Non-deterministic Semantics for Dynamic Topological Logic

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Abstract

Dynamic Topological Logic ($\mathcal{DTL}$) is a combination of $\mathcal{S}4$, under its topological interpretation, and the temporal logic $\mathcal{LTL}$ interpreted over the natural numbers. $\mathcal{DTL}$ is used to reason about properties of dynamical systems based on topological spaces. Semantics are given by dynamic topological models, which are tuples $\langle X, \mathcal{T}, f, V\rangle$, where $\langle X, \mathcal{T}\rangle$ is a topological space, f a function on X and V a truth valuation assigning subsets of X to propositional variables.

Our main result is that the set of valid formulas of $\mathcal{DTL}$ over spaces with continuous functions is recursively enumerable. We do this by defining alternative semantics for $\mathcal{DTL}$. Under the standard semantics, $\mathcal{DTL}$ is not complete for Kripke frames. However, we introduce the notion of a non-deterministic quasimodel, where the function f is replaced by a binary relation g assigning to each world multiple temporal successors. We place restrictions on the successors so that the logic remains unchanged; under these alternative semantics, $\mathcal{DTL}$ becomes Kripke-complete. We then apply model-search techniques to enumerate the set of all valid formulas.

1 Introduction

Dynamic Topological Logic ($\mathcal{DTL}$) is a propositional tri-modal system introduced in [9, 2] for reasoning about topological dynamics; that is, about the action of a continuous function f on a topological space $\langle X, \mathcal{T}\rangle$. The interpretation of formulas of $\mathcal{DTL}$ involves not only points $x \in X$, but also their orbit

$$\{x, f(x), f^2(x), ...\}$$

and their neighborhoods in $\mathcal{T}$.

The language uses propositional variables, Classical Boolean connectives and three modalities: $\Box$ from $\mathcal{S}4$, interpreted as the topological interior, and the temporal operators $\bigcirc$ and $*$ of Linear Temporal Logic ([12]), which are interpreted as 'next' and 'henceforth', respectively.

Every class $\mathcal{C}$ of dynamic topological systems induces a logic consisting of those formulas of $\mathcal{DTL}$ that are valid on all systems in $\mathcal{C}$. Many logics arising in this form have been studied; the following are some of the main results which are known.

1. The fragment $\mathcal{DTL}^{\bigcirc}$:

   $\mathcal{DTL}^{\bigcirc}$ is the fragment of $\mathcal{DTL}$ which uses only $\Box$ and $\bigcirc$. This fragment is finitely axiomatizable and has the finite model property, both when f is taken to be a continuous function ([2]) and when it is a homeomorphism ([9]). Interestingly enough, the logic over arbitrary spaces does not coincide with the logic over $\mathbb{R}$ ([15]) but it does coincide with the logic over $\mathbb{Q}$ ([8]).

2. The fragment $\mathcal{DTL}_0$:

   In this fragment all three modalities are used but temporal modalities may not appear in the scope of $\Box$. $\mathcal{DTL}_0$ is finitely axiomatizable and decidable ([9]).

3. The fragment $\mathcal{DTL}_1$:

   $\mathcal{DTL}_1$ is the fragment of $\mathcal{DTL}$ where all three modalities are used but $*$ may not appear in the scope of $\Box$. This fragment is powerful enough to encode some undecidable problems and hence is undecidable. However, it is recursively enumerable ([5]), and the logic coincides over arbitrary spaces, locally finite spaces and $\mathbb{R}^2$ ([5, 14, 4]). This implies that all these logics are also equal on the smaller fragments $\mathcal{DTL}^{\bigcirc}$ and $\mathcal{DTL}_0$.

4. Spaces with homeomorphisms:

   The set of valid formulas of $\mathcal{DTL}_1$ over spaces with homeomorphisms is not recursively enumerable ([6]), and hence the same is true for the set of valid formulas of the full language. Furthermore, the logics over arbitrary spaces, Aleksandroff spaces and $\mathbb{R}^n$ for $n \geq 0$ are all distinct ([6, 15]).

5. Full $\mathcal{DTL}$ with arbitrary continuous functions:

   It is known that for full $\mathcal{DTL}$, the logics over arbitrary spaces, Aleksandroff spaces and $\mathbb{R}^n$ are all distinct ([9, 4]). Over almost disjoint spaces (where all open sets are closed) the logic is decidable, even when f is taken to be a homeomorphism ([7]).

   $\mathcal{DTL}$ is over arbitrary spaces is undecidable; this follows from the fact that $\mathcal{DTL}_1$ is already undecidable. However, it is recursively enumerable; this is the main result we will present here.

The layout of this paper is as follows. In §3 we will define non-deterministic quasimodels for $\mathcal{DTL}$, where the function of a dynamic topological system is replaced by a binary relation g, so that each point may have several immediate temporal successors. However, we will place restrictions on g so that the logic remains sound, and in §4 show that a dynamic topological model can always be reconstructed from a non-deterministic quasimodel.

The central result is that $\mathcal{DTL}$ is complete for the class of locally finite Kripke frames under the new semantics. Our strategy for proving this will be to generate truth-preserving binary relations between Kripke frames and dynamic topological models. The relations we will use are called *ω-simulations* and are developed in §6.

We then apply techniques very similar to those in [5] to show that $\mathcal{DTL}$ is recursively enumerable. There, Kruskal's Tree Theorem is used to prove that a certain model-search algorithm always reports failure in finite time when a non-satisfiable formula of $\mathcal{DTL}_1$ is given as imput. In §7 we use non-deterministic semantics to develop a variation of this which can be applied to arbitrary formulas of the language.

2 Dynamic Topological Logic

The language of $\mathcal{DTL}$ is built from propositional variables in a countably infinite set **Var** using the Boolean connectives $\wedge$ and $\neg$ (all other connectives are to be defined in terms of these) and the three unary modal operators $\Box$ ('interior'), $\bigcirc$ ('next') and $*$ ('henceforth'). We write $\Diamond$ as a shorthand for $\neg\Box\neg$. Formulas of this language are interpreted on dynamical systems over topological spaces, or *dynamic topological systems*.

Definition 2.1. *A* dynamic topological system *is a triple* $\mathfrak{S} = \langle X, \mathcal{T}, f\rangle$ *, where* $\langle X, \mathcal{T}\rangle$ *is a topological space and* $f : X \to X$ *is a continuous function.*

A valuation on $\mathfrak{S}$ *is a relation* $V \subseteq \mathbf{Var} \times X$*. A dynamic topological system equipped with a valuation is a* dynamic topological model*.*

The valuation V is extended inductively to arbitrary formulas as follows:

$$\begin{array}{lcl} V(\alpha\wedge\beta) & = & V(\alpha)\cap V(\beta) \\ V(\neg\alpha) & = & X\setminus V(\alpha) \\ V(\Box\alpha) & = & V(\alpha)^{\circ} \\ V(\bigcirc\alpha) & = & f^{-1}V(\alpha) \\ V(*\alpha) & = & \bigcap\limits_{n\geq 0} f^{-n}V(\alpha). \end{array}$$

$\mathcal{DTL}$ distinguishes arbitrary spaces from finite spaces and even from locally finite spaces (those where every point has a neighborhood with finitely many points).

More generally, $\mathcal{DTL}$ distinguishes arbitrary topological spaces from Aleksandroff spaces ([9]); that is, spaces where arbitrary intersections of open sets are open ([1]). All locally finite spaces are Aleksandroff spaces.

Nevertheless, we will show how locally finite spaces can be used to represent a larger class dynamic topological systems; to do this, we will define alternative semantics for $\mathcal{DTL}$.

3 Non-deterministic quasimodels

We will denote the set of subformulas of φ by $\mathrm{sub}(\varphi)$, and define

$$\mathrm{sub}_{\pm}(\varphi) = \mathrm{sub}(\varphi) \cup \neg\mathrm{sub}(\varphi).$$

If we identify ψ with $\neg\neg\psi$, one can think of $\mathrm{sub}_{\pm}(\varphi)$ as being closed under negation.

A set of formulas $\mathbf{t} \subseteq \mathrm{sub}_{\pm}(\varphi)$ is a *φ-type* if, for all $\psi \in \mathrm{sub}_{\pm}(\varphi)$,

$$\psi \notin \mathbf{t} \Leftrightarrow \neg\psi \in \mathbf{t}$$

and for all $\psi_1 \wedge \psi_2 \in \mathrm{sub}_{\pm}(\varphi)$,

$$\psi_1 \wedge \psi_2 \in \mathbf{t} \Leftrightarrow \psi_1 \in \mathbf{t} \text{ and } \psi_2 \in \mathbf{t}.$$

The set of φ-types will be denoted by $\mathbf{type}(\varphi)$.

