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Commutative integral bounded residuated lattices with an added involution

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To Franco Montagna on his 60th birthday

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ABSTRACT

A symmetric residuated lattice is an algebra $\boldsymbol{A} = (A, \vee, \wedge, *, \to, \sim, 1, 0)$ such that $(A, \vee, \wedge, *, \to, 1, 0)$ is a commutative integral bounded residuated lattice and the equations $\sim\sim x = x$ and $\sim (x \vee y) = \sim x \wedge \sim y$ are satisfied. The aim of the paper is to investigate the properties of the unary operation ε defined by the prescription $\varepsilon x = \sim x \to 0$. We give necessary and sufficient conditions for ε being an interior operator. Since these conditions are rather restrictive (for instance, on a symmetric Heyting algebra ε is an interior operator if and only the equation $(x \to 0) \vee ((x \to 0) \to 0) = 1$ is satisfied) we consider when an iteration of ε is an interior operator. In particular we consider the chain of varieties of symmetric residuated lattices such that the n iteration of ε is a boolean interior operator. For instance, we show that these varieties are semisimple. When $n = 1$, we obtain the variety of symmetric stonean residuated lattices. We also characterize the subvarieties admitting representations as subdirect products of chains. These results generalize and in many cases also simplify, results existing in the literature.

0. Introduction

Heyting algebras endowed with an involution were introduced by Moisil in 1942 [23], as the algebraic models of an expansion of intuitionistic propositional calculus by means of a De Morgan negation. These algebras have been extensively investigated by A. A. Monteiro under the name of *symmetric Heyting algebras* [25]. They were also considered by Sankappanavar [26], independently of the previous work. In [12], Esteva, Godo, Hájek and Navara, also independently of previous work, considered pseudocomplemented BL-algebras with an added involution. This line of research was continued in [10], where subvarieties of pseudocomplemented BL-algebras with involution were introduced, and in [14], where the more general case of MTL-algebras was considered.

As Heyting algebras, BL-algebras and MTL-algebras were introduced as algebraic counterparts of logical systems (see Section 1 for details), and the three form subvarieties of the variety $\mathcal{BRL}$ of *commutative integral bounded residuated lattices*, or *residuated lattices*, for short. Residuated lattices were introduced in the 1930s as models of the divisibility properties of ideals in commutative rings, and recently acquired importance as the algebraic counterparts of certain substructural and fuzzy logics [16,18,13].

In all the papers referred to above, but [26], the introduction of the involution had strong logical motivations. Moisil [23] considered that $\varepsilon x := \sim x \to 0$ could be interpreted as a modal operator of necessity, where $\sim$ denotes the added involution and $\to$ denotes the intuitionistic implication. In [12], an involution is introduced on pseudocomplemented BL-algebras in order to define a dual multiplicative disjunction since the defined negation is very weak (what is called a Gödel

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negation in the fuzzy logic literature). An involutive pseudocomplemented BL-algebra is defined as a pseudocomplemented BL-algebra $\boldsymbol{A}$ equipped with an order reversing involution $\sim$ such that εx (where now $\rightarrow$ means the implication operation in BL-algebras) satisfy certain equations. In particular it is required that εx be complemented for all $x \in A$. It follows that ε is an interior operator. Remarkably, this coincides with Moisil's interpretation of ε as a modal necessity, i. e., an interior operator from the algebraic point of view. In [14] the involution is added to MTL$_\triangle$-algebras, that is, MTL-algebras already equipped with a complemented interior operator $\triangle$.

The aim of this paper is to investigate the operator ε in residuated lattices. Our approach is algebraic. Since varieties of residuated lattices correspond to axiomatic extensions of some substructural and fuzzy logics, our results have logical interpretations that we are not going to discuss in this paper.

After the introduction and preliminaries on residuated lattices, in Section 2 we define the variety of symmetric residuated lattices and the ε operator. We investigate congruences and conditions for ε being an interior operator. In Section 3 we consider interior operators obtained by iterating ε. In particular we consider the chain of varieties of symmetric residuated lattices such that the n iteration of ε is a boolean interior operator. For instance, we show that these varieties are semisimple. When $n = 1$, we obtain the variety of symmetric stonean residuated lattices. We also characterize the subvarieties admitting representations as subdirect products of chains.

These results generalize and in many cases also simplify the results given in the papers [25,12].

1. Preliminaries

1.1. Residuated lattices

Recall that an *integral residuated lattice-ordered commutative monoid*, or *residuated lattice* for short, is an algebra $\boldsymbol{A} = (A, \vee, \wedge, *, \rightarrow, 1)$ of type (2, 2, 2, 2, 0) such that $\langle A, *, 1\rangle$ is a commutative monoid, $\langle A, \vee, \wedge, 1\rangle$ is a lattice with greatest element 1, and the following residuation condition holds:

$$x * y \leq z \quad \text{if and only if } x \leq y \rightarrow z, \tag{1.1}$$

where x, y, z denote arbitrary elements of A and $\leq$ is the order given by the lattice structure.

Although we are assuming familiarity with the theory of residuated lattices, as developed, for instance in [20,22],[1] we list some well-known properties for further reference.

