# Coherence in Linear Predicate Logic 

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#### Abstract

Coherence with respect to Kelly-Mac Lane graphs is proved for categories that correspond to the multiplicative fragment without constant propositions of classical linear first-order predicate logic without or with mix. To obtain this result, coherence is first established for categories that correspond to the multiplicative conjunction-disjunction fragment with first-order quantifiers of classical linear logic, a fragment lacking negation. These results extend results of [7] and [8], where coherence was established for categories of the corresponding fragments of propositional classical linear logic, which are related to proof nets, and which could be described as star-autonomous categories without unit objects.


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## 0 Introduction

The goal of this paper is to prove coherence for categories that correspond to the multiplicative fragment without propositional constants (nullary connectives) of classical linear first-order predicate logic without or with mix. (In this fragment the modal operators! and? are left out.) The propositional logic corresponding to this fragment is the fragment of linear logic caught by proof nets. Coherence for categories that correspond to this propositional logic, called proof-net categories, was proved in [8], where it is also demonstrated that the notion of proof-net category is the right notion of star-autonomous category without unit objects (and where references to related work may be found; see [1] for a general categorial introduction to propositional linear logic). The notion of proof-net category is here extended with assumptions concerning first-order quantifiers, and this yields the notion of category that corresponds to the fragment of linear
predicate logic mentioned in the first sentence. We prove coherence for these categories. Coherence in [8] and here is understood as in Kelly's and Mac Lane's coherence result of [12] for symmetric monoidal closed categories. It is coherence with respect to the same kind of graphs.

Such coherence results are very useful, because they enable us to decide easily equality of canonical arrows. The coherence results of this paper are interesting also for general proof theory. They deal with a plausible notion of identity of proofs in logic (see [7], Chapter 1).

The addition of first-order quantification to categories treated in [8] does not bring anything new with respect to the graphs. Before they involved propositional letters, and now they involve predicate letters. Individual variables do not bring anything to these graphs. All the new arrows for quantifiers have identity graphs (see Section 1.5). Although it seems we have made an inessential addition (based on a trivial adjunction; see Section 1.4), we don't know how to reduce simply the coherence result proved in this paper to the coherence result of [8]. The proofs in this paper extend and modify those of [8], but they require considerable additional effort.

We omit the multiplicative propositional constants from our treatment because they present special problems for coherence (though their addition too may be based on a trivial adjunction; see [7], Section 7.9). Anyway, primitive constant propositions may be out of place in predicate logic.

It may seem that in the absence of the multiplicative propositional constant $\top$ we will not be able express that a formula is a theorem with sequents of the form $A \vdash B$, for $A$ and $B$ formulae. These are the sequents of categorial proof theory, and of this paper. For a theorem $B$ we should have a derivable sequent $\top \vdash B$, but in the absence of $\top$ we shall have instead the sequents $A \vdash A \wedge B$ derivable for every formula $A$.

We omit also the additive (lattice) connectives from our treatment. They would lead to the same kind of problems for coherence that arise for classical or intuitionistic conjunctive-disjunctive logic with quantifiers added. We shall deal with this matter on another occasion.

The categories corresponding to the fragment of linear predicate logic that we cover are here presented equationally, in an axiomatic, regular and surveyable manner. These axiomatic equations should correspond to the combinatorial building blocks of identity of proofs in this fragment of logic, as in knot theory the Reidemeister moves are the combinatorial building blocks of identity of knots and links (see [3]).

In this paper, our approach to categories corresponding to first-order predicate logic is quite syntactical. We deal mainly with freely generated categories, which are a categorial presentation of syntax. Objects are formulae, and arrows are proofs, or deductions, i.e. equivalence classes of derivations. At the level of objects, our first-order language is quite standard. After these freely generated categories are introduced, other concrete categories belonging to the classes in which our categories are free may be taken as models - of proofs, rather than
formulae. The only models of this kind that we consider in this paper are categories whose arrows are graphs. Our coherence results may be understood as completeness results with respect to these models. What is shown complete is the axiomatization of equality between arrows in the freely generated category. Previous treatments of first-order quantifiers in categorial logic, which started with the work of Lawvere (see Section 1.4, and references given there), are less syntactical in spirit than ours, even when they claim that they belong to categorial proof theory.

As we said above, the proofs in this paper are based on proofs to be found in [8], which are themselves based on proofs in [7]. We will eschew repeating this previously published material with all its details, and so our paper will not be self-contained. To make it self-contained would yield a rather sizable book, overlapping excessively with [8] and [7]. We suppose the reader is acquainted up to a point with [7] (at least Sections 3.2-3, 7.6-8 and 8.4) and with [8] (at least Chapters 2 and 6 ). Although to avoid unnecessary lengthy repetitions we sometimes presuppose the reader knows the previous material, and we make only remarks concerning additions and changes, we have in general strived to make our text as self-contained as possible, so the reader can get an idea of what we do from this text only.

In the first part of the paper we deal with categories that correspond to the multiplicative conjunction-disjunction fragment with first-order quantifiers of classical linear logic. Coherence proved for these categories extends results proved for the corresponding propositional fragment in [7] (Chapter 7). When we add the multiplicative propositional constants to the corresponding propositional categories we may obtain the linearly distributive categories of [4] (see also [1], Section 2), or categories with more equations presupposed, which coherence requires (see [7], Section 7.9).

In this first part we introduce in detail the categorial notions brought by quantifiers. The greatest novelty here may be the treatment accorded to renaming of free individual variables, which is not usually considered as a primitive rule of inference (see Sections 1.2, 1.8 and 2.2). Taking this renaming as primitive enables us to have categorial axioms that are regular, surveyable and easy to handle. We prove in the first part a categorial cut-elimination result, which says that every arrow is equal to a cut-free one. The proof of this result requires a preparation involving change of individual variables. This preparation gives a categorial form to ideas of Gentzen and Kleene.

In the second part of the paper we add negation (the only connective missing from the first part), and proceed to prove coherence following the direction of [8] (Chapter 2). In the third part, we add the mix principle, and indicate what adjustments should be made in the proofs of the previously obtained results in order to obtain coherence also in the presence of mix. The exposition in the first part, where we introduce new matters concerning quantifiers, is in general more detailed than the exposition in the second and third part, where we rely even more heavily on previously published results, and where we suppose that
the reader has already acquired some dexterity.

## 1 Coherence of QDS

### 1.1 The language $\mathcal{L}$

Let $\mathcal{P}$ be a set whose elements we call predicate letters, for which we use $P$, $R, \ldots$, sometimes with indices. To every member of $\mathcal{P}$ we assign a natural number $n \geq 0$, called its arity. For every $n \geq 0$, we assume that we have infinitely many predicate letters in $\mathcal{P}$ of arity $n$.

To build the first-order language $\mathcal{L}$ generated by $\mathcal{P}$, we assume that we have infinitely many individual variables, which we call simply variables, and for which we use $x, y, z, u, v, \ldots$, sometimes with indices. Let $\mathbf{x}_{n}$ stand for a sequence of variables of length $n \geq 0$. The atomic formulae of $\mathcal{L}$ are all of the form $P \mathbf{x}_{n}$ for $P$ a member of $\mathcal{P}$ of arity $n$. We assume throughout this paper that $\xi \in\{\wedge, \vee\}$ and $Q \in\{\forall, \exists\}$. The symbols $\wedge$ and $\vee$ are used here for the multiplicative conjunction and disjunction connectives (for which $\otimes$ and the inverted ampersand are used in [10]).

The formulae of $\mathcal{L}$ are defined inductively by the following clauses:
every atomic formula is a formula;
if $A$ and $B$ are formulae, then $(A \xi B)$ is a formula;
if $A$ is a formula and $x$ is a variable, then $Q_{x} A$ is a formula.
As usual, we will omit the outermost parentheses of formulae and take them for granted. We call $Q_{x}$ a quantifier prefix. (The advantage of the quantifier prefix $\forall_{x}$ over the more usual $\forall x$ is that in $\forall_{x} x=x$ we do not need parentheses, or a dot before $x=x$, for which a need is felt in $\forall x x=x$; in this paper, where we use the schematic letter $Q$ for quantifiers, we want the quantifier prefix $Q_{x}$ to be clearly distinguished from a formula $P x$.) For formulae we use $A, B, C, \ldots$, sometimes with indices.

The notions of free and bound occurrences of variables in a formula are understood as usual, and, as usual, we say that $x$ is free in $A$ when there is at least one free occurrence of $x$ in $A$. We say that $x$ is bound in $A$ when $Q_{x}$ occurs in $A$ (though the quantifier prefix need not bind any occurrence of $x$, as in $\forall_{x} P y$ ).

The variable $y$ is said to be free for substitution for $x$ in $A$ when $x$ is not free in a subformula of $A$ of the form $Q_{y} B$. We write $A_{y}^{x}$ for the result of uniformly substituting $y$ for the free occurrences of $x$ in $A$, provided that, as usual, $y$ is free for substitution for $x$ in $A$.

### 1.2 The category QDS

The category QDS, which we introduce in this section, corresponds to the multiplicative conjunction-disjunction fragment with first-order quantifiers of
classical linear logic. This category extends with quantifiers (this is where $\mathbf{Q}$ comes from) the propositional category DS of [7] (Section 7.6).

The objects of the category QDS are the formulae of $\mathcal{L}$. To define the arrows of QDS, we define first inductively a set of expressions called the arrow terms of QDS. Every arrow term will have a type, which is an ordered pair of formulae of $\mathcal{L}$. We write $f: A \vdash B$ when the arrow term $f$ is of type $(A, B)$. Here $A$ is the source, and $B$ the target of $f$. For arrow terms we use $f, g, h, \ldots$, sometimes with indices. Intuitively, the arrow term $f$ is the code of a derivation of the conclusion $B$ from the premise $A$ (which explains why we write $\vdash$ instead of $\rightarrow$ ).

For all formulae $A, B$ and $C$ of $\mathcal{L}$, for every variable $x$, and for all formulae $D$ of $\mathcal{L}$ in which $x$ is not free, the following primitive arrow terms:

$$
\begin{array}{rrr}
\mathbf{1}_{A}: A \vdash A, \\
\hat{b}_{A, B, C} & : A \wedge(B \wedge C) \vdash(A \wedge B) \wedge C, & \check{b}_{A, B, C}: A \vee(B \vee C) \vdash(A \vee B) \vee C, \\
\hat{b}_{A, B, C}^{\leftarrow}:(A \wedge B) \wedge C \vdash A \wedge(B \wedge C), & \check{b}_{A, B, C}^{\leftarrow}:(A \vee B) \vee C \vdash A \vee(B \vee C), \\
\hat{c}_{A, B}: A \wedge B \vdash B \wedge A, & \check{c}_{A, B}: B \vee A \vdash A \vee B, \\
d_{A, B, C}: A \wedge(B \vee C) \vdash(A \wedge B) \vee C \\
\iota_{A}^{\forall_{x}}: \forall_{x} A \vdash A, & \iota_{A}^{\exists_{x}}: A \vdash \exists_{x} A, \\
\gamma_{D}^{\forall_{x}}: D \vdash \forall_{x} D, & \gamma_{D}^{\exists_{x}}: \exists_{x} D \vdash D, \\
\hat{\theta}_{A, D}^{\forall_{x} \rightarrow}: \forall_{x}(A \vee D) \vdash \forall_{x} A \vee D, & \hat{\theta}_{A, D}^{\exists_{x} \leftarrow}: \exists_{x} A \wedge D \vdash \exists_{x}(A \wedge D)
\end{array}
$$

are arrow terms. (Intuitively, these are the axioms of our logic with the codes of their trivial derivations.)

Next we have the following inductive clauses:
if $f: A \vdash B$ and $g: B \vdash C$ are arrow terms, then $(g \circ f): A \vdash C$ is an arrow term;
if $f_{1}: A_{1} \vdash B_{1}$ and $f_{2}: A_{2} \vdash B_{2}$ are arrow terms,
then $\left(f_{1} \xi f_{2}\right): A_{1} \xi A_{2} \vdash B_{1} \xi B_{2}$ is an arrow term;
if $f: A \vdash B$ is an arrow term,
then $Q_{x} f: Q_{x} A \vdash Q_{x} B$ and $[f]_{y}^{x}: A_{y}^{x} \vdash B_{y}^{x}$ are arrow terms,
provided $A_{y}^{x}$ and $B_{y}^{x}$ are defined. (Intuitively, the operations on arrow terms $\circ, \xi, Q_{x}$ and []$_{y}^{x}$ are codes of the rules of inference of our logic.) This defines the arrow terms of QDS. As we do with formulae, we will omit the outermost parentheses of arrow terms.

The types of the arrow terms $\iota_{A}^{\forall_{x}}$ and $\gamma_{D}^{\forall_{x}}$ are related to the logical principles of universal instantiation and universal generalization respectively (this is where $\iota$ and $\gamma$ come from). The logical principle related to the type of $\iota_{A}^{\exists_{x}}$, and not of $\gamma_{D}^{\exists_{x}}$, is sometimes called existential generalization, but for the sake of duality we use $\iota$ and $\gamma$ with the existential quantifier as with the universal quantifier.

The logical principles of the types of $\check{\theta}_{A, D}^{\forall_{x} \rightarrow}$ and $\hat{\theta}_{A, D}^{\exists_{x}} \leftarrow$ are distributivity principles. The first, which is the intuitionistically spurious constant domain principle, is the converse of distribution of disjunction over universal quantification, and the second is distribution of conjunction over existential quantification. We define below arrow terms with the converse types, which are both intuitionistically valid (cf. also the end of the section).

With $Q_{x}$ and []$_{y}^{x}$ we are given infinite families of operations, indexed by variables. We call [ $]_{y}^{x}$ renaming of free variables, or for short just renaming. The operations $Q_{x}$ and $\xi$ are total, but composition $\circ$ and renaming are not total operations on arrow terms. The result $[f]_{y}^{x}$ of applying renaming []$_{y}^{x}$ to $f: A \vdash B$ is defined iff $A_{y}^{x}$ and $B_{y}^{x}$ are defined.

Note that renaming is not substitution. The arrow terms $\left[\mathbf{1}_{A}\right]_{y}^{x}: A_{y}^{x} \vdash A_{y}^{x}$ and $\mathbf{1}_{A_{y}^{x}}: A_{y}^{x} \vdash A_{y}^{x}$ are different arrow terms. Note that $[g \circ f]_{y}^{x}$ may be defined though $[f]_{y}^{x}$ and $[g]_{y}^{x}$ are not defined (for example, with $f$ being $\iota_{R x y}^{\exists_{y}}$ : $R x y \vdash \exists_{y} R x y$ and $g$ being $\iota_{\exists_{y} R x y}^{\exists_{x}}: \exists_{y} R x y \vdash \exists_{x} \exists_{y} R x y$, where $\left(\exists_{y} R x y\right)_{y}^{x}$ is not defined). Note also that $[f]_{y}^{x}$ and $[g]_{y}^{x}$ may be defined and composable without $[g \circ f]_{y}^{x}$ being defined (for example, with $f$ being $\iota_{P y}^{\forall_{y}}: \forall_{y} P y \vdash P y$ and $g$ being $\iota_{P x}^{\exists_{x}}: P x \vdash \exists_{x} P x$, where $g \circ f$ is not defined).

Renaming [ $]_{y}^{x}$ is usually implicitly considered in proof theory as a derivable rule when it is applied to $f: A \vdash B$ with $x$ not free either in $A$ or in $B$. For example, for $x$ not free in $D$, we have

$$
\frac{\frac{D \vdash B}{D \vdash \forall_{x} B}}{D \vdash B_{y}^{x}}
$$

It is also implicit in a derivable rule, which we have in the presence of implication:

$$
\frac{\frac{A \vdash B}{\vdash A \rightarrow B}}{\frac{\vdash \forall_{x}(A \rightarrow B)}{\vdash A_{y}^{x} \rightarrow B_{y}^{x}}} \frac{A_{y}^{x} \vdash B_{y}^{x}}{\vdash}
$$

We assume renaming here in the absence of implication. Renaming corresponds to a structural rule of logic, in Gentzen's terminology.

Next we define inductively the set of equations of QDS, which are expressions of the form $f=g$, where $f$ and $g$ are arrow terms of QDS of the same type. These equations hold whenever both sides are defined. For example, in the equation $(\forall \gamma$ nat) below we assume that both sides are defined, which introduces the proviso that $x$ is free neither in $A$ nor $B$. An analogous proviso
is introduced already by (cat 2) below, where we assume that $f$ and $g$, as well as $g$ and $h$, are composable. We will always assume these provisos, but we will usually not mention them explicitly. These tacit provisos are carried by the conventions of the notation for arrow terms and conditions concerning substitution of variables in formulae.

Intuitively, these equations should catch a plausible notion of identity of proofs, proofs being understood as equivalence classes of derivations. Coherence results should justify our calling this notion of equality plausible. A justification may also be provided by purely syntactical results, like cut elimination, and other similar normal-form results. The two justifications may, but need not, coincide (see [7], Chapter 1). In this paper, we concentrate on coherence results for the justification, but as a tool for demonstrating this coherence we establish cut-elimination and normal-form results. The latter results also provide a partial justification: they show the sufficiency of the assumed equations. We do not consider here (like in [6]) the question whether all these equations are also necessary for these or related syntactical results. (Such a question should first be precisely phrased.)

In the long list of axiomatic equations below, only the quantificational equations and the renaming equations at the end are new. The preceding propositional DS equations are taken from [7] and [8] (Section 2.1), while the first two categorial equations are, of course, omnipresent. We stipulate first that all the instances of $f=f$ and of the following equations are equations of QDS:

## Categorial equations:

(cat 1) $\quad f \circ \mathbf{1}_{A}=\mathbf{1}_{B} \circ f=f: A \vdash B$,
(cat 2) $\quad h \circ(g \circ f)=(h \circ g) \circ f$,
DS equations:

$$
\mathbf{1}_{A} \xi \mathbf{1}_{B}=\mathbf{1}_{A \xi B}
$$

$$
\left(g_{1} \circ f_{1}\right) \xi\left(g_{2} \circ f_{2}\right)=\left(g_{1} \xi g_{2}\right) \circ\left(f_{1} \xi f_{2}\right)
$$

for $f: A \vdash D, g: B \vdash E$ and $h: C \vdash F$,

$$
\begin{align*}
& (\stackrel{\xi}{b} \rightarrow n a t) \quad((f \xi g) \xi h) \circ \stackrel{\xi}{b} \overrightarrow{A, B, C}=\stackrel{\xi}{b} \vec{D}, E, F \cdot(f \xi(g \xi h)), \\
& (\hat{c} \text { nat }) \quad(g \wedge f) \circ \hat{c}_{A, B}=\hat{c}_{D, E^{\circ}}(f \wedge g), \\
& (\check{c} n a t) \quad(g \vee f) \circ \check{c}_{B, A}=\check{c}_{E, D^{\circ}}(f \vee g) \text {, } \\
& \text { (d nat) } \quad((f \wedge g) \vee h) \circ d_{A, B, C}=d_{D, E, F} \circ(f \wedge(g \vee h)) \text {, } \\
& \stackrel{\xi}{b} \overrightarrow{A, B, C} \text { 。 } \circ \stackrel{\xi}{b_{A, B, C}}=\mathbf{1}_{(A \xi B) \xi C}, \quad \stackrel{\xi}{b_{A, B, C}} \circ \stackrel{\xi}{b} \vec{A}, B, C=10 \mathbf{1}_{A \xi(B \xi C)}, \tag{array}
\end{align*}
$$

$(\hat{c} \hat{c}) \quad \hat{c}_{B, A} \circ \hat{c}_{A, B}=\mathbf{1}_{A \wedge B}$,
( $\check{c} \check{c}) \quad \check{c}_{A, B} \circ \check{c}_{B, A}=\mathbf{1}_{A \vee B}$,
(b̂c) $\quad\left(\mathbf{1}_{B} \wedge \hat{c}_{C, A}\right) \circ \hat{b}_{B, C, A}{ }^{\circ} \hat{c}_{A, B \wedge C} \circ \hat{b}_{A, B, C}^{\overleftarrow{c}} \circ\left(\hat{c}_{B, A} \wedge \mathbf{1}_{C}\right)=\hat{b}_{B, A, C}^{\overleftarrow{c}}$,
$(\breve{b} \check{c}) \quad\left(\mathbf{1}_{B} \vee \check{c}_{A, C}\right) \circ \check{b}_{B, C, A}^{\leftarrow} \circ \check{c}_{B \vee C, A} \circ \check{b}_{A, B, C} \overleftarrow{C}^{\circ}\left(\check{c}_{A, B} \vee \mathbf{1}_{C}\right)=\check{b}_{B, A, C}^{\overleftarrow{ }}$,
$(d \wedge) \quad\left(\hat{b_{A, B, C}^{\overleftarrow{ }}} \vee^{\overleftarrow{ }} \mathbf{1}_{D}\right) \circ d_{A \wedge B, C, D}=d_{A, B \wedge C, D} \circ\left(\mathbf{1}_{A} \wedge d_{B, C, D}\right) \circ \hat{b}_{A, B, C \vee D}^{\overleftarrow{,}}$,
$(d \vee) \quad d_{D, C, B \vee A} \circ\left(\mathbf{1}_{D} \wedge \check{b}_{C, B, A}\right)=\check{b} \overleftarrow{D \wedge C, B, A}{ }^{\circ}\left(d_{D, C, B} \vee \mathbf{1}_{A}\right) \circ d_{D, C \vee B, A}$,
for $d_{C, B, A}^{R}={ }_{d f} \check{c}_{C, B \wedge A} \circ\left(\hat{c}_{A, B} \vee \mathbf{1}_{C}\right) \circ d_{A, B, C} \circ\left(\mathbf{1}_{A} \wedge \check{c}_{B, C}\right) \circ \hat{c}_{C \vee B, A}$ : $(C \vee B) \wedge A \vdash C \vee(B \wedge A)$,
(dî) $\quad d_{A \wedge B, C, D}^{R} \circ\left(d_{A, B, C} \wedge \mathbf{1}_{D}\right)=d_{A, B, C \wedge D} \circ\left(\mathbf{1}_{A} \wedge d_{B, C, D}^{R}\right) \circ \hat{b}_{A, B \vee C, D}^{\leftarrow}$,
(d $d$ ) $\quad\left(\mathbf{1}_{D} \vee d_{C, B, A}\right) \circ d_{D, C, B \vee A}^{R}=\check{b}_{D, C \wedge B, A^{\circ}}^{\overleftarrow{ }}\left(d_{D, C, B}^{R} \vee \mathbf{1}_{A}\right) \circ d_{D \vee C, B, A}$,

## Quantificational equations:

(Q1) $\quad Q_{x} \mathbf{1}_{A}=\mathbf{1}_{Q_{x} A}$,
(Q2) $\quad Q_{x}(g \circ f)=Q_{x} g \circ Q_{x} f$,
for $f: A \vdash B$,

| ( $\forall \iota$ nat) | $f \circ \iota_{A}^{\forall_{x}}=\iota_{B}^{\forall_{s}} \circ \forall_{x} f$, | $(\exists \iota n a t)$ | $\exists_{x} f \circ \iota_{A}^{\exists_{x}}=\iota_{B}^{\exists_{x}} \circ f$, |
| :---: | :---: | :---: | :---: |
| $(\forall \gamma n a t)$ | $\forall_{x} f \circ \gamma_{A}^{\gamma_{x}}=\gamma_{B}^{\forall_{x}} \circ f$, | $(\exists \gamma n a t)$ | $f \circ \gamma_{A}^{\exists x}=\gamma_{B}^{\exists x} \circ \exists_{x} f$, |
| $(\forall \beta)$ | $\iota_{A}^{\forall_{A}} \circ \gamma_{A}^{\forall_{x}}=1_{A}$, | $(\exists \beta)$ | $\gamma_{A}^{\exists x} \circ \iota_{A}^{\exists x}=\mathbf{1}_{A}$, |
| $(\forall \eta)$ | $\left.\forall_{x}\right\rangle_{A}^{\forall_{A}} \circ \gamma_{\forall_{x} A}^{\forall_{x}}=\mathbf{1}_{\forall_{x} A}$, | $(\exists \eta)$ |  |

for $\ddot{\theta}_{A, D}^{\forall_{x} \leftarrow}={ }_{d f} \forall_{x}\left(\iota_{A}^{\forall_{x}} \vee \mathbf{1}_{D}\right) \circ \gamma_{\forall_{x}}^{\forall_{x}} A \vee D: \forall_{x} A \vee D \vdash \forall_{x}(A \vee D)$,
$\hat{\theta}_{A, D}^{\exists \exists_{x}}={ }_{d f} \gamma_{\exists_{x} A \wedge D}^{\exists_{x}}{ }^{\circ} \exists_{x}\left(\iota_{A}^{\exists_{x}} \wedge \mathbf{1}_{D}\right): \exists_{x}(A \wedge D) \vdash \exists_{x} A \wedge D$ and
$(\xi, Q) \in\{(\mathrm{V}, \forall),(\wedge, \exists)\}$,


## Renaming equations:

for $x, y, z$ and $v$ mutually different variables and
$\alpha_{A_{1}, \ldots, A_{n}}$ a primitive arrow term except $\iota_{A}^{Q_{x}}$,
$($ ren $\alpha) \quad\left[\alpha_{A_{1}, \ldots, A_{n}}\right]_{y}^{x}=\alpha_{\left(A_{1}\right)_{y}^{x}, \ldots,\left(A_{n}\right)_{y}^{x}}$,
(ren $\circ$ ) $[g \circ f]_{y}^{x}=[g]_{y}^{x} \circ[f]_{y}^{x}$,
(ren $\xi$ ) $\left[f_{1} \xi f_{2}\right]_{y}^{x}=\left[f_{1}\right]_{y}^{x} \xi\left[f_{2}\right]_{y}^{x}$,
$($ ren $Q) \quad\left[Q_{z} f\right]_{y}^{x}=Q_{z}[f]_{y}^{x}$,
(ren 1) $[f]_{x}^{x}=f$,
(ren 2) $[f]_{y}^{x}=f$, if $x$ is free neither in the source nor in the target of $f$,
(ren 3) $\quad\left[[f]_{v}^{z}\right]_{y}^{x}=\left[[f]_{y}^{x}\right]_{v}^{z}$,
(ren 4) $\left[[f]_{y}^{z}\right]_{y}^{x}=\left[[f]_{y}^{x}\right]_{y}^{z}$,
(ren 5) $\quad\left[[f]_{x}^{z}\right]_{y}^{x}=\left[[f]_{y}^{z}\right]_{y}^{x}$,
(ren 6) $\quad\left[[f]_{x}^{y}\right]_{y}^{x}=[f]_{y}^{x}$.
This concludes the list of axiomatic equations stipulated for QDS. To define all the equations of QDS it remains only to say that the set of these equations is closed under symmetry and transitivity of equality and under the rules

$$
\begin{gathered}
(\circ \text { cong }) \frac{f=f^{\prime} \quad g=g^{\prime}}{g \circ f=g^{\prime} \circ f^{\prime}} \quad(\xi \text { cong }) \frac{f_{1}=f_{1}^{\prime} \quad f_{2}=f_{2}^{\prime}}{f_{1} \xi f_{2}=f_{1}^{\prime} \xi f_{2}^{\prime}} \\
(Q \text { cong }) \frac{f=f^{\prime}}{Q_{x} f=Q_{x} f^{\prime}} \\
(\text { ren cong }) \\
\frac{f=f^{\prime}}{[f]_{y}^{x}=\left[f^{\prime}\right]_{y}^{x}}
\end{gathered}
$$

On the arrow terms of QDS we impose the equations of QDS. This means that an arrow of QDS is an equivalence class of arrow terms of QDS defined with respect to the smallest equivalence relation such that the equations of QDS are satisfied (see [7], Section 2.3, for details).

