2}\right]$ and $u_{2} \in\left[\psi_{2}, \phi_{1}\right]$ is straight forward as the map $A_{G}$ is monotone and completely continuous. The proof of existence of a third solution follows as in the case of Laplacian(see Theorem 3.9 of [5]), but we shall briefly describe the underlying idea here. Using the given condition $A_{G}\left(\phi_{2}\right)<\phi_{2}$ we note that $\psi_{1} \leq u_{1}<\phi_{2}$. Also,

$$
\begin{align*}
-\Delta_{p} u_{1}-\frac{\lambda}{u_{1}^{\delta}} & =\lambda G\left(u_{1}\right) \text { in } \Omega \\
-\Delta_{p} \phi_{2}-\frac{\lambda}{\phi_{2}^{\delta}} & \geq \lambda G\left(\phi_{2}\right) \text { in } \Omega  \tag{3.11}\\
u_{1}=\phi_{2} & =0 \text { on } \partial \Omega .
\end{align*}
$$

Since $u_{1}<\phi_{2}$ and $G$ is strictly increasing, using Theorem 2.3 of [7] (or by Theorem 2.7.1 of Pucci and Serrin [13]) we have $\frac{\partial u_{1}}{\partial \nu}>\frac{\partial \phi_{2}}{\partial \nu}$, or $\phi_{2}-u_{1} \geq c_{1} e(x)$ for some positive constant $c_{1}$. Similarly for some constant $c_{2}>0$ we can show that $u_{2}-\psi_{2}>c_{2} e(x)$. Now the open balls,

$$
\left.B_{k}=\left\{z \in C_{e}(\bar{\Omega})\right\}:\left\|z-u_{k}\right\|_{e}<c_{k}\right\}
$$

for $k=1,2$ lie entirely inside $X_{1}=\left[\psi_{1}, \phi_{2}\right]$ and $X_{2}=\left[\psi_{2}, \phi_{1}\right]$ respectively. Thus we prove that $X_{i}$ for $i=1,2$ have non-empty interior and appeal to the fixed point theorem of Amann [2] to conclude the existence of a third solution $u_{3} \in\left[\psi_{1}, \phi_{1}\right] \backslash\left(\left[\psi_{1}, \phi_{2}\right] \cup\left[\psi_{2}, \phi_{1}\right]\right)$.

Next we wish to understand under what hypothesis the conditions $A_{G}\left(\psi_{2}\right)>\psi_{2}$ and $A_{G}\left(\phi_{2}\right)<\phi_{2}$ are valid. Let $\psi_{2}$ be a sub-solution of (3.10), then $\psi_{2}$ is also a weak solution of the BVP

$$
\left.\begin{array}{r}
-\Delta_{p} \psi_{2}-\frac{\lambda}{\psi_{2}^{\delta}}=\lambda \tilde{G}(x) \text { in } \Omega  \tag{3.12}\\
\psi_{2}=0 \text { on } \partial \Omega
\end{array}\right\}
$$

for some function $\tilde{G}$ defined on $\Omega$. Since $\psi_{2}$ is a subsolution we have $\tilde{G}(x) \leq G \circ \psi_{2}(x)$. In most of the applications, when $\psi_{2}$ is known the function $\tilde{G}$ happens to be continuous in $\Omega$. If we write $\omega=A_{G}\left(\psi_{2}\right)$, then

$$
\left.\begin{array}{r}
-\Delta_{p} \omega-\frac{\lambda}{\omega^{\delta}}=\lambda G\left(\psi_{2}\right) \text { in } \Omega  \tag{3.13}\\
\omega=0 \text { on } \partial \Omega
\end{array}\right\}
$$

If $\Omega=B_{R}$ is a ball and $1<p \leq 2$, Theorem 1.1 provides a sufficient condition that ensures the hypothesis of the three solution theorem. This result is stated as the following proposition. Similar results hold true for $\phi_{2}$ as well.

Proposition 3.6. Let $1<p \leq 2$ and $\Omega=B_{R}$. Suppose $\psi_{2}$ and $A_{G}\left(\psi_{2}\right)$ are radially decreasing functions in $B_{R}$. If $\tilde{G}(x)$ is a continuous radial function in $B_{R}$, then we have $A_{G}\left(\psi_{2}\right)>\psi_{2}$.

Next we shall state another condition which can be useful to prove the three solution theorem.