Under the assumption of integrality (which means that the neutral element of the monoid reduct coincides with the greatest element of $\boldsymbol{A}$) one has that

$$x \leq y \quad \text{if and only if } x \rightarrow y = 1, \tag{1.2}$$

$$x * y \leq x \wedge y, \tag{1.3}$$

$$\text{if } x \leq y, \quad \text{then } x * z \leq y * z,\ y \rightarrow z \leq x \rightarrow z \quad \text{and} \quad z \rightarrow x \leq z \rightarrow y, \tag{1.4}$$

$$x * (y \vee z) = (x * y) \vee (x * z). \tag{1.5}$$

Residuated lattices form a variety. Indeed, the residuation condition can be replaced by the following identities (see [22]):

RL$_1$ $(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$,
RL$_2$ $(x * (x \rightarrow y)) \vee y = y$,
RL$_3$ $(x \wedge y) \rightarrow y = 1$.

A *bounded* residuated lattice is a residuated lattice $\boldsymbol{A}$ equipped with a constant 0 that is the bottom of the lattice. In this case, 0 turns out to be an absorbent element for $*$, and a derived unary operation $\neg$ is defined by $\neg\, x = x \rightarrow 0$. As usual this operation is called the *negation operation*. It is well known, and easy to prove, that the following equation holds in bounded residuated lattices:

$$\neg\,(x \vee y) = \neg\, x \wedge \neg\, y, \tag{1.6}$$

$$x \leq \neg\neg\, x, \tag{1.7}$$

$$\neg\neg\neg\, x = \neg\, x. \tag{1.8}$$

By a *residuated chain* (*bounded residuated chain*) we mean a residuated lattice (bounded residuated lattice) $\boldsymbol{A}$ whose natural order is total, i. e., given $x, y \in A$, $x \leq y$ or $y \leq x$.

Recall that a *t-norm* is a binary operation $*$ on the real segment [0, 1] which is associative, commutative, order preserving and satisfying $1 * x = x$ and $0 * x = 0$ for each $x \in [0, 1]$. With each t-norm we can associate a binary operation $\rightarrow$ defined as follows:

$$x \rightarrow y := \sup\{z \in [0, 1] : z * x \leq y\}.$$

1 Our main reference for residuated latices will be [22] because some of the results in the mentioned paper cannot be found in the recent book [16].

It is well known (see, for instance, [3]) that $*$ and $\to$ satisfy the residuation condition (1.1) if and only if $*$ is left-continuous with respect to the first variable (and the usual topology of [0, 1]).

Thus left-continuous t-norms on the real segment [0, 1] with its usual order provide examples of bounded residuated chains. They generate the subvariety $\mathbb{MTL}$ of $\mathbb{BRL}$, that provides the algebraic models of *monoidal t-norm based logic* [11, 21]. The subvariety $\mathbb{MTL}$ is characterized by the following *prelinear equation*:

$$(x \to y) \vee (y \to x) = 1. \tag{1.9}$$

This implies that *a bounded residuated lattice* $\boldsymbol{A}$ *belongs to* $\mathbb{MTL}$ *if and only if* $\boldsymbol{A}$ *is a subdirect product of bounded residuated chains* [11]. The elements of $\mathbb{MTL}$ are called *MTL-algebras*.

The subvariety $\mathbb{BL}$ of $\mathbb{MTL}$ generated by the *continuous t*-norms is characterized by the following *divisibility equation*

$$x \wedge y = x * (x \to y) \tag{1.10}$$

[7,11] and has been extensively considered in Hájek's monograph [18]. The elements of $\mathbb{BL}$ are called *BL-algebras*.

The subvariety $\mathbb{G}$ of $\mathbb{BL}$ characterized by the equation $x * x = x$ coincides with the variety of Heyting algebras satisfying the prelinearity equation (1.9). The algebras in $\mathbb{G}$ are called *linear Heyting algebras* in [25]. Following the nomenclature of [18], we shall call them *Gödel algebras*.

A bounded residuated lattice is called *involutive* (or *integral, commutative Girard monoid* [20]) when its negation $\neg$ is involutive, i. e., when the following equation holds:

$$x = \neg\neg x = (x \to 0) \to 0.$$

MV-algebras, the algebras of Łukasiewicz infinite-valued logic, coincide with the involutive BL-algebras (see, for instance, [18]). The subvariety of $\mathbb{BL}$ formed by the MV-algebras will be denoted by $\mathbb{MV}$.

By a *(weak) negation* on [0, 1] we mean a function $neg\colon [0,1] \to [0,1]$ that is order reversing and satisfies $neg(1) = 0$ and $x \leq neg(neg(x))$ for all $x \in [0, 1]$. Given a negation neg, define the following binary operation on [0, 1]:

$$x * y = \begin{cases} x \wedge y & \text{if } x > neg(y), \\ 0 & \text{if } x \leq neg(y). \end{cases}$$

It follows from [8, Theorem 2 (i)] that $*$ is a left-continuous t-norm such that $\neg x = x \to 0 = neg(x)$ for all $x \in [0, 1]$, where $\to$ is the residual of $*$. The real segment [0, 1] with the bounded residuated lattice determined by this left-continuous t-norm, is *the standard WNM-algebra determined by the negation neg* [11].