The equations $(\xi 1),(\xi 2),(Q 1)$ and $(Q 2)$ are called functorial equations. They say that $\xi$ is a biendofunctor and $Q_{x}$ an endofunctor of QDS (i.e. 2-endofunctor and 1-endofunctor respectively, in the terminology of [7], Section 2.4). The equations with "nat" in their names are called naturality equations. The naturality equations above say that $\stackrel{\xi}{b}, c_{c}^{\epsilon}, d$ and $\iota^{Q_{x}}$ are natural transformations $\left(b^{\leftarrow}\right.$ is a natural transformation too, due to $(\hat{\xi} \xi \bar{b})$ ). The naturality equation ( $Q \gamma$ nat) says that $\gamma^{Q_{x}}$ has some properties of a natural transformation, but one side of ( $Q \gamma n a t$ ) is not always defined when the other is. We will see later (in Section 1.4) where $\gamma^{Q_{x}}$ gives rise to a natural transformation. As $\gamma^{Q_{x}}$, so $\check{\theta}^{\forall_{x} \rightarrow}$ and $\hat{\theta}^{\exists}{ }^{x} \leftarrow$ have some properties of natural transformations due to the equations


In spite of the equations (ren $\mathbf{1}$ ) and (ren $\circ$ ), which are also functorial equations, renaming combined with substitution applied to formulae does not give an endofunctor of QDS, because we do not have totally defined functions. If the renaming operations were not assumed as primitive operations on arrow terms for defining QDS (we will see in Sections 1.8 and 2.2 that $\left[\iota_{A}^{Q_{x}}\right]_{y}^{x}$ could
be assume instead), then we would have problems in formulating assumptions that give the following equation of QDS:

$$
[f]_{y}^{x} \circ\left[\iota_{A}^{\forall_{x}}\right]_{y}^{x}=\left[\iota_{B}^{\forall_{x}}\right]_{y}^{x} \circ \forall_{x} f
$$

which follows from ( $\forall \iota$ nat), (ren cong) and (ren $\circ$ ); we would not know what to write for $[f]_{y}^{x}$.

The following equation:

$$
(\forall \gamma \iota) \quad \gamma_{A}^{\forall_{x} \circ \iota_{A}^{\forall_{x}}=\mathbf{1}_{\forall_{x} A}, ~}
$$

holds in QDS. (By our tacit presupposition, introduced before presenting the equations of QDS, the variable $x$ is here not free in $A$.) This equation is derived as follows:

$$
\begin{aligned}
\gamma_{A}^{\forall_{x}} \circ \iota_{A}^{\forall_{x}} & =\forall_{x} \iota_{A}^{\forall_{x}} \circ \gamma_{\forall_{x} A}^{\forall_{x}}, & & \text { by }(\forall \gamma n a t), \\
& =\mathbf{1}_{\forall_{x} A}, & & \text { by }(\forall \eta) .
\end{aligned}
$$

In an analogous manner, we derive in QDS the equation:

$$
(\exists \gamma \iota) \quad \iota_{A}^{\exists_{x}} \circ \gamma_{A}^{\exists_{x}}=\mathbf{1}_{\exists_{x} A} .
$$

With the help of the equations ( $Q \gamma \iota$ ) we derive easily the following equations analogous to ( $Q 1$ ):

$$
\begin{array}{ll}
(Q \iota) & Q_{x} \iota_{A}^{Q_{x}}=\iota_{Q_{x} A}^{Q_{x}} \\
(Q \gamma) & Q_{x} \gamma_{A}^{Q_{x}}=\gamma_{Q_{x} A}^{Q_{x}} .
\end{array}
$$

Note that $(Q \iota)$ can replace $(Q \eta)$ for axiomatizing the equations of QDS, but $(Q \gamma)$ cannot do so, because for it we presuppose that $x$ is not free in $A$, which we do not presuppose for $(Q \eta)$.

Note that $(\forall \eta)$ and $(\exists \eta)$ could be replaced respectively by the equations

$$
\begin{array}{ll}
(\forall \text { ext }) & \forall x_{x}\left(\iota_{A}^{\left.\forall_{x} \circ f\right) \circ \gamma_{B}^{\forall_{x}}=f: B \vdash \forall_{x} A,}\right. \\
(\exists \text { ext }) & \gamma_{B}^{\exists x} \circ \exists_{x}\left(g \circ \iota_{A}^{\exists_{x}}\right)=g: \exists_{x} A \vdash B,
\end{array}
$$

which are easily derived with $(Q \gamma n a t)$ and $(Q \eta)$. (In both of these equations we tacitly presuppose that $x$ is not free in $B$.)

By relying on $(Q \beta)$ and ( $Q \gamma \iota$ ) we can easily derive the following equations if $x$ is free neither in $A$ nor in $D$ :

$$
\begin{aligned}
& \check{\theta}_{A, D}^{\forall_{x} \rightarrow}=\left(\gamma_{A}^{\forall_{x}} \vee \mathbf{1}_{D}\right) \circ \iota_{A \vee D}^{\forall_{x}}, \\
& \hat{\theta}_{A, D}^{\exists_{x} \overleftarrow{D}}=\iota_{A \wedge D}^{\exists_{x}} \circ\left(\gamma_{A}^{\exists_{x}} \wedge \mathbf{1}_{D}\right) .
\end{aligned}
$$

Note that if $x$ is free in $A$ and not free in $D$, then in QDS we do not have arrows of the types converse to the types of the following distributivity arrows:

$$
\begin{aligned}
& \hat{\theta}_{A, D}^{\forall_{x} \leftarrow}={ }_{d f} \forall_{x}\left(\iota_{A}^{\forall_{x}} \wedge \mathbf{1}_{D}\right) \circ \gamma_{\forall_{x} A \wedge D}^{\forall_{x}}: \forall_{x} A \wedge D \vdash \forall_{x}(A \wedge D), \\
& \hat{\theta}_{A, D}^{\exists_{x} \rightarrow}={ }_{d f} \gamma_{\exists_{x} A \vee D}^{\exists_{x}} \exists_{x}\left(\iota_{A}^{\exists_{x}} \vee \mathbf{1}_{D}\right): \exists_{x}(A \vee D) \vdash \exists_{x} A \vee D,
\end{aligned}
$$

which are analogous to the arrows $\overleftarrow{\theta}_{A, D}^{\forall_{x} \leftarrow}$ and $\hat{\theta}_{A, D}^{\exists x \rightarrow}$ respectively. (That these arrows do not exist in QDS is shown via cut elimination in GQDS; see Sections 1.5-9.) So we cannot have a prenex normal form for formulae, i.e. objects.

### 1.3 Change of bound variables

We call change of bound variables what could as well be called renaming of bound variables, because we do not want to confuse this renaming with renaming of free variables. We define in QDS the following arrows, which formalize change of bound variables ( $\tau$ might come from "transcribe"):

$$
\begin{aligned}
& \tau_{A, u, v}^{\forall_{x}}={ }_{d f} \forall_{v}\left[\iota_{A_{u}^{x}}^{\forall_{u}}\right]_{v}^{u} \circ \gamma_{\forall_{u} A_{u}^{x}}^{\forall_{v}^{x}}: \forall_{u} A_{u}^{x} \vdash \forall_{v} A_{v}^{x}, \\
& \tau_{A, v, u}^{\exists_{x}}={ }_{d f} \gamma_{\exists_{u} A_{u}^{x}}^{\exists_{v}} \circ \exists_{v}\left[\iota_{A_{u}^{x}}^{\exists_{u}^{x}}\right]_{v}^{u}: \exists_{v} A_{v}^{x} \vdash \exists_{u} A_{u}^{x},
\end{aligned}
$$

provided $u$ and $v$ are not free in $A$.
Note that $\tau_{A, u, v}^{Q_{x}}$ is the same arrow term as $\tau_{A_{y}^{x}, u, v}^{Q_{y}}$ for $y$ not free in $A$, and a fortiori for $y$ neither free nor bound in $A$. The variable $x$ in $\tau_{A, u, v}^{Q_{x}}$ is just a place holder, which can always be replaced by an arbitrary new variable.

We can derive the following equations of QDS:
$(Q \tau \operatorname{ren}) \quad\left[\tau_{A, u, v}^{Q_{x}}\right]_{z}^{y}=\tau_{A_{z}^{y}, u, v}^{Q_{x}}$, if $x$ is not $y$ or $z$
(if $y$ is free in $A$ and $z$ is $u$ or $v$, then the right-hand side of ( $Q \tau$ ren) is undefined),

$$
\begin{array}{ll}
(Q \tau \text { nat }) & Q_{v}[f]_{v}^{x} \circ \tau_{A, u, v}^{Q_{x}}=\tau_{B, u, v}^{Q_{x}} \circ Q_{u}[f]_{u}^{x}, \\
(Q \tau \text { ref }) & \tau_{A, u, u}^{Q_{x}}=\mathbf{1}_{Q_{u} A_{u}^{x}}, \\
(Q \tau \text { sym }) & \tau_{A, v, u}^{Q_{x}} \circ \tau_{A, u, v}^{Q_{x}}=\mathbf{1}_{Q_{u} A_{u}^{x}}, \\
(Q \tau \text { trans }) & \tau_{A, v, w}^{Q_{x}} \circ \tau_{A, u, v}^{Q_{x}}=\tau_{A, u, w}^{Q_{x}} .
\end{array}
$$

From the equation $\left(Q \tau\right.$ sym) we see that $\tau_{A, u, v}^{Q_{x}}$ and $\tau_{A, v, u}^{Q_{x}}$ are inverse to each other. We can also derive the following equations of QDS:

$$
\begin{align*}
& \iota_{A_{v}^{x}}^{\forall_{v}^{x}} \circ \tau_{A, u, v}^{\forall_{x}}=\left[\forall_{A_{u}^{x}}^{\forall_{u}^{x}}\right]_{v}^{u}, \\
& \tau_{A, v, u}^{\exists{ }^{\exists}} \circ \iota_{A_{v}^{x}}^{\exists_{v}}=\left[\iota_{A_{u}^{x}}^{\exists_{u}^{x}}\right]_{v}^{u}, \\
& \tau_{A, u, v}^{\forall_{x}} \circ \gamma_{A}^{\forall_{u}}=\gamma_{A}^{\forall_{v}} \text {, } \\
& \gamma_{A}^{\exists u} \circ \tau_{A, v, u}^{\exists x}=\gamma_{A}^{\exists v}, \\
& \left(\tau_{A, u, v}^{\forall_{x}} \vee \mathbf{1}_{D}\right) \circ \ddot{\theta}_{A_{u}^{x}, D}^{\forall_{u} \rightarrow}=\check{\theta}_{A_{v}^{x}, D}^{\forall_{v} \rightarrow} \circ \tau_{A \vee D, u, v}^{\forall_{x}}, \\
& \tau_{A \wedge D, u, v}^{\exists x} \circ \hat{\theta}_{A_{u}^{x}, D}^{\exists_{u} \leftarrow}=\hat{\theta}_{A_{v}^{x}, D}^{\exists_{v} \leftarrow} \circ\left(\tau_{A, u, v}^{\exists x} \wedge \mathbf{1}_{D}\right) .
\end{align*}
$$

To derive $(\forall \tau \check{\theta})$ we derive

$$
\check{\theta}_{A_{v}^{x, D}}^{\forall_{v} \leftarrow} \circ\left(\tau_{A, u, v}^{\forall_{x}} \vee \mathbf{1}_{D}\right)=\tau_{A \vee D, u, v}^{\forall_{x}} \circ \check{\theta}_{A_{u}^{x}, D}^{\forall_{u} \leftarrow}
$$

with the help of $(\forall \tau \iota)$ and $(\forall \tau \gamma)$. We proceed analogously for $(\exists \tau \hat{\theta})$.
Note that $\tau^{\forall_{x}}$ and $\tilde{\theta}^{\forall_{x} \leftarrow}$, as well as $\tau^{\exists_{x}}$ and $\hat{\theta}^{\exists_{x}} \rightarrow$, have analogous definitions. The following two equations of QDS are analogous to the equations ( $Q \tau \iota$ ):

$$
\begin{aligned}
& \iota_{A \vee D}^{\forall x} \circ \check{\theta}_{A, D}^{\forall_{x} \leftarrow}=\iota_{A}^{\forall} \vee \mathbf{1}_{D}, \\
& \hat{\theta}_{A, D}^{\exists x \rightarrow} \circ \iota_{A \wedge D}^{\exists_{x}}=\iota_{A}^{\exists_{x}} \wedge \mathbf{1}_{D} .
\end{aligned}
$$

As a consequence of these two equations we have

$$
\begin{align*}
& \left(\iota_{A}^{\forall_{x}} \vee \mathbf{1}_{D}\right) \circ \tilde{\theta}_{A, D}^{\forall_{x} \rightarrow}=\iota_{A \vee D}^{\forall_{x}}, \\
& \hat{\theta}_{A, D}^{\exists_{x} \leftarrow} \circ\left(\iota_{A}^{\exists_{x}} \wedge \mathbf{1}_{D}\right)=\iota_{A \wedge D}^{\exists_{x}}
\end{align*}
$$

### 1.4 Quantifiers and adjunction

Lawvere's presentation of predicate logic in categorial terms (see [15], [16] and [17], Appendix A.1), and presentations that follow him more or less closely (see, for instance, [21], [5], [19] and [11], Chapter 4), are less syntactical than ours. They do not pay close attention to syntax. If this syntax were to be supplied precisely, then a language without variables, like Quine's variable-free language for predicate logic (see [20], and references therein), called predicate functor logic, would be more appropriate. Our first-order language is on the contrary quite standard. It should be mentioned also that Lawvere's approach is more general, whereas we concentrate on first-order logic.

Lawvere characterized quantifiers in intuitionistic logic through an adjoint situation. In Lawvere's characterization of quantifiers, functors from which the universal and existential quantifiers arise are respectively the right and left adjoints of a functor that is an instance of a functor Lawvere calls substitution. This instance of substitution is a kind of inverse of projection. This sort of substitution is not to be confused with substitution of individual variables as usually conceived in first-order predicate logic, where free individual variables are uniformly replaced by individual terms.

We will now present the adjoint situations involving the quantifiers of QDS. The gist of these adjunctions may be derived from Lawvere's ideas, but our approach is more syntactical. In this syntactical approach substitution is not mentioned. (What we call renaming plays no role in it.)

Let QDS $^{-x}$ be the full subcategory of QDS whose objects are all formulae of $\mathcal{L}$ in which $x$ is not free. From QDS $^{-x}$ to QDS there is an obvious inclusion functor, which we call $E$. (It behaves like identity on objects and on arrows.) The functor $E$ is full and faithful. By restricting the codomain of the functors $Q_{x}$ from QDS to QDS we obtain the functors $Q_{x}$ from $\mathbf{Q D S}$ to $\mathbf{Q D S}^{-x}$. Then
the functor $\forall_{x}$ is right adjoint to $E$, and $\exists_{x}$ is left adjoint to $E$. Consider first the adjunction involving $\forall_{x}$ and $E$. In this adjunction, the arrows $\iota^{\forall_{x}}$ make the counit and the arrows $\gamma^{\forall_{x}}$ the unit natural transformation, while $(\forall \beta)$ and $(\forall \eta)$ are the triangular equations of this adjunction. In the adjunction involving $\exists_{x}$ and $E$, the arrows $\iota^{\exists_{x}}$ make the unit and the arrow $\gamma^{\exists_{x}}$ the counit natural transformation, while $(\exists \beta)$ and $(\exists \eta)$ are the triangular equations. In other words, the full subcategory QDS $^{-x}$ of QDS is both coreflective and reflective in QDS. The equations ( $Q \gamma \iota$ ) of Section 1.2 follow from Theorem 1 and its dual in [18] (Section IV.3). From these theorems we also obtain that $\forall_{x} A$ and $A$ are isomorphic for every object $A$ of $\mathbf{Q D S}^{-x}$.

Note that these two adjunctions, due to the presence of the equations ( $Q \gamma \iota$ ), or $(Q \iota)$, or $(Q \gamma)$, of Section 1.2, are trivial adjunctions in the following sense. If $f$ and $g$ of the same type are arrow terms of QDS made only of $\mathbf{1}, \circ, Q_{x}, \iota^{Q_{x}}$ and $\gamma^{Q_{x}}$, then $f=g$ in QDS (see [6], Sections 4.6.2 and 4.11). Whenever a full subcategory of a category $\mathcal{C}$ is coreflective or reflective in $\mathcal{C}$, we have a trivial adjunction in the same sense.

Note that the adjunctions involving $E$ and $Q_{x}$ do not deliver the distributivity arrows $\check{\theta}^{\forall x} \rightarrow$ and $\hat{\theta}^{\exists_{x}} \leftarrow$. Lawvere was able to define $\hat{\theta}^{\exists x} \leftarrow$ in the presence of intuitionistic implication, while the constant domain arrows $\check{\theta}^{\forall} \rightarrow$ are not present, and not desired in intuitionistic logic. In an analogous way, the adjunctions of product and coproduct with the diagonal functor do not deliver distributivity isomorphisms of coproduct over product and of product over coproduct in categories that are both cartesian and cocartesian. In cartesian closed categories, where we have the exponential functor, from which intuitionistic implication arises, we obtain distributivity isomorphisms of product over coproduct, but distributivity isomorphisms of coproduct over product are missing.

So Lawvere's thesis that logical constants are characterized completely by adjoint situations should be taken with a grain of salt. Conjunction, which corresponds to product, is characterized by right-adjointness to the diagonal functor when it is alone, or when it is accompanied by intuitionistic implication. When conjunction and disjunction, which corresponds to coproduct, are alone, then the two adjunctions with the diagonal functor do not suffice. Some distributivity arrows, which we would have in the presence of implication, are missing. The situation is analogous with quantifiers and the distributivity arrows $\ddot{\theta}^{\forall x} \rightarrow$ and $\hat{\theta}^{\exists x} \leftarrow$ in intuitionistic logic.

The situation is different in classical logic, where duality reigns. Both of the distributivity arrows $\check{\theta}^{\forall} \rightarrow$ and $\hat{\theta}^{\exists x} \leftarrow$ are definable in the presence of negation (see Section 2.7; negation yields implication and "coimplication"). Both, when defined, happen to be isomorphisms in this paper, and should be such in classical logic, but neither the distribution of disjunction over conjunction nor the distribution of conjunction over disjunction should be isomorphisms in classical logic, as we argued in [7].

### 1.5 The category GQDS

In this section we enlarge the results of Section 7.7 of [7], on which our exposition will heavily rely. We introduce a category called GQDS, which extends with quantifiers the category GDS of [7]. In GQDS we will be able to perform in a manageable manner the Gentzenization of QDS (this is where G comes from).

Let a formula of $\mathcal{L}$ be called diversified when every predicate letter occurs in it at most once. A type $A \vdash B$ is called diversified when $A$ and $B$ are diversified, and an arrow term is diversified when its type is diversified.

It is easy to verify that for every arrow $f: A \vdash B$ of QDS there is a diversified arrow term $f^{\prime}: A^{\prime} \vdash B^{\prime}$ of QDS such that $f$ is obtained by substituting uniformly predicate letters for some predicate letters in $f^{\prime}: A^{\prime} \vdash B^{\prime}$. Namely, $f$ is a letter-for-letter substitution instance of $f^{\prime}$ (cf. [7], Sections 3.3 and 7.6).

Our aim is to show that QDS is a diversified preorder, which means that if $f_{1}, f_{2}: A \vdash B$ are diversified arrow terms, then $f_{1}=f_{2}$ in QDS. For that purpose we introduce an auxiliary category GQDS where the $\stackrel{\xi}{b}^{\rightarrow}$ arrows, $\stackrel{\xi}{b}^{\leftarrow}$ arrows and $\stackrel{\xi}{c}$ arrows are identity arrows. We will prove that GQDS is a preorder, which means that for all arrow terms $f_{1}$ and $f_{2}$ of the same type we have $f_{1}=f_{2}$ in GQDS. That GQDS is a preorder will imply that QDS is a diversified preorder.

From the fact that QDS is a diversified preorder one can infer that there is a faithful functor $G$ from QDS to the category Rel, whose objects are finite ordinals and whose arrows are relations between these ordinals (see [7], Sections 2.9 and 7.6). This functor $G$ is defined as for DS in [7] with the understanding that predicate letters now stand for propositional letters; we have moreover that $G Q_{x} A=G A$ (so that $G A$ is the number of occurrences of predicate letters in the formula $A$ ), the arrow $G \alpha$ for $\alpha$ being $\iota_{A}^{Q_{x}}, \gamma_{A}^{Q_{x}}, \breve{\theta}_{A, D}^{\forall_{x} \rightarrow}$ and $\hat{\theta}_{A, D}^{\exists x \leftarrow}$ is an identity arrow, while $G Q_{x} f=G[f]_{y}^{x}=G f$. The theorem that $G$ is a faithful functor is called QDS Coherence.