Proposition 3.7. Let $1<p<\infty$ and $\Omega$ is an arbitrary bounded open set with smooth boundary. Assume that $\tilde{G}(x)<G\left(\psi_{2}(x)\right)$ for all $x \in \Omega$. Then $A_{G}\left(\psi_{2}\right)>\psi_{2}$.

Proof. Proof is a straightforward application of the strong comparison principle in Theorem 2.3 of [7] or Proposition 4 of [12].

We demonstrate the three solution theorem for an elliptic equation with singularity through an example.
Example 3.1 Ko et al. [9] have considered the boundary value problem

$$
\left.\begin{array}{rl}
-\Delta_{p} u & =\lambda \frac{F(u)}{u^{\delta}} \text { in } \Omega  \tag{3.14}\\
u & =0 \text { on } \partial \Omega \\
u & >0 \text { in } \Omega
\end{array}\right\}
$$

where $1<p<\infty, \delta \in(0,1), \lambda$ is a positive parameter and $\Omega$ is a bounded domain in $\mathbb{R}^{N}, N \geq 1$, with smooth boundary. It is also assumed that $F \in C^{1}([0, \infty))$ is a nondecreasing function with $F(u)>0$ for all $u \geq 0$ and $\lim _{u \rightarrow \infty} \frac{F(u)}{u^{c+p-1}}=0$. With a few more technical assumptions on $F$, in [9], authors have established the existence of two positive solutions $u_{1}, u_{2}$ of (3.14) by constructing two pairs of sub-super solutions ( $\psi_{1}, \phi_{1}$ ) and $\left(\psi_{2}, \phi_{2}\right)$ whenever $\lambda \in\left(\lambda_{*}, \lambda^{*}\right)$ as given in the Theorem 1.3 of [9]. A model problem was given by $F(u)=e^{\frac{\alpha u}{\alpha+u}}$ for $\alpha \gg 1$. We urge the readers to go through the cited reference to know about the exact definition of $\lambda_{*}, \lambda^{*}$ and sub-supersolutions $\psi_{i}$ and $\phi_{i}$.

We intend to modify the construction of the subsolution $\psi_{2}$ given in [9] and by abuse of notation we call the new subsolution also as $\psi_{2}$. This reconstruction of $\psi_{2}$ is necessary to use Proposition 3.7 and we conclude the example by showing (3.14) has a third solution $u_{3}$ when $\lambda \in\left(\lambda_{*}, \lambda^{*}\right)$. For ease of notation let us assume that $F(0)=1$. Now we re-write the equation (3.14) as (3.10) by taking $G(u):=\frac{F(u)-F(0)}{u^{\circ}}$. Clearly, Theorem 1.2 would guarantee the existence of a third solution $u_{3} \in\left[\psi_{1}, \phi_{1}\right] \backslash\left(\left[\psi_{1}, \phi_{2}\right] \cup\left[\psi_{2}, \phi_{1}\right]\right)$ of (3.14) if $A_{G}\left(\phi_{2}\right)<\phi_{2}$ and $A_{G}\left(\psi_{2}\right)>\psi_{2}$. For our purpose of establishing $A_{G}\left(\psi_{2}\right)>\psi_{2}$, as mentioned before we slightly modify the construction of $\psi_{2}$ given in [9]. We first fix a $\lambda \in\left(\lambda_{*}, \lambda^{*}\right)$ and
define $H(u):=\frac{h(u)}{1+\epsilon_{\lambda}}$ where $h(u)$ is given in Page no-7 of [9]. Here $\epsilon_{\lambda}$ is chosen in such a way that $\frac{\lambda}{1+\epsilon_{\lambda}}$ still lies within the interval $\left(\lambda_{*}, \lambda^{*}\right)$. We now follow the construction of subsolution $\psi_{2}$ in [9] except for equation number (5) in page 8 . If we modify this particular equation (5) in [9], with $-\Delta_{p} u=\lambda H(u)$ in $\Omega ;\left.u\right|_{\partial \Omega}=0$ and redo the calculations then the resulting sub-solution verifies the strict inequality $A_{G}\left(\psi_{2}\right)>\psi_{2}$. From the definition of $\phi_{2}$ in [9], clearly $A_{G}\left(\phi_{2}\right)<\phi_{2}$.