Remark 1.1. Let $\boldsymbol{A}$ be a bounded residuated lattice. If $x \wedge \neg x = 0$, then $\neg x$ is the pseudocomplement of x as a lattice: $y \wedge x = 0$ if and only if $y \leq \neg x$. Indeed, suppose $y \wedge x = 0$. Then by (1.3), $y * x = 0$, and since $\neg x = x \to 0$, we have $y \leq \neg x$. This justifies to call *pseudocomplemented residuated lattices* the bounded residuated lattices satisfying $x \wedge \neg x = 0$.

The variety of pseudocomplemented residuated lattices will be denoted by $\mathbb{PRL}$.

An *implicative filter* or *i-filter* of a residuated lattice $\boldsymbol{A}$ is a subset $F \subseteq A$ satisfying the following conditions: F_1: $1 \in F$; F_2: If $x \in F$ and $x \leq y$, then $y \in F$; F_3: If x, y are in F, then $x * y \in F$. Alternatively, an implicative filter can be defined by properties F_1 and F_4: If $x \in F$ and $x \to y \in F$, then $y \in F$. An implicative filter is *proper* if $F \neq A$, i. e., if $0 \notin F$. We quote, for further reference, the following well-known result (see, for instance, [22]).

Theorem 1.2. *Let* $\boldsymbol{A} \in \mathbb{RL}$. *Then one has:*

(i) *For each congruence* ϑ *of* $\boldsymbol{A}$, *the set* $F(\vartheta) := \{x \in A : (x, 1) \in \vartheta\}$ *is an i-filter.*
(ii) *For each i-filter* G *of* $\boldsymbol{A}$, *the binary relation* $\theta(G)$ *defined by* $(x, y) \in \theta(G)$ *if and only if* $x \to y \in G$ *and* $y \to x \in G$ *is a congruence of* $\boldsymbol{A}$.
(iii) *For each i-filter* G, $G = F(\theta(G))$, *and for each congruence* ϑ, $\vartheta = \theta(F(\theta))$. $\square$

We shall write A/F to denote the quotient algebra $A/\theta(F)$, and x/F to denote the equivalence class of an element $x \in A$.

Recall that an element x of a bounded lattice $\boldsymbol{L}$ is *complemented* if there is an element y such that $x \vee y = 1$ and $x \wedge y = 0$. Such a y is called a *complement* of x. The set of complemented elements of $\boldsymbol{L}$ will be denoted by $B(\boldsymbol{L})$. The following results are borrowed from [22, pp. 10–12].

Lemma 1.3. *The following properties are true in every bounded residuated lattice* $\boldsymbol{A}$:

(i) *If* $x \in A$ *has a complement, the complement must coincide with* $\neg x$,
(ii) $B(\boldsymbol{A}) = \{x \in A : \neg x \vee x = 1\}$,
(iii) *If* $z \in B(\boldsymbol{A})$, *then for each* $x \in A$, $x * z = x \wedge z$,
(iv) $B(\boldsymbol{A})$ *is the universe of a subalgebra of* $\boldsymbol{A}$, *which is a boolean algebra that we shall denote by* $\boldsymbol{B}(\boldsymbol{A})$.

A *stonean filter* of a bounded distributive lattice $\boldsymbol{L}$ is a filter F generated by the complemented elements of $\boldsymbol{L}$: $x \in F$ if and only if there is $z \in F \cap B(\boldsymbol{L})$ such that $z \leq x$.

Since $x * y \leq x \wedge y$ it follows that *i*-filters are lattice filters, but the converse is not true in general. The next result is well known. We offer a proof for completeness.

Lemma 1.4. *Every stonean filter F of a bounded residuated lattice $\boldsymbol{A}$ is an i-filter of $\boldsymbol{A}$, and $(x, y) \in \theta(F)$ if and only if there is $z \in F \cap B(\boldsymbol{A})$ such that $z \wedge x = z \wedge y$.*

Proof. Since F satisfies F_1 and F_2, we need to show F_3. Suppose $x, y \in F$. Then there are $v, w \in B(\boldsymbol{A}) \cap F$ such that $v \leq x$, $w \leq y$, and then $x * y \geq v * w = v \wedge w \in F$. Hence, by F_2, $x * y \in F$ and F_3 is satisfied. Suppose there is $z \in B(\boldsymbol{A}) \cap F$ such that $z \wedge x = z \wedge y$. By (iii) in Lemma 1.3, we have that $z * x \leq y$, hence $z \to (x \to y) = 1 \in F$, hence $x \to y \in F$. Interchanging x and y we also obtain that $y \to x \in F$. Therefore $(x, y) \in \theta(F)$. Now suppose that $(x, y) \in \theta(F)$. Then there are $u, v \in B(\boldsymbol{A}) \cap F$ such that $u \leq x \to y$ and $v \leq y \to x$. Hence $z = u \wedge v \in B(\boldsymbol{A}) \cap F$, and by (iii) in Lemma 1.3 we have $x \wedge z \leq x * (x \to y) \leq y$ and $y \wedge z \leq y * (y \to x) \leq x$, and these inequalities imply that $z \wedge x = z \wedge y$. □