Let $\mathcal{L}^{d i v}$ be the set of diversified formulae of $\mathcal{L}$. Consider the smallest equivalence relation $\equiv$ on $\mathcal{L}^{d i v}$ that satisfies

$$
\begin{aligned}
& A \xi(B \xi C) \equiv(A \xi B) \xi C \\
& A \xi B \equiv B \xi A \\
& \text { if } A_{1} \equiv B_{1} \text { and } A_{2} \equiv B_{2}, \text { then } A_{1} \xi A_{2} \equiv B_{1} \xi B_{2} \\
& \text { if } A \equiv B, \text { then } Q_{x} A \equiv Q_{x} B
\end{aligned}
$$

and let $[A]$ be the equivalence class of a diversified formula $A$ with respect to this equivalence relation. We call $[A]$ a form set (which follows the terminology of [7], Section 7.7). We use $X, Y, Z, \ldots$, sometimes with indices, for form sets. It is clear that the form set $[A]$ can be named by any of the members of the equivalence class $[A]$. In these names we may delete parentheses tied to $\xi$ in the immediate scope of $\xi$. A subform set of a form set $X$ is a form set $[A]$ for $A$ a subformula of a formula in $X$.

Let the objects of the category GQDS be the form sets we have just introduced. The arrow terms of GQDS are defined as the arrow terms of QDS save that their indices are form sets instead of formulae. The equations of GQDS are defined as those of QDS save that we add the equations

$$
\begin{aligned}
& \stackrel{\xi}{b}_{X, Y, Z}=\stackrel{\xi}{b}_{X, Y, Z}=\mathbf{1}_{X \xi Y \xi Z} \\
& \stackrel{\xi}{c}_{X, Y}=\mathbf{1}_{X \xi Y}
\end{aligned}
$$

This defines the category GQDS. From the fact that GQDS is a preorder we infer that QDS is a diversified preorder as in [7] (Sections 3.3, 7.6, beginning of 7.7 and end of 7.8).

We define by induction a set of terms for the arrows of GQDS, which we call Gentzen terms. They are defined as in [7] (Section 7.7), save that to the Gentzen operations cut $X_{X}, \wedge_{X_{1}, X_{2}}$ and $\vee_{X_{1}, X_{2}}$ we add the following Gentzen operations, where $={ }_{d n}$ is read "denotes":

$$
\begin{aligned}
& \frac{f: X_{y}^{x} \wedge Z \vdash U}{\forall_{x, X}^{L} f={ }_{d n} f \circ\left(\left[\stackrel{\rightharpoonup}{X}_{X}^{\forall_{X}}\right]_{y}^{x} \wedge \mathbf{1}_{Z}\right): \forall_{x} X \wedge Z \vdash U} \\
& \frac{f: X_{y}^{x} \vdash U}{\forall_{x, X}^{L} f={ }_{d n} f \circ\left[\iota_{X}^{\forall_{X}}\right]_{y}^{x}: \forall_{x} X \vdash U} \\
& \frac{f: U \vdash X_{u}^{x} \vee Z}{\forall_{x, X}^{R} f={ }_{d n}\left(\tau_{X_{v}^{x}, u, x}^{\forall_{v}} \vee \mathbf{1}_{Z}\right) \circ \check{\theta}_{X_{u}^{x}, Z}^{\forall_{u} \rightarrow} \circ \forall_{u} f \circ \gamma_{U}^{\forall_{u}}: U \vdash \forall_{x} X \vee Z} \\
& \frac{f: U \vdash X_{u}^{x}}{\forall_{x, X}^{R} f={ }_{d n} \tau_{X_{v}^{x}, u, x}^{\forall_{v}} \circ \forall_{u} f \circ \gamma_{U}^{\forall_{u}}: U \vdash \forall_{x} X} \\
& \frac{f: U \vdash X_{y}^{x} \vee Z}{\exists_{x, X}^{R} f={ }_{d n}\left(\left[\iota_{X}^{\exists}\right]_{y}^{x} \vee \mathbf{1}_{Z}\right) \circ f: U \vdash \exists_{x} X \vee Z} \\
& \frac{f: U \vdash X_{y}^{x}}{\exists_{x, X}^{R} f={ }_{d n}\left[\iota_{X}^{\exists_{x}}\right]_{y}^{x} \circ f: U \vdash \exists_{x} X} \\
& \frac{f: X_{u}^{x} \wedge Z \vdash U}{\exists_{x, X}^{L} f={ }_{d n} \gamma_{U}^{\exists_{u}} \circ \exists_{u} f \circ \hat{\theta}_{X_{u}^{x}, Z}^{\exists_{u} \leftarrow} \circ\left(\tau_{X_{v}^{x}, x, u}^{\exists_{v}} \wedge \mathbf{1}_{Z}\right): \exists_{x} X \wedge Z \vdash U} \\
& \frac{f: X_{u}^{x} \vdash U}{\exists_{x, X}^{L} f={ }_{d n} \gamma_{U}^{\exists_{u}} \circ \exists_{u} f \circ \tau_{X_{v}^{x}, x, u}^{\exists_{v}}: \exists_{x} X \vdash U}
\end{aligned}
$$

$$
\frac{f: X \vdash Y}{[f]_{y}^{x}: X_{y}^{x} \vdash Y_{y}^{x}}
$$

The usual proviso for the eigenvariable in connection with $\forall^{R}$ and $\exists^{L}$ is imposed by the tacit provisos concerning $\gamma_{U}^{Q_{u}}, \check{\theta}_{X_{u}^{x}, Z}^{\forall_{u} \rightarrow}, \hat{\theta}_{X_{u}^{x}, Z}^{\exists \exists^{x} \leftarrow}, \tau_{X_{v}^{x}, u, x}^{\forall_{v}}$ and $\tau_{X_{v}^{x}, x, u}^{\exists v}$. This proviso says that $u$, which is called the eigenvariable, is not free in the types of $\forall_{x, X}^{R} f$ and $\exists_{x, X}^{L} f$; i.e., $u$ is free neither in the sources nor in the targets.

The types of all the subterms of a Gentzen term make a derivation tree usual in Gentzen systems. (An example may be found in Section 1.10.)

It is easy to show that every arrow of GQDS is denoted by a Gentzen term. For that we rely on the Gentzenization Lemma of Section 7.7 of [7], together with the following equations of GQDS:

$$
\begin{aligned}
\iota_{X}^{\forall_{x}} & =\forall_{x, X}^{L} \mathbf{1}_{X} \\
\gamma_{U}^{\forall_{x}} & =\forall_{x, U}^{R} \mathbf{1}_{U} \\
\forall_{x} f & =\forall_{x, Y}^{R} \forall_{x, X}^{L} f, \text { for } f: X \vdash Y,
\end{aligned}
$$

and the dual equations involving $\exists$ instead of $\forall$.
With the help of the equations $(\forall \tau \iota)$ (together with renaming), $(\forall \check{\theta} \iota)$, ( $\forall \iota$ nat) and $(\forall \beta)$ (see Sections 1.2-3) we derive the following equations of GQDS:

$$
\begin{array}{ll}
(\forall \beta \text { red }) \quad & \left(\left[\iota_{X}^{\forall_{x}}\right]_{u}^{x} \vee \mathbf{1}_{Z}\right) \circ \forall_{x, X}^{R} f=f: U \vdash X_{u}^{x} \vee Z, \\
& {\left[\iota_{X}^{\forall_{x}}\right]_{u}^{x} \circ \forall_{x, X}^{R} f=f: U \vdash X_{u}^{x} .}
\end{array}
$$

With the help of the naturality of $\check{\theta}^{\forall} \rightarrow,(\forall \tau$ nat $),($ ren 1$),(\forall \iota),(\forall \tau \iota),(\forall \check{\theta} \iota)$, ( $\forall \iota$ nat) and $(\forall \beta)$ (see Sections $1.2-3$ ) we derive the following equations of GQDS:

$$
\begin{array}{ll}
(\forall \eta \text { red }) & \forall_{x, X}^{R}\left(\left(\left[\iota_{X}^{\forall_{x}}\right]_{u}^{x} \vee \mathbf{1}_{Z}\right) \circ g\right)=g: U \vdash \forall_{x} X \vee Z, \\
& \forall_{x, X}^{R}\left(\left[\iota_{X}^{\forall_{x}}\right]_{u}^{x} \circ g\right)=g: U \vdash \forall_{x} X .
\end{array}
$$

We derive analogously the dual equation of GQDS involving $\exists$ instead of $\forall$, which are called ( $\exists \beta$ red) and ( $\exists \eta$ red $)$.

### 1.6 Variable-purification

For proving the results of the following sections we need to replace arbitrary Gentzen terms by Gentzen terms in whose type no variable is both free and bound. This is the same kind of condition that Kleene had to satisfy in [13] (Section 78) in order to prove cut elimination in the predicate calculus. The condition is implicit in Gentzen's [9], because he did not use the same letters for free and bound variables.

A variable $x$ is free in the type of $f: X \vdash Y$ when $x$ is free either in $X$ or in $Y$. We say that $x$ participates free in $f$ when $x$ is free in the type of some subterm of $f$. We have analogous definitions with "free" replaced by "bound". We say that $x$ participates in $f$ when $x$ participates either free or bound in $f$. A Gentzen term of GQDS is variable-pure when no variable participates in it both free and bound.

By changing only bound variables one can transform an arbitrary Gentzen term that is not variable-pure into a variable-pure Gentzen term. (We could as well talk of renaming of bound variables, but, as we said at the beginning of Section 1.3, we do not want to confuse this renaming with the renaming of free variables.) The initial term and the resulting term need not be of the same type, and hence need not be equal, but they will be equal up to an isomorphism, as we shall see below.

Kleene's purification was done for a sequent where there was no variable both free and bound, and his aim was to obtain a derivation for it in which no variable is both free and bound. For that he could not just change bound variables, but he needed also to change free variables. Our aim is different, and we can change only bound variables.

We have the following equations in GQDS:

$$
\begin{aligned}
& \left(Q^{L} \tau\right) \quad \begin{array}{l}
Q_{x, X}^{L} f=Q_{y, X_{y}^{x}}^{L} f \circ\left(\tau_{X_{v}^{x}, x, y}^{Q_{v}} \wedge \mathbf{1}_{Z}\right), \\
Q_{x, X}^{L} f=Q_{y, X_{y}^{x}}^{L} f \circ \tau_{X_{v}^{x}, x, y}^{Q_{v}}, \\
\left(Q^{R} \tau\right) \quad Q_{x, X}^{R} f=\left(\tau_{X_{v}^{x}, y, x}^{Q_{v}} \vee \mathbf{1}_{Z}\right) \circ Q_{y, X_{y}^{x}}^{R} f, \\
Q_{x, X}^{R} f=\tau_{X_{v}^{x}, y, x}^{Q_{v}} \circ Q_{y, X_{y}^{x}}^{R} f .
\end{array}, l
\end{aligned}
$$

To prove these equations we use essentially the equations ( $Q \tau \iota$ ) and ( $Q \tau$ trans) of Section 1.3.

We define $\tau$-terms inductively with the following clauses:

$$
\begin{aligned}
& \tau_{X, u, v}^{Q_{x}} \text { is a } \tau \text {-term; } \\
& \text { if } f \text { is a } \tau \text {-term and } S \text { is a quantifier prefix, then } f \xi \mathbf{1}_{Y} \text { and } S f \text { are } \tau \text {-terms. }
\end{aligned}
$$

The unique subterm $\tau_{X, u, v}^{Q_{x}}$ of a $\tau$-term is called its head. Then for every $\tau$-term $h$ there is a $\tau$-term $h^{\prime}$ such that the following equations hold in GQDS:
( $\xi h$ ) equations:

$$
\begin{aligned}
& \xi_{X_{1}, X_{2}}(f \circ h, g)=\xi_{X_{1}^{\prime}, X_{2}}(f, g) \circ h^{\prime}, \\
& \xi_{X_{1}, X_{2}}(h \circ f, g)=h^{\prime} \circ \xi_{X_{1}^{\prime}, X_{2}}(f, g),
\end{aligned}
$$

(cut h) equations:

$$
\begin{aligned}
& \operatorname{cut}_{X}(f \circ h, g)=\operatorname{cut}_{X}(f, g) \circ h^{\prime} \\
& \operatorname{cut}_{X}(f, g \circ h)=\operatorname{cut}_{X}(f, g) \circ h^{\prime}
\end{aligned}
$$

$$
\begin{aligned}
& \operatorname{cut}_{X}(h \circ f, g)=h^{\prime} \circ \operatorname{cut}_{X}(f, g), \\
& \operatorname{cut}_{X}(f, h \circ g)=h^{\prime} \circ \operatorname{cut}_{X}(f, g), \\
& \operatorname{cut}_{X}\left(f \circ h_{1}, h_{2} \circ g\right)=\operatorname{cut}_{X^{\prime}}(f, g),
\end{aligned}
$$

(Qh) equations: for $S \in\{L, R\}$,

$$
\begin{aligned}
Q_{x, X}^{S}(f \circ h) & =Q_{x, X^{\prime}}^{S} f \circ h^{\prime} \\
Q_{x, X}^{S}(h \circ f) & =h^{\prime} \circ Q_{x, X^{\prime}}^{S} f
\end{aligned}
$$

(ren h) equations:

$$
\begin{aligned}
& {[f \circ h]_{y}^{x}=[f]_{y}^{x} \circ h^{\prime},} \\
& {[h \circ f]_{y}^{x}=h^{\prime} \circ[f]_{y}^{x} .}
\end{aligned}
$$

In these equations $X^{\prime}$ is either $X$ or a different form set. The $\tau$-terms $h_{1}$ and $h_{2}$ in the last (cut $h$ ) equation differ in their heads, which are inverse to each other (see Section 1.3).

To derive the $(\xi h)$ equations and the first four (cut $h$ ) equations we use essentially functorial and naturality equations (see [7]). For the last (cut h) equation we also use ( $Q \tau \operatorname{sim}$ ), and for the $(Q h)$ and (ren $h$ ) equations we use essentially ( $Q \tau$ ren $),(Q \iota$ nat) and $(Q \tau$ nat) (see Sections 1.2-3).

By applying the equations of GQDS mentioned in this section, we can establish the following.

Variable-Purification Lemma. For every Gentzen term $f: X \vdash Y$ there is a variable-pure Gentzen term $f^{\prime}: X^{\prime} \vdash Y^{\prime}$ such that in GQDS

$$
f=h_{2} \circ f^{\prime} \circ h_{1}
$$

where $h_{1}$ and $h_{2}$ are compositions of $\tau$-terms or $\mathbf{1}_{X}$ or $\mathbf{1}_{Y}$.
Let us explain up to a point how we achieve that.
Let $x$ be new for $f$ when $x$ does not participate in $f$ (see the beginning of the section) and does not occur as an index in the Gentzen operations of renaming that occur in $f$. Let $x_{1}, \ldots, x_{n}$ be all the variables that participate bound in $f$. Then take the variables $x_{1}^{\prime}, \ldots, x_{n}^{\prime}$ all new for $f$, and apply first the equations $\left(Q^{L} \tau\right)$ and $\left(Q^{R} \tau\right)$ with $x$ being $x_{i}$ and $y$ being $x_{i}^{\prime}$. In Gentzen terms, $\mathbf{1}_{X}$ occurs only with $X$ atomic, and so for every variable that participates bound in $f$ there is a Gentzen operation by which it was introduced. It remains then to apply the equations $(\xi h)$, (cut $h)$, $(Q h)$ and (ren $h)$. Note that $h_{1}$ and $h_{2}$ depend only on the type of $f$ and on the choice of the variables $x_{1}^{\prime}, \ldots, x_{n}^{\prime}$.

### 1.7 Renaming of eigenvariables

In this section we prove the equations of GQDS of the following form:

$$
\left(Q^{S} \text { ren }\right) \quad Q_{x, X}^{S} f=Q_{x, X}^{S}[f]_{v}^{u}
$$

for $Q^{S} \in\left\{\forall^{R}, \exists \exists^{L}\right\}$, with $u$ the eigenvariable and $v$ a variable not free in the type of $f$ (see the beginning of the preceding section). That $v$ is not free in the type of $f$ is satisfied a fortiori when $v$ is new for the left-hand side (see the end of the preceding section). The equations ( $Q^{S}$ ren) say that GQDS covers the renaming of eigenvariables by new variables, which is a technique derived from [9] (Section III.3.10). We need the equations ( $Q^{S}$ ren) to prove the results of Sections 1.9-10.

We derive now the equation $\left(\forall^{R}\right.$ ren $)$ for $f: U \vdash X_{u}^{x} \vee Z$ :

$$
\begin{aligned}
& \forall_{x, X}^{R} f=\left(\left(\tau_{X_{w}^{x}, v, x}^{\forall_{w}^{x}} \circ \tau_{X_{w}^{x}, u, v}^{\forall_{w}^{x}}\right) \vee \mathbf{1}_{Z}\right) \circ \check{\theta}_{X_{u}^{x}, Z}^{\forall_{u}} \circ \forall_{u}\left[[f]_{w}^{u}\right]_{u}^{w} \circ \gamma_{U}^{\forall_{u}}, \text { by }(\text { ren } 6),(\text { ren } 2) \\
& \quad \text { and }(\forall \tau \text { trans }), \\
&=\left(\tau_{X_{w}^{x}, v, x}^{\forall_{w}^{x}} \vee \mathbf{1}_{Z}\right) \circ \check{\theta}_{X_{v}^{x}, Z}^{\forall_{x}^{x}} \circ \forall_{w}\left[[f]_{w}^{u}\right]_{v}^{w} \circ \tau_{U, u, v}^{\forall_{w}} \circ \gamma_{U}^{\forall_{u}}, \text { by }(\forall \tau \check{\theta}) \text { and }(\forall \tau \text { nat }), \\
&= \forall_{x, X}^{R}[f]_{v}^{u}, \text { by }(\text { ren } 5),(\text { ren 2) and }(\forall \tau \gamma) \text { (see Sections 1.2-3) } .
\end{aligned}
$$

The equation ( $\exists^{L} r e n$ ) is derived analogously.
We can prove also the equations

$$
\left(Q^{T} \text { ren }\right) \quad Q_{x, X}^{T} f=Q_{x, X}^{T}[f]_{z}^{y}
$$

for $Q^{T} \in\left\{\forall^{L}, \exists^{R}\right\}$, with $f$ either of the type $X_{y}^{x} \wedge Z \vdash U$ or $X_{y}^{x} \vdash U$ or of the type $U \vdash X_{y}^{x} \vee Z$ or $U \vdash X_{y}^{x}$, provided $y$ is not free in the type of $Q_{x, X}^{T} f$. So, though $y$ is not here an eigenvariable, it could have been one.

To derive the equation ( $\forall^{L}$ ren) for $f: X_{y}^{x} \wedge Z \vdash U$ we have

$$
\begin{aligned}
\forall_{x, X}^{L} f & =\left[f \circ\left(\left[\iota_{X}^{\forall_{x}}\right]_{y}^{x} \wedge \mathbf{1}_{Z}\right)\right]_{z}^{y}, \quad \text { by }(\text { ren } 2), \\
& =[f]_{z}^{y} \circ\left(\left[\iota_{X}^{\forall_{x}}\right]_{z}^{x} \wedge \mathbf{1}_{Z}\right)=\forall_{x, X}^{L}[f]_{z}^{y},
\end{aligned}
$$

by using, together with other renaming equations, (ren 5) and (ren 2) if $y$ is different from $x$, since $y$ is then not free in the type of $\iota_{X}^{\forall_{x}}$, and by using (ren 1 ) if $y$ is $x$. The equation ( $\exists^{R}$ ren) is derived analogously.

### 1.8 Elimination of renaming

We can establish the following proposition for GQDS.
Renaming Elimination. For every variable-pure and cut-free Gentzen term $t$ there is a variable-pure, cut-free and renaming-free Gentzen term $t^{\prime}$ such that $t=t^{\prime}$.

Here cut-free means of course that no instance of the Gentzen operation cut ${ }_{X}$ occurs in $t$ and $t^{\prime}$, and renaming-free means that none of the Gentzen operations [ $]_{y}^{x}$ occurs in $t^{\prime}$.

The proof of Renaming Elimination is based on the following equations of GQDS:

$$
\begin{aligned}
& {\left[\xi_{X_{1}, X_{2}}(f, g)\right]_{y}^{x}=\xi_{X_{1}^{\prime}, X_{2}^{\prime}}\left([f]_{y}^{x},[g]_{y}^{x}\right)} \\
& {\left[Q_{z, X}^{S} f\right]_{y}^{x}=Q_{z, X^{\prime}}^{S}[f]_{y}^{x} \text {, if } z \text { is neither } x \text { nor } y .}
\end{aligned}
$$

To eliminate all occurrences of renaming we eliminate one by one innermost occurrences of renaming, i.e. occurrences of renaming within the scope of which there is no renaming. Variable-purity ensures that the proviso of the second equation is not an obstacle.

We will use Renaming Elimination for the proof of the Cut-Elimination Theorem for GQDS in the next section. For that we need a strengthened version of Renaming Elimination, in which it is specified that the Gentzen term $t^{\prime}$ is exactly analogous to $t$ : only indices of its identity arrows and of its Gentzen operations may change.

For $f: X \vdash Y$ and $g: Y \vdash X$ such that $x$ is not free in $X$ in GQDS we have

$$
\begin{aligned}
{[f]_{y}^{x} } & =\operatorname{cut}_{\forall_{x} Y}\left(\forall_{x, Y}^{R} f, \forall_{x, Y}^{L} \mathbf{1}_{Y_{y}^{x}}\right), \\
{[g]_{y}^{x} } & =\operatorname{cut}_{\exists_{x} Y}\left(\exists_{x, Y}^{R} \mathbf{1}_{Y_{y}^{x}}, \exists_{x, X}^{L} g\right) .
\end{aligned}
$$

So particular instances of renaming (and Gentzen and Kleene did not envisage implicitly more than that) can be easily eliminated provided we want to tolerate cut. (In the presence of implication we could eliminate all instances of renaming in the presence of cut, as we mentioned in Section 1.2.) Our aim however is to eliminate both cut and renaming.

If we delete "variable-pure" from Renaming Elimination, then this proposition cannot be proved. A counterexample, analogous to a counterexample in [13] (Section 78, Example 4), is the following:

$$
\left[\forall_{x, \forall_{y} R x y}^{L} \forall_{y, R u y}^{L} \mathbf{1}_{R u z}\right]_{y}^{u}: \forall_{x} \forall_{y} R x y \vdash R y z .
$$

From this Gentzen term we can eliminate renaming only by introducing cut, as above.

Kleene in [13] also needed variable-purity to eliminate cut. But his counterexample, mentioned above, which is analogous to our counterexample, would not be a counterexample in the presence of renaming.

### 1.9 Cut elimination

Our aim in this section is to establish the following theorem for GQDS.
Cut-Elimination Theorem. For every variable-pure Gentzen term there is a variable-pure and cut-free Gentzen term $t^{\prime}$ such that $t=t^{\prime}$.

The proof of this theorem is obtained by modifying and expanding the proof of the Cut-Elimination Theorem for GDS in [7] (Section 7.7). We presuppose below the terminology introduced in this previous proof.

The $Q$-rank of $\operatorname{cut}_{Q_{x} X}(f, g)$ is $n_{1}+n_{2}$ when $f$ has a subterm $Q_{x, X}^{R} f^{\prime}$ of depth $n_{1}$ and $g$ has a subterm $Q_{x, X}^{L} g^{\prime}$ of depth $n_{2}$. The rank of a topmost cut cut ${ }_{X}(f, g)$ is either its $\wedge$-rank, or $\vee$-rank, or $p$-rank, or $Q$-rank depending on $X$.

The complexity of a topmost cut cut $X_{X}(f, g)$ is $(m, n)$ where $m \geq 1$ is the sum of the number of predicate letters and occurrences of quantifier prefixes in $X$ and $n \geq 0$ is the rank of this cut. Every form set of the form $Q_{x} X$ is considered to be both of colour $\wedge$ and colour $\vee$.