Definition 3.1 (typed Kripke frame)**.** *Let φ be a formula in the language of $\mathcal{DTL}$. A φ-*typed Kripke frame *is a triple $\mathfrak{F} = \langle W, R, t\rangle$, where W is a set, R a transitive, reflexive relation on W and t a function assigning a φ-type $t(w)$ to each $w \in W$ such that*

$$\Box\psi \in t(w) \Leftrightarrow \forall v\,(Rwv \Rightarrow \psi \in t(v))\,.$$

It is easy to see that this is equivalent to the dual condition that

$$\Diamond\psi \in t(w) \Leftrightarrow \exists v\,(Rwv \text{ and } \psi \in t(v))\,.$$

Kripke frames give sound and complete semantics for for $\mathcal{S}4$ ([3]), but here we are disregarding the temporal modalities by giving valuations of these formulas a priori rather than by their usual meaning. One would then be tempted to equip the Kripke frame with a transition function in order to interpret temporal operators directly. However, this would give us a class of models for which $\mathcal{DTL}$ is incomplete; instead, we will allow each world to have multiple temporal successors via a 'sensible' relation g, as we will define below. For our purposes, a *continuous relation* on a topological space is a relation under which the preimage of any open set is open.

Definition 3.2 (sensible relation)**.** *Let φ be a formula of $\mathcal{DTL}$ and $\langle W, R, t\rangle$ a φ-typed Kripke frame.*

Suppose that $\mathbf{t}, \mathbf{s} \in \mathbf{type}(\varphi)$. The ordered pair $(\mathbf{t}, \mathbf{s})$ is sensible *if*

1. *for all $\bigcirc\psi \in \mathrm{sub}(\varphi)$, $\bigcirc\psi \in \mathbf{t} \Leftrightarrow \psi \in \mathbf{s}$ and*
2. *for all $*\psi \in \mathrm{sub}(\varphi)$, $*\psi \in \mathbf{t} \Leftrightarrow (\psi \in \mathbf{t}$ and $*\,\psi \in \mathbf{s})\,.$*

Likewise, a pair (w, v) of worlds in W is sensible if $(t(w), t(v))$ is sensible.

A continuous relation $g \subseteq W \times W$ such that $g(w) \neq \varnothing$ for all $w \in W$ is sensible *if every pair in g is sensible.*

Further, g is ω-sensible if for all $*\psi \in \mathrm{sub}(\varphi)$,

$$\neg *\psi \in t(w) \Leftrightarrow \exists v \in W \text{ and } N \geq 0 \text{ such that } \neg\psi \in t(v) \text{ and } g^N wv.$$

It is a good idea to examine what continuity means in a Kripke frame. Suppose $\mathfrak{F}$ is as in Definition 3.1 and g is continuous. Pick $w, v \in W$ so that gwv. Since the set $R(v) = \{u : Rvu\}$ is open, we know that $R(w) \subseteq g^{-1}R(v)$. In other words, if Rww', there exists v' such that Rvv' and $gw'v'$, so that the following square can always be completed:

$$
\begin{tikzcd}
w' \arrow[r, dashed, "g"] & v' \\
w \arrow[u, "R"] \arrow[r, "g"] & v. \arrow[u, dashed, "R"']
\end{tikzcd}
$$

We are now ready to define our non-deterministic semantics for $\mathcal{DTL}$.

Definition 3.3 (non-deterministic quasimodel)**.** *A* φ-typed non-deterministic quasimodel *is a tuple* $\mathfrak{D} = \langle W, R, t, g \rangle$ *, where* $\langle W, R, t \rangle$ *is a* φ*-typed Kripke frame and* g *is an* ω*-sensible relation on* W*.*

$\mathfrak{D}$ satisfies φ *if there exists* $w_* \in W$ *such that* $\varphi \in t(w_*)$*.*

Non-deterministic quasimodels are similar to dynamic Kripke frames ([2, 4, 5]) except for the fact that g is now a relation instead of a function. However, note that we do not allow any subformulas of φ to be left undecided by g; that is, if $\bigcirc\psi \in \mathrm{sub}_{\pm}(\varphi)$ and $w \in W$, then either $\psi \in t(v)$ for all temporal successors v of w or $\neg\psi \in t(v)$ for all such v. This is necessary in order to preserve soundness.

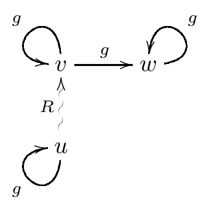

Figure 1: The formula $\varphi = *\Box p \to \Box * p$ is valid on all finite topological models; in fact, it is valid on all topological models based on a locally finite topological space ([9]). However, φ can be refuted in a non-deterministic quasimodel with only three worlds. Take $W = \{u, v, w\}$ and let R, g be as shown in the diagram above (closing R under reflexivity). Assign types to u, v and w in such a way that $p, *\Box p, \neg\Box * p \in t(u)$, $p, \neg * p \in t(v)$ and $\neg p \in t(w)$. Then, $\varphi \notin t(u)$, hence we have a non-deterministic quasimodel satisfying $\neg\varphi$. This, we will see below, shows that φ is not a theorem of $\mathcal{DTL}$.

4 Generating dynamic topological models from non-deterministic quasimodels

A dynamic topological model can be constructed from any non-deterministic quasimodel. Evidently, a non-deterministic quasimodel is not always a dynamic topological model, since dynamic topological models require a transition function rather than a relation. Instead, we will build a topological space whose points are infinite sequences of worlds.

4.1 Realizing sequences

A *path* in $\mathfrak{D}$ is any finite or infinite sequence $\langle w_n \rangle$ such that $g w_n w_{n+1}$.

An infinite path $\vec{w} = \langle w_n \rangle_{n \geq 0}$ is *realizing* if for all $n \geq 0$ and $\neg * \psi \in t(w_n)$ there exists $K \geq n$ such that $\neg \psi \in t(w_K)$.

Denote the set of realizing paths by W^g. We will construct dynamic topological models from non-deterministic quasimodels by topologizing this set. All other paths will be thrown away, since in such paths, $*$ could not be interpreted according to its intended meaning.

The main transformation we will consider on W^g will be the 'shift' operator, defined by $\sigma\left(\langle w_n \rangle_{n \geq 0}\right) = \langle w_{n+1} \rangle_{n \geq 0}$. This simply removes the first element in the sequence.

For our construction to work we must guarantee that there are 'enough' realizing paths, in the sense of the following definition.

Definition 4.1 (extensive subset)**.** *Let φ be a formula of $\mathcal{DTL}$,*

$$\mathfrak{D} = \langle W, R, t, g \rangle$$

be a φ-typed non-deterministic quasimodel and $Y \subseteq W^g$.

Then, Y is extensive *if*

1. *Y is closed under σ;*

2. *any finite path $\langle w_0, w_1, .., w_N \rangle$ in $\mathfrak{D}$ can be extended to an infinite path $\vec{w} = \langle w_n \rangle_{n \geq 0} \in Y$.*

Lemma 4.2. *If $\mathfrak{D} = \langle W, R, t, g \rangle$ is a non-deterministic quasimodel, W^g is extensive.*

Proof. It is obvious that W^g is closed under σ.

Let $\langle w_0, ..., w_N \rangle$ be a finite path and $\psi_0, ..., \psi_I$ be all formulas such that $\neg * \psi_i \in t(w_N)$. Because g is ω-sensible, we know that there exist K_I and $v_I \in g^{K_I}(w_N)$ such that $\neg \psi_I \in t\left(v_I\right)$. We can then define $w_{N+1}, ..., w_{N+K_I} = v_I$ in such a way that $g w_n w_{n+1}$ for all $n < N + K_I$. Now consider ψ_{I-1}. If $\neg \psi_{I-1} \in t(w_n)$ for some $n \leq K_I$, there is nothing to do and we can set $K_{I-1} = 0$.

Otherwise, $\neg * \psi_{I-1} \in t(w_{N+K_I})$ and we can pick K_{I-1} and v_{I-1} such that $g^{K_{I-1}} v_{I-1} v_I$. Then, define $w_{N+K_I+1}, ..., w_{N+K_I+K_{I+1}} = v_{I-1}$ as before.

Continuing inductively, we can define $\{w_n\}_{n \leq N+K}$, where $K = \sum_{i \leq I} K_i$ and for all $I \leq i$, $\neg \psi_i \in t(w_{N+k})$ for some $k \leq K$. We can then repeat the process starting with $\{w_n\}_{n \leq N+K}$, and continue countably many times to get a path $\{w_n\}_{n \geq 0}$. It is then easy to see that this path is realizing. [1] $\square$

[1] We must ensure that $K > 0$ at each step so that the sequence increases in length and the end result is an infinite path, but this can always be done since $g(w) \neq \varnothing$ for all w.

Lemma 4.3. *Let $\mathfrak{D} = \langle W, R, t, g \rangle$ be a φ-typed non-deterministic quasimodel, $\langle w_n \rangle_{n \le N}$ a finite path and v_0 be such that Rw_0v_0.*

Then, there exists a path $\langle v_n \rangle_{n \le N}$ such that, for $n \le N$, Rw_nv_n.