1.2. Stonean residuated lattices

A *stonean residuated lattice* is a bounded residuated lattice satisfying the Stone equation

$$\neg x \vee \neg\neg x = 1. \tag{1.11}$$

Lemma 1.5. *Each stonean residuated lattice is pseudocomplemented.*

Proof. Since $\boldsymbol{A}$ is stonean, by (1.6) and (1.8)

$$0 = \neg 1 = \neg(\neg x \vee \neg\neg x) = \neg\neg\neg x \wedge \neg\neg x = \neg x \wedge \neg\neg x,$$

and now, by (1.7) $\neg x \wedge x \leq \neg x \wedge \neg\neg x = 0$. □

The following construction provides plenty of examples of stonean residuated lattices. Let $\boldsymbol{L}$ be an arbitrary bounded lattice, and let $\boldsymbol{L}'$ be the lattice obtained by adding a new top element $\top$ to $\boldsymbol{L}$. Then $\boldsymbol{L}'$ becomes a residuated lattice if we define the operation $*$ as $x * y = 0$ if $x, y \in L$, and $\top * x = x * \top = x$ for all $x \in L \cup \{\top\}$. If we add a new bottom $\perp$ to $\boldsymbol{L}'$, then it becomes a stonean residuated lattice if we extend the definition of $*$ by setting $x * \perp = \perp * x = \perp$ for all $x \in \{\perp\} \cup L'$. In particular, this shows that stonean residuated lattices do not need to satisfy any lattice equation, like distributivity or modularity.

It is known that in a *distributive* residuated lattice, Eq. (1.11) is equivalent to the following

$$\neg (x \wedge y) = \neg x \vee \neg y \tag{1.12}$$

(see, for instance, [2,17]). The next lemma shows that for residuated lattices, (1.11) implies (1.12) without assuming the distributivity of the underlying lattice.

Lemma 1.6. *A stonean residuated lattice $\boldsymbol{A}$ satisfies Eq. (1.12).*

Proof. Observe first that by (1.5), for any pair of elements of a residuated lattice we have $(x \wedge y) * (\neg x \vee \neg y) = ((x \wedge y) * \neg x) \vee ((x \wedge y) * \neg y) = 0$. Suppose that $\boldsymbol{A}$ is a stonean residuated lattice and take $x, y, z \in A$ such that $z * (x \wedge y) = 0$. Then $z \leq \neg (x \wedge y)$ and by Lemma 1.5, $z \wedge (x \wedge y) = 0$. Hence taking into account Remark 1.1, we have that $z \wedge x \leq \neg y$, and then $z \wedge x \wedge \neg\neg y = 0$, i. e., $z \wedge \neg\neg y \leq \neg x$. Hence, taking into account (iii) of Lemma 1.3, we have $z = z * 1 = z * (\neg y \vee \neg\neg y) = (z * \neg y) \vee (z * \neg\neg y) = (z \wedge \neg y) \vee (z \wedge \neg\neg y) \leq \neg y \vee \neg x$. This completes the proof. □

Theorem 1.7. *The following are equivalent conditions for a bounded (integral, commutative) residuated lattice $\boldsymbol{A}$:*

(i) *$\boldsymbol{A}$ is stonean,*
(ii) *$\boldsymbol{A} \in \mathbb{PRL}$ and satisfies Eq. (1.12),*
(iii) *$B(\boldsymbol{A}) \supseteq \neg(A) := \{\neg x : x \in A\}$.*

Proof. In a bounded residuated lattice satisfying $x \wedge \neg x = 0$, (1.12) implies the Stone identity. Indeed: $1 = \neg 0 = \neg (x \wedge \neg x) = \neg x \vee \neg\neg x$. Hence it follows from Lemmas 1.5 and 1.6 that (i) and (ii) are equivalent. It follows from (i) in Lemma 1.3 that $\neg x$ is complemented if and only if $\neg x \vee \neg\neg x = 1$. Hence (i) and (iii) are equivalent. □

Lemma 1.8. *Let $\boldsymbol{C}$ be a bounded residuated chain. Then $\boldsymbol{C}$ is pseudocomplemented if and only if $\boldsymbol{C}$ is stonean.*

Proof. If $\boldsymbol{A}$ is pseudocomplemented, then by Remark 1.1 we have that $\neg x = 0$ for each $x > 0$, therefore $\neg\neg x = 1$ for each $x > 0$, and this obviously imply that $\boldsymbol{A}$ is stonean. Conversely, if $\boldsymbol{A}$ is stonean, then by Lemma 1.5 $x \wedge \neg x = 0$. □

Remark 1.9. Since MTL-algebras are subdirect products of bounded residuated chains, it follows that a MTL-algebra $\boldsymbol{A}$ is stonean if and only if it is pseudocomplemented. Consequently, the variety of stonean MTL-algebras coincides with the variety $\mathbb{SMTL}$ as defined in [11].

Let $\boldsymbol{A}$ be the boolean algebra with two atoms with a new top element added. It is an example of a Heyting algebra (and hence, a pseudocomplemented residuated lattice) which is not stonean. Free stonean residuated lattices are considered in [5].

2. Symmetric residuated lattices

Following the nomenclature introduced by A. A. Monteiro for the case of Heyting algebras (see [25]), by a *symmetric residuated lattice* we shall mean a *bounded* residuated lattice $\boldsymbol{A}$ equipped with a unary operation $\sim$ satisfying the following conditions:

M_1 $\sim\sim x = x$,
M_2 $\sim(x \vee y) = \sim x \wedge \sim y$,
M_3 $\sim(x \wedge y) = \sim x \vee \sim y$.

We shall denote by $\mathbb{SRL}$ the variety of symmetric residuated lattices.