In the proof we have the following additional cases. We consider only the most complicated cases, and leave out the remaining simpler cases, which are dealt with analogously.
$(\forall 1)$ If our topmost cut is

$$
\operatorname{cut}_{\forall_{x} X}\left(\forall_{x, X}^{R} f, \forall_{x, X}^{L} g\right): U \wedge Y \vdash Z \vee W
$$

for $f: U \vdash X_{u}^{x} \vee Z$ and $g: X_{v}^{x} \wedge Y \vdash W$, with complexity $(m, 0)$ where $m>1$, then we use the equation

$$
\operatorname{cut}_{\forall_{x} X}\left(\forall_{x, X}^{R} f, \forall_{x, X}^{L} g\right)=\operatorname{cut}_{X_{v}^{x}}\left([f]_{v}^{u}, g\right)
$$

in which the cut on the right-hand side is of lower complexity than the topmost cut on the left-hand side. To derive this equation we use essentially the equations ( $\forall \beta$ red) (see Section 1.5) together with naturality and functorial equations. We proceed analogously when the topmost cut we start from is

$$
\operatorname{cut}_{\exists_{x} X}\left(\exists_{x, X}^{R} f, \exists_{x, X}^{L} g\right)
$$

Suppose for the cases below that $X$ is of colour $\wedge$.
$(\forall 2)$ If our topmost cut is

$$
\operatorname{cut}_{X}\left(\forall_{x, V}^{R} f, g\right): U \wedge Y \vdash \forall_{x} V \vee Z \vee W
$$

for $f: U \vdash X \vee V_{u}^{x} \vee Z$ and $g: X \wedge Y \vdash W$, with complexity $(m, n)$ where $m, n \geq 1$, then we use the equation

$$
\operatorname{cut}_{X}\left(\forall_{x, V}^{R} f, g\right)=\forall_{x, V}^{R} \text { cut }_{X}\left([f]_{v}^{u}, g\right)
$$

with $v$ being a variable new for the left-hand side. By the strengthened version of Renaming Elimination from the preceding section, there is a variable-pure, cutfree and renaming-free Gentzen term $f^{\prime}$ such that $[f]_{v}^{u}=f^{\prime}$, and the complexity of $\operatorname{cut}_{X}\left(f^{\prime}, g\right)$ is $(m, n-1)$. This equation is derived as follows.

$$
\begin{align*}
\operatorname{cut}_{X}\left(\forall_{x, V}^{R} f, g\right)= & \operatorname{cut}_{X}\left(\forall_{x, V}^{R}[f]_{v}^{u}, g\right), \text { by }\left(\forall^{R} \text { ren) for } v\right. \text { new for the left-hand } \\
& \text { side (see Section 1.7), } \\
= & \forall_{x, V}^{R}\left(\left(\left[\iota_{V}^{\forall_{y}}\right]_{v}^{y} \vee \mathbf{1}_{Z \vee W}\right) \circ \operatorname{cut}_{X}\left(\forall_{x, V}^{R}[f]_{v}^{u}, g\right)\right) \text {, by ( } \forall \eta \text { red) } \tag{seeSection1.5}
\end{align*}
$$

$$
\begin{aligned}
& =\forall_{x, V}^{R}\left(\operatorname{cut}_{X}\left(\left(\left[\iota_{V}^{\forall_{x}}\right]_{v}^{x} \vee \mathbf{1}_{Z}\right) \circ \forall_{x, V}^{R}[f]_{v}^{u}, g\right), \quad \begin{array}{l}
\text { by functorial and } \\
\\
=\forall_{x, V}^{R}\left(\operatorname{cut}_{X}\left([f]_{v}^{u}, g\right)\right), \text { by }(\forall \beta r e d) .
\end{array}\right.
\end{aligned}
$$

We needed to rename the eigenvariable $u$ by a new $v$ in order to ensure that the proviso for the eigenvariable is satisfied in the second line for the $\forall_{x, V}^{R}$ operation newly introduced.
$(\forall 3)$ If our topmost cut is

$$
\operatorname{cut}_{X}\left(\forall_{x, V}^{L} f, g\right): \forall_{x} V \wedge U \wedge Y \vdash Z \vee W
$$

for $f: V_{y}^{x} \wedge U \vdash X \vee Z$ and $g: X \wedge Y \vdash W$, then we use the straightforward equation

$$
\operatorname{cut}_{X}\left(\forall_{x, V}^{L} f, g\right)=\forall_{x, V}^{L} \operatorname{cut}_{X}(f, g)
$$

We have also the straightforward equation

$$
\operatorname{cut}_{X}\left(\exists_{x, V}^{R} f, g\right)=\exists_{x, V}^{R} \text { cut }_{X}(f, g)
$$

and the equation

$$
\operatorname{cut}_{X}\left(\exists_{x, V}^{L} f, g\right)=\exists_{x, V}^{L} \text { cut }_{X}\left([f]_{v}^{u}, g\right),
$$

proved like the analogous equation in case $(\forall 2)$. These equations enable us to settle the remaining cases when $X$ is of colour $\wedge$. When $X$ is of colour $\vee$, we proceed in a dual manner.

By Renaming Elimination, we need not consider cases when in our topmost cut $\operatorname{cut}_{X}(f, g)$ either $f$ or $g$ is of the form $[h]_{z}^{y}$.

### 1.10 Invertibility in GQDS

The results we are going to prove in this section correspond to inverting rules in derivations, i.e. passing from conclusions to premises. This invertibility is guaranteed by the possibility to permute rules, i.e. change their order in derivations, and we show for that permuting that it is covered by the equations of GQDS. (Permutation of rules is a theme treated in [14], but without considering equations between derivations.)

Besides the equations mentioned in [7] (beginning of Section 7.8) we will need the equations of GQDS of the following form

$$
\left(\xi Q^{S}\right) \quad \xi_{X_{1}, X_{2}}\left(Q_{x, X}^{S} f_{1}, f_{2}\right)=Q_{x, X}^{S} \xi_{X_{1}, X_{2}}\left(f_{1}^{\prime}, f_{2}\right)
$$

for $X_{y}^{x}$ being a subform set of the source or target of $f_{1}$, and $f_{1}^{\prime}$ being $f_{1}$ when $Q^{S} \in\left\{\forall^{L}, \exists^{R}\right\}$, and $\left[f_{1}\right]_{v}^{y}$ with $v$ new for the left-hand side when $Q^{S} \in\left\{\forall^{R}, \exists^{L}\right\}$. These equations are either straightforward to derive, or when $Q^{S} \in\left\{\forall^{R}, \exists \exists^{L}\right\}$
we derive them by imitating the derivation of the equation of case $(\forall 2)$ of the preceding section, with the help of the equations ( $Q \beta$ red) and ( $Q \eta$ red) (see the end of Section 1.5).

We will also need for the end of the section the following equations, whose derivations are not difficult to find:
for $Q^{S} \in\left\{\forall^{L}, \exists^{R}\right\}$,

$$
\begin{array}{ll}
\left(Q^{S} Q^{S}\right) & Q_{y, Y}^{S} Q_{x, X}^{S} f=Q_{x, X}^{S} Q_{y, Y}^{S} f \\
\left(\exists^{R} \forall^{L}\right) & \exists_{y, Y}^{R} \forall_{x, X}^{L} f=\forall_{x, X}^{L} \exists_{y, Y}^{R} f
\end{array}
$$

for $Q \in\{\forall, \exists\}, S \in\{L, R\}$, and the proviso for the eigenvariable being satisfied,

$$
\begin{array}{ll}
\left(Q^{R} Q^{L}\right) & Q_{y, Y}^{R} Q_{x, X}^{L} f=Q_{x, X}^{L} Q_{y, Y}^{R} f \\
\left(\exists^{S} \forall^{S}\right) & \exists_{y, Y}^{S} \forall_{x, X}^{S} f=\forall_{x, X}^{S} \exists_{y, Y}^{S} f
\end{array}
$$

The Invertibility Lemmata for $\wedge$ and $\vee$ are formulated as in [7] (Section 7.8). Only $\operatorname{let}(X)$ is the set of predicate letters occurring in the form set $X$. These lemmata hold also when we replace throughout "cut-free Gentzen term" by "variable-pure, cut-free and renaming-free Gentzen term". They are proved as in [7], with additional cases covered by the equations $\left(\xi Q^{S}\right)$.

The following invertibility lemmata are easy consequences of the equations ( $\forall \eta$ red) and ( $\exists \eta$ red) (see the end of Section 1.5).

Invertibility Lemma for $\forall^{R}$. If $f$ is a variable-pure Gentzen term of the type $U \vdash \forall_{x} X \vee Z$ or $U \vdash \forall_{x} X$, then there is a variable-pure Gentzen term $f^{\prime}$ of the type $U \vdash X_{u}^{x} \vee Z$ or $U \vdash X_{u}^{x}$ respectively such that $\forall_{x, X}^{R} f^{\prime}=f$.

Invertibility Lemma for $\exists^{L}$. If $f$ is a variable-pure Gentzen term of the type $\exists_{x} X \wedge Z \vdash U$ or $\exists_{x} X \vdash U$, then there is a variable-pure Gentzen term $f^{\prime}$ of the type $X_{u}^{x} \wedge Z \vdash U$ or $X_{u}^{x} \vdash U$ respectively such that $\exists_{x, X}^{L} f^{\prime}=f$.

Before formulating the remaining invertibility lemmata for $\forall^{L}$ and $\exists^{R}$ we must introduce a number of notions concerning occurrences of variables within the types of variable-pure, cut-free and renaming-free Gentzen terms. Although many of the notions introduced make sense also for other Gentzen terms, we need these notions only in the context of variable-pure, cut-free and renamingfree Gentzen terms. The essential assertions using these notions, which we need for our results, need not hold for all Gentzen terms.

Let $\alpha$ and $\beta$, sometimes with indices, stand for occurrences of individual variables in a form set, and let $\gamma$, sometimes with indices, stand for an occurrence of a quantifier prefix in a form set. Let $\alpha_{1}$ and $\alpha_{2}$ be different occurrences of the variable $x$ in the form set $X$, and let $\gamma$ be an occurrence of $Q_{x}$ in $X$. Then we say that $\alpha_{1}$ and $\alpha_{2}$ are simultaneously bound by $\gamma$ when $X$ has a subform set $\gamma Y$ such that $\alpha_{1}$ and $\alpha_{2}$ are free in $Y$.

We say that a predicate letter $P$ occurs in the type of the Gentzen term $f: X_{1} \vdash X_{2}$ when it occurs in $X_{1}$ or $X_{2}$. Because of diversification (see the beginning of Section 1.5), every $n$-ary predicate letter $P$ that occurs in the type of $f$ occurs exactly once in $X_{i}$ in a subform set $P \alpha_{1}^{i} \ldots \alpha_{n}^{i}$ of $X_{i}$, for $i \in\{1,2\}$. Here $\alpha_{j}^{1}$ and $\alpha_{j}^{2}$, for $1 \leq j \leq n$, are not necessarily occurrences of the same variable. We say that the pair $\left(P \alpha_{1}^{1} \ldots \alpha_{n}^{1}, P \alpha_{1}^{2} \ldots \alpha_{n}^{2}\right)$ is a formula couple of $f$, and we say that $\left(\alpha_{j}^{1}, \alpha_{j}^{2}\right)$ is a couple of $f$. We say that it is a $P_{j}$-couple of $f$ when we want to stress from which formula couple and from which place $j$ in it originates. If $\left(\alpha^{1}, \alpha^{2}\right)$ is a couple of $f$, then $\alpha^{i}$, for $i \in\{1,2\}$, is free in ( $\alpha^{1}, \alpha^{2}$ ) when it is free in $X_{i}$, and analogously with "free" replaced by "bound", "universally bound" and "existentially bound".

A left bridge between the different couples $\left(\alpha^{1}, \alpha^{2}\right)$ and $\left(\beta^{1}, \beta^{2}\right)$ of $f$ such that $\alpha^{1}$ and $\beta^{1}$ are occurrences of the same variable $x$ is an occurrence $\gamma$ of a quantifier prefix $Q_{x}$ in $X_{1}$ such that $\alpha_{1}$ and $\beta_{1}$ are simultaneously bound by $\gamma$. We define analogously a right bridge by replacing $\alpha^{1}, \beta^{1}$ and $X_{1}$ with $\alpha^{2}, \beta^{2}$ and $X_{2}$ respectively. A bridge is a left bridge or a right bridge.

Two different couples of $f$ are bridgeable when there is a bridge between them (there might be both bridges). For $n \geq 2$, a sequence of couples

$$
\left(\alpha_{1}^{1}, \alpha_{1}^{2}\right),\left(\alpha_{2}^{1}, \alpha_{2}^{2}\right), \ldots,\left(\alpha_{n}^{1}, \alpha_{n}^{2}\right)
$$

of $f$ such that $\left(\alpha_{j}^{1}, \alpha_{j}^{2}\right)$ and $\left(\alpha_{j+1}^{1}, \alpha_{j+1}^{2}\right)$, where $1 \leq j \leq n-1$, are bridgeable is called a bridgeable chain of couples. For every bridgeable chain of couples we can find at least one sequence of bridges $\gamma_{1}, \ldots, \gamma_{n-1}$ that ensure its bridgeability.

We say that $\left(\alpha_{1}^{1}, \alpha_{1}^{2}\right)$ and $\left(\alpha_{n}^{1}, \alpha_{n}^{2}\right)$ are clustered when there is a bridgeable chain of couples in which $\left(\alpha_{1}^{1}, \alpha_{1}^{2}\right)$ and $\left(\alpha_{n}^{1}, \alpha_{n}^{2}\right)$ are respectively the first and last member. A sequence of bridges ensuring the bridgeability of this bridgeable chain of couples is said to ensure the clustering of $\left(\alpha_{1}^{1}, \alpha_{1}^{2}\right)$ and $\left(\alpha_{n}^{1}, \alpha_{n}^{2}\right)$.

A set $C$ of couples of $f$ is a cluster of $f$ when there is a couple $\left(\alpha^{1}, \alpha^{2}\right)$ in $C$ such that for every couple $\left(\beta^{1}, \beta^{2}\right)$ of $f$ different from $\left(\alpha^{1}, \alpha^{2}\right)$ we have that $\left(\beta^{1}, \beta^{2}\right) \in C$ iff $\left(\beta^{1}, \beta^{2}\right)$ is clustered with $\left(\alpha^{1}, \alpha^{2}\right)$. Let $P$ be a predicate letter occurring in the type of a cut-free Gentzen term $f$, and let $S u b_{P}(f)$ be the set of subterms of $f$ in whose type $P$ occurs. For every member $f^{\prime}$ of $S u b_{P}(f)$ there is a formula couple $\left(P \alpha_{1}^{1} \ldots \alpha_{n}^{1}, P \alpha_{1}^{2} \ldots \alpha_{n}^{2}\right)$ of $f^{\prime}$.

The set of all the couples $\left(\alpha^{1}, \alpha^{2}\right)$ such that there is a member $f^{\prime}$ of $S u b_{P}(f)$ with $\left(\alpha^{1}, \alpha^{2}\right)$ the $P_{j}$-couple of $f^{\prime}$ is called an arc of $f$. For example, in

$$
\frac{\mathbf{1}_{R u y}: R u(y) \vdash R u(y)}{\frac{\forall_{x, R u x}^{L} \mathbf{1}_{R u y}: \forall_{x} R u \times \vdash R u(y)}{\wedge_{R u y, P y}\left(\forall_{x, R u x}^{L} \mathbf{1}_{R u y}, \mathbf{1}_{P y}\right): \forall_{x} R u \times \wedge P(y) \vdash R u(y) \wedge P(y)}} \frac{\mathbf{1}_{P y}: P(y) \vdash P(y)}{\exists_{z, R u z \wedge P z}^{R} \wedge_{R u y, P y}\left(\forall_{x, R u x}^{L} \mathbf{1}_{R u y}, \mathbf{1}_{P y}\right): \forall_{x} R u(x) \wedge P(y) \vdash \exists_{z}(R u(z) \wedge P(Z)}
$$

the encircled occurrences of variables connected by lines make two arcs of the variable-pure, cut-free and renaming-free Gentzen term in the last line.

We say that an arc is a $P_{j}$-arc when we want to stress from which formula couples and from which place $j$ in them it originates. In our example above, we have an $R_{2}$-arc and a $P_{1}$-arc. The bottom of the $P_{j}$-arc of $f$ is the $j$-th coordinate $\left(\alpha_{j}^{1}, \alpha_{j}^{2}\right)$ of the formula couple $\left(P \alpha_{1}^{1} \ldots \alpha_{n}^{1}, P \alpha_{1}^{2} \ldots \alpha_{n}^{2}\right)$ of $f$. In the example above, the bottom of the $R_{2}$-arc is $(x, z)$ in the last line and the bottom of the $P_{1}$-arc is $(y, z)$ in the last line.

Two arcs of $f$ are clustered when their bottoms are clustered. In the example above, the two arcs are clustered. If we delete the last line, then we obtain two arcs of

$$
\wedge_{R u y, P y}\left(\forall_{x, R u x}^{L} \mathbf{1}_{R u y}, \mathbf{1}_{P y}\right)
$$

that are not clustered.
A set $\mathcal{A}$ of arcs of $f$ is an arc-cluster of $f$ when there is an arc $a$ in $\mathcal{A}$ such that for every arc $b$ of $f$ different from $a$ we have that $b \in \mathcal{A}$ iff $b$ is clustered with $a$. The bottoms of the arcs in an arc-cluster of $f$ make a cluster of $f$. We call this cluster the bottom cluster of the arc-cluster. In our example, we have an arc-cluster, whose bottom cluster is made of the bottom of the $R_{2}$-arc and the bottom of the $P_{1}$-arc.

All occurrences of variables that are free in the couples of an arc, or of an arc-cluster, of a cut-free and renaming-free Gentzen term $f$ are occurrences of the same variable. (Here cut-freedom and renaming-freedom is essential.) This variable is called the free variable of the arc, or of the arc-cluster. The free variable of the arc-cluster in our example is $y$.

For $S \in\{L, R\}$, suppose we have a cut-free and renaming-free Gentzen term $f$ that has a subterm $Q_{x, X}^{S} g$ for $x$ free in $X$ and $X_{y}^{x}$ occurring in the type of $g$. We say that $Q_{x, X}^{S} g$ belongs to an arc-cluster of $f$ when the occurrences of $y$ in $X_{y}^{x}$ in the type of $g$ that have replaced $x$ in $X$ belong to couples in this arc-cluster. We say that $Q_{x, X}^{S} g$ belongs to a cluster of $f$ when it belongs to an arc-cluster whose bottom cluster is this cluster.

The subterms of $f$ that belong to an arc-cluster of $f$ are called the gates of that arc-cluster, and analogously with "arc-cluster" replaced by "cluster". A gate belonging to a cluster may correspond to bridges in bridgeable chains of couples in this cluster, but it need not correspond to such a bridge. In our example above, the Gentzen term in the last line

$$
\exists_{z, R u z \wedge P z}^{R} \wedge_{R u y, P y}\left(\forall_{x, R u x}^{L} \mathbf{1}_{R u y}, \mathbf{1}_{P y}\right)
$$

corresponds to a bridge, but $\forall_{x, R u x}^{L} \mathbf{1}_{R u y}$ does not.
A gate is called an eigengate when it is either of the $\forall^{R}$ or of the $\exists^{L}$ type. As a consequence of the proviso for the eigenvariable, we obtain that if a cluster has an eigengate $Q_{x, X}^{S} g$, then every other gate of that cluster is a subterm of $g$. This implies that every cluster has at most one eigengate. As another consequence of the proviso for the eigenvariable, we have the following remark.

Eigengate Remark. If we have an eigengate in a cluster, then there is no couple $\left(\alpha^{1}, \alpha^{2}\right)$ in this cluster such that $\alpha^{i}$ is free and $\alpha^{3-i}$ is bound, for $i \in\{1,2\}$.

A couple $\left(\alpha^{1}, \alpha^{2}\right)$ can be of the following six kinds, depending on whether the occurrences of variables in it are universally bound, free or existentially bound:

| $\alpha^{1}$ | $\alpha^{2}$ | name of kind |
| :---: | :---: | :---: |
| universally bound | universally bound | $(\forall, \forall)$ |
| universally bound | free | $(\forall, \emptyset)$ |
| universally bound | existentially bound | $(\forall, \exists)$ |
| free | free | $(\emptyset, \emptyset)$ |
| free | existentially bound | $(\emptyset, \exists)$ |
| existentially bound | existentially bound | $(\exists, \exists)$ |

The kinds not mentioned-namely, $(\emptyset, \forall),(\exists, \forall)$ and $(\exists, \emptyset)$-are not possible.
If a $(\emptyset, \emptyset)$ couple occurs in a cluster, then this cluster is a singleton. If a $(\forall, \forall)$ couple or a $(\exists, \exists)$ couple occurs in a cluster, then this cluster has an eigengate. Together with $(\forall, \forall)$ couples in a cluster we can find only $(\forall, \forall)$ couples and $(\forall, \exists)$ couples, and analogously together with $(\exists, \exists)$ couples in a cluster we can find only $(\exists, \exists)$ couples and $(\forall, \exists)$ couples. This is a consequence of the Eigengate Remark and of the fact that a cluster can have only one eigengate. Couples of the $(\forall, \exists),(\forall, \emptyset)$ and $(\emptyset, \exists)$ kind, can be joined together in a cluster without eigengate. In the bottom cluster in our example above, we have a $(\forall, \exists)$ couple $(x, z)$ and a $(\emptyset, \exists)$ couple $(y, z)$. This cluster has no eigengate.

A Gentzen term $f$ is eigendiversified when it is variable-pure, cut-free and renaming-free, and, moreover, for every arc-cluster of $f$ that has an eigengate the free variable of this arc-cluster is different from the free variable of any other arc-cluster of $f$. (Eigendiversification is inspired by [9], Section III.3.10.) We have the following for GQDS.

Eigendiversification Lemma. For every variable-pure, cut-free and renamingfree Gentzen term $f$ there is an eigendiversified Gentzen term $f^{\prime}$ such that $f=f^{\prime}$.

Proof. This lemma is proved by replacing the free variable of an arc-cluster of $f$ that has an eigengate by a variable new for $f$. By doing that for every arccluster of $f$ that has an eigengate we obtain $f^{\prime}$, which differs from $f$ just in the indices of identity arrows and of Gentzen operations. The equations ( $Q^{S}$ ren) of Section 1.7 guarantee that $f^{\prime}=f$.

Take for example a subterm of $f$ of the form $\forall_{x, X}^{R} g: Y \vdash \forall_{x} X \vee Z$ for $g: Y \vdash$ $X_{u}^{x} \vee Z$, and suppose that in $f^{\prime}$ we have instead at the same place a subterm $\forall_{x, X}^{R} g^{\prime}: Y \vdash \forall_{x} X \vee Z$ for $g^{\prime}: Y \vdash X_{u^{\prime}}^{x} \vee Z$ where $u^{\prime}$ is new for $f$. Since $u$ is not free in the type of $\forall_{x, X}^{R} g$, so $u$ is not free in the type of $g^{\prime}$, and by ( $\forall^{R}$ ren) we have that $\forall_{x, X}^{R} g^{\prime}=\forall_{x, X}^{R}\left[g^{\prime}\right]_{u}^{u^{\prime}}$. By the strengthened version of Renaming Elimination (see Section 1.8), we obtain that $\left[g^{\prime}\right]_{u}^{u^{\prime}}=g$.

Next we give the following inductive definition of the notion of subform of a form set, which extends the notion of subform set (see the beginning of Section 1.5):
$X$ is a subform of $X$;
if $Y$ is a subform of $X_{y}^{x}$, then $Y$ is a subform of $Q_{x} X$;
if $X$ and $Y$ are subforms of $X^{\prime}$ and $Y^{\prime}$ respectively, then $X \xi Y$ is a subform
of $X^{\prime} \xi Y^{\prime}$;
if $X$ is a subform of $Y$, then $X$ is a subform of $Y \xi Z$.
(Note that $Y \xi Z$ is the same form set as $Z \xi Y$.)
We say that a Gentzen term is $\forall_{x} X$-regular when it does not have subterms of one of the following two forms:
(a) $\forall_{z, Z}^{R} \forall_{x, X}^{L} g$ for $g$ of the type $X_{y}^{x} \wedge U \vdash Z_{y}^{z} \vee V$, or of one of the three types obtained by omitting $\wedge U$ or $\vee V$,
(b) $\exists_{z, Z}^{L} \forall_{x, X}^{L} g$ for $g$ of the type $X_{y}^{x} \wedge Z_{y}^{z} \wedge Y \vdash V$ or $X_{y}^{x} \wedge Z_{y}^{z} \vdash V$.