Proof. This follows from continuity of g by an easy induction on N. □

4.2 Limit models

If φ is a formula of $\mathcal{L}$ and $\mathfrak{D} = \langle W, R, g, t \rangle$ is a non-deterministic quasimodel, the relation R induces a topology on W, as we have seen before, by letting open sets be those which are upward closed under R. Likewise, R induces a very different topology on W^g, in a rather natural way:

Lemma 4.4. *For each $\vec{w} \in W^g$ and $N \ge 0$ define*

$$R_N(\vec{w}) = \left\{ \{v_n\}_{n \ge 0} \in W^g : \forall n \le N, Rw_nv_n \right\}.$$

Then, the set $\mathcal{B}^R = \{R_N(\vec{w}) : \vec{w} \in W^g, N \ge 0\}$ forms a topological basis on W^g.

Proof. Recall that a collection $\mathcal{B}$ of subsets of X is a basis if

1. $\bigcup\limits_{B \in \mathcal{B}} B = X$;
2. whenever $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there exists $B_3 \subseteq B_1 \cap B_2$ such that $x \in B_3$.

To check the first property, note that it is obvious that, given any path $\vec{w} \in W^g$, there is a basic set containing it (namely, $R_0(\vec{w})$). Hence $W^g = \bigcup_{\vec{w} \in W^g} R_0(\vec{w})$.

As for the second, assuming that $\vec{w} \in R_{N_0}(\vec{w}_0) \cap R_{N_1}(\vec{w}_1)$, one can see that $R_{\max(N_0,N_1)}(\vec{w}) \subseteq R_{N_0}(\vec{w}_0) \cap R_{N_1}(\vec{w}_1)$ using the transitivity of R. □

Definition 4.5. *The topology $\mathcal{T}^R$ on W^g is the topology generated by the basis $\mathcal{B}^R$.*

Now that we have equipped W^g with a topology, we need a continuous transition function on it to have a dynamic topological system.

Lemma 4.6. *The 'shift' operator $\sigma : W^g \to W^g$ is continuous under the topology $\mathcal{T}^R$.*

Proof. Let $\vec{w} = \{w_n\}_{n \ge 0}$ be a realizing path and $R_N(\sigma(\vec{w}))$ be a neighborhood of $\sigma(\vec{w})$. Then, if $\vec{v} \in R_{N+1}(\vec{w})$, Rw_nv_n for all $n \le N+1$, so $Rw_{n+1}v_{n+1}$ for all $n \le N$ and $\sigma(\vec{v}) \in R_N(\sigma(\vec{w}))$. Hence $\sigma(R_{N+1}(\vec{w})) \subseteq R_N(\sigma(\vec{w}))$, and σ is continuous. □

Finally, we will use t to define a truth valuation: if p is a propositional variable, set $V^t(p) = \{\vec{w} \in W^g : p \in t(w_0)\}$.

We are now ready to assign a dynamic topological model to every non-deterministic quasimodel:

Definition 4.7 (limit model)**.** *Given a non-deterministic quasimodel* $\mathfrak{D} = \langle W, R, g, t\rangle$, *define* $\lim \mathfrak{D} = \left\langle W^g, \mathcal{T}^R, \sigma, V^t \right\rangle$ *to be the* limit model *of* $\mathfrak{D}$.

Of course this model is only useful if V^t corresponds with t on all subformulas of φ, not just propositional variables. Fortunately, this turns out to be the case.

Lemma 4.8. *Let* $Y \subseteq W^g$ *be extensive,* $\vec{w} = \{w_n\}_{n\geq 0} \in Y$ *and* $\psi \in \mathrm{sub}_{\pm}(\varphi)$. *Then,*

$$\langle \lim \mathfrak{D} \upharpoonright Y, \vec{w}\rangle \models \psi \text{ if and only if } \psi \in t(w_0).$$

Proof. The proof goes by standard induction of formulas. The induction steps for Boolean operators are trivial; here we will only treat the cases for the modal operators.

Case 1: $\psi = \Box\alpha$. If $\psi \in t(w_0)$, take the neighborhood $R_0(w_0)$ of $\vec{w}$. We can then see that

$$\begin{array}{rcl} \Box\alpha \in t(w_0) & \Rightarrow & \forall v\,(Rw_0v \Rightarrow \alpha \in t(v)) \\ & \Rightarrow & \forall \vec{v} \in R_0(\vec{w}), \alpha \in t(v_0) \\ \text{IH} & \Rightarrow & \forall \vec{v} \in R_0(\vec{w}), \langle \lim \mathfrak{D} \upharpoonright Y, \vec{v}\rangle \models \alpha \\ & \Rightarrow & \langle \lim \mathfrak{D} \upharpoonright Y, \vec{w}\rangle \models \Box\alpha. \end{array}$$

On the other hand, if $\psi \notin t(w_0)$, any neighborhood $U_{\vec{w}}$ of $\vec{w}$ contains a sub-neighborhood $R_N(\vec{w})$ for some $N \geq 0$ (because these sets generate the topology). Then, by Lemma 4.3, there exists a path $\langle v_0, ..., v_N\rangle \subseteq W$ such that Rw_nv_n, gv_nv_{n+1} and $\neg\alpha \in t(v_0)$. Because Y is extensive, $\{v_n\}_{0\leq n\leq N}$ can be extended to a realizing path $\vec{v} \in Y$. Then $\vec{v} \in U_{\vec{w}}$, and by induction hypothesis we have that $\langle \lim \mathfrak{D} \upharpoonright Y, \vec{v}\rangle \models \neg\alpha$.

Since $U_{\vec{w}}$ was arbitrary, we conclude that $\langle \lim \mathfrak{D} \upharpoonright Y, \vec{w}\rangle \models \neg\Box\alpha$.

Case 2: $\psi = \bigcirc\alpha$. This case follows from the fact that (w_0, w_1) is sensible.

Case 3: $\psi = *\alpha$. Because $\vec{w}$ is a realizing path, we have that if $\neg * \alpha \in t(w_0)$, $\neg\alpha \in t(w_N)$ for some $N \geq 0$. We can use the induction hypothesis to conclude that $\left\langle \lim \mathfrak{D} \upharpoonright Y, \sigma^N(\vec{w})\right\rangle \not\models \alpha$ and so

$$\langle \lim \mathfrak{D} \upharpoonright Y, \vec{w}\rangle \not\models *\alpha.$$

Otherwise, $*\alpha \in t(w_0)$. For all n, (w_n, w_{n+1}) is sensible so $\alpha \in t(w_n)$ and $\langle \lim \mathfrak{D} \upharpoonright Y, \sigma^n(\vec{w})\rangle \models \alpha$; hence $\langle \lim \mathfrak{D} \upharpoonright Y, (\vec{w})\rangle \models *\alpha$. $\Box$

We are now ready to prove the main theorem of this section, which in particular implies that our semantics are sound for $\mathcal{DTL}$.

Theorem 4.9. *Let φ be a formula of $\mathcal{L}$, and suppose φ is satisfied in a non-deterministic quasimodel $\mathfrak{D} = \langle W, R, g, t\rangle$. Then, there exists $\vec{w}^* \in W^g$ such that $\langle \lim \mathfrak{D}, \vec{w}^*\rangle \models \varphi$.*

Proof. Pick $w^* \in W$ such that $\varphi \in t(w^*)$. By Lemma 4.2, w^* can be included in a realizing path $\vec{w}^*$. It follows from Lemma 4.8 that $\langle \lim \mathfrak{D}, \vec{w}^*\rangle \models \varphi$. □

5 Local Kripke frames

In this section we establish a basic framework for describing small substructures of Kripke frames. We wish to work with locally finite frames, and often it is convenient to give explicit bounds on the size of neighborhoods. These bounds will depend on the length of φ, denoted $|\varphi|$.

Definition 5.1 (local Kripke frame)**.** *A* local φ-typed Kripke frame *is a tuple $\mathfrak{a} = \langle w_{\mathfrak{a}}, W_{\mathfrak{a}}, R_{\mathfrak{a}}, t_{\mathfrak{a}}\rangle$, where $\langle W_{\mathfrak{a}}, R_{\mathfrak{a}}, t_{\mathfrak{a}}\rangle$ is a φ-typed Kripke frame and $w_{\mathfrak{a}} \in W_{\mathfrak{a}}$ is such that $R_{\mathfrak{a}} w_{\mathfrak{a}} v$ for all $v \in W_{\mathfrak{a}}$.*

The reader may recognize local Kripke frames as being nothing more than Kripke frames with a root. The reason we call them 'local' here is that, for our purposes, $\mathfrak{a}$ will represent a neighborhood of $w_{\mathfrak{a}}$, which may be a world in a larger Kripke frame $\mathfrak{A}$. The frame $\mathfrak{A}$ will often be disconnected and, hence, have no candidate for a root. The lower-case letters used to denote local Kripke frames are meant to be suggestive of this local character.

We will write $t(\mathfrak{a})$ instead of $t_{\mathfrak{a}}(w_{\mathfrak{a}})$.

5.1 Tree-like Kripke frames

Given a local Kripke frame $\mathfrak{a} = \langle w, W, R, t\rangle$, the relation R induces an equivalence relation $\sim_R$ on W given by $w \sim_R v \Leftrightarrow Rwv$ and Rvw.