Each involutive residuated lattice $\boldsymbol{A}$ becomes a symmetric residuated lattice if we define $\sim x = \neg x = x \to 0$ for all $x \in A$. The symmetric residuated lattice so obtained will be denoted by $\boldsymbol{A}_\sim$.

For every $\boldsymbol{A} \in \mathbb{SRL}$, define, for each $x \in A$,

$$\varepsilon x := \neg \sim x, \tag{2.13}$$

and

$$K(\boldsymbol{A}) := \{x \in A : \neg x = \sim x\}. \tag{2.14}$$

We shall denote by $\mathbb{SPRL}$ the variety of *symmetric pseudocomplemented residuated lattices*.

Lemma 2.1. *The following properties hold in every $\boldsymbol{A} \in \mathbb{SPRL}$, where x, y denote arbitrary elements of A:*

(i) $x \vee \sim \varepsilon x = 1$,
(ii) $\varepsilon(x \to y) \leq \sim y \to \sim x$.

Proof. Since $\sim x \wedge \varepsilon x = 0$, (i) follows from M_1 and M_3. To prove (ii), note that $\sim \varepsilon(x \to y) \leq x \to \sim \varepsilon(x \to y)$. Hence by (i) and (1.4):

$$1 = (x \to y) \vee (x \to \sim \varepsilon(x \to y)) \leq x \to (y \vee \sim \varepsilon(x \to y)).$$

Therefore taking into account (1.2), (1.4) and M_2, we have:

$$x \leq y \vee \sim \varepsilon(x \to y) = \sim (\sim y \wedge \varepsilon(x \to y) \leq \sim (\sim y * \varepsilon(x \to y)).$$

Consequently $\sim y * \varepsilon(x \to y)) \leq \sim x$, and (ii) follows from (1.1). $\square$

By an *ε-filter* of a symmetric residuated lattice $\boldsymbol{A}$ we mean an *i*-filter F of $\boldsymbol{A}$ which is closed under ε: $x \in F$ implies $\varepsilon x \in F$.

The next theorem is an immediate consequence of Theorem 1.2 and Lemma 2.1, and it generalizes [25, Théorème 4.11].

Theorem 2.2. *Let $\boldsymbol{A} \in \mathbb{SRL}$. Then:*

(i) *For each congruence ϑ of $\boldsymbol{A}$, $F(\vartheta)$ is an ε-filter, and $\vartheta = \theta(F(\vartheta))$.*
(ii) *If $\boldsymbol{A} \in \mathbb{SPRL}$, then for each ε-filter G of $\boldsymbol{A}$, $\theta(G)$ is a congruence of $\boldsymbol{A}$, and $G = F(\theta(G))$.* $\square$

Note that the condition that $\boldsymbol{A}$ be pseudocomplemented is sufficient for the validity of (ii) in the above theorem, but it is not necessary. As a trivial example, take any involutive residuated lattice $\boldsymbol{A}$ which is not a boolean algebra. Then $\boldsymbol{A}$ is not pseudocomplemented, and $\boldsymbol{A}_\sim$ ε-filters and *i*-filters coincide.

A symmetric residuated lattice is called *normal* (cf [23,25]) provided it satisfies the following equation:

$$\neg x = \neg x \wedge \sim x. \tag{2.15}$$

Notice that $\boldsymbol{A} \in \mathbb{SRL}$ is normal if and only if $\varepsilon x \leq x$ for all $x \in A$.

The variety of normal symmetric residuated lattices will be denoted by $\mathbb{NSRL}$.

Recall that a *Kleene algebra* [2,25] is a distributive lattice equipped with a unary operation $\sim$ satisfying equations M_1, M_2, M_3 and

$$\sim x \wedge x = (\sim x \wedge x) \wedge (\sim y \vee y). \tag{2.16}$$

It was shown in [6, Lemma 1.1] that in a Kleene algebra, $x \wedge y = 0$ implies $y \leq \sim x$. Then we have the following generalization of a result of A. Monteiro for symmetric Heyting algebras [25, Chap. III, Cor. 6.2]:

Proposition 2.3. *All symmetric pseudocomplemented residuated distributive lattices satisfying* (2.16) *are normal.*

In [25, Pages 89–90] an example of a normal symmetric Heyting algebra which does not satisfy (2.16) is given.

Since all symmetric pseudocomplemented chains satisfy the distributivity equation and Eq. (2.16), we have:

Corollary 2.4. *All subdirect products of symmetric pseudocomplemented residuated chains are Kleene algebras, and therefore normal.*

We shall return to the relations between Kleene and normality conditions in Corollary 3.13.