We can then prove the following lemma.
Lemma $\forall^{L}$. If $f_{1}: \forall_{x} X \wedge U_{1} \vdash Z_{1}$ is an eigendiversified Gentzen term and $f_{2}: X_{u}^{x} \wedge U_{2} \vdash Z_{2}$ is a variable-pure Gentzen term such that $U_{1}$ and $Z_{1}$ are subforms of $U_{2}$ and $Z_{2}$ respectively, then $f_{1}$ is $\forall_{x} X$-regular. The same holds if in all the types above we omit $\wedge U_{1}$ and $\wedge U_{2}$, or just $\wedge U_{1}$.

Proof. Suppose $f_{1}$ is not $\forall_{x} X$-regular. We will consider only the case when $f_{1}$ has a subterm of the form $\forall_{z, Z}^{R} \forall_{x, X}^{L} g$ for $g: X_{y}^{x} \wedge U \vdash Z_{y}^{z} \vee V$. When $f_{1}$ has a subterm of the form mentioned in the remaining cases of $(a)$ or in case $(b)$, we proceed analogously. Let $Z^{\prime}$ be the subform set of the target $Z_{1}$ of $f_{1}$ containing exactly the same predicate letters as $Z$, and let $\gamma$ be the occurrence of $\forall_{x}$ at the beginning of $\forall_{x} X$ in the source of $f_{1}$. By the assumption that $f_{1}$ is eigendiversified, for $\alpha$ an occurrence of $x$ in $X$ and $\beta$ an occurrence of $z$ in $Z^{\prime}$, either
(1) we have a couple $(\alpha, \beta)$ of $f_{1}$, or
(2) we have two clustered couples $\left(\alpha, \alpha^{\prime}\right)$ and $\left(\beta^{\prime}, \beta\right)$ of $f_{1}$ with a sequence of bridges $\gamma_{1}, \ldots, \gamma_{n-1}$ different from $\gamma$ that ensure their clustering.

If we have clustered couples as in (2), but $\gamma$ occurs in $\gamma_{1}, \ldots, \gamma_{n-1}$, then let $\gamma_{j}$, for $1 \leq j \leq n-1$, be the rightmost occurrence of $\gamma$ in $\gamma_{1}, \ldots, \gamma_{n-1}$. The bridge $\gamma_{j}$ is between $\left(\alpha_{j}, \beta_{j}\right)$ and $\left(\alpha_{j+1}, \beta_{j+1}\right)$, and $\alpha_{j+1}$ is an occurrence of $x$ in $X$ in the source of $f_{1}$. If $j=n-1$, then we have (1), and if $j<n-1$, then we have (2).

By Renaming Elimination and the Cut-Elimination Theorem (see Sections $1.8-9)$, we may assume that $f_{2}$ is cut-free and renaming-free. If we have (1), then
we should have a $(\emptyset, \forall)$ couple of $f_{2}$, which is impossible. If we have (2), then in $\left(\alpha, \alpha^{\prime}\right)$ we have that $\alpha$ is universally bound. Since $U_{1}$ and $Z_{1}$ are subforms of $U_{2}$ and $Z_{2}$ respectively, there should be a bridgeable chain of couples of $f_{2}$ whose first member is of the kind $(\emptyset, \emptyset)$ or $(\emptyset, \exists)$, and whose last member is of the kind $(\forall, \forall)$. The bridges ensuring the bridgeability of this chain of couples of length $n$ correspond to $\gamma_{1}, \ldots, \gamma_{n-1}$. However, a bridgeable chain of couples of the kind above cannot exist, as we said after the Eigengate Remark. (All couples in a bridgeable chain of couples belong to the same cluster.)

There is an analogous lemma that should be called Lemma $\exists^{R}$. It involves $\exists^{R}$ instead of $\forall^{L}$ (which engenders the notion of $\exists_{x} X$-regularity). We can now finally state the following lemmata.

InVERTIBILITY LEMMA FOR $\forall^{L}$. If $f_{1}: \forall_{x} X \wedge U_{1} \vdash Z_{1}$ is an eigendiversified Gentzen term, and there is a variable-pure Gentzen term $f_{2}: X_{y}^{x} \wedge U_{2} \vdash Z_{2}$ where $U_{1}$ and $Z_{1}$ are subforms of $U_{2}$ and $Z_{2}$ respectively, then there is an eigendiversified Gentzen term $f_{1}^{\prime}: X_{y}^{x} \wedge U_{1} \vdash Z_{1}$ such that $\forall_{x, X}^{L} f_{1}^{\prime}=f_{1}$. The same holds if in all the types above we omit $\wedge U_{1}$ and $\wedge U_{2}$, or just $\wedge U_{1}$.

Invertibility Lemma for $\exists R$. If $f_{1}: Z_{1} \vdash \exists_{x} X \vee U_{1}$ is an eigendiversified Gentzen term, and there is a variable-pure Gentzen term $f_{2}: Z_{2} \vdash X_{y}^{x} \vee U_{2}$ where $U_{1}$ and $Z_{1}$ are subforms of $U_{2}$ and $Z_{2}$ respectively, then there is an eigendiversified Gentzen term $f_{1}^{\prime}: Z_{1} \vdash X_{y}^{x} \vee U_{1}$ such that $\exists_{x, X}^{R} f_{1}^{\prime}=f_{1}$. The same holds if in all the types above we omit $\vee U_{1}$ and $\vee U_{2}$, or just $\vee U_{1}$.

These two lemmata are proved by induction on the complexity of $f_{1}$ with the help of the equations $\left(\xi Q^{S}\right),\left(Q^{S} Q^{S}\right),\left(\exists^{R} \forall^{L}\right),\left(Q^{R} Q^{L}\right)$ and $\left(\exists^{S} \forall^{S}\right)$, from the beginning of the section, and Lemmata $\forall^{L}$ and $\exists^{R}$. Without the Lemmata $\forall^{L}$ and $\exists^{R}$ we would not be able to apply the equations $\left(Q^{R} Q^{L}\right)$ and $\left(\exists^{S} \forall^{S}\right)$. If we end up with $f_{1}^{\prime}$ of the type $X_{y^{\prime}}^{x} \wedge U_{1} \vdash Z_{1}$, or $Z_{1} \vdash X_{y^{\prime}}^{x} \vee U_{1}$ respectively, where $y^{\prime}$ is different from $y$, then we proceed in the spirit of the proof of the Eigendiversification Lemma by using the equations ( $Q^{T}$ ren) of Section 1.7.

### 1.11 Proof of QDS Coherence

We are now ready to prove that GQDS is a preorder (see the beginning of Section 1.5). Suppose we have two arrow terms of GQDS of the same type. These arrow terms are equal to two Gentzen terms $f_{1}$ and $f_{2}$ by the new Gentzenization Lemma of Section 1.5. Let $x_{1}, \ldots, x_{k}$ be the variables occurring bound in the types of $f_{1}$ and $f_{2}$. By the Variable-Purification Lemma of Section 1.6, we have that

$$
f_{i}=h_{2}^{i} \circ f_{i}^{\prime} \circ h_{1}^{i}, \quad \text { for } i \in\{1,2\},
$$

where $f_{i}^{\prime}$ is variable-pure, while $h_{1}^{i}$ and $h_{2}^{i}$ are isomorphisms of GQDS. By choosing the same new variables $x_{1}^{\prime}, \ldots, x_{k}^{\prime}$ both for $f_{1}$ and $f_{2}$, we obtain that $h_{j}^{1}=h_{j}^{2}$ for $j \in\{1,2\}$. So if $f_{1}^{\prime}=f_{2}^{\prime}$, we will be able to derive $f_{1}=f_{2}$.

The Gentzen terms $f_{1}^{\prime}$ and $f_{2}^{\prime}$ are variable-pure, and hence by Renaming Elimination and the Cut-Elimination Theorem (see Sections 1.8-9) we can assume that they are cut-free and renaming-free. By the Eigendiversification Lemma of the preceding section, we can assume that they are moreover eigendiversified.

Let the quantity of a Gentzen term of GQDS be the sum of the number of predicate letters in its source (which is equal to the number of predicate letters in its target) with the number of occurrences of quantifier prefixes in its source and target. Then we proceed by induction on the quantity of $f_{2}^{\prime}$, which is equal to the quantity of $f_{1}^{\prime}$, in order to show that $f_{1}^{\prime}=f_{2}^{\prime}$. In the basis of this induction, if $n=1$, then $f_{1}^{\prime}=f_{2}^{\prime}=\mathbf{1}_{A}$, where $A$ is atomic. In the induction step we apply the invertibility lemmata of the preceding section (cf. [7], end of Section 7.9).

So GQDS is a preorder. And, as we explained at the beginning of Section 1.5 , we have proved thereby QDS Coherence.

## 2 Coherence of $\mathrm{QPN}^{\urcorner}$

### 2.1 The categories QPN $\urcorner$ and QPN

In this section we introduce the category $\mathbf{Q P N}\urcorner$ (here $\mathbf{P N}$ comes from "proof net"), which corresponds to the multiplicative fragment without propositional constants of classical linear first-order predicate logic without mix. This category extends with quantifiers the propositional category $\mathbf{P N}\urcorner$ of [8] (Section 2.2).

The category $\mathbf{Q P N}\urcorner$ is defined as the category QDS in Section 1.2 save that we make the following additions and changes. Instead of the language $\mathcal{L}$ of Section 1.1 we have the language $\mathcal{L}_{\neg}$, which differs from $\mathcal{L}$ by having the additional unary connective $\neg$. So in the definition of formula we have the additional clause
if $A$ is a formula, then $\neg A$ is a formula.
The objects of the category $\mathbf{Q P N}{ }^{\urcorner}$are the formulae of $\mathcal{L}_{\neg}$.
Let $\mathbf{x}_{n}$ stand for the sequence $x_{1}, \ldots, x_{n}$ when $n \geq 1$, and for the empty sequence when $n=0$. For $A$ a formula, let $A_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}$ stand for $A_{y_{1} \ldots y_{n}}^{x_{1} \ldots x_{n}}$ when $n \geq 1$, and for $A$ when $n=0$; so $A_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}$ is the result of a series of $n$ substitutions. We use $Q_{\mathbf{x}_{n}}$ as an abbreviation for $Q_{x_{n}} \ldots Q_{x_{1}}$ when $n \geq 1$, and for the empty sequence when $n=0$.

When $A$ is a formula containing free exactly the mutually different variables $\mathbf{x}_{n}$ in order of first occurrence counting from the left, we say that $\mathbf{x}_{n}$ is the freevariable sequence of $A$. For example, the free-variable sequence of $\forall_{y}(P y x \wedge$ $\left.\exists_{x} R z x z\right)$ is $x, z$ (provided $x, y$ and $z$ are all mutually different).

To define the arrow terms of $\mathbf{Q P N}\urcorner$, in the inductive definition we had for the arrow terms of $\mathbf{Q D S}$ we replace $\mathcal{L}$ by $\mathcal{L}_{\neg}$ and assume in addition that for all formulae $A$ and $B$ of $\mathcal{L}_{\neg}$, and for $\mathbf{x}_{n}$ being the free-variable sequence of $B$, the following primitive arrow terms

$$
\begin{aligned}
& \Delta_{B, A}^{\forall}: A \vdash A \wedge \forall_{\mathbf{x}_{n}}(\neg B \vee B), \\
& \Sigma_{B, A}^{\exists}: \exists_{\mathbf{x}_{n}}(B \wedge \neg B) \vee A \vdash A
\end{aligned}
$$

are arrow terms of $\mathbf{Q P N}\urcorner$. In other words, $\forall_{\mathbf{x}_{n}}(\neg B \vee B)$ is the universal closure of $\neg B \vee B$, and $\exists_{\mathbf{x}_{n}}(B \wedge \neg B)$ is the existential closure of $B \wedge \neg B$. We assume throughout the remaining text that $\Xi \in\{\Delta, \Sigma\}$.

We call the first index $B$ of $\Delta_{B, A}^{\forall}$ and $\Sigma_{B, A}^{\exists}$ the crown index, and the second index $A$ the stem index. The right conjunct $\forall_{\mathbf{x}_{n}}(\neg B \vee B)$ in the target of $\Delta_{B, A}^{\forall}$ is the crown of $\Delta_{B, A}^{\forall}$, and the left disjunct $\exists_{\mathbf{x}_{n}}(B \wedge \neg B)$ in the source of $\Sigma_{B, A}^{\exists}$ is the crown of $\Sigma_{B, A}^{\exists}$. We have analogous definitions of crown and stem indices, and crowns, for $\Sigma^{\forall}, \Delta^{\prime \forall}, \Sigma^{\prime \forall}, \Delta^{\exists}, \Sigma^{\nexists}$ and $\Delta^{\prime \exists}$, which will be introduced later. (The symbol $\Delta$ should be associated with the Latin dexter, because in $\Delta_{B, A}^{\forall}, \Delta_{B, A}^{\forall}$, $\Delta_{B, A}^{\exists}$ and $\Delta_{B, A}^{\ni}$ the crown is on the right-hand side of the stem; analogously, $\Sigma$ should be associated with sinister.)

Before we define the arrows of $\mathbf{Q P N}\urcorner$, we introduce a number of abbreviations:

$$
\begin{array}{ll}
\text { for } n=0 \text { and } \alpha \in\{\iota, \gamma\}, & \alpha_{B}^{Q_{\mathbf{x}_{n}}}={ }_{d f} \mathbf{1}_{B}, \\
\text { for } n>0, & \\
\text { for }(\alpha, Q) \in\{(\iota, \forall),(\gamma, \exists)\}, & \alpha_{B}^{Q_{\mathbf{x}_{n}}}={ }_{d f} \alpha_{B}^{Q_{\mathbf{x}_{n-1}}} \circ \alpha_{Q_{\mathbf{x}_{n-1}} B}^{Q_{x_{n}}}: Q_{\mathbf{x}_{n}} B \vdash B, \\
\text { for }(\alpha, Q) \in\{(\iota, \exists),(\gamma, \forall)\}, & \alpha_{B}^{Q_{\mathbf{x}_{n}}}={ }_{d f} \alpha_{Q_{\mathbf{x}_{n-1}} B}^{Q_{x_{n}}}{ }^{\circ} \alpha_{B}^{Q_{\mathbf{x}_{n-1}}}: B \vdash Q_{\mathbf{x}_{n}} B, \\
\text { for } n=0, & {[f]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}={ }_{d f} f,} \\
\text { for } n>0, & {[f]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}={ }_{d f}\left[[f]_{\mathbf{y}_{n-1}}^{\mathbf{x}_{n-1}}\right]_{y_{n}}^{x_{n}},}
\end{array}
$$

for $\mathbf{u}_{n}$ and $\mathbf{v}_{n}$ not free in $A$,

$$
\begin{aligned}
& \tau_{A, \mathbf{u}_{n}, \mathbf{v}_{n}}^{\forall_{\mathbf{x}_{n}}}={ }_{d f} \forall_{\mathbf{v}_{n}}\left[\iota_{A_{\mathbf{u}_{n}}^{\mathbf{u}_{n}}}^{\forall_{\mathbf{u}_{n}}}\right]_{\mathbf{v}_{n}}^{\mathbf{u}_{n}} \circ \gamma_{\forall_{\mathbf{u}_{n}} A_{\mathbf{u}_{n}}^{\mathbf{x}_{n}}}^{\forall_{\mathbf{v}_{n}}}: \forall_{\mathbf{u}_{n}} A_{\mathbf{u}_{n}}^{\mathbf{x}_{n}} \vdash \forall_{\mathbf{v}_{n}} A_{\mathbf{v}_{n}}^{\mathbf{x}_{n}}, \\
& \tau_{A, \mathbf{v}_{n}, \mathbf{u}_{n}}^{\exists_{\mathbf{x}_{n}}}={ }_{d f} \gamma_{\exists_{\mathbf{u}_{n}} A_{\mathbf{u}_{n}}^{\mathbf{x}_{n}}}^{\exists} \circ \exists_{\mathbf{v}_{n}}\left[\iota_{A_{\mathbf{u}_{n}}}^{\exists_{\mathbf{u}_{n}}}\right]_{\mathbf{v}_{n}}^{\mathbf{u}_{n}}: \exists \exists_{\mathbf{v}_{n}} A_{\mathbf{v}_{n}}^{\mathbf{x}_{n}} \vdash \exists_{\mathbf{u}_{n}} A_{\mathbf{u}_{n}}^{\mathbf{x}_{n}} .
\end{aligned}
$$

In QDS, for $n=0$ we have $\tau_{A, \mathbf{u}_{n}, \mathbf{v}_{n}}^{Q_{\times_{n}}}=\mathbf{1}_{A}$, and for $n>0$ we have

$$
\begin{aligned}
& \tau_{A, \mathbf{u}_{n}, \mathbf{v}_{n}}^{\forall_{\mathbf{x}_{n}}}=\tau_{\forall_{\mathbf{v}_{n-1}} A, u_{n}, v_{n}}^{\forall_{x_{n}}} \circ \forall_{u_{n}} \tau_{A, \mathbf{u}_{n-1}, \mathbf{v}_{n-1}}^{\forall_{\mathbf{x}_{n-1}}}, \\
& \tau_{A, \mathbf{v}_{n}, \mathbf{u}_{n}}^{\exists_{\mathbf{x}_{n}}}=\exists_{u_{n}} \tau_{A, \mathbf{v}_{n-1}, \mathbf{u}_{n-1}}^{\exists_{\mathbf{x}_{n-1}}}{ }^{\circ} \tau_{\exists_{\mathbf{v}_{n-1}} A, v_{n}, u_{n}}^{\exists_{x_{n}}}
\end{aligned}
$$

For $\mathbf{x}_{n}$ being the free-variable sequence of $B$, we have also the abbreviations

$$
\begin{aligned}
& \Sigma_{B, A}^{\forall}={ }_{d f} \hat{c}_{A, \forall \forall_{x_{n}}(\neg B \vee B)} \circ \Delta_{B, A}^{\forall}: A \vdash \forall_{\mathbf{x}_{n}}(\neg B \vee B) \wedge A, \\
& \Delta_{B, A}^{\exists}={ }_{d f} \Sigma_{B, A}^{\exists} \circ \check{c}_{\exists_{x_{n}}(B \wedge \neg B), A}: A \vee \exists_{\mathbf{x}_{n}}(B \wedge \neg B) \vdash A, \\
& \hat{\Delta}_{B, A}={ }_{d f}\left(\mathbf{1}_{A} \wedge \iota_{\neg B \vee B}^{\forall_{x_{n}}}\right) \circ \Delta_{B, A}^{\forall}: A \vdash A \wedge(\neg B \vee B), \\
& \check{\Sigma}_{B, A}={ }_{d f} \Sigma_{B, A}^{\exists} \circ\left(\iota_{B \wedge \neg B}^{\exists} \exists_{\mathbf{x}_{n}} \vee \mathbf{1}_{A}\right):(B \wedge \neg B) \vee A \vdash A, \\
& \hat{\Delta}_{B, A}^{\prime}={ }_{d f}\left(\mathbf{1}_{A} \wedge \check{c}_{B, \neg B}\right) \circ \hat{\Delta}_{B, A}: A \vdash A \wedge(B \vee \neg B), \\
& \check{\Sigma}_{B, A}^{\prime}={ }_{d f} \check{\Sigma}_{B, A} \circ\left(\hat{c}_{\neg B, B} \vee \mathbf{1}_{A}\right):(\neg B \wedge B) \vee A \vdash A .
\end{aligned}
$$

To define the arrows of $\mathbf{Q P N}\urcorner$ we assume in the inductive definition we had for the equations of QDS the following additional axiomatic equations:

| $\begin{aligned} & \left(\Delta^{\forall} n a t\right) \\ & \left(\Sigma^{\exists} n a t\right) \end{aligned}$ | $\begin{aligned} & \left(f \wedge \mathbf{1}_{\forall_{x_{n}}(\neg B \vee B)}\right) \circ \Delta_{B, A}^{\forall}=\Delta_{B, D}^{\forall} \circ f, \\ & \left.f \circ \Sigma_{B, A}^{\exists}=\Sigma_{B, D}^{\exists} \circ \mathbf{1}_{\exists_{x_{n}}(B \wedge \neg B)}^{\forall} \vee f\right), \end{aligned}$ |
| :---: | :---: |
| $\left(\hat{b} \Delta^{\forall}\right)$ | $\hat{b}_{A, B, \forall_{x_{n}}(\neg C \vee C)}^{\leftarrow} \Delta_{C, A \wedge B}^{\forall}=\mathbf{1}_{A} \wedge \Delta_{C, B}^{\forall}$, |
| $\left(\Sigma^{\text {a }}{ }^{\text { }}\right.$ ) | $\Sigma_{C, B \vee A}^{\exists} \circ \check{b}_{\exists_{x_{n}}(C \wedge \neg C), B, A}^{\leftarrow}=\Sigma_{C, B}^{\exists} \vee \mathbf{1}_{A}$, |
| $\left(d \Sigma^{\forall}\right)$ | $d_{\forall_{\times_{n}}(\neg A \vee A), B, C} \circ \Sigma_{A, B \vee C}^{\forall}=\Sigma_{A, B}^{\forall} \vee \mathbf{1}_{C}$, |
| $\left(d \Delta^{\exists}\right)$ | $\Delta_{A, C \wedge B}^{\exists}{ }^{\circ} d_{C, B, \exists_{x_{n}}(A \wedge \neg A)}=\mathbf{1}_{C} \wedge \Delta_{A, B}^{\exists}$, |
| ( $\check{\Sigma} \hat{\Delta}$ ) | $\check{\Sigma}_{A, A} \circ d_{A, \neg A, A} \circ \hat{\Delta}_{A, A}=\mathbf{1}_{A}$, |
| ( $\Sigma^{\prime} \hat{\Delta}^{\prime}$ ) | $\check{\Sigma}_{A, \neg A}^{\prime} \circ d_{\neg A, A, \neg A} \circ \hat{\Delta}_{A, \neg A}^{\prime}=\mathbf{1}_{\neg A}$, |
| $\left(\right.$ ren $\Xi^{Q}$ ) | $\left[\Xi_{B, A}^{Q}\right]_{y}^{x}=\Xi_{B, A_{y}^{x}}^{Q}$, for $\Xi^{Q} \in\left\{\Delta^{\forall}, \Sigma^{\exists}\right\}$, |
| $(\Delta \tau)$ | $\Delta_{B_{\mathbf{v}_{n}}^{\times_{n}}, A}^{\forall}=\left(\mathbf{1}_{A} \wedge \tau_{\neg B \vee B, \mathbf{u}_{n}, \mathbf{v}_{n}}^{\forall \times_{\times_{n}}}\right) \circ \Delta_{B_{\mathbf{u}_{n}}^{\times_{n}, A}}^{\forall}$, |
| $(\Sigma \tau)$ | $\Sigma_{B_{\mathbf{v}_{n}}^{\times_{n}}, A}^{\exists}=\Sigma_{B_{\mathbf{u}_{n}}^{\times_{n}}, A^{\circ}}{ }^{\circ}\left(\tau_{B \wedge \neg B, \mathbf{v}_{n}, \mathbf{u}_{n}}^{\exists} \vee \mathbf{1}_{A}\right)$. |

The equation (ren $\alpha$ ) of Section 1.2 does not hold when $\alpha$ is $\Delta^{\forall}$ or $\Sigma^{\exists}$, but instead we have the equations (ren $\Xi^{Q}$ ) above. This defines the category $\left.\mathbf{Q P N}\right\urcorner$.