The equivalence class of a world w is usually called the *cluster* of w. We will denote it by $[w]_R$, or simply $[w]$ if this does not lead to confusion. R then induces a partial order on $W/\sim_R$ defined by $R[w][v] \Leftrightarrow Rwv$. If this partial order forms a tree (that is, if whenever $R[u][w]$ and $R[v][w]$, then either $R[u][v]$ or $R[v][u]$), we will say that $\mathfrak{a}$ is *tree-like*.

Given a tree-like local Kripke frame $\mathfrak{a}$, we can define $\mathrm{hgt}(\mathfrak{a})$ and $\mathrm{wdt}(\mathfrak{a})$ as the height and width of $\mathfrak{a}/\sim_R$. Likewise, we will define the *depth* of $\mathfrak{a}$, $\mathrm{dpt}(\mathfrak{a})$, to be the maximum number of elements in a single cluster of $W_{\mathfrak{a}}$.

Definition 5.2 (norm of a Kripke frame)**.** *Let $\mathfrak{a}$ be a tree-like local Kripke frame. Define the* norm *of $\mathfrak{a}$, denoted $\|\mathfrak{a}\|$, by*

$$\|\mathfrak{a}\| = \max(\mathrm{hgt}(\mathfrak{a}), \mathrm{wdt}(\mathfrak{a}), \mathrm{dpt}(\mathfrak{a})).$$

We will use the norm of a local Kripke frame as a measure of its size rather than the more obvious $|W_{\mathfrak{a}}|$, because it is often more manageable. However, it is clear that one can use the norm of a frame to find bounds for the number of worlds in it (and vice-versa).

For the rest of this paper, all local Kripke frames will be assumed to be tree-like.

5.2 Binary relations between local Kripke frames

Many times it will be useful to compare different local Kripke frames and express relations between them. The following binary relations are essential and will appear throughtout the text:

Definition 5.3 (reduction of local Kripke frames)**.** *Say that* $\mathfrak{b}$ reduces *to* $\mathfrak{a}$ *(or, alternately,* $\mathfrak{a}$ embeds *into* $\mathfrak{b}$*), denoted* $\mathfrak{a} \trianglelefteq \mathfrak{b}$*, if there exists an injective function* $e : W_{\mathfrak{a}} \to W_{\mathfrak{b}}$ *such that, for all* $w, v \in W_{\mathfrak{a}}$*,* $R_{\mathfrak{a}}wv \Leftrightarrow R_{\mathfrak{b}}e(w)e(v)$*,* $t_{\mathfrak{a}}(w) = t_{\mathfrak{b}}(e(w))$ *and* $e\left(w_{\mathfrak{a}}\right) = w_{\mathfrak{b}}$*.*

Roughly, if $\mathfrak{a} \triangleleft \mathfrak{b}$, $\mathfrak{b}$ contains worlds which could be removed without altering $t(\mathfrak{b})$, and hence $\mathfrak{b}$ could be replaced by $\mathfrak{a}$ for most purposes. We will make this precise later in this section.

Definition 5.4 (subframe)**.** *For* $v \in W_{\mathfrak{a}}$*, set* $\mathfrak{a}^v = \langle v, W^v_{\mathfrak{a}}, R^v_{\mathfrak{a}}, t^v_{\mathfrak{a}} \rangle$ *, where* $W^v_{\mathfrak{a}} = \{w \in W_{\mathfrak{a}} : R_{\mathfrak{a}}vw\}$ *and* $R^v_{\mathfrak{a}}, t^v_{\mathfrak{a}}$ *are the corresponding restrictions to* $W^v_{\mathfrak{a}}$*.*

Then define $\mathfrak{b} \preceq \mathfrak{a}$ *if* $\mathfrak{b} = \mathfrak{a}^v$ *for some* $v \in W_{\mathfrak{a}}$*; we will say* $\mathfrak{b}$ *is a* subframe *of* $\mathfrak{a}$*.*

If $\mathfrak{a} \preceq \mathfrak{b}$ and $\mathfrak{b} \preceq \mathfrak{a}$, we will write $\mathfrak{a} \sim \mathfrak{b}$. Likewise, $\mathfrak{a} \prec \mathfrak{b}$ means that $\mathfrak{a} \preceq \mathfrak{b}$ but not vice-versa, while $\mathfrak{a} \prec_1 \mathfrak{b}$ means that $\mathfrak{a} \prec \mathfrak{b}$ and there is no intermediate local Kripke frame $\mathfrak{c}$ such that $\mathfrak{a} \prec \mathfrak{c} \prec \mathfrak{b}$.

Suppose $\mathfrak{b} \preceq \mathfrak{a}$. If we think of $\mathfrak{a}$ as a neighborhood of $w_{\mathfrak{a}}$, then $\mathfrak{b}$ represents an open subset of $W_{\mathfrak{a}}$ (which does not necessarily contain $w_{\mathfrak{a}}$). If it does contain $w_{\mathfrak{a}}$, then $\mathfrak{b} \sim \mathfrak{a}$, and the two represent the same open set but maybe 'centered' at a different point.

Definition 5.5 (subframe representatives)**.** *Let* $\mathfrak{a}$ *be a local Kripke frame. A* set of subframe representatives for $\mathfrak{a}$ *is a set of representatives of the equivalence classes of* $\{\mathfrak{b} : \mathfrak{b} \prec_1 \mathfrak{a}\}$ *under* $\sim$*.*

In the following definition and throughout the paper, a binary relation g between Kripke frames is *non-confluent* if whenever gwv, $gw'v'$ and Rvv', it follows that Rww', so that we can fill in the dotted arrow in the following diagram:

$$\begin{tikzcd} w' \arrow[r, "g"] & v' \\ w \arrow[u, dashed, "R"] \arrow[r, "g"] & v. \arrow[u, "R"'] \end{tikzcd}$$

Definition 5.6 (termporal successor)**.** *Say* $\mathfrak{a}$ *is an* temporal successor *of* $\mathfrak{b}$*, denoted* $\mathfrak{a} \rightrightarrows \mathfrak{b}$*, if there exists a non-confluent sensible relation* $g \subseteq W_{\mathfrak{a}} \times W_{\mathfrak{a}}$ *such that* $gw_{\mathfrak{a}}w_{\mathfrak{b}}$*.*

Lemma 5.7. *Let* φ *be any formula of* $\mathcal{DTL}$*, and* $\mathfrak{a} \rightrightarrows \mathfrak{b}$ *be local Kripke frames.*

Then, there exists $\mathfrak{d} \trianglelefteq \mathfrak{b}$ *such that* $\mathfrak{a} \rightrightarrows \mathfrak{d}$ *and* $\|\mathfrak{d}\| \leq \|\mathfrak{a}\| + |\varphi|$ *.*

Proof. We will skip the proof. The general idea is that if $\|\mathfrak{b}\| > \|\mathfrak{a}\| + |\varphi|$, then $W_\mathfrak{b}$ contains worlds which could be deleted, giving us $\mathfrak{d}$ such that $\mathfrak{a} \rightrightarrows \mathfrak{d} \lhd \mathfrak{b}$. Repeating this enough times we can attain the desired bound. $\square$

5.3 The space of bounded frames

Definition 5.8 ($\mathfrak{I}_K(\varphi)$)**.** *Let φ be any formula of $\mathcal{DTL}$ and $K \geq 0$. Define $I_K(\varphi)$ to be the set of all local, tree-like Kripke frames $\mathfrak{a}$ such that $\|\mathfrak{a}\| \leq (K+1)|\varphi|$.*

Now, consider $I_\omega(\varphi) = \bigcup_{k\geq 0} I_k(\varphi)$ (evidently this is the set of all finite local tree-like frames). Define $\mathfrak{I}_\omega(\varphi) = \langle I_\omega(\varphi), \succeq, \rightrightarrows, t\rangle$, where $t(\mathfrak{a}) = \mathbf{t}_\mathfrak{a}$.

$\mathfrak{I}_\omega(\varphi)$ is a φ-typed Kripke frame[2] with a sensible relation $\rightrightarrows$; however, it is not necessarily ω-sensible, so $\mathfrak{I}_\omega(\varphi)$ is not a non-deterministic quasimodel as it stands. It does contain substructures which are, as we will see in the next section.

5.4 Building local Kripke frames from subframes

Often we will want to construct a local Kripke frame from smaller pieces. Here we will define the basic operation we will use to do this, and establish the conditions that the pieces must satisfy.

Definition 5.9. *Let φ be a formula of $\mathcal{DTL}$, $T \subseteq \mathbf{type}(\varphi)$ and $A \subseteq I_\omega(\varphi)$.*

For each $\mathbf{t} \in T$ define $[T \oplus A]_\mathbf{t} = \langle w, W, R, t\rangle$ by setting $w = \mathbf{t}$,

$$W = T \cup \coprod_{\mathfrak{a}\in A} W_\mathfrak{a},$$

$$R = (T \times W) \cup \coprod_{\mathfrak{a}\in A} R_\mathfrak{a}$$

and

$$t(w) = \begin{cases} w & \text{if } w \in T \\ t_\mathfrak{a}(w) & \text{if } w \in W_\mathfrak{a}. \end{cases}$$

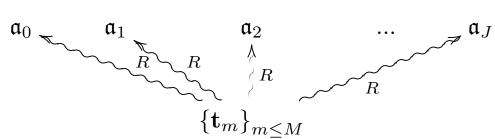

Figure 2: If $A = \{\mathfrak{a}_j\}_{j\leq J}$ and $T = \{\mathbf{t}_m\}_{m\leq M}$, $[T \oplus A]_{\mathbf{t}_0}$ has $\mathbf{t}_0$ as a root and each $\mathfrak{a}_j$ as a subframe.