Lemma 2.5. *The following properties hold for every $\boldsymbol{A} \in \mathbb{NSRL}$:*

(i) *If $x \in K(\boldsymbol{A})$, then $\neg\neg\, x = x$,*
(ii) *$K(\boldsymbol{A})$ is closed under $\neg$ and $\sim$,*
(iii) *$\{x \in A : \varepsilon x = x\} = K(\boldsymbol{A})$,*
(iv) *$B(\boldsymbol{A}) \subseteq K(\boldsymbol{A}) \subseteq \neg(A)$.*

Proof. Taking into account normality, $\sim x = \neg\, x$ implies $\neg\neg\, x = \neg \sim x \leq x$, and since $x \leq \neg\neg\, x$, we have (i). Notice that by (i), $\neg\, x = \sim x$ implies

$$\sim\sim x = x = \neg\neg\, x = \neg \sim x \tag{2.17}$$

and $\neg\neg\, x = x = \sim\sim x = \sim \neg\, x$. Hence $\sim x$ and $\neg\, x$ both belong to $K(\boldsymbol{A})$, and (ii) is proved. It follows from (2.17) that $\neg x = \sim\, x$ implies $\varepsilon x = x$. On the other hand, if $\varepsilon x = x$, then $\neg \sim x = x = \sim\sim x$. Hence $\sim x \in K(\boldsymbol{A})$, and by (ii), $x \in K(\boldsymbol{A})$, and the proof of (iii) is completed. Suppose that $z \in B(\boldsymbol{A})$. Then $1 = \neg\, z \vee z \leq \sim z \vee z$, and by M_1 and M_2, we also have $z \wedge \sim z = 0$. Hence $\sim z$ is a complement of z, and from (i) in Lemma 1.3 we can conclude that $B(\boldsymbol{A}) \subseteq K(\boldsymbol{A})$. Finally, the inclusion $K(\boldsymbol{A}) \subseteq \neg(A)$ is an immediate consequence of (i). $\square$

Lemma 2.6. *Let $\boldsymbol{A}$ be a normal symmetric residuated lattice. If $\{x_i\}_{i\in I}$ is a family of elements of A such that $\bigwedge_{i\in I} x_i$ exists in $\boldsymbol{A}$, then $\bigwedge_{i\in I} \varepsilon x_i$ also exists, and*

$$\varepsilon\left(\bigwedge_{i\in I} x_i\right) = \bigwedge_{i\in I} \varepsilon x_i.$$

Proof. Let $x = \bigwedge_{i\in I} x_i$. Then $\sim x = \bigvee_{i\in I} \sim x_i$, and $\varepsilon x = \neg \bigvee_{i\in I} \sim x_i = \bigwedge_{i\in I} \neg \sim x_i = \bigwedge_{i\in I} \varepsilon x_i$. $\square$

As a particular case of the above lemma, we have that $\varepsilon(x \wedge y) = \varepsilon x \wedge \varepsilon y$. This implies that $K(\boldsymbol{A})$ is closed under $\wedge$, and by M_2 and (ii) in Lemma 2.5, $K(\boldsymbol{A})$ is also closed under $\vee$. Summing up, we have:

Theorem 2.7. *For each $\boldsymbol{A} \in \mathbb{NSRL}$, $K(\boldsymbol{A})$ is a sublattice of $\boldsymbol{A}$, closed under $\sim$ and $\neg$. Moreover $\neg\neg\, z = z$ for all $z \in K(\boldsymbol{A})$.* $\square$

Let $\boldsymbol{P} = (P, \leq)$ be a poset. A subset $S \subseteq P$ is called *upper relatively complete* provided that for each $x \in P$, $\{s \in S : s \leq x\}$ has a greatest element.

By an *interior operator* on a bounded lattice $\boldsymbol{L}$ we mean a function $I\colon L \to L$ satisfying the conditions I_1: $I\,1 = 1$, I_2: $Ix \leq x$, I_3: $I(x \wedge y) = Ix \wedge Iy$, I_4: $IIx = Ix$, where x, y denote arbitrary elements of L.

The following result, that we quote for further reference, is well known (see, for instance, [2]).

Lemma 2.8. *Let $\boldsymbol{L}$ be a bounded lattice. If I is an interior operator on $\boldsymbol{L}$, then $x \in I(L)$ if and only if $Ix = x$, $I(L)$ is an upper relatively complete sublattice of $\boldsymbol{L}$ and for each $x \in L$, Ix is the greatest element in $\{z \in I(L) : z \leq x\}$. Conversely, given an upper relatively complete sublattice $\boldsymbol{S}$ of $\boldsymbol{L}$, the prescription $I_S x = \max(\{s \in S : s \leq x\})$ defines an interior operator on $\boldsymbol{L}$, such that $I_S(L) = S$.* $\square$

Theorem 2.9. *The following are equivalent conditions for every $\boldsymbol{A} \in \mathbb{NSRL}$:*

(i) *ε is an interior operator on $\boldsymbol{A}$,*
(ii) *the equation $\varepsilon\varepsilon x = \varepsilon x$ holds in $\boldsymbol{A}$,*
(iii) *the equation $\varepsilon(\neg\, x) = \neg\, x$ holds in $\boldsymbol{A}$,*
(iv) *$K(\boldsymbol{A}) = \neg(A)$.*

Proof. Clearly ε satisfies conditions I_1, I_2 (normality) and I_3 (Lemma 2.6). Therefore (i) and (ii) are equivalent conditions. Interchanging x and $\sim x$ we obtain the equivalence between (ii) and (iii), and taking into account (iv) of Lemma 2.5 we obtain the equivalence between (iii) and (iv). $\square$

Clearly symmetric residuated lattices, such that ε is an interior operator, form a subvariety of $\mathbb{NSRL}$ that we shall denote by $\mathbb{INT}$.

The following result is an immediate consequence of (iv) of the above theorem.