In this list of axiomatic equations the equations ( $\check{\Sigma} \hat{\Delta})$ and ( $\check{\Sigma}^{\prime} \tilde{\Delta}^{\prime}$ ) are taken as they stand from [8] (Section 2.2), where they were used to axiomatize the category $\mathbf{P N}\urcorner$. The preceding first six axiomatic equations of $\mathbf{P N}\urcorner$ are obtained from the first six axiomatic equations of $\mathbf{Q P N}\urcorner$ above by replacing $\Delta^{\forall}$ and $\Sigma^{\exists}$ with $\hat{\Delta}$ and $\check{\Sigma}$ respectively, and by deleting quantifier prefixes. It is clear that we can derive these axiomatic equations of $\mathbf{P N}\urcorner$ in $\mathbf{Q P N}\urcorner$, and hence we have in $\mathbf{Q P N}\urcorner$ all the equations of $\mathbf{P N}\urcorner$, with $A, B, C, \ldots$ being formulae of the language $\mathcal{L}_{\neg}$. The really new axiomatic equations of $\left.\mathbf{Q P N}\right\urcorner$ are only the last displayed (ren $\Xi^{Q}$ ) and $(\Xi \tau)$.

We have in $\mathbf{Q P N}\urcorner$ the additional abbreviations

$$
\begin{aligned}
& \Delta_{B, A}^{\prime \forall}={ }_{d f}\left(\mathbf{1}_{A} \wedge \forall_{\mathbf{x}_{n}} \check{c}_{B, \neg B}\right) \circ \Delta_{B, A}^{\forall}: A \vdash A \wedge \forall_{\mathbf{x}_{n}}(B \vee \neg B), \\
& \Sigma_{B, A}^{\prime \exists}={ }_{d f} \Sigma_{B, A}^{\exists}{ }^{\ominus}\left(\exists_{\mathbf{x}_{n}} \hat{c}_{\neg B, B} \vee \mathbf{1}_{A}\right): \exists_{\mathbf{x}_{n}}(\neg B \wedge B) \vee A \vdash A, \\
& \Sigma_{B, A}^{\prime \forall}={ }_{d f} \hat{c}_{A, \forall_{\mathbf{x}_{n}}(B \vee \neg B)} \circ \Delta_{B, A}^{\prime \forall}: A \vdash \forall_{\mathbf{x}_{n}}(B \vee \neg B) \wedge A, \\
& \Delta_{B, A}^{\prime \exists}={ }_{d f} \Sigma_{B, A}^{\prime \exists} \circ \check{c}_{\exists_{\mathbf{x}_{n}}(\neg B \wedge B), A}: A \vee \exists_{\mathbf{x}_{n}}(\neg B \wedge B) \vdash A,
\end{aligned}
$$

and as in [8] (see Section 2.2) we have also the abbreviations

$$
\begin{aligned}
& \hat{\Sigma}_{B, A}={ }_{d f} \hat{c}_{A, \neg B \vee B} \circ \hat{\Delta}_{B, A}: A \vdash(\neg B \vee B) \wedge A, \\
& \check{\Delta}_{B, A}={ }_{d f} \check{\Sigma}_{B, A} \circ \check{c}_{B \wedge \neg B, A}: A \vee(B \wedge \neg B) \vdash A, \\
& \hat{\Sigma}_{B, A}^{\prime}={ }_{d f} \hat{c}_{A, B \vee \neg B} \circ \hat{\Delta}_{B, A}^{\prime}: A \vdash(B \vee \neg B) \wedge A, \\
& \check{\Delta}_{B, A}^{\prime}={ }_{d f} \check{\Sigma}_{B, A}^{\prime} \circ \check{c}_{\neg B \wedge B, A}: A \vee(\neg B \wedge B) \vdash A .
\end{aligned}
$$

Note that the equations $(\Xi \tau)$ say that we could define our arrows $\Delta_{B, A}^{\forall}$ and $\Sigma_{B, A}^{\exists}$ in terms of such arrows with the proviso that the set of variables free in the stem index $A$ and the set variables free in the crown index $B$ are disjoint.

In QDS and $\mathbf{Q P N}\urcorner$ for $f: B \vdash \forall_{\mathbf{x}_{n}} A$ and $g: \exists_{\mathbf{x}_{n}} A \vdash B$ such that the variables $\mathbf{x}_{n}$ are not free in $B$ we have the equations

$$
\begin{aligned}
& \forall_{\mathbf{x}_{n}}\left(\iota_{A}^{\forall_{\mathbf{x}_{n}}} \circ f\right) \circ \gamma_{B}^{\forall_{\mathbf{x}_{n}}}=f, \\
& \gamma_{B}^{\exists_{\mathbf{x}_{n}}} \circ \exists_{\mathbf{x}_{n}}\left(g \circ \iota_{A}^{\exists_{\mathbf{x}_{n}}}\right)=g,
\end{aligned}
$$

which generalize the equations $(\forall e x t)$ and $(\exists e x t)$ of Section 1.2. These equations, together with equations of QDS analogous to the equations $(Q \tau \iota)$ of Section 1.3 and the isomorphism of $\tau^{Q_{\mathbf{u}_{n}}}$, entail the following cancellation implications:
$(\forall \iota$ canc $)$ if $\left[\iota_{A}^{\forall_{\mathbf{x}_{n}}}\right]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}} \circ f_{1}=\left[\iota_{A}^{\forall_{x_{n}}}\right]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}} \circ f_{2}$, then $f_{1}=f_{2}$,
$(\exists \iota$ canc $)$ if $g_{1} \circ\left[\iota_{A}^{\exists_{\mathbf{x}_{n}}}\right]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}=g_{2} \circ\left[\iota_{A}^{\exists_{\mathbf{x}_{n}}}\right]_{\mathbf{y}_{n}}^{\mathbf{x}_{n}}$, then $g_{1}=g_{2}$,
provided the variables $\mathbf{y}_{n}$ are not free in the source of $f_{1}$ and $f_{2}$ and in the target of $g_{1}$ and $g_{2}$.

In $\mathbf{Q P N}\urcorner$ we have stem-increasing equations analogous to the stem-increasing equations of [8] (Section 2.5; $\hat{\Delta}$ and $\Sigma \Sigma$ are replaced by $\Delta^{\forall}$ and $\Sigma^{\exists}$ respectively, which entails further adjustments). The equations $\left(\hat{b} \Delta^{\forall}\right),\left(\hat{b} \Sigma^{\exists}\right),\left(d \Sigma^{\forall}\right)$ and $\left(d \Delta^{\exists}\right)$ are such stem-increasing equations (when read from right to left), and there are further such equations for all the arrows $\Xi_{B, A}^{Q}$ and $\Xi_{B, A}^{\prime}$.

We have in $\mathbf{Q P N}\urcorner$ the following additional stem-increasing equations:

$$
\begin{aligned}
& \left(\forall \Delta^{\forall}\right) \quad \forall_{x} \Delta_{B, A}^{\forall}=\forall_{x}\left(\iota_{A}^{\forall x} \wedge \mathbf{1}_{\forall_{\mathbf{x}_{n}}(\neg B \vee B)}\right) \circ \gamma_{\forall_{x} A \wedge \forall_{\mathbf{x}_{n}}(\neg B \vee B)}^{\forall_{x}}{ }^{\circ} \Delta_{B, \forall_{x} A}^{\forall}, \\
& \left(\exists \Delta^{\forall}\right) \\
& \exists_{x} \Delta_{B, A}^{\forall}=\hat{\theta}_{A, \forall_{\mathbf{x}_{n}}(\neg B \vee B)}^{\exists_{x}}{ }^{\circ} \Delta_{B, \exists_{x} A}^{\forall}, \\
& \left(\forall \Sigma^{\exists}\right) \quad \forall_{x} \Sigma_{B, A}^{\exists}=\Sigma_{B, \forall_{x} A}^{\exists} \circ \check{c}_{\exists_{\mathbf{x}_{n}}(B \wedge \neg B), \forall_{x} A} \circ \check{\theta}_{A, \exists_{\mathbf{x}_{n}}(B \wedge \neg B)}^{\forall_{x}}{ }^{\circ} \forall_{x} \check{c}_{A, \exists_{\mathbf{x}_{n}}(B \wedge \neg B)}, \\
& \left(\exists \Sigma^{\exists}\right) \quad \exists_{x} \Sigma_{B, A}^{\exists}=\Sigma_{B, \exists_{x} A}^{\exists}{ }^{\circ} \gamma_{\exists_{x_{n}}(B \wedge \neg B) \vee \exists_{x} A}^{\exists}{ }^{\exists} \exists_{x}\left(\mathbf{1}_{\exists_{x_{n}}(B \wedge \neg B)} \vee \iota_{A}^{\exists_{x}}\right),
\end{aligned}
$$

which are derived with the help of the implications ( $Q \iota$ canc), QDS equations, QDS Coherence and the naturality of $\Delta^{\forall}$ and $\Sigma^{\exists}$ in their stem index.

We introduce next a category called QPN, for which we will establish in Section 2.6 that it is equivalent to the category QPN $\urcorner$. The category QPN is for us an auxiliary category (though it is closer to the formulation of linear logic in [10]). We prove coherence for this category in Section 2.5, and from that and the equivalence of $\mathbf{Q P N}\urcorner$ and $\mathbf{Q P N}$ we infer coherence for $\mathbf{Q P N}\urcorner$ in Section 2.7. The category $\mathbf{Q P N}$ is very much like $\mathbf{Q P N}\urcorner$ save that in its objects the negation connective $\neg$ is prefixed only to atomic formulae. The arrow terms $\Delta_{B, A}^{\forall}$ and $\Sigma_{B, A}^{\exists}$ are primitive only for the crown index $B$ being an atomic formula. Here is a more formal definition of QPN.

For $\mathcal{P}$ being the set of letters that we used to generate $\mathcal{L}$ and $\mathcal{L}_{\neg}$ in Sections 1.1 and 2.1 , let $\mathcal{P}^{\urcorner}$be the set of predicate letters $\{\neg P \mid P \in \mathcal{P}\}$. The arity of the new predicate letter $\neg P$ is the same as the arity of $P$. The objects of QPN are the formulae of the first-order language $\mathcal{L}^{\neg P}$ generated from $\left.\mathcal{P} \cup \mathcal{P}\right\urcorner$ in the same way as $\mathcal{L}$ was generated from $\mathcal{P}$ in Section 1.1.

To define the arrow terms of $\mathbf{Q P N}$, in the inductive definition we had for the arrow terms of QDS we replace $\mathcal{L}$ by $\mathcal{L}^{\neg P}$, and we assume in addition that for every formula $A$ of $\mathcal{L}^{\neg P}$, for every predicate letter $P \in \mathcal{P}$ of arity $n$, and for $\mathbf{x}_{n^{\prime}}^{\prime}$ being the free-variable sequence of $P \mathbf{x}_{n}$,

$$
\begin{aligned}
& \Delta_{P \mathbf{x}_{n}, A}^{\forall}: A \vdash A \wedge \forall_{\mathbf{x}_{n^{\prime}}^{\prime}}\left(\neg P \mathbf{x}_{n} \vee P \mathbf{x}_{n}\right), \\
& \Sigma_{P \mathbf{x}_{n}, A}^{\exists}: \exists_{\mathbf{x}_{n^{\prime}}^{\prime}}\left(P \mathbf{x}_{n} \wedge \neg P \mathbf{x}_{n}\right) \vee A \vdash A
\end{aligned}
$$

are primitive arrow terms of QPN.
To define the arrows of QPN, we assume as additional axiomatic equations in the inductive definition we had for the equations of QDS all the additional axiomatic equations assumed above for $\mathbf{Q P N}$, but restricted to the arrow terms $\Delta_{P \mathbf{x}_{n}, A}^{\forall}$ and $\Sigma_{P \mathbf{x}_{n}, A}^{\exists}$ whose crown index is atomic. This defines the category QPN.

### 2.2 Development for QDS, QPN $\urcorner$ and QPN

If $\beta$ is a primitive arrow term of $\mathbf{Q P N}\urcorner$ except $\mathbf{1}_{B}$, then we call $\beta$-terms of $\mathbf{Q P N}\urcorner$ the set of arrow terms defined inductively as follows: $\beta$ is a $\beta$-term; if $f$ is a $\beta$-term, then for every $A$ in $\mathcal{L}_{\neg}$ and all variables $x$ and $y$ we have that $\mathbf{1}_{A} \xi f, f \xi \mathbf{1}_{A}, Q_{x} f$ and $[f]_{y}^{x}$ are $\beta$-terms, provided $[f]_{y}^{x}$ is defined.

In a $\beta$-term the subterm $\beta$ is called the head of this $\beta$-term. For example, the head of the $\Delta_{B, C}^{\forall}$-term $\mathbf{1}_{A} \wedge \forall_{x}\left(\left[\Delta_{B, C}^{\forall}\right]_{z}^{y} \vee \mathbf{1}_{E}\right)$ is $\Delta_{B, C}^{\forall}$.

We define 1-terms like $\beta$-terms; we just replace $\beta$ in the definition above by $\mathbf{1}_{B}$. So $\mathbf{1}$-terms are headless.

An arrow term of the form $f_{n} \circ \ldots \circ f_{1}$, where $n \geq 1$, with parentheses tied to $\circ$ associated arbitrarily, such that for every $i \in\{1, \ldots, n\}$ we have that $f_{i}$ is composition-free is called factorized. In a factorized arrow term $f_{n} \circ \ldots \circ f_{1}$ the
arrow terms $f_{i}$ are called factors. A factor that is a $\beta$-term for some $\beta$ is called a headed factor. A factorized arrow term is called headed when each of its factors is either headed or a 1 -term. A factorized arrow term $f_{n} \circ \ldots \circ f_{1}$ is called developed when $f_{1}$ is a 1 -term and if $n>1$, then every factor of $f_{n} \circ \ldots \circ f_{2}$ is headed. Analogous definitions of $\beta$-term and developed arrow term can be given for QDS.

We have the following lemma for QDS.
Development Lemma. For every arrow term $f$ there is a developed arrow term $f^{\prime}$ such that $f=f^{\prime}$.

Proof. This lemma would be easy to prove by using the categorial and functorial equations together with the equation (ren $\circ$ ) of Section 1.2 if for (ren $\circ$ ), as for the other of these equations, we had that the right-hand side is defined whenever the left-hand side is defined. Since this need not be the case, we must first eliminate renaming. This can be achieved by relying on the category GQDS, cut elimination and the result of Section 1.8.

We adapt the definitions of $\beta$-term and developed arrow term to the category GQDS of Section 1.5. For every arrow term $g$ of GQDS there is a Gentzen term $g^{\prime}$ denoting the arrow $g$. By the Variable-Purification Lemma of Section 1.6, we have that $g^{\prime}$ is equal to $h_{2} \circ g^{\prime \prime} \circ h_{2}$ where $h_{1}$ and $h_{2}$ are compositions of $\tau$-terms and $g^{\prime \prime}$ is a variable-pure Gentzen term, which by Renaming Elimination and the Cut-Elimination Theorem (see Sections 1.8-9) we may assume to be cut-free and renaming-free. Then by using the categorial and functorial equations it is easy to obtain from $h_{2} \circ g^{\prime \prime} \circ h_{2}$ a developed arrow term $g^{\prime \prime \prime}$ of GQDS equal to the initial arrow term $g$.

For an arbitrary arrow term $f: A \vdash B$ of QDS we find a diversified arrow term $f^{\prime}: A^{\prime} \vdash B^{\prime}$ of QDS such that $f$ is a letter-for-letter substitution instance of $f^{\prime}$ (see the beginning of Section 1.5). As in [7] (Sections 3.2-3) we pass by a functor $H_{\mathcal{G}}$ from $f$ to the arrow term $H_{\mathcal{G}} f$ of GQDS, which, as we have shown above, is equal to a developed arrow term $\left(H_{\mathcal{G}} f\right)^{\prime \prime \prime}$ of GQDS. By applying a functor $H$ in the opposite direction we obtain a developed arrow term $H\left(\left(H_{\mathcal{G}} f\right)^{\prime \prime \prime}\right)$ of QDS, which we call $h$. The type of $h$ is $A^{\prime \prime} \vdash B^{\prime \prime}$, where $A^{\prime \prime}$ and $B^{\prime \prime}$ belong to the same form sets as $A^{\prime}$ and $B^{\prime}$ respectively. So by QDS Coherence we have that $f^{\prime}=j_{2} \circ h \circ j_{1}$, where $j_{1}$ and $j_{2}$ are headed factorized arrow terms of QDS whose heads are of the $\stackrel{\xi}{b}$ and $\stackrel{\xi}{c}$ kind. We obtain the arrow term of QDS equal to $f$ as a letter-for-letter substitution instance of $j_{2} \circ h \circ j_{1}$.

By relying on various renaming equations of QDS, we can prove a Refined Development Lemma for QDS, which differs from the Development Lemma by requiring that in the developed arrow term $f^{\prime}$ renaming occurs only in subterms of the form $\left[\iota_{A}^{Q_{x}}\right]_{y}^{x}$ for $x$ different from $y$ and free in $A$. With the help of the Refined Development Lemma for QDS, the stem-increasing equations of QPN $\urcorner$
(see the preceding section), together with the naturality of $\Delta^{\forall}$ and $\Sigma^{\exists}$ in their stem index and the equations (ren $\Xi^{Q}$ ), we can prove the Refined Development Lemma, and hence also the Development Lemma, for the categories $\mathbf{Q P N}{ }^{\urcorner}$and QPN too.

The Refined Development Lemma is not only important because of the applications it will find latter in this paper. It is also important because we can conclude from it that renaming, except in $\left[\iota_{A}^{Q_{x}}\right]_{y}^{x}$ for $x$ different from $y$ and free in $A$, is eliminable in $\mathbf{Q D S}, \mathbf{Q P N}\urcorner$ and $\mathbf{Q P N}$. This elimination of renaming is not straightforward, but is achieved in a roundabout way, involving cut elimination. The eliminability of renaming may perhaps serve to explain why it is neglected as a primitive rule of inference in logic.

### 2.3 Some properties of QDS

In this section we establish some results concerning the category QDS of Section 1.2 , which we will use to prove coherence for $\mathbf{Q P N}$ and $\mathbf{Q P N}\urcorner$. First we introduce a definition.

Suppose $X$ is the $n$-th occurrence of a predicate letter (counting from the left) in a formula $A$ of $\mathcal{L}$, and $Y$ is the $m$-th occurrence of the same predicate letter in a formula $B$ of $\mathcal{L}$. Then we say that $X$ and $Y$ are tied in an arrow $f: A \vdash B$ of QDS when $(n-1, m-1) \in G f$ (see Section 1.5 ; note that to find the $n$-th occurrence we count starting from 1 , but the ordinal $n>0$ is $\{0, \ldots, n-1\}$ ). It is easy to establish that every occurrence of a predicate letter in $A$ is tied to exactly one occurrence of the same letter in $B$, and vice versa. This is related to matters about diversification mentioned at the beginning of Section 1.5.

For the lemma below, let $X$ in $A$ and $Y$ in $B$ be occurrences of the same predicate letter tied in an arrow $f: A \vdash B$ of QDS, and let $S^{A}$ and $S^{B}$ be two finite (possibly empty) sequences of quantifier prefixes. Then by an easy induction on the complexity of $f$ we can prove the following, which generalizes Lemma 2 of Section 2.4 of [8].
$\wedge \vee$ Lemma. It is impossible that $A$ has a subformula $S^{A} X \mathbf{x}_{n} \wedge A^{\prime}$ or $A^{\prime} \wedge S^{A} X \mathbf{x}_{n}$ while $B$ has a subformula $S^{B} Y \mathbf{y}_{n} \vee B^{\prime}$ or $B^{\prime} \vee S^{B} Y \mathbf{y}_{n}$.

For the next lemma, for $i \in\{1,2\}$ let $X_{i}$ in $A$ and $Y_{i}$ in $B$ be occurrences of the predicate letter $P_{i}$ tied in an arrow $f: A \vdash B$ of $\mathbf{Q D S}$ (here $P_{1}$ and $P_{2}$ may also be the same predicate letter).
$\vee \wedge$ Lemma. For every $i, j \in\{1,2\}$, it is impossible that $A$ has a subformula $X_{i} \mathbf{y}_{n} \vee X_{3-i} \mathbf{z}_{m}$ while $B$ has a subformula $Y_{j} \mathbf{u}_{k} \wedge Y_{3-j} \mathbf{v}_{l}$.

This lemma, exactly analogous to Lemma 3 of Section 2.4 of [8], is a corollary of lemmata exactly analogous to Lemmata 3D and 3C of Section 2.4 of [8], which are easily proved by induction on the complexity of the arrow term $f$.

As a matter of fact, the $\wedge \vee$ and $\vee \wedge$ Lemmata above could be proved by supposing the contrary and deleting quantifiers and individual variables together with arrow terms and operations on arrow terms involving them, which would yield arrow terms contradicting Lemma 2 and Lemma 3 respectively of Section 2.4 of [8]. The $\wedge \vee$ Lemma is related to the acyclicity condition of proof nets, while the $\vee \wedge$ Lemma is related to the connectedness condition (see [8], Sections 2.4, 7.1, and references therein).

Next we can prove the following lemma.
$P-Q-R$ Lemma. Let $f: A \vdash B$ be an arrow of QDS, let $X_{i}$ for $i \in\{1,2,3\}$ be occurrences of the predicate letters $P, Q$ and $R$, respectively, in $A$, and let $Y_{i}$ be occurrences of $P, Q$ and $R$, respectively, in $B$, such that $X_{i}$ and $Y_{i}$ are tied in $f$. Let, moreover, $X_{2} \mathbf{x}_{q}^{2} \vee X_{3} \mathbf{x}_{r}^{3}$ be a subformula of $A$ and $Y_{1} \mathbf{y}_{p}^{1} \wedge Y_{2} \mathbf{y}_{q}^{2}$ a subformula of $B$. Then there is a $d_{P \mathbf{z}_{p}^{1}, Q \mathbf{z}_{q}^{2}, R \mathbf{z}_{r}^{3}}$-term $h: A^{\prime} \vdash B^{\prime}$ such that $X_{i}^{\prime}$ are occurrences of $P, Q$ and $R$, respectively, in the source $P \mathbf{z}_{p}^{1} \wedge\left(Q \mathbf{z}_{q}^{2} \vee R \mathbf{z}_{r}^{3}\right)$ of the head of $h$ and $Y_{i}^{\prime}$ are occurrences of $P, Q$ and $R$, respectively, in the target $\left(P \mathbf{z}_{p}^{1} \wedge Q \mathbf{z}_{q}^{2}\right) \vee R \mathbf{z}_{r}^{3}$ of the head of $h$, such that for some arrows $f_{x}: A \vdash A^{\prime}$ and $f_{y}: B^{\prime} \vdash B$ of QDS we have $f=f_{y} \circ h \circ f_{x}$ in QDS, and $X_{i}$ is tied to $X_{i}^{\prime}$ in $f_{x}$, while $Y_{i}^{\prime}$ is tied to $Y_{i}$ in $f_{y}$.

This lemma, exactly analogous to the $p-q-r$ Lemma of [8] (Section 2.4), is proved like this previous lemma by relying on the Gentzenization of GQDS.

### 2.4 Some properties of QPN

In this section, by relying on the results of the preceding section, we establish some results concerning the category QPN introduced at the end of Section 2.1, which we will find useful for calculations later on. For these results we need to introduce the following.

Let QDS ${ }^{\neg P}$ be the category defined as QDS save that it is generated not by $\mathcal{P}$, but by $\mathcal{P} \cup \mathcal{P}\urcorner$ (see the end of Section 2.1). So the objects of $\mathbf{Q D S}^{\neg P}$ are the formulae of $\mathcal{L}^{\neg P}$, i.e. the objects of $\mathbf{Q P N}$. For $A$ and $B$ formulae of $\mathcal{L}^{\neg P}$, we define when an occurrence of the predicate letter $P$ in $A$ is tied to an occurrence of $P$ in $B$ in an arrow $f: A \vdash B$ of $\mathbf{Q D S}^{\neg P}$ analogously to what we had at the beginning of the preceding section.