[2] Note that the accessibility relation is written as $\succeq$, so that $\Box\psi$ holds in $\mathfrak{a}$ if ψ holds in $\mathfrak{b}$ for all $\mathfrak{b} \preceq \mathfrak{a}$, even though it is more standard to write the accessibility relation in the opposite direction. We believe it is natural to adopt this convention because $\|\mathfrak{b}\| \leq \|\mathfrak{a}\|$ whenever $\mathfrak{b} \preceq \mathfrak{a}$.

Definition 5.10 (admitting pair). *Let φ be a formula of $\mathcal{DTL}$, $T \subseteq \mathbf{type}(\varphi)$, $\mathbf{t} \in T$, $\mathfrak{c} \in I_\omega(\varphi)$ and $A \subseteq I_\omega(\varphi)$.*

The triple $\langle T, A, \mathbf{t} \rangle$ admits $\mathfrak{b}$ *if $(t(\mathfrak{b}), \mathbf{t})$ is sensible and either*

1. *there is $\mathfrak{a} \in A$ such that $\mathfrak{b} \rightrightarrows \mathfrak{a}$ or*
2. *(a) for each $w \in [w_{\mathfrak{b}}]$ there exists $\mathbf{s} \in T$ such that $(t(w), \mathbf{s})$ is sensible and*

   (b) there is a set of subframe representatives B for $\mathfrak{b}$ and an injection $\iota : B \to A$ such that for each $\mathfrak{c} \in B$, $\mathfrak{c} \rightrightarrows \iota(\mathfrak{c})$.

Lemma 5.11. *If $\mathfrak{a}$ and $\mathfrak{b}$ are local Kripke frames, B is a set of subframe representatives for $\mathfrak{b}$ and the triple $\langle t_{\mathfrak{b}}[w_{\mathfrak{b}}], B, t(\mathfrak{b}) \rangle$ admits $\mathfrak{a}$, then $\mathfrak{a} \rightrightarrows \mathfrak{b}$.*

Proof. The proof is straightforward and we omit it here. □

Definition 5.12 (coherence). *Let φ be a formula of $\mathcal{DTL}$, $T \subseteq \mathbf{type}(\varphi)$, $\mathbf{t} \in T$ and $A \subseteq I_\omega(\varphi)$.*

Then,

1. *the pair $\langle T, A \rangle$ is* coherent *if, for all $\Box\psi \in \mathrm{sub}_\pm(\varphi)$ and $\mathbf{t} \in T$, $\Box\psi \in \mathbf{t}$ if and only if $\psi \in \mathbf{s}$ for all $\mathbf{s} \in T$ and $\Box\psi \in t(\mathfrak{a})$ for all $\mathfrak{a} \in A$ and*
2. *if $\mathfrak{c} \in I_\omega(\varphi)$, the pair $\langle T, A \rangle$ is* coherent for $\mathfrak{c}$ *if it is coherent and admits $\mathfrak{c}$.*

Coherent pairs are useful for constructing local Kripke frames.

Lemma 5.13. *Let φ be a formula of $\mathcal{DTL}$, $T \subseteq \mathbf{type}(\varphi)$, $A \subseteq I_\omega(\varphi)$ and $\mathbf{t} \in T$.*

Then,

1. *$[T \oplus A]_{\mathbf{t}}$ is a local Kripke frame if and only if $\langle T, A \rangle$ is coherent and*
2. *$[T \oplus A]_{\mathbf{t}}$ is a local Kripke frame and $\mathfrak{c} \rightrightarrows \mathfrak{a}$ whenever $\langle T, A \rangle$ is coherent for $\mathfrak{c}$.*

Proof. To prove 1, one can check that the coherence conditions correspond exactly to the condition in Definition 3.1. The second claim follows from the first and Lemma 5.11. □

6 Simulating topological models

In this section we will study simulations, which are the basic tool for extracting non-deterministic quasimodels from dynamic topological models.

If $\mathfrak{M} = \langle X, \mathcal{T}, f, V \rangle$ is a dynamic topological model and $x \in X$, assign a φ-type $\tau(x)$ to x given by $\tau(x) = \{\psi \in \mathrm{sub}_\pm(\varphi) : x \in V(\psi)\}$. We will also define $\tau^\Diamond(x) = \{\psi \in \mathrm{sub}_\pm(\varphi) : \Diamond\psi \in \tau(x)\}$.

Analogously, if $\mathfrak{F} = \langle W, R, t \rangle$ is a φ-typed Kripke frame and $w \in W$, set $t^\Diamond(w) = \{\psi \in \mathrm{sub}_\pm(\varphi) : \Diamond\psi \in t(w)\}$.

6.1 Simulations

Definition 6.1 (simulation)**.** *Let φ be a formula of $\mathcal{DTL}$,*

$$\mathfrak{M} = \langle X, \mathcal{T}, f, V \rangle$$

a dynamic topological model and $\mathfrak{F} = \langle W, R, t \rangle$ a φ-typed Kripke frame.

A continuous relation $\chi \subseteq W \times X$ is a simulation *if, for all $x \in X$,*

$$x \in \chi(w) \Rightarrow \tau(x) = t(w).$$

We will call the latter property *type-preservation.*

A simulation can be thought of as a one-way bisimulation; a topological bisimulation would be an open, continuous map which preserves valuations of propositional variables. Simulations can be used to capture much of the purely topological information about $\mathfrak{M}$. However, temporal behavior is disregarded here; for this we need simulations on non-deterministic quasimodels, not just Kripke frames, and these simulations must respect the transition function.

Definition 6.2 (ω-simulation)**.** *Let φ be a formula of $\mathcal{DTL}$ and $\mathfrak{M} = \langle X, \mathcal{T}, f, V \rangle$ a dynamic topological model. Let $\mathfrak{F} = \langle W, R, t \rangle$ be a Kripke frame and g a sensible relation on W.*

Suppose $\chi \subseteq W \times X$ is a simulation.

Then, χ is an ω-simulation if $f\chi \subseteq \chi g$.

While g is not required to be ω-sensible on $\mathfrak{F}$, we can use χ to extract a non-deterministic quasimodel from $\mathfrak{F}$.

Lemma 6.3. *Suppose $\mathfrak{F} = \langle W, R, t \rangle$ is a φ-typed Kripke frame with a sensible relation g, $\mathfrak{M} = \langle X, \mathcal{T}, f, V \rangle$ is a dynamic topological model and $\chi \subseteq W \times X$ is an ω-simulation.*

Then, $\mathfrak{F} \upharpoonright \mathrm{dom}(\chi)$ is a non-deterministic quasimodel.

Proof. We only need to prove that $g \upharpoonright \mathrm{dom}(\chi)$ is ω-sensible.

Let $w \in \mathrm{dom}(\chi)$, $\neg * \psi \in t(w)$ and $x \in \chi(w)$.

Then, $\langle \mathfrak{M}, x \rangle \models \neg * \psi$, so for some $N > 0$, $f^N(x) \models \neg\psi$; but $f\chi \subseteq \chi g$, so there exists $v \in W$ such that $f^N(x) \in \chi(v)$ (hence $v \in \mathrm{dom}(\chi)$) and $g^N wv$.

Thus $\neg\psi \in t(v)$, which is what we wanted. □

Suppose that χ is an ω-simulation and $x^* \in X$ is such that $\langle \mathfrak{M}, x^* \rangle \models \varphi$. If there exists $w^* \in W$ such that $x^* \in \chi(w^*)$, then clearly $\mathfrak{Q}$ satisfies φ, since $\varphi \in t(w^*)$.

Thus, if we show that, given a dynamic topological model $\mathfrak{M} = \langle X, \mathcal{T}, f, V \rangle$, there exists a non-deterministic quasimodel $\mathfrak{Q} = \langle W, R, g, t \rangle$ with a surjective ω-simulation $\chi \subseteq W \times X$, this would imply that, given any satisfiable formula, it can be satisfied in a non-deterministic quasimodel.

In fact, we will show that $\mathfrak{I}_\omega(\varphi)$ (defined in Section 5.3) contains a 'canonical' quasimodel in the sense that there always exists a surjective ω-simulation from $I_\omega(\varphi)$ to X.