Corollary 2.10. *If $\boldsymbol{A}$ is an involutive bounded residuated lattice and $\sim$ is an involution on $\boldsymbol{A}$ such that $\boldsymbol{A}_\sim \in \mathbb{INT}$, then $\sim = \neg$, and ε is the identity.* $\square$

The above corollary applies in particular to MV-algebras. We shall denote by $\mathbb{MV}_\sim$ *the subvariety of $\mathbb{INT}$ formed by the symmetric MV-algebras such that the involution $\sim$ coincides with the MV-negation $\neg$.*

Corollary 2.11. *Let $\boldsymbol{A} \in \mathbb{NSPRL}$. Then $\boldsymbol{A} \in \mathbb{INT}$ if and only if $\boldsymbol{A}$ is stonean.*

Proof. If $\boldsymbol{A}$ is stonean, then by (iii) in Theorem 1.7, $B(\boldsymbol{A}) \supseteq \neg(A)$, hence by (iv) in Lemma 2.5 we have $K(\boldsymbol{A}) = \neg(A)$. Therefore (iv) in Theorem 2.9 implies that ϵ is an interior operator. Conversely, suppose that ε is an interior operator and let $x \in A$. Taking into account M_2, (iv) in Theorem 2.9 and (ii) in Lemma 2.5 we have:

$$1 = \sim (\neg x \wedge \neg\neg x) = \sim \neg x \vee \sim \neg\neg x = \neg\neg x \vee \neg\neg\neg x = \neg\neg x \vee \neg x,$$

and the Stone equation (1.11) holds in $\boldsymbol{A}$. □

Corollary 2.12. *Let $\boldsymbol{A}$ be a normal symmetric Heyting algebra. Then $\boldsymbol{A} \in \mathbb{INT}$ if and only if the Stone equation* (1.11)*) holds in $\boldsymbol{A}$.* □

Corollary 2.13. *If a BL-chain $\boldsymbol{A}$ admits an involution $\sim$ such that ε is an interior operator, then $\boldsymbol{A}$ is either pseudocomplemented or $\boldsymbol{A}$ is involutive.*

Proof. It is well known [19,7,11,9] that if $\boldsymbol{A}$ is a BL-chain that is neither pseudocomplemented nor involutive, then it is a proper ordinal sum with a first component $[0, a]$ ($a \neq 1$) such that $\neg$ is involutive over $(0, a)$ and $\neg x = 0$ for all $x > a$. Then $\neg(A) = \{1\} \cup [0, a)$. A simple computation proves that it is impossible to define an involution on A that coincides with $\neg$ on $\neg(A)$ since in $[0, a)$ $\neg$ is not continuous in 0 with respect to order topology while all involutions must be continuous with respect to the order topology of a chain. Now the result follows from (iv) in Theorem 2.9. □

The hypothesis that the BL-algebra $\boldsymbol{A}$ is totally ordered cannot be omitted in the above corollary, as the following example shows.

Example 2.14. Let $\boldsymbol{A}_1$ be a normal symmetric Gödel algebra such that $B(\boldsymbol{A}_1) \subsetneq A_1$ and let $\boldsymbol{A}_2 \in \mathbb{MV}_\sim$ be such that $B(\boldsymbol{A}_2) \subsetneq A_2$. Then $\boldsymbol{A} = \boldsymbol{A}_1 \times \boldsymbol{A}_2$ is a non-pseudocomplemented and non-involutive symmetric BL-algebra belonging to $\mathbb{INT}$. □

In contrast with the case of BL-chains, the next example shows that there are symmetric MTL-chains in $\mathbb{INT}$ that are neither involutive nor pseudocomplemented.

Example 2.15. Let $\boldsymbol{A}$ be the standard WNM-algebra determined by the following negation on the real segment $[0, 1]$:

$$\neg(x) = \begin{cases} 1 - x, & \text{if } x \in [0, \frac{1}{4}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{3}{4}, 1], \\ \frac{2}{3}, & \text{if } x \in (\frac{1}{4}, \frac{1}{3}], \\ \frac{1}{4}, & \text{if } x \in (\frac{2}{3}, \frac{3}{4}]. \end{cases}$$

Since $\neg(A) = [0, \frac{1}{4}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{3}{4}, 1]$, by (iv) in Theorem 2.9 there are as many involutions $\sim$ on $[0, 1]$ as possible continuous strictly decreasing bijections from $[\frac{1}{4}, \frac{1}{3}]$ into $[\frac{2}{3}, \frac{3}{4}]$, such that ε is an interior operator;hence infinitely many. The simplest such involution is $\sim x = 1 - x$. □

3. From ε to boolean interior operators

It follows from Theorem 1.7(iii) and Theorem 2.9 (iv), and Lemma 2.5(iv) that ε is an interior operator such that $K(\boldsymbol{A}) = B(\boldsymbol{A})$ if and only if $\boldsymbol{A}$ is a normal symmetric stonean lattice. As mentioned in the Introduction, in the literature normal symmetric residuated lattices $\boldsymbol{A}$ endowed with an interior operator Δ such that $\Delta(\boldsymbol{A}) = B(\boldsymbol{A})$ are considered. The aim of this section is to investigate when such Δ can be obtained as a power of ε. We start with a result on limits of powers of ε, of independent interest.