We say that a finite (possibly empty) sequence $S$ of quantifier prefixes is foreign to a formula $B$ when the set of variables occurring in $S$ is disjoint from the set of free variables of $B$. If $S$ is foreign to $B$, then there is an isomorphism $j_{S, B}: B \vdash S B$ of $\mathbf{Q D S}{ }^{\neg P}$; defined in terms of $\gamma^{\forall}$ and $\iota^{\exists}$. The inverse of $j_{S, B}$ is the arrow $j_{S, B}^{\leftarrow}: S B \vdash B$ of $\mathbf{Q D S}{ }^{\neg P}$ defined in terms of $\iota^{\forall}$ and $\gamma^{\exists}$. By QDS Coherence, these isomorphisms are unique.

We introduce next a generalization of the arrow terms $\Xi_{B, A}^{Q}$ obtained by letting the arrow terms $\Xi_{B, A}^{Q}$ "absorb" various QDS arrow terms. These generalized term have the right form for the first two technical results below-the
$\Xi^{Q}$-Permutation Lemmata.
For $S$ and $S\urcorner$ two independent finite sequences of quantifier prefixes both foreign to $P \mathbf{x}_{n}$, and for $I$ being the sequence of indices $\left.P \mathbf{x}_{n}, A, \mathbf{y}_{m}, S, S\right\urcorner$, we have

$$
\begin{aligned}
& \left.A \vdash A \wedge \forall \mathbf{y}_{m}(S\urcorner \neg P \mathbf{x}_{n} \vee S P \mathbf{x}_{n}\right), \\
& \Sigma_{I}^{\exists}={ }_{d f} \Sigma_{P \mathbf{x}_{n}, A}^{\exists} \circ\left(\left(\gamma_{\exists_{\mathbf{x}_{n}}\left(P \mathbf{x}_{n} \wedge \neg P \mathbf{x}_{n}\right)}^{\exists \mathbf{y}_{\mathbf{x}_{2}}}{ }^{\circ} \exists_{\mathbf{y}_{m}}\left(\iota_{P \mathbf{x}_{n} \wedge \neg P \mathbf{x}_{n}}^{\exists_{\mathbf{x}_{n}}}{ }^{\circ}\left(j_{S, P \mathbf{x}_{n}}^{\overleftarrow{ }} \wedge j_{S\urcorner, \neg P \mathbf{x}_{n}}\right)\right)\right) \vee \mathbf{1}_{A}\right): \\
& \exists_{\mathbf{y}_{m}}\left(S P \mathbf{x}_{n} \wedge S \neg \neg P \mathbf{x}_{n}\right) \vee A \vdash A .
\end{aligned}
$$

The analogous abbreviations

$$
\begin{aligned}
& \left.\Sigma_{I}^{\forall}: A \vdash \forall_{\mathbf{y}_{m}}(S\urcorner \neg P \mathbf{x}_{n} \vee S P \mathbf{x}_{n}\right) \wedge A, \\
& \left.\Delta_{I}^{\exists}: A \vee \exists_{\mathbf{y}_{m}}\left(S P \mathbf{x}_{n} \wedge S\right\urcorner \neg P \mathbf{x}_{n}\right) \vdash A, \\
& \left.\Delta_{I}^{\prime \forall}: A \vdash A \wedge \forall_{\mathbf{y}_{m}}\left(S P \mathbf{x}_{n} \vee S\right\urcorner \neg P \mathbf{x}_{n}\right), \\
& \left.\Sigma_{I}^{\prime \exists}: \exists \exists_{\mathbf{y}_{m}}(S\urcorner \neg P \mathbf{x}_{n} \wedge S P \mathbf{x}_{n}\right) \vee A \vdash A, \\
& \left.\Sigma_{I}^{\prime \forall}: A \vdash \forall_{\mathbf{y}_{m}}\left(S P \mathbf{x}_{n} \vee S\right\urcorner \neg P \mathbf{x}_{n}\right) \wedge A, \\
& \left.\Delta_{I}^{\ni}: A \vee \exists_{\mathbf{y}_{m}}(S\urcorner \neg P \mathbf{x}_{n} \wedge S P \mathbf{x}_{n}\right) \vdash A
\end{aligned}
$$

are defined in terms of $\Delta^{\forall}$ and $\Sigma^{\exists}$ like the analogous abbreviations of Section 2.1. The right conjunct $\forall_{\mathbf{y}_{m}}\left(S \neg \neg P \mathbf{x}_{n} \vee S P \mathbf{x}_{n}\right)$ in the target of $\Delta_{I}^{\forall}$ is the crown of $\Delta_{I}^{\forall}$, and analogously for $\Sigma_{I}^{\exists}$ and the other abbreviations, replacing "right" by "left", "conjunct" by "disjunct", and "target" by "source", as appropriate. Note that here $\mathbf{y}_{m}$ is an arbitrary sequence of variables, and not necessarily the free-variable sequence of $P \mathbf{x}_{n}$ as in $\Xi_{P \mathbf{x}_{n}, A}^{Q}$ (see Section 2.1).

The definition of $\Delta_{I}^{\forall}$ above is of the form $f \circ \Delta_{P \mathbf{x}_{n}, A}^{\forall}$. By QDS Coherence, instead of the arrow term $f$ of $\mathbf{Q D S}^{\neg P}$ we could have used for this definition any other arrow term $g$ of $\mathbf{Q D S}{ }^{P}$ of the same type as $f$ such that $G f=G g$, since we have $g=f$, and analogously for $\Sigma_{I}^{\exists}$, etc.

Let $\Xi, \Theta \in\left\{\Delta, \Delta^{\prime}, \Sigma, \Sigma^{\prime}\right\}$, and let a $\Xi_{I}^{Q}$-term be defined as a $\beta$-term in Section 2.2 save that $\beta$ is replaced by $\Xi_{I}^{Q}$, and the clause "if $f$ is a $\Xi_{I}^{Q}$-term, then $[f]_{y}^{x}$ is a $\Xi_{I}^{Q}$-term" is omitted. Then we have the following analogue of the $\hat{\Xi}$-Permutation Lemma of [8] (Section 2.5).
$\Xi^{\forall}$-Permutation Lemma. Let $g: C \vdash D$ be $a \Xi_{P_{\mathbf{x}_{n}}, A, \mathbf{y}_{m}, S, S^{\neg}}^{\forall}$-term of QPN such that $X_{1}$ and $\neg X_{2}$ are respectively the occurrences within $D$ of the predicate letters $P$ and $\neg P$ in the crown of the head $\Xi_{\left.P \mathbf{x}_{n}, A, \mathbf{y}_{m}, S, S\right\urcorner}^{\forall}$ of $g$, and let $f: D \vdash E$ be an arrow term of $\mathbf{Q D S}^{\neg P}$ such that we have an occurrence $Y_{1}$ of $P$ and an occurrence $\neg Y_{2}$ of $\neg P$ within a subformula of $E$ of the form
 quences of quantifier prefixes, and $X_{i}$ is tied to $Y_{i}$ for $i \in\{1,2\}$ in $f$. Then

 term $f^{\prime}: C \vdash D^{\prime}$ of $\mathbf{Q D S}{ }^{\neg P}$ such that in $\mathbf{Q P N}$ we have $f \circ g=g^{\prime} \circ f^{\prime}$.

Proof. We proceed in principle as for the proof of the $\hat{\Xi}$-Permutation Lemma in [8], with some adjustments and additions. We appeal to the Refined Development Lemma for $\mathbf{Q P N}{ }^{\urcorner}$(see Section 2.2), and we use the $\wedge \vee$ Lemma of the preceding section to ascertain that cases involving "problematic" $d_{\left.A, S P \mathbf{x}_{n}, S\right\urcorner \neg P \mathbf{x}_{n}-}$ terms or $d_{A, S \neg \neg P \mathbf{x}_{n}, S P \mathbf{x}_{n}}$-terms in the developed arrow term $f$ are excluded.

We rely then on equations analogous to the equations mentioned in the proof of the $\hat{\Xi}$-Permutation Lemma, where $\hat{\Xi}_{p, A}$ is replaced by $\Xi_{I}^{\forall}$, which entails further adjustments. Such equations, which are either stem-increasing, or related to the stem-increasing equations, or are simply consequences of definitions, are established with the help of the implications ( $Q \iota$ canc) together with the equations $(\Xi \tau)$ (see Section 2.1) and QDS Coherence. We rely also on the remark we made before the lemma concerning the alternative definitions of $\Xi_{I}^{\forall}$.

We have a dual lemma, called the $\Xi^{\exists}$-Permutation Lemma, analogous to the氕-Permutation Lemma of [8] (Section 2.5), which involves $\Xi_{I}^{\exists}$-terms instead of $\Xi_{I}^{\forall}$-terms.

Next we have a lemma analogous to the $p-\neg p-p$ Lemma of [8] (Section 2.5).
$P-\neg P-P$ Lemma. Let $X_{1}, \neg X_{2}$ and $X_{3}$ be occurrences of the predicate letters $P$, $\neg P$ and $P$, respectively, in a formula $A$ of $\mathcal{L}^{\neg p}$, and let $Y_{1}, \neg Y_{2}$ and $Y_{3}$ be occurrences of $P, \neg P$ and $P$, respectively, in a formula $B$ of $\mathcal{L}^{\neg p}$. Let $g_{1}: A^{\prime} \vdash A$ be a $\Xi_{P \mathbf{x}_{n}, A, \mathbf{y}_{m}, S, S \neg-t e r m}^{\forall}$ of $\mathbf{Q P N}$ such that $\forall_{\mathbf{y}_{m}}\left(S \neg \neg X_{2} \mathbf{x}_{n} \vee S X_{3} \mathbf{x}_{n}\right)$ or $\forall_{\mathbf{y}_{m}}\left(S X_{3} \mathbf{x}_{n} \vee\right.$ $\left.S\urcorner \neg X_{2} \mathbf{x}_{n}\right)$ is the crown of the head of $g_{1}$, let $g_{2}: B \vdash B^{\prime}$ be a $\Theta_{P \mathbf{x}_{n}^{\prime}, A^{\prime}, \mathbf{y}_{m^{\prime}}^{\prime}, S^{\prime}, S^{ᄀ^{\prime}}}^{\exists}$ term of $\mathbf{Q P N}$ such that $\exists_{\mathbf{y}_{m^{\prime}}^{\prime}}\left(S^{\prime} Y_{1} \mathbf{x}_{n}^{\prime} \wedge S^{\left.\neg^{\prime} \neg Y_{2} \mathbf{x}_{n}^{\prime}\right) \text { or } \exists_{\mathbf{y}_{m^{\prime}}^{\prime}}\left(S^{\neg^{\prime}} \neg Y_{2} \mathbf{x}_{n}^{\prime} \wedge S^{\prime} Y_{1} \mathbf{x}_{n}^{\prime}\right)}\right.$ is the crown of the head of $g_{2}$, and let $f: A \vdash B$ be an arrow term of $\mathbf{Q D S}{ }^{\neg P}$ such that $X_{i}$ and $Y_{i}$ are tied in for $i \in\{1,2,3\}$. Then $g_{2} \circ f \circ g_{1}$ is equal in QPN to an arrow term of $\mathbf{Q D S}{ }^{\neg P}$.

The proof of this lemma is analogous to the proof of the $p-\neg p-p$ Lemma in [8]. We use the $P-Q-R$ Lemma of the preceding section and the $\Xi^{Q}$-Permutation Lemmata instead of the $p-q-r$ Lemma and the $\stackrel{\xi}{\Xi}$-Permutation Lemmata, and we apply the equation $(\check{\Sigma} \hat{\Delta})$ of Section 2.1.

We establish in the same manner the $\neg P-P-\neg P$ Lemma, analogous to the $\neg p-p-\neg p$ Lemma of [8] (Section 2.5). The formulation of the $\neg P-P-\neg P$ Lemma is obtained from that of the $P-\neg P-P$ Lemma by replacing the sequence $P, \neg P, P$ by the sequence $\neg P, P, \neg P$, which entails that $S$ and $S^{\neg}$, as well as $S^{\prime}$ and $S^{\neg^{\prime}}$, are permuted. The $\neg P-P-\neg P$ Lemma is proved by applying the $P-Q-R$ Lemma, the $\Xi^{Q}$-Permutation Lemmata and the equation $\left(\Sigma^{\prime} \hat{\Delta}^{\prime}\right)$ of Section 2.1.

### 2.5 QPN Coherence

In [8] (Section 2.3) one can find a detailed definition of a category called Br, whose objects are finite ordinals, and whose arrows are graphs sometimes called Kelly-Mac Lane graphs (because of [12]). These graphs may also be found in [2] (from whose author the name of $B r$ is derived). We define a functor $G$ from $\mathbf{Q P N}\urcorner$ or $\mathbf{Q P N}$ into $B r$ as we defined in [8] (Section 2.3) an identically named functor from the categories $\mathbf{P} \mathbf{N}^{\urcorner}$and $\mathbf{P N}$ into $B r$, without paying attention to variables and quantifier prefixes. This means that $G Q_{x} A=G A$ (so that $G A$ is the number of occurrences of predicate letters in the formula $A$ ), the arrow $G \alpha$ for $\alpha$ being $\iota_{A}^{Q_{x}}, \gamma_{A}^{Q_{x}}, \check{\theta}_{A, D}^{\forall_{x} \rightarrow}$ and $\hat{\theta}_{A, D}^{\exists x} \overleftarrow{D}$ is an identity arrow, $G Q_{x} f=G[f]_{y}^{x}=G f$, while $G \Delta_{B, A}^{\forall}$ and $G \Sigma_{B, A}^{\exists}$ are like $G \hat{\Delta}_{B, A}$ and $G \check{\Sigma}_{B, A}$ respectively. The category Rel mentioned in Section 1.5 is a subcategory of $B r$, and $G$ restricted to the QDS part of QPN $\urcorner$ and QPN coincides with the functor $G$ from QDS to Rel.

The theorems that the functors $G$ from $\mathbf{Q P N}\urcorner$ or $\mathbf{Q P N}$ into $B r$ are faithful functors are called QPN $\urcorner$ Coherence and QPN Coherence respectively. We establish first QPN Coherence, and $\mathbf{Q P N}\urcorner$ Coherence will be derived from it in Section 2.7.

We prove QPN Coherence by proceeding as for the proof of PN Coherence in [8] (Section 2.7), through lemmata analogous to the Confrontation and Purification Lemmata. Roughly speaking, the analogue of the Confrontation Lemma says that a $\Delta_{P \mathbf{x}_{n}, A}^{\forall}$-term, called a $\Delta^{\forall}$-factor, and a $\Sigma_{P \mathbf{y}_{n}, B^{\prime}}^{\exists}$-term, called a $\Sigma^{\exists}$-factor, mutually tied in a direct manner through the crowns, which are called confronted factors, can be permuted with the help of stem-increasing and naturality equations so that they are ready to get eliminated by applying the $P-\neg P-P$ and $\neg P-P-\neg P$ Lemmata of Section 2.4. The analogue of the Purification Lemma states that this elimination can be pursued until we obtain an arrow term without confronted factors, such an arrow term being called pure.

For the proof of these analogues of the Confrontation and Purification Lemmata we need the Refined Development Lemma for QPN of Section 2.2. We also need the stem-increasing equations for $\Delta^{\forall}$ and $\Sigma^{\exists}$ (see Section 2.1) and the naturality of $\Delta^{\forall}$ and $\Sigma^{\exists}$ in the stem index. Where in the proof of the Purification Lemma in [8] (Section 2.7) we appealed to Lemma 3, we now appeal to the $\vee \wedge$ Lemma of Section 2.3. Instead of the $p-\neg p-p$ and $\neg p-p-\neg p$ Lemmata we now have the $P-\neg P-P$ and $\neg P-P-\neg P$ Lemmata.

The equation $(\Delta \tau)$ of Section 2.1 is essential, together with the stem-increasing equations and the naturality of $\Delta^{\forall}$ and $\Sigma^{\exists}$ in the stem index, to guarantee that if there is a $\Delta^{\forall}$-factor in a pure arrow term $f$, then

$$
f=f^{\prime} \circ \Delta_{P \mathbf{x}_{n}, A}^{\forall}
$$

for any sequence of variables $\mathbf{x}_{n}$. The equation $(\Sigma \tau)$ of Section 2.1 is needed to establish an analogous equation for $\Sigma^{\exists}$-factors. By pushing in this manner to the extreme right the $\Delta^{\forall}$-factors remaining in a pure arrow term, and to the
extreme left the remaining $\Sigma^{\exists}$-factors, and by relying on QDS Coherence, we establish QPN Coherence.

### 2.6 The equivalence of $\mathrm{QPN}^{\urcorner}$and QPN

To prove that $\mathbf{Q P N}\urcorner$ and $\mathbf{Q P N}$ are equivalent categories we proceed as in [8] (Section 2.6), with the following adjustments and additions.

When we define the functor $F$ from $\mathbf{Q P N}\urcorner$ to $\mathbf{Q P N}$ we have the following new clauses on objects:

$$
\begin{aligned}
& F A=A, \quad \text { for } A \text { of the form } P \mathbf{x}_{n} \text { or } \neg P \mathbf{x}_{n}, \\
& F Q_{x} A=Q_{x} F A, \\
& F \neg \forall_{x} A=\exists_{x} F \neg A, \\
& F \neg \exists_{x} A=\forall_{x} F \neg A .
\end{aligned}
$$

On arrows we have first new clauses analogous to the old clauses where $\alpha$ is $\iota^{Q_{x}}$, $\gamma^{Q_{x}}, \check{\theta}^{\forall_{x} \rightarrow}$ and $\hat{\theta}^{\exists x} \leftarrow$, while $\hat{\Delta}$ and $\check{\Sigma}$ are replaced by $\Delta^{\forall}$ and $\Sigma^{\exists}$, the letter $p$ is replaced by $P \mathbf{x}_{n}$, and some further adjustments are made. We have moreover the following new clauses:
if $x$ is free in $B$,

$$
\begin{aligned}
F \Delta_{\forall_{x} B, A}^{\forall}=\left(\mathbf { 1 } _ { A } \wedge \left(\forall _ { \mathbf { x } _ { n - 1 } } \left(\check { c } _ { \exists _ { x } F \neg B , \forall _ { x } F B } \circ \check { \theta } _ { F B , \exists _ { x } F \neg B } ^ { \forall _ { x } \rightarrow } \forall _ { x } \left(\check{c}_{F B, \exists_{x} F \neg B} \circ\right.\right.\right.\right. \\
\left.\left.\left.\left.\left(\iota_{F \neg B}^{\exists_{x}} \vee \mathbf{1}_{F B}\right)\right)\right) \circ h\right)\right) \circ F \Delta_{B, A}^{\forall},
\end{aligned}
$$

where

$$
h: \forall_{x_{n-1}} \ldots \forall_{x_{i+1}} \forall_{x} \forall_{x_{i}} \ldots \forall_{x_{1}}(F \neg B \vee F B) \vdash \forall_{\mathbf{x}_{n-1}} \forall_{x}(F \neg B \vee F B)
$$

is an isomorphism of $\mathbf{Q D S}^{\neg P}$ (see the preceding section) generalizing isomorphisms of the type $\forall_{x} \forall_{y} C \vdash \forall_{y} \forall_{x} C$,
if $x$ is not free in $B$,

$$
F \Delta_{\forall_{x} B, A}^{\forall}=\left(\mathbf{1}_{A} \wedge \forall_{\mathbf{x}_{n}}\left(\iota_{F \neg B}^{\exists_{x}} \vee \gamma_{F B}^{\forall_{x}}\right)\right) \circ F \Delta_{B, A}^{\forall},
$$

if $x$ is free in $B$, for $h$ as above,

$$
F \Delta_{\exists_{x} B, A}^{\forall}=\left(\mathbf{1}_{A} \wedge\left(\forall_{\mathbf{x}_{n-1}}\left(\tilde{\theta}_{F \neg B, \exists_{x} F B}^{\forall \forall_{x} \rightarrow} \circ \forall_{x}\left(\mathbf{1}_{F \neg B} \vee \iota_{F B}^{\exists x}\right)\right) \circ h\right)\right) \circ F \Delta_{B, A}^{\forall},
$$

if $x$ is not free in $B$,

$$
F \Delta_{\exists_{x} B, A}^{\forall}=\left(\mathbf{1}_{A} \wedge \forall_{\mathbf{x}_{n}}\left(\gamma_{F \neg B}^{\forall} \vee \iota_{F B}^{\exists_{x}}\right)\right) \circ F \Delta_{B, A}^{\forall},
$$

and dual clauses for $F \Sigma_{\forall_{x} B, A}^{\exists}$ and $F \Sigma_{\exists_{x} B, A}$,
$F Q_{x} f=Q_{x} F f$,
$F[f]_{y}^{x}=[F f]_{y}^{x}$.

This defines the functor $F$.
For $f$ an arrow term of $\mathbf{Q P N}{ }^{\urcorner}$we have that $G F f$ coincides with $G f$, where $G$ in $G F f$ is the functor $G$ from $\mathbf{Q P N}$ to $B r$, and $G$ in $G f$ is the functor $G$ from $\mathbf{Q P N}\urcorner$ to $B r$ (see the beginning of the preceding section). To show that, it is essential to check that $G F \Delta_{B, A}^{\forall}$ and $G F \Sigma_{B, A}^{\exists}$ coincide with $G \Delta_{B, A}^{\forall}$ and $G \Sigma_{B, A}^{\exists}$ respectively, which is done by induction on the complexity of the crown index $B$.

Then we can easily verify that $F$, as defined above, is indeed a functor. If $f=g$ in $\mathbf{Q P N}{ }^{\urcorner}$, then $G f=G g$, and hence, as we have just seen, $G F f=G F g$. By QPN Coherence of the preceding section, we conclude that $F f=F g$ in QPN. (To verify that the functor $F$ from $\mathbf{P N}{ }^{\urcorner}$to $\mathbf{P N}$ in Section 2.6 of [8] is a functor we could have proceeded analogously, by establishing PN Coherence first, before introducing the functor $F$. We did not need the functor $F$ to prove PN Coherence. This would make the exposition in [8] somewhat simpler, and better organized.)