Lemma 6.4. *Given any dynamical topological model $\mathfrak{M} = \langle X, \mathcal{T}, f, V\rangle$, there exists a unique maximal simulation $\chi^* \subseteq I_\omega(\varphi) \times X$.*

Proof. The proof uses Zorn's Lemma in a straightforward fashion and we skip it. For uniqueness, if χ^* and χ^+ are two maximal simulations, one can check that $\chi^* \cup \chi^+$ is also a simulation, which must equal both χ^* and χ^+ by maximality. □

Given a simulation $\chi \subseteq I_\omega(\varphi) \times X$, we will denote $\chi \restriction I_k(\varphi)$ by χ_k.

Our goal is to prove that χ^* gives us a surjective ω-simulation. The following lemma will be essential in proving this.

Lemma 6.5 (simulation extensions)**.** *Let $\mathfrak{M} = \langle X, \mathcal{T}, f, V\rangle$ be a dynamic topological model and $\chi \subseteq I_\omega(\varphi) \times X$ be a simulation.*

If $T \subseteq \mathbf{type}(\varphi)$ and $A \subseteq I_\omega(\varphi)$ are coherent (as in Definition 5.12) and there is a set $E \subseteq X$ such that, for all $\mathfrak{a} \in A$, $E \subseteq \overline{\chi(\mathfrak{a})}$ and, for all $\mathbf{t} \in T$, $E \subseteq \overline{E \cap V\left(\bigwedge \mathbf{t}\right)}$, then

$$\zeta = \chi \cup \left\{ \left([T \oplus A]_{\tau(x)}, x\right) : x \in E \text{ and } \tau(x) \in T \right\}$$

is also a simulation.

Proof. Note, first, that ζ is type-preserving. We must prove it is also continuous.

Consider an arbitrary open set $U \subseteq X$. If $U \cap E = \varnothing$, $\zeta^{-1}(U) = \chi^{-1}(U)$, which is open because χ is continuous.

Otherwise, $\zeta^{-1}(U) = \chi^{-1}(U) \cup \zeta^{-1}(E \cap U)$.

We must prove that if $\mathfrak{b} \in \zeta^{-1}(E \cap U)$ and $\mathfrak{c} \preceq \mathfrak{b}$, then $\mathfrak{c} \in \chi^{-1}(U) \cup \zeta^{-1}(E \cap U)$. First assume that $\mathfrak{c} \prec \mathfrak{b}$. In this case, $\mathfrak{c} \preceq \mathfrak{a}$ for some $\mathfrak{a} \in A$.

Since

$$E \subseteq \overline{\chi(\mathfrak{a})},$$

there exists some $y \in U \cap \chi(\mathfrak{a})$. Because U is open and χ is continuous, there also exists $z \in U \cap \chi(\mathfrak{c})$, as we wanted.

Otherwise, $\mathfrak{c} \sim \mathfrak{b}$. Now, $E \subseteq \overline{E \cap V\left(\bigwedge t(\mathfrak{c})\right)}$, so there exists $y \in E \cap U$ such that $\tau(y) = t(\mathfrak{c})$. By the definition of ζ, $y \in \chi(\mathfrak{c})$, so $\mathfrak{c} \in \zeta^{-1}(U)$.

We conclude that ζ is continuous, and therefore a simulation. □

Proposition 6.6. *For any dynamic topological model $\mathfrak{M}$, χ_0^* is surjective.*

Proof. We must define slightly stronger bounds on frames for our proof to go through; namely, define *small* and *very small* as follows:

1. Say $\mathfrak{a}$ is *very small* if $\|\mathfrak{a}\| \leq \left|t^\Diamond(\mathfrak{a})\right|$ and $\mathrm{hgt}(\mathfrak{a}) < \left|t^\Diamond(\mathfrak{a})\right|$.

2. Say $\mathfrak{a}$ is *small* if $\|\mathfrak{a}\| \leq \left|t^\Diamond(\mathfrak{a})\right|$ and there is a very small $\mathfrak{b} \preceq \mathfrak{a}$ such that $t^\Diamond(\mathfrak{b}) = t^\Diamond(\mathfrak{a})$.

Note that if $\mathfrak{a}$ is small, then $\mathfrak{a} \in I_0(\varphi)$. We will prove that χ^* is surjective even when restricted to the set of small frames.

Suppose $\chi \subseteq I_\omega(\varphi) \times X$ is any simulation.

Say a point $x \in X$ is *bad* if there is no small $\mathfrak{a}$ such that $x \in \chi(\mathfrak{a})$.

We claim that χ is not maximal if there are bad points.

To see this, let E be an arbitrary subset of X. Define

$$\mathrm{Bad}(E) = \{\tau(x) : x \in E \text{ is bad}\}.$$

Assume that $\mathrm{Bad}(X) \neq \varnothing$.

Then, we can pick out an open set U_* which minimizes $|\mathrm{Bad}(U)| + \left|\chi_0^{-1}(U)\right|$, where U ranges over all open sets that contain bad points. Notice that $\mathrm{Bad}(E)$ and $\chi_0^{-1}(E)$ are finite, since they are subsets of $\mathbf{type}(\varphi)$ and $I_0(\varphi)$, which are finite.

Note also that, for such a U_*, whenever $U \subseteq U_*$ is open and contains bad points, we know that $\mathrm{Bad}(U) = \mathrm{Bad}(U_*)$ and $\chi_0^{-1}(U) = \chi_0^{-1}(U_*)$ (otherwise U_* would not be optimal).

We will construct a simulation $\zeta \supseteq \chi$ such that $\zeta_0 \neq \chi_0$.

Let Ψ be the set of all formulas $\psi \in \mathrm{sub}_\pm(\varphi)$ such that $\Diamond\psi \in \bigcup \mathrm{Bad}(U_*)$ but $\psi \notin \bigcup \mathrm{Bad}(U_*)$.

For each $\psi \in \Psi$, U contains a point y such that $\langle \mathfrak{M}, y\rangle \models \psi$. Note that y cannot be bad, so there exists a small frame $\mathfrak{c}$ such that $y \in \chi_0(\mathfrak{c})$, and hence a very small $\mathfrak{a} \preceq \mathfrak{c}$. Set $\mathfrak{a} = \mathfrak{a}_\psi$, and $A = \{\mathfrak{a}_\psi : \psi \in \Psi\}$.

Pick a minimal, non-empty $T \subseteq \mathrm{Bad}(U_*)$ such that T and A are coherent. By Lemma 5.13, $\mathfrak{a}_* = [T \oplus A]_{\mathfrak{t}_*}$ is local Kripke frame, and we can set

$$\zeta = \chi \cup \{(\mathfrak{b}, y) : y \in U_*, \mathfrak{b} \sim \mathfrak{a} \text{ and } \tau(y) = t(\mathfrak{b})\}.$$

By Lemma 6.5, ζ is a simulation. It remains to show that $\mathfrak{a}_*$ is small.

Note first that, since all elements of A are very small, for all $\mathfrak{a} \in A$, $\mathrm{hgt}(\mathfrak{a}) < \left|t^\Diamond(\mathfrak{a})\right|$.

This shows that $\mathrm{hgt}(\mathfrak{a}_*) \leq \left|t^\Diamond(\mathfrak{a}_*)\right|$, and equality holds only if $t^\Diamond(\mathfrak{b}) = t^\Diamond(\mathfrak{a}_*)$ for some $\mathfrak{b} \in A$; this gives us the very small frame $\mathfrak{b} \preceq \mathfrak{a}_*$.

Similarly, $\mathrm{wdt}(\mathfrak{a}) \leq \left|t^\Diamond(\mathfrak{a}_*)\right|$ for all $\mathfrak{a} \in A$, and $\mathfrak{a}_*$ has at most $\left|t^\Diamond(\mathfrak{a}_*)\right|$ immediate successors, so $\mathrm{wdt}(\mathfrak{a}_*) \leq \left|t^\Diamond(\mathfrak{a}_*)\right|$. One can easily see that $\mathrm{dpt}(\mathfrak{a}_*) \leq |\varphi|$.

It follows that $\mathfrak{a}_* \in I_0(\varphi)$. Since U_* contained bad points, $\chi \subsetneq \zeta$, as desired. $\square$

Proposition 6.7. *For all $K \geq 0$, $f\chi^*_K \subseteq \chi^*_{K+1} g$.*

Note that, as a consequence of this, $f\chi^* \subseteq \chi^* g$ and thus χ^* is an ω-simulation.

Proof. The proof follows much the same structure as that of Proposition 6.6.

Suppose $\mathfrak{a} \rightrightarrows \mathfrak{b}$. Define *small relative to* $\mathfrak{a}$ and *very small relative to* $\mathfrak{a}$ as follows:

1. Say $\mathfrak{b}$ is *very small relative to* $\mathfrak{a}$ if $\|\mathfrak{b}\| \le \|\mathfrak{a}\| + |\varphi|$ and $\mathrm{hgt}(\mathfrak{a}) < \|\mathfrak{a}\| + |\varphi|$.

2. Say $\mathfrak{b}$ is *small relative to* $\mathfrak{a}$ if $\|\mathfrak{b}\| \le \|\mathfrak{a}\| + |\varphi|$ and there is $\mathfrak{d} \preceq \mathfrak{b}$ which is very small relative to $\mathfrak{a}$.

Let χ be a simulation.