We summarize the above discussion in the following remark.
Remark 3.8. The boundary value problem (3.14) admits three solutions whenever $\lambda \in$ $\left(\lambda_{*}, \lambda^{*}\right)$.

Towards the completion of our work we came across two recent manuscripts [1] and [3] where a problem similar to Example 3.1 is considered for $p-q$ Laplacian. In both papers, the authors focus on the construction of two pairs of sub-super solutions either in a ball or in a general domain. Arora [3] also establishes a three solution theorem for $p-q$ Laplacian with the help of the strong comparison principle given in Proposition 6 of [11]. Though Example 3.1 can be treated as a special case of the work of Arora, we wish to conclude our paper emphasizing that in the light of the strong comparison principle in a ball (Theorem 1.1), our three solution theorem is applicable for more general elliptic boundary value problems.

## References

[1] Ananta Acharya, Ujjal Das, and Ratnasingham Shivaji. Existence and multiplicity results for pqlaplacian boundary value problems. Electronic Journal of Differential Equations, Special Isuue 01(2021), pp. 293-300, 2021.
[2] Herbert Amann. Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces. SIAM review, 18(4):620-709, 1976.
[3] Rakesh Arora. Multiplicity results for nonhomogeneous elliptic equations with singular nonlin- earities. arXiv preprint arXiv:2109.03274, 2021.
[4] Mabel Cuesta and Peter Takáč. A strong comparison principle for positive solutions of degen- erate elliptic equations. Differential and Integral Equations, 13(4-6):721-746, 2000.
[5] R Dhanya, Eunkyung Ko, and Ratnasingham Shivaji. A three solution theorem for singular nonlinear elliptic boundary value problems. Journal of Mathematical Analysis and Applications, 424(1):598-612, 2015.
[6] Francesco Esposito and Berardino Sciunzi. On the höpf boundary lemma for quasilinear problems involving singular nonlinearities and applications. Journal of Functional Analysis, 278(4):108346, 2020.
[7] Jacques Giacomoni, Ian Schindler, and Peter Takáč. Sobolev versus hölder local minimizers and existence of multiple solutions for a singular quasilinear equation. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 6(1):117-158, 2007.
[8] Jacques Giacomoni, Ian Schindler, and Peter Takáč. Singular quasilinear elliptic systems and hölder regularity. Advances in Differential Equations, 20(3/4):259-298, 2015
[9] Eunkyung Ko, Eun Kyoung Lee, and Ratnasingham Shivaji. Multiplicity results for classes of infinite positone problems. Zeitschrift für Analysis und ihre Anwendungen, 30(3):305-318, 2011.
[10] M Lucia and S Prashanth. Strong comparison principle for solutions of quasilinear equations. Proceedings of the American Mathematical Society, 132(4):1005-1011, 2004.
[11] Nikolaos S Papageorgiou, Vicenţiu D Rădulescu, and Dušan D Repovš. Nonlinear nonhomoge- neous singular problems. Calculus of Variations and Partial Differential Equations, 59(1):1-31, 2020.
[12] Nikolaos S Papageorgiou and George Smyrlis. A bifurcation-type theorem for singular nonlinear elliptic equations. Methods and Applications of Analysis, 22(2):147-170, 2015.
[13] Patrizia Pucci and James B Serrin. The maximum principle, volume 73. Springer Science and Business Media, 2007.
[14] Hal L Smith. Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems: an introduction to the theory of competitive and cooperative systems. Num- ber 41. American Mathematical Soc., 2008. 8


[^0]:    ${ }^{1}$ School of Mathematics, Indian Institute of Science Education and Research, Trivandrum (IISER-TVM), Maruthamala PO, Vithura, Thiruvananthapuram, Kerala, 695 551, INDIA.
    e-mail: dhanya.tr@iisertvm.ac.in
    ${ }^{2}$ School of Mathematics, Indian Institute of Science Education and Research, Trivandrum (IISER-TVM), Maruthamala PO, Vithura, Thiruvananthapuram, Kerala, 695 551, INDIA
    e-mail: indulekhams17@iisertvm.ac.in
    ${ }^{2}$ School of Mathematics, Indian Institute of Science Education and Research, Trivandrum (IISER-TVM), Maruthamala PO, Vithura, Thiruvananthapuram, Kerala, 695 551, INDIA
    e-mail: ritabrata20@iisertvm.ac.in
    *Corresponding author.