We define a functor $F\urcorner$ from $\mathbf{Q P N}$ to $\mathbf{Q P N}\urcorner$ by stipulating that $F\urcorner A=A$ and $F\urcorner f=f$. To show that $\mathbf{Q P N}\urcorner$ and $\mathbf{Q P N}$ are equivalent categories via the functors $F$ and $F\urcorner$ we proceed as in [8] (Section 2.6) with the following additions. We have the following auxiliary definitions in $\mathbf{Q P N}\urcorner$, for $\mathbf{x}_{n}$ being the free-variable sequence of $A$ (see Section 2.1), and $\mathbf{y}_{m}$ being this sequence with $x$ omitted (if $x$ is free in $A$, then $m=n-1$; otherwise, $\mathbf{x}_{n}$ is $\mathbf{y}_{m}$ and $m=n$ ):

$$
\begin{aligned}
& q_{A}^{\forall_{x} \rightarrow}={ }_{d f} \Sigma_{\forall_{x} A, \exists_{x} \neg A}^{\prime \exists} \circ\left(\iota_{\neg \forall_{x} A \wedge \forall_{x} A}^{\exists_{\mathbf{y}_{m}}} \vee \mathbf{1}_{\exists_{x} \neg A}\right) \circ d_{\neg \forall_{x} A, \forall_{x} A, \exists_{x} \neg A} \text { 。 } \\
& \left(\mathbf{1}_{\neg \forall_{x} A} \wedge\left(\check{\theta}_{A, \exists_{x} \neg A}^{\forall_{x} \rightarrow} \circ \forall_{x}\left(\left(\mathbf{1}_{A} \vee \iota_{\neg A}^{\exists_{x}}\right) \circ \iota_{A \vee \neg A}^{\forall_{\mathbf{x}_{n}}}\right) \circ \gamma_{\forall_{\mathbf{x}_{n}}(A \vee \neg A)}^{\forall_{x}}\right)\right) \circ \Delta_{A, \neg \forall_{x} A}^{\prime \forall} \text { : } \\
& \neg \forall_{x} A \vdash \exists_{x} \neg A, \\
& q_{A}^{\forall_{x} \leftarrow}={ }_{d f} \Sigma_{A, \neg \forall_{x} A}^{\exists} \circ\left(\left(\gamma_{\exists_{\mathbf{x}_{n}}(\neg A \wedge A)}^{\exists_{x}} \circ \exists_{x}\left(\iota_{\neg A \wedge A}^{\exists_{\mathbf{x}_{n}}} \circ\left(\mathbf{1}_{\neg A} \wedge \iota_{A}^{\forall_{x}}\right)\right) \circ \hat{\theta}_{\neg A, \forall_{x} A}^{\exists_{x} \leftarrow}\right) \vee \mathbf{1}_{\neg \forall_{x} A}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \exists_{x} \neg A \vdash \neg \forall_{x} A,
\end{aligned}
$$

and we have analogous definitions of

$$
\begin{aligned}
& q_{A}^{\exists_{x} \rightarrow}: \neg \exists_{x} A \vdash \forall_{x} \neg A, \\
& q_{A}^{\exists_{x} \leftarrow}: \forall_{x} \neg A \vdash \neg \exists_{x} A .
\end{aligned}
$$

It can be shown that $q_{A}^{Q_{x} \rightarrow}$ is an isomorphism, with inverse $q_{A}^{Q_{x}} \leftarrow$.
Next in the inductive definitions of the isomorphisms $i_{A}: A \vdash F A$ and $i_{A}^{-1}$ : $F A \vdash A$ we have the following clauses in addition to clauses in [8] (Section 2.6):

$$
\begin{array}{ll}
\quad i_{A}=i_{A}^{-1}=\mathbf{1}_{A}, & \text { if } A \text { is } P_{\mathbf{x}_{n}} \text { or } \neg P_{\mathbf{x}_{n}}, \\
i_{Q_{x} A}=Q_{x} i_{A}, & i_{Q_{x} A}^{-1}=Q_{x} i_{A}^{-1},
\end{array}
$$

$$
\begin{array}{ll}
i_{\neg \forall_{x} A}=\exists_{x} i_{\neg A \circ} \circ q_{A}^{\forall_{x} \rightarrow}, & i_{\neg \forall_{x} A}^{-1}=q_{A}^{\forall_{x} \leftarrow \circ} \circ \exists_{x} i_{\neg A}^{-1}, \\
i_{\neg \exists_{x} A}=\forall_{x} i_{\neg A \circ} \circ q_{A}^{\exists_{x} \rightarrow}, & i_{\neg \exists_{x} A}^{-1}=q_{A}^{\exists_{x} \leftarrow \circ \forall_{x} i_{\neg A}^{-1} .} .
\end{array}
$$

We can then extend the proof of the Auxiliary Lemma of Section 2.6 of [8] in order to establish that for $f: A \vdash B$ we have in $\mathbf{Q P N}\urcorner$ the equation $f=i_{B}^{-1} \circ F f \circ i_{A}$. In this extended proof, for the isomorphism $n_{B}^{\leftarrow}: B \vdash \neg \neg B$ of $\mathbf{Q P N}\urcorner$, we need the following equation of $\mathbf{Q P N}\urcorner$ :

$$
\left(\Delta^{\forall} n\right) \quad \Delta_{\neg B, A}^{\forall}=\left(\mathbf{1}_{A} \wedge \forall_{\mathbf{x}_{n}}\left(n \overleftarrow{B} \vee \mathbf{1}_{\neg B}\right)\right) \circ \Delta_{B, A}^{\prime},
$$

analogous to the equation ( $\hat{\Delta} n$ ) of [8] (Section 2.6, Proof of the Auxiliary Lemma). To derive ( $\Delta^{\forall} n$ ) we use, analogously to what we had before for the derivation of $(\hat{\Delta} n)$, the stem-increasing equation $\left(\forall \Delta^{\forall}\right)$ of Section 2.1, the naturality of $\Delta^{\forall}$ in the stem index, the $\neg P-P-\neg P$ Lemma of the preceding section and QDS Coherence. (In the derivation of $(\hat{\Delta} n)$ in the printed text of [8], Section 2.6, Proof of the Auxiliary Lemma, "(with $p$ replaced by $A$ )" is a misprint for "(with $p$ replaced by $B$ )".) We derive similarly an equation analogous to the equation ( $\hat{\Delta} r$ ) of $[8]$ (Section 2.6, Proof of the Auxiliary Lemma) involving $\Delta^{\forall}$.

We need also the following equations of $\mathbf{Q P N}\urcorner$, analogous to the clauses defining $F \Delta_{\forall_{x} B, A}^{\forall}$ above:
if $x$ is free in $B$,

$$
\begin{aligned}
& \Delta_{\forall_{x} B, A}^{\forall}=\left(\mathbf { 1 } _ { A } \wedge \left(\forall _ { \mathbf { x } _ { n - 1 } } \left(( q _ { B } ^ { \forall } \leftarrow \vee \mathbf { 1 } _ { \forall _ { x } B } ) \circ \check { c } _ { \exists _ { x } \neg B , \forall _ { x } B } \circ \check { \theta } _ { B , \exists _ { x } \neg B } ^ { \forall _ { x } } \circ \forall _ { x } \left(\check{c}_{B, \exists_{x} \neg B} \circ\right.\right.\right.\right. \\
&\left(\iota_{\neg B}^{\left.\left.\left.\left.\left.\exists \exists_{x} \vee \mathbf{1}_{B}\right)\right)\right) \circ h\right)\right) \circ \Delta_{B, A}^{\forall},}\right.
\end{aligned}
$$

where

$$
h: \forall_{x_{n-1}} \ldots \forall_{x_{i+1}} \forall_{x} \forall_{x_{i}} \ldots \forall_{x_{1}}(\neg B \vee B) \vdash \forall_{\mathbf{x}_{n-1}} \forall_{x}(\neg B \vee B)
$$

is an isomorphism of $\mathbf{Q D S}{ }{ }^{P}$,
if $x$ is not free in $B$,

$$
\Delta_{\forall_{x} B, A}^{\forall}=\left(\mathbf{1}_{A} \wedge \forall_{\mathbf{x}_{n}}\left(\left(q_{B}^{\forall_{x} \leftarrow} \vee \mathbf{1}_{\forall_{x} B}\right) \circ\left(\iota_{\neg B}^{\exists_{x}} \vee \gamma_{B}^{\forall x}\right)\right)\right) \circ \Delta_{B, A}^{\forall} .
$$

The idea for the derivation of these equations is the same as the idea for the derivation of $\left(\Delta^{\forall} n\right)$ above. We need also equations analogous to the clauses defining $F \Delta_{\exists_{x} B, A}^{\forall}$ above.

To show that in QPN ${ }^{\urcorner}$

$$
[f]_{y}^{x}=i_{B_{y}^{x}}^{-1} \circ F[f]_{y}^{x} \circ i_{A_{y}^{x}}
$$

we need the equation $\left[i_{A}\right]_{y}^{x}=i_{A_{y}^{x}}$ of $\left.\mathbf{Q P N}\right\urcorner$, which is established by induction on the complexity of $A$. For this induction we use the equation

$$
\left[\hat{\Delta}_{B, A}\right]_{y}^{x}=\hat{\Delta}_{B_{y}^{x}, A_{y}^{x}}
$$

and analogous equations of $\mathbf{Q P N}\urcorner$. The last displayed equation is established with the help of the equations (ren $\left.\Delta^{\forall}\right)$ and $(\Delta \tau)$ of Section 2.1. We need also the equation

$$
\left[q_{A}^{\forall_{z} \rightarrow}\right]_{y}^{x}=q_{A_{y}^{x}}^{\forall_{z} \rightarrow}
$$

of $\mathbf{Q P N}\urcorner$, for which we use the equations ( $\Xi \tau$ ) (see Section 2.1) and QDS Coherence. This suffices to establish that the categories $\mathbf{Q P N}\urcorner$ and $\mathbf{Q P N}$ are equivalent.

### 2.7 QPN $\urcorner$ Coherence

As we said at the beginning of Section 2.5, $\mathbf{Q P N}\urcorner$ Coherence is the theorem that the functor $G$ from $\mathbf{Q P N}{ }^{\urcorner}$to the category $B r$ is faithful. We can then prove $\mathbf{Q P N}\urcorner$ Coherence as follows.

Proof of $\mathbf{Q P N}\urcorner$ Coherence. Suppose that for $f$ and $g$ arrows of $\mathbf{Q P N}\urcorner$ of the same type we have $G f=G g$. Then, as we noted after the definition of the functor $F$ from $\mathbf{Q P N}\urcorner$ to $\mathbf{Q P N}$ in the preceding section, we have $G F f=G F g$, and hence $F f=F g$ in QPN by QPN Coherence of Section 2.5. It follows that $f=g$ in $\mathbf{Q P} \mathbf{N}^{\urcorner}$by the equivalence of the categories $\mathbf{Q P N}{ }^{\urcorner}$and $\mathbf{Q P N}$ established in the preceding section.

With $\mathbf{Q P N}{ }^{\urcorner}$Coherence we can establish easily equations of $\left.\mathbf{Q P N}\right\urcorner$ whose derivation may otherwise be quite demanding. We have, for example, the following equations in $\mathbf{Q P N}\urcorner$ :

$$
\begin{aligned}
& \hat{\theta}_{A, D}^{\forall_{x} \rightarrow}=\left(\left(\forall_{x}\left(\check{\Delta}_{D, A} \circ d_{A, D, \neg D}^{R}\right) \circ \hat{\theta}_{A \vee D, \neg D}^{\forall_{x} \leftarrow}\right) \vee \mathbf{1}_{D}\right) \circ d_{\forall_{x}(A \vee D), \neg D, D} \circ \hat{\Delta}_{D, \forall_{x}(A \vee D)}, \\
& \hat{\theta}_{A, D}^{\exists_{x} \leftarrow}=\check{\Delta}_{D, \exists_{x}(A \wedge D)}^{\prime} \circ d_{\exists_{x}(A \wedge D), \neg D, D}^{R} \circ\left(\left(\check{\theta}_{A \wedge D, \neg D}^{\exists} \rightarrow \exists_{x}\left(d_{A, D, \neg D} \circ \hat{\Delta}_{D, A}^{\prime}\right)\right) \wedge \mathbf{1}_{D}\right)
\end{aligned}
$$

(see the end Section 1.2 for the definitions of $\hat{\theta}_{A \vee D, \neg D}^{\forall} \leftarrow{ }^{\forall}$ and $\breve{\theta}_{A \wedge D, \neg D}^{\exists x \rightarrow}$ ). These equations say that the distributivity arrow terms $\tilde{\theta}_{A, D}^{\forall_{x} \rightarrow}$ and $\hat{\theta}_{A, D}^{\exists x \leftarrow}$ are definable in $\mathbf{Q P N}{ }^{\urcorner}$in terms of the remaining primitive arrow terms and operations on arrow terms. If these distributivity arrow terms are taken as defined when we introduce $\mathbf{Q P N}{ }^{\urcorner}$, then the equations $(Q \stackrel{\xi}{\theta} \theta)$ of Section 1.2 become superfluous as axioms - they can be derived from the remaining axiomatic equations.

We define a contravariant endofunctor of $\mathbf{Q P N}\urcorner$, i.e. a functor from $\mathbf{Q P N}\urcorner$ to $\mathbf{Q P} \mathbf{N}^{\circ p}$, in the following manner, for $f: A \vdash B$,

$$
\neg f={ }_{d f} \check{\Sigma}_{B, \neg A^{\prime}}^{\prime} d_{\neg B, B, A} \circ\left(\mathbf{1}_{\neg B} \wedge\left(f \vee \mathbf{1}_{\neg A}\right)\right) \circ \hat{\Delta}_{A, \neg B}^{\prime}: \neg B \vdash \neg A,
$$

and we verify that this is indeed a contravariant functor by proceeding as in [8] (Section 2.8, where there is also an alternative definition of $\neg f$ ). In the course of this verification, we establish easily with the help of $\mathbf{Q P N}\urcorner$ Coherence that $\Xi^{Q}$ is a dinatural transformation in the crown index (see [18], Section IX.4, for the notion of dinatural transformation).

## 3 Coherence of QMDS and QMPN ${ }^{\urcorner}$

### 3.1 QMDS Coherence

The category QMDS is defined as the category QDS in Section 1.2 save that we have the additional primitive arrow terms

$$
m_{A, B}: A \wedge B \vdash A \vee B
$$

for all formulae $A$ and $B$ of $\mathcal{L}$, and we assume the following additional equations:

$$
\begin{array}{ll}
(m \text { nat }) & (f \vee g) \circ m_{A, B}=m_{D, E} \circ(f \wedge g), \quad \text { for } f: A \vdash D \text { and } g: B \vdash E, \\
(\hat{b} m) & m_{A \wedge B, C} \circ \hat{b}_{A, B, C}=d_{A, B, C} \circ\left(\mathbf{1}_{A} \wedge m_{B, C}\right), \\
(\check{b} m) & \check{b}_{C, B, A} \circ m_{C, B \vee A}=\left(m_{C, B} \vee \mathbf{1}_{A}\right) \circ d_{C, B, A}, \\
(c m) & m_{B, A} \circ \hat{c}_{A, B}=\check{c}_{B, A} \circ m_{A, B} .
\end{array}
$$

The proof-theoretical principle underlying $m_{A, B}$ is called mix (see the Gentzen operation below, and [7], Section 8.1, where references are given).

To obtain the functor $G$ from QMDS to the category Rel (see Section 1.5), or to the category $B r$ (see Section 2.5), we extend the definition of the functor $G$ from QDS to Rel by adding the clause that says that $G m_{A, B}$ is an identity arrow. To prove that this functor $G$ is faithful-this result is called QMDS Coherence - we extend the proof of QDS Coherence of the first part of this paper.

The Gentzenization of QMDS is obtained with the category GQMDS, which has an additional Gentzen operation

$$
\frac{f: U \vdash Z}{\operatorname{mix}(f, g)=_{d n}(f \vee g) \circ m_{U, Y}: U \vdash Y \vdash Z \vee W}
$$

The Cut-Elimination Theorem is proved for GQMDS by enlarging the proof we had for GQDS in Section 1.9 with an additional case dealt with in [7] (Section 8.4). The preparation for this Cut-Elimination Theorem involving variablepurity is not impeded by the presence of mix.

To prove the invertibility lemmata we need for GQMDS we rely on the following equations of GQMDS:

$$
\left(\operatorname{mix} Q^{S}\right)
$$

$$
\operatorname{mix}\left(Q_{x, X}^{S} f_{1}, f_{2}\right)=Q_{x, X}^{S} \operatorname{mix}\left(f_{1}^{\prime}, f_{2}\right)
$$

for $f_{1}^{\prime}$ being as for $\left(\xi Q^{S}\right)$ in Section 1.10 and $S \in\{L, R\}$. These equations are either straightforward to derive, or when $Q^{S} \in\left\{\forall^{R}, \exists^{L}\right\}$ we derive them by imitating the derivation of the equation of case $(\forall 2)$ of Section 1.9, with the help of the equations $(Q \beta$ red $)$ and ( $Q \eta$ red) (see the end of Section 1.5). To prove the new Invertibility Lemmata for $\wedge$ and $\vee$ we enlarge the proofs of such
invertibility lemmata we had for GQDS in Section 1.10 with cases involving mix covered by the remarks preceding the Invertibility Lemma for mix in [7] (Section 8.4). The proofs of the new Invertibility Lemmata for $\forall^{R}$ and $\exists^{L}$ are taken over unchanged. To prove the new Invertibility Lemmata for $\forall^{L}$ and $\exists^{R}$ we use in addition the equations ( $\operatorname{mix} \forall^{L}$ ) and ( $\operatorname{mix} \exists^{R}$ ) respectively.

We need moreover a new Invertibility Lemma for mix, analogous to the lemma with the same name in [7] (Section 8.4). The proof of this new lemma is based on the proof in [7] and on the equations $\left(\operatorname{mix} Q^{S}\right)$. This suffices to establish QMDS Coherence.

### 3.2 QMPN $\urcorner$ Coherence

We introduce now the category $\mathbf{Q M P N}{ }^{\urcorner}$, which corresponds to the multiplicative fragment without propositional constants of classical linear first-order predicate logic with mix. The category $\mathbf{Q M P N}\urcorner$ is defined as the category QPN $\urcorner$ in Section 2.1 save that we have the additional primitive arrow terms $m_{A, B}: A \wedge B \vdash A \vee B$ for all formulae $A$ and $B$ of $\mathcal{L}_{\neg}$, and we assume as additional equations ( $m$ nat) , $(\hat{b} m),(\check{b} m)$ and $(c m)$ of the preceding section. To obtain the functor $G$ from $\mathbf{Q M P N}\urcorner$ to the category $B r$ we extend what we had for the functor $G$ from $\mathbf{Q P N}\urcorner$ to $B r$ (see Section 2.5) with the clause that says that $G m_{A, B}$ is an identity arrow. The theorem asserting that this functor is faithful is called QMPN $\urcorner$ Coherence.

The category QMPN is defined as the category QPN at the end of Section 2.1 save that we have the additional primitive arrow terms $m_{A, B}$ for all objects $A$ and $B$ of QPN, and we assume the additional equations ( $m$ nat), ( $\hat{b} m$ ), ( $\check{b} m$ ) and $(c m)$. We can prove that $\mathbf{Q M P N}\urcorner$ and $\mathbf{Q M P N}$ are equivalent categories as in Section 2.6, with trivial additions.

The proof of QMPN $\urcorner$ Coherence is then reduced to the proof of QMPN Coherence, and the latter proof can be obtained quite analogously to what we have in [8] (Sections 6.1-2). Here are some remarks concerning additions and changes.

The problem here is that the $\wedge \vee$ Lemma of Section 2.3, which was used for proving the $\Xi^{\forall}$-Permutation Lemma for QPN in Section 2.4, does not hold for QMDS (the $\vee \wedge$ Lemma of Section 2.3 holds for QMDS). We can nevertheless prove a modified version of the $\Xi^{\forall}$-Permutation Lemma, where we assume that $Y_{1}$ and $Y_{2}$ occur within a subformula of $E$ of the form $\neg P \mathbf{x}_{n}^{\prime} \wedge\left(Y_{1} \mathbf{x}_{n}^{\prime} \vee \neg Y_{2} \mathbf{x}_{n}^{\prime}\right)$ or $P \mathbf{x}_{n}^{\prime} \wedge\left(\neg Y_{2} \mathbf{x}_{n}^{\prime} \vee Y_{1} \mathbf{x}_{n}^{\prime}\right)$. For the proof of this modified version of the $\Xi^{\forall}$ Permutation Lemma we rely on some auxiliary results, which we will now consider.

Let us call quasi-atomic formulae of $\mathcal{L}$ all formulae of the form $S P \mathbf{x}_{n}$ for $S$ a finite sequence of quantifier prefixes, i.e. formulae in which $\wedge$ and $\vee$ do not occur. For $X$ a particular occurrence of a predicate letter in a formula $A$ such that there is a subformula of the form $B \xi C$ or $C \xi B$ of $A$ where $C$ is a quasi-
atomic formula in which $X$ occurs, let $A^{-X}$ be obtained from $A$ by replacing the particular subformula $B \xi C$ or $C \xi B$ by $B$. When $X$ occurs in $A$ as we have just said we say that $X$ is deletable from $A$.

For $i \in\{1,2\}$, let $A_{i}$ be a formula of $\mathcal{L}$, let $X_{i}$ be an occurrence of the predicate letter $P$ deletable in $A_{i}$, and let $X_{1}$ and $X_{2}$ be tied in the arrow $f: A_{1} \vdash A_{2}$ of QMDS (see the beginning of Section 2.3 for the meaning of "tied"). The new version of Lemma 1 of Section 6.1 of [8] then says that there is an arrow term $f^{-P}: A_{1}^{-X_{1}} \vdash A_{2}^{-X_{2}}$ of QMDS such that $G f^{-P}$ is obtained from $G f$ by deleting the pair corresponding to $\left(X_{1}, X_{2}\right)$. In the proof of Lemma 1 of Section 6.1 in the printed version of [8] there is an omission. The last sentence of the first paragraph should be replaced by: "If $x_{i}$ is not a proper subformula of the subformula $B_{j}$, then $d_{B_{1}, q, B_{3}}^{-q}$ is $m_{B_{1}, B_{3}}$ or $f^{-q}$ is $\mathbf{1}_{A_{i}^{-x_{i}}}$." The proof of the new version of Lemma 1 is then analogous to the old proof with the addition in the induction step that when $f$ is $g \xi h$ or $h \xi g$, then $f^{-P}$ is $g$ not only when $h$ is equal to $\mathbf{1}_{P \mathbf{x}_{n}}$, but also when it is of a type $B_{1} \vdash B_{2}$ such that $B_{i}$ is a quasiatomic subformula of $A_{i}$ in which $X_{i}$ occurs. Note that this deleting lemma does not hold for QDS, because we cannot cover $d_{B_{1}, S P \mathbf{x}_{n}, B_{3}}^{-P}$.

A context $Z$ is obtained from a formula of $\mathcal{L}$ by replacing a particular occurrence of an atomic subformula with a place holder $\square$. We write $Z(A)$ for the formula obtained by putting the formula $A$ at the place of $\square$ in $Z$, and we write $Z(f)$ for the arrow term obtained by putting the arrow term $f$ at the place of $\square$ in $Z$ and $\mathbf{1}_{B}$ at the place of every atomic formula $B$ in $Z$. For $X$ and $Y$ contexts, let $f: X\left(P \mathbf{x}_{n}\right) \wedge \forall_{\mathbf{y}_{m}} B \vdash Y\left(P \mathbf{x}_{n} \wedge B\right)$ be an arrow term of QMDS such that $\mathbf{y}_{m}$ are all the variables free in $B$, the displayed occurrences of $P$ in the source and target are tied in $f$, and the same holds for the $k$-th occurrence of predicate letter (counting from the left) in the displayed occurrences of $B$ in the source and target, for every $k$. Then, by successive applications of the new version of Lemma 1, we obtain the arrow term $f^{-B}: X\left(P \mathbf{x}_{n}\right) \vdash Y\left(P \mathbf{x}_{n}\right)$ of QMDS such that the displayed occurrences of $P$ in the source and target are tied in $f$.

Let $f^{\dagger}: X\left(P \mathbf{x}_{n} \wedge \forall_{\mathbf{y}_{m}} B\right) \vdash Y\left(P \mathbf{x}_{n} \wedge \forall_{\mathbf{y}_{m}} B\right)$ be the arrow term of QMDS obtained from $f^{-B}$ by replacing $P \mathbf{x}_{n}$ by $P \mathbf{x}_{n} \wedge \forall_{\mathbf{y}_{m}} B$ in the indices of the primitive arrow terms of $f^{-B}$ at places corresponding to the occurrences displayed in the source and target (see [8], Section 6.1, for an example). The new version of Lemma $2 \wedge$ of Section 6.1 of [8] should state the following:

Let $f$ and $f^{\dagger}$ be as above. Then there is an arrow term

$$
h_{X}: X\left(P \mathbf{x}_{n}\right) \wedge \forall_{\mathbf{y}_{m}} B \vdash X\left(P \mathbf{x}_{n} \wedge \forall_{\mathbf{y}_{m}} B\right)
$$

of QDS such that $f=Y\left(\mathbf{1}_{P \mathbf{x}_{n}} \wedge \iota_{B}^{\forall_{\mathbf{y}_{m}}}\right) \circ f^{\dagger} \circ h_{X}$ in $\mathbf{Q M D S}$.
In the proof of this new lemma, when we define inductively $h_{X}$, besides clauses analogous to the old clauses, we should have the additional clauses:

$$
\begin{aligned}
h_{\forall_{x} Z} & =\forall_{x} h_{Z} \circ \hat{\theta}_{Z\left(P \mathbf{x}_{n}\right), \forall_{\mathbf{y} m} B}^{\forall_{x}}, \\
h_{\exists_{x} Z} & =\exists_{x} h_{Z} \circ \hat{\theta}_{Z\left(P \mathbf{x}_{n}\right), \forall_{y_{m}} B}^{\exists}
\end{aligned}
$$

(see the end Section 1.2 for the definition of $\hat{\theta}_{Z\left(P \mathbf{x}_{n}\right), \forall_{\mathbf{y}_{m} B}}^{\forall_{x} \leftarrow}$ ).
There is an analogous new version of Lemma $2 \bigvee$ of Section 6.1 of [8]. The proof of QMPN Coherence then proceeds as in Section 6.2 of [8]. This suffices to establish $\mathbf{Q M P N}\urcorner$ Coherence.

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