Say $x \in X$ *fails* for $\mathfrak{a}$ if $f^{-1}(x) \cap \chi(\mathfrak{a}) \neq \varnothing$, but there is no $\mathfrak{b}$ which is small relative to $\mathfrak{a}$ such that $x \in \chi(\mathfrak{b})$.

We claim that if χ contains points that fail for any $\mathfrak{a}$, then χ is not maximal.

Suppose there exists $\mathfrak{a}_*$ such that some point fails for $\mathfrak{a}_*$, and $\mathfrak{a}_*$ is minimal with this property.

For $E \subseteq X$ and $\mathfrak{a} \in I_\omega(\varphi)$ define

$$\mathrm{Fail}(E) = \{\tau(x) : x \in E \text{ fails for some } \mathfrak{c} \sim \mathfrak{a}_*\}.$$

Pick an open set U_* which minimizes $|\mathrm{Fail}(U)| + \left|\chi^{-1}_{K+1}(U)\right|$, where U ranges over all open sets which contain points that fail for $\mathfrak{a}_*$.

As before, let Ψ be the set of all formulas $\psi \in \mathrm{sub}_\pm(\varphi)$ such that $\Diamond\psi \in \bigcup \mathrm{Fail}(U_*)$ but $\psi \notin \bigcup \mathrm{Fail}(U_*)$. To each element ψ of Ψ assign a very small frame $\mathfrak{b}_\psi \in \chi^{-1}(U_*)$ such that $\psi \in t(\mathfrak{b}_\psi)$. These frames exist by Lemma 6.6.

Pick any $\mathbf{t}_* \in \mathrm{Fail}(U_*)$ such that $(t(\mathfrak{a}_*), \mathbf{t}_*)$ is sensible.

We must find T and B' such that the triple $\langle T, B', \mathbf{t}_* \rangle$ admits $\mathfrak{a}_*$. Here we will consider two cases.

Case 1. Suppose there is $\mathfrak{c} \sim \mathfrak{a}_*$ such that no point of U_* fails for $\mathfrak{c}$. In this case, there must exist $\mathfrak{d}_* \in \chi^{-1}_{K+1}(U)$ which is very small relative to $\mathfrak{c}$.

If this holds, set $B' = \{\mathfrak{d}_*\}$ and $T = \{\mathbf{t}_*\}$.

Case 2. Suppose Case 1 does not hold. Note that in this case, given any $\mathfrak{c} \sim \mathfrak{a}_*$, there is some point z in U_* which fails for $\mathfrak{c}$, and hence there is $\mathbf{t}_\mathfrak{c} = \tau(z) \in \mathrm{Fail}(U_*)$ such that $(t(\mathfrak{c}), \mathbf{t}_\mathfrak{c})$ is sensible. Set $T = \{\mathbf{t}_\mathfrak{c} : \mathfrak{c} \sim \mathfrak{a}_*\}$.

Let C be a set of subframe representatives for $\mathfrak{a}_*$. For each $\mathfrak{c} \in C$, no point of U_* fails for $\mathfrak{c}$ (because we picked $\mathfrak{a}_*$ to be minimal). However, $f^{-1}(U_*)$ is open and therefore contains points in $\chi(\mathfrak{c})$, so there must exist a frame $\mathfrak{b}_\mathfrak{c}$ which is small relative to $\mathfrak{c}$ (and, by passing to a subframe if necessary, we can pick it to be very small relative to $\mathfrak{c}$).

Then define $B' = \{\mathfrak{b}_\mathfrak{c} : \mathfrak{c} \in C\}$.

In either of the two cases set $B = B' \cup \{\mathfrak{b}_\psi\}_{\psi \in \Psi}$.

Then, by Lemma 5.13,

$$\mathfrak{b}_* = [\,T \oplus B\,]_{\mathbf{t}_*}$$

is a local Kripke frame and $\mathfrak{a}_* \rightrightarrows \mathfrak{b}_*$ by Lemma 5.11.

We can then set

$$\zeta = \chi \cup \{(\mathfrak{d}, y) : y \in U_*, \mathfrak{d} \sim \mathfrak{b}_* \text{ and } \tau(y) = t(\mathfrak{d})\}.$$

One can then show as before that $\mathfrak{b}_*$ is small for $\mathfrak{a}_*$, so ζ is a simulation which properly contains χ. Therefore, χ is not maximal. $\square$

We are now ready to give a completeness proof of non-deterministic semantics for $\mathcal{DTL}$.

Definition 6.8. *Given a dynamic topological model $\mathfrak{M}$ satisfying φ, define $\mathfrak{M}/\varphi = \mathfrak{I}_\omega(\varphi) \upharpoonright \mathrm{dom}(\chi^*)$.*

Theorem 6.9. *If $\mathfrak{M}$ is a dynamic topological model satisfying φ, then $\mathfrak{M}/\varphi$ is a non-deterministic quasimodel satisfying φ.*

Proof. By Proposition 6.7, χ^* is an ω-simulation, so by Lemma 6.3, $\mathfrak{M}/\varphi$ is a φ-typed non-deterministic quasimodel.

Pick $x_* \in X$ such that $\langle \mathfrak{M}, x_* \rangle \models \varphi$. By Proposition 6.6, χ^* is surjective, so there exists $\mathfrak{a}_* \in I_\omega(\varphi)$ such that $x_* \in \chi^*(\mathfrak{a}_*)$; hence $\varphi \in t(\mathfrak{a}_*)$.

This shows that that $\mathfrak{M}/\varphi$ satisfies φ, as desired. □

7 A model-search procedure

Non-deterministic quasimodels can be used to give a recursive enumeration of all valid formulas of $\mathcal{DTL}$. The general strategy is to generate finite 'chunks' of φ-typed non-deterministic quasimodels; if the search for chunks of arbitrary size terminates, φ is not satisfiable. Otherwise we can construct a non-deterministic quasimodel for φ and hence a model.

We will use Kruskal's Tree Theorem to give a recursive enumeration of all valid formulas of $\mathcal{DTL}$. Most of what follows is an adaptation of a proof in [5] that $\mathcal{DTL}_1$, the fragment of $\mathcal{DTL}$ where $*$ is not allowed to appear in the scope of $\Box$, is recursively enumerable. We will use non-deterministic quasimodels to generalize this result to full $\mathcal{DTL}$.

Recall that a pair $\langle S, \leq \rangle$ is a well-partial order if, for any infinite sequence $\langle s_n \rangle_{n \geq 0} \subseteq S$, there exist indices $M_0 < M_1$ such that $s_{M_0} \leq s_{M_1}$.

A labeled tree is a triple $\langle T, \leq, L \rangle$, where $\langle T, \leq \rangle$ is a tree and $L : T \to \Lambda$ is a labeling function to some fixed set Λ of labels.

If $\mathfrak{T}_0$ and $\mathfrak{T}_1$ are labeled trees and Λ is partially ordered, an embedding between $\mathfrak{T}_0$ and $\mathfrak{T}_1$ is a function $e : T_0 \to T_1$ which is an embedding as trees and such that $L_0 \leq_\Lambda L_1 e$. If such an embedding exists, we say that $\mathfrak{T}_0 \leq \mathfrak{T}_1$. We will always assume that embeddings map roots to roots.

Theorem 7.1 (Kruskal)**.** *The set of finite trees with labels in a well partially-ordered set is well-partially ordered.*

Proof. Kruskal's original proof can be found in [11]. □

We wish to apply Kruskal's Theorem to elements of $I_\omega(\varphi)$.

Lemma 7.2. *Let $\mathfrak{a}_0, \mathfrak{a}_1, ..., \mathfrak{a}_n, ...$ be an infinite sequence of finite, tree-like local Kripke frames.*

Then, there exist $M_1 < M_2$ such that $\mathfrak{a}_{M_1} \trianglelefteq \mathfrak{a}_{M_2}$.

Proof. This is a straightforward application of Kruskal's Tree Theorem; we only need to represent tree-like local Kripke frames by labeled trees.

Namely, to each tree-like local Kripke frame $\mathfrak{a} = \langle w, W, R, t\rangle$ assign a labeled tree $\mathfrak{T}_\mathfrak{a} = \langle T_\mathfrak{a}, \leq_\mathfrak{a}, l_\mathfrak{a}\rangle$ given by $\mathfrak{T}_\mathfrak{a} = W/\sim_R$ and $\leq_\mathfrak{a} = R/\sim_R$. If $[w] = \{w_i\}_{i\leq I}$, let $l([w]) = \langle t(w_0), ..., t(w_I)\rangle$.

It is known that the set of finite sequences of elements of a finite set is well-partially ordered by the 'subsequence' relation and hence we can apply Kruskal's Tree Theorem. It is not hard to see that if $\mathfrak{T}_{\mathfrak{a}_{M_1}} \leq \mathfrak{T}_{\mathfrak{a}_{M_2}}$, then $\mathfrak{a}_{M_1} \trianglelefteq \mathfrak{a}_{M_2}$. $\square$

Definition 7.3 (eventuality; realization time)**.** *Let $\mathfrak{D} = \langle W, R, g, t\rangle$ be a non-deterministic quasimodel.*

An eventuality *is any formula of the form $\neg * \psi \in \mathrm{sub}_\pm(\varphi)$.*

Given a path $\vec{w} \in W^g$, $N \geq 0$ and an eventuality $\neg * \psi \in t(w_N)$, define the realization time *of $\neg * \psi$ at N, denoted $\rho_N^{\neg*\psi}(\vec{w})$, to be the least $K \geq N$ such that $\neg * \psi \in t(w_K)$. In case that no such K exists set $\rho_N^{\neg*\psi}(\vec{w}) = \infty$.*

Likewise, define

$$\rho_N(\vec{w}) = \left\{\rho_N^{\neg*\psi}(\vec{w}) : \neg * \psi \in t(w_N)\right\}.$$

Let $\rho_N^\infty(\vec{w})$ be the maximum element of $\rho_N(\vec{w})$ and $\rho_N^{<\infty}(\vec{w})$ be the maximum finite element. In case that one of the sets being considered is empty, take zero instead of the maximum.

Definition 7.4 (efficiency)**.** *Let $\vec{\mathfrak{a}} = \langle \mathfrak{a}_n\rangle$ be a finite or infinite path of local Kripke frames.*

An inefficiency *in $\vec{\mathfrak{a}}$ is a triple $N \leq M_1 < M_2$ such that $\mathfrak{a}_{M_1} \trianglelefteq \mathfrak{a}_{M_2}$, $M_2 < \rho_N^\infty(\vec{\mathfrak{a}})$ and $\rho_N(\vec{\mathfrak{a}}) \cap (M_1, M_2) = \varnothing$.*

A finite or infinite path $\langle \mathfrak{a}_n\rangle$ is efficient *if it contains no inefficiencies and, for all $n \geq 0$, $\|\mathfrak{a}_{n+1}\| \leq \|\mathfrak{a}_n\| + |\varphi|$.*

Roughly, the previous definition says that if the same state occurs twice in a row in an efficient path, some eventuality must have been realized in the middle. Otherwise, the path between them gives us a sort of loop which we could simply skip. Furthermore, efficiency gives us a way to guarantee that a path is realizing.

Lemma 7.5. *For all $\mathfrak{a} \in I_\omega(\varphi)$, there is a realizing path beginning on $\mathfrak{a}$ if and only if there is an efficient path beginning on $\mathfrak{a}$.*

Proof. First suppose that we have an efficient path $\vec{\mathfrak{a}} = \langle \mathfrak{a}_n\rangle_{n\geq 0}$. We claim that $\vec{\mathfrak{a}}$ is realizing.

Let $N \geq 0$. We will show that $\rho_N^\infty(\vec{\mathfrak{a}}) < \infty$, and therefore that all eventualities of $\mathfrak{a}_N$ are realized.

By Lemma 7.2, there exist indices $\rho_N^{<\infty}(\vec{\mathfrak{a}}) \leq M_1 < M_2$ such that $\mathfrak{a}_{M_1} \trianglelefteq \mathfrak{a}_{M_2}$.

Since we know that $\vec{\mathfrak{a}}$ is efficient, this cannot produce an inefficiency. But no eventualities of $\mathfrak{a}_N$ occur between M_1 and M_2, so we must have $\rho_N^\infty(\vec{\mathfrak{a}}) \leq M_2$, and all eventualities of $\mathfrak{a}_N$ are thus realized by time M_2.

Since N was arbitrary, it follows that the path is realizing.

For the other direction, suppose $\vec{\mathfrak{a}} = \langle \mathfrak{a}_n\rangle_{n\geq 0}$ is a realizing path. We claim that we can obtain an efficient path from $\vec{\mathfrak{a}}$ by removing all inefficiencies.

Let us first show how to remove a single inefficiency.

Suppose $\mathfrak{a}_{M_1} \trianglelefteq \mathfrak{a}_{M_2}$ and this produces an inefficiency.

Then, clearly $\mathfrak{a}_{M_1} \rightrightarrows \mathfrak{a}_{M_2+1}$, and we get a sequence

$$\mathfrak{a}_0 \rightrightarrows ...\mathfrak{a}_{M_1} \rightrightarrows \mathfrak{a}_{M_2+1} \rightrightarrows ...$$

which we can then reduce using Lemma 5.7 to a sequence

$$\mathfrak{a}_0 \rightrightarrows \mathfrak{a}'_1 \rightrightarrows ...\mathfrak{a}'_n \rightrightarrows \mathfrak{a}'_{n+1} \rightrightarrows ...$$

with $\|\mathfrak{a}'_{n+1}\| \leq \|\mathfrak{a}'_n\| + |\varphi|$.

Now, to obtain an efficient sequence, we can apply this process countably many times: first we ensure that no inefficient loops start at time 0, then at time 1, etc. The end result is well-defined because for all $n \geq 0$, the n^{th} element in the sequence stabilizes in finite time. This gives us an efficient realizing sequence. □

Definition 7.6 (extension function)**.** *Let $\mathfrak{D} = \langle W, R, g, t \rangle$ be a non-deterministic quasimodel for a formula φ.*

An extension function *is a function $\epsilon : W \to W^g$, where $\epsilon(w) = \langle \epsilon_n(w) \rangle_{n \geq 0}$ satisfies $\epsilon_0(w) = w$.*

Extension functions give us a canonical way to include points in realizing sequences. If we have an extension function on a typed Kripke frame, this gives us a way to guarantee that the transition relation is ω-sensible.

Definition 7.7 (family of paths)**.** *A* family of paths *is a pair $\mathfrak{P} = \langle A, \epsilon \rangle$, where $A \subseteq I_\omega(\varphi)$ is open and ϵ is an extension function assigning a realizing path in A to each $\mathfrak{a} \in A$.*

Likewise, a partial family of paths of depth N *is a pair $\mathfrak{P}^N = \left\langle A^N, \epsilon^N \right\rangle$, where $A \subseteq I_N(\varphi)$ and, for all $k \leq N$ and $\mathfrak{a} \in A \cap I_k(\varphi)$, $\epsilon^N(\mathfrak{a})$ is a path of length $N - k + 1$ in A such that $\epsilon_0^N(\mathfrak{a}) = \mathfrak{a}$.*

In either case, $\mathfrak{P}$ is efficient *if for all $\mathfrak{a} \in A$, $\epsilon(\mathfrak{a})$ is efficient. $\mathfrak{P}$* satisfies φ *if there exists $\mathfrak{a}_* \in A \cap I_0(\varphi)$ such that $\varphi \in t(\mathfrak{a}_*)$.*

Lemma 7.8. *A formula φ is satisfiable if and only if there exists an efficient family of paths $\mathfrak{P}$ satisfying φ.*

Proof. If $\mathfrak{M}$ is a model satisfying φ, then we can use Lemma 7.5 to assign an efficient path $\epsilon(\mathfrak{a})$ to each $\mathfrak{a} \in \mathrm{dom}(\chi^*)$.

This gives us an efficient family of paths $\langle \mathrm{dom}(\chi^*), \epsilon \rangle$.

Conversely, it is easy to see that if we have an efficient family of paths, we also have a non-deterministic quasimodel; since ϵ gives us realizing paths in A, it follows that the relation $\rightrightarrows$ is ω-sensible on $\mathfrak{I}_\omega(\varphi) \upharpoonright A$. □

Theorem 7.9. *The set of all valid formulas of $\mathcal{DTL}$ is recursively enumerable.*

Proof. The strategy is to enumerate all efficient partial families of paths. This can be done, since there are only finitely many partial families of paths of any fixed depth N. We claim that φ is valid if and only there exists $N \geq 0$ such that no efficient family of paths of depth N satisfies $\neg\varphi$.

If $\neg\varphi$ is satisfiable, by Lemma 7.8, there exists an efficient family of paths $\mathfrak{P}$ satisfying $\neg\varphi$. This immediately gives us an infinite sequence of partial families $\mathfrak{P}^n = \mathfrak{P} \upharpoonright I_n(\varphi)$ satisfying $\neg\varphi$.

Conversely, if the search does not terminate, we can use König's Lemma to find an increasing sequence $\langle \mathfrak{P}^n \rangle_{n \geq 0}$, satisfying $\neg\varphi$, where $\mathfrak{P}^n = \mathfrak{P}^{n+1} \upharpoonright I_n(\varphi)$ for all n.

We can then define $A = \bigcup_{n \geq 0} A^n$, and for $K \geq 0$ and $\mathfrak{a} \in A^K$, set $\epsilon_n(\mathfrak{a}) = \epsilon_n^{K+n}(\mathfrak{a})$. Clearly the path $\epsilon(\mathfrak{a})$ is efficient.

Thus, $\mathfrak{P} = \langle A, \epsilon \rangle$ gives us an efficient family of paths satisfying $\neg\varphi$, as desired. □

Unfortunately, the procedure we have just described does not suggest an obvious proof system for $\mathcal{DTL}$, and the above model-search algorithm is the only recursive enumeration of valid fomulas that we offer here. One axiomatization is suggested in [10]; the question of its completeness remains open.

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