Theorem 3.1. *Let $\boldsymbol{A}$ be a normal symmetric residuated lattice, and consider the following properties:*

(i) $\varepsilon_\infty x := \bigwedge_{n \geq 1} \varepsilon^n x$ *exists for all* $x \in A$.
(ii) *There exists an interior operator ε' such that, for all $x \in A$,*

$$\varepsilon' x = x \quad \text{if and only if } \varepsilon x = x.$$

(iii) $K(\boldsymbol{A})$ *is upper relatively complete.*

Then (i) *implies* (ii)*, and* $\varepsilon' = \varepsilon_\infty$. (ii) *and* (iii) *are equivalent, and $\varepsilon' x$ is the greatest element in $\{z \in K(\boldsymbol{A}) : z \leq x\}$.*

Proof. Suppose that (i) holds. Given $x \in A$, let $z = \varepsilon_\infty x$. By Lemma 2.6,

$$\varepsilon z = \varepsilon \left(\bigwedge_{n \in \boldsymbol{N}} \varepsilon^n x \right) = \bigwedge_{n \in \boldsymbol{N}} \varepsilon^{n+1} x = z. \tag{3.18}$$

Now it is easy to see, by induction on n, that $\varepsilon^n z = z$. Then it follows that $\varepsilon_\infty \varepsilon_\infty x = \varepsilon_\infty x$. By normality, $\varepsilon_\infty x \leq x$, and by Lemma 2.6, $\varepsilon_\infty (x \wedge y) = \varepsilon_\infty x \wedge \varepsilon_\infty y$. Since clearly $\varepsilon_\infty 1 = 1$, we have proved that ε_∞ is an interior operator on $\boldsymbol{A}$. It is obvious that $\varepsilon x = x$ implies $\varepsilon_\infty x = x$, and it follows from (3.18)) that $\varepsilon_\infty x = x$ implies $\varepsilon x = x$. Therefore (i) implies (ii). The equivalence between (ii) and (iii) follows from (iii) in Lemma 2.5 and Lemma 2.8. □

Corollary 3.2. *If $\boldsymbol{A} \in \mathbb{NSRL}$ is complete as a lattice, then all of* (i), (ii) *and* (iii) *in Theorem* 3.1 *hold.* □

Interesting examples of application of the above corollary are provided by *standard normal symmetric chains*, i. e., symmetric residuated lattices defined over the real unit interval by a left-continuous t-norm, its residual and an involution satisfying normality. In many of these structures ε is not an interior operator, but ε_∞ is always defined: either there exists some $n \in \omega$ such that $\varepsilon^n = \varepsilon_\infty$ (as in Example 3.3 below) or $\varepsilon_\infty \neq \varepsilon^n$ for all $n \in \omega$ (as in the case of the product or Łukasiewicz standard chains with a normal involution).

The following example shows that in the lack of completeness, condition (i) in Theorem 3.1 can be stronger than conditions (ii) and (iii).

Example 3.3. Let $\{a_i\}_{i \geq 0}$ be a strictly increasing sequence in $[0, 1]$ whose limit is $\frac{1}{2}$ and $a_0 = 0$. The standard WNM-algebra determined by the following negation function:

$$\neg\, x = \begin{cases} 1, & \text{if } x = 0, \\ 1 - a_i, & \text{if } x \in (a_{i-1}, a_i],\ i \geq 1, \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ a_i, & \text{if } x \in (1 - a_{i+1}, 1 - a_i],\ i \geq 0, \end{cases}$$

becomes a normal symmetric residuated lattice $\boldsymbol{A}$ when endowed with the involution $\sim x = 1 - x$ on $[0, 1]$. Let $\boldsymbol{C}$ be the subalgebra of $\boldsymbol{A}$ with universe $C = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$. It is clear that on $\boldsymbol{C}$, $K(\boldsymbol{C}) = \{0, 1\}$, hence it is trivially upper relatively complete. Since for $x \in (0, \frac{1}{2})$, $\bigwedge_{n \geq 1} \varepsilon^n(x)$ does not exist, ε_∞ is not defined on $\boldsymbol{C}$. On the other hand, ε' as defined in (ii) of Theorem 3.1 exists, since $K(\boldsymbol{C}) = \{0, 1\}$. □

Another case in which we can assert the existence of an interior operator ε' on a normal symmetric residuated lattice $\boldsymbol{A}$ such that ε' and ε have the same invariant elements, is when there is a positive integer n such that the following equation holds in $\boldsymbol{A}$:

$$\varepsilon^{n+1}x = \varepsilon^n x. \tag{3.19}$$

Clearly every *finite* normal symmetric residuated lattice satisfies Eq. (3.19) for a suitable integer $n \geq 1$.

Notice that since $\varepsilon x \leq x$, (3.19) is equivalent to the equation $\varepsilon^{2n}x = \varepsilon^n x$, and as in the proof of Theorem 3.1, it follows that the last equation holds if and only if ε^n is an interior operator. Specially interesting is the case in which also the equality $K(\boldsymbol{A}) = B(\boldsymbol{A})$ holds, that we are going to consider next.

Given $n \geq 1$, an *n-boolean interior symmetric residuated lattice* is a normal symmetric residuated lattice $\boldsymbol{A}$ such that $K(\boldsymbol{A}) = B(\boldsymbol{A})$ and Eq. (3.19) holds in $\boldsymbol{A}$.

It is easy to see that n-boolean interior symmetric lattices form a subvariety of $\mathbb{NSRL}$, characterized by equations (3.19) and

$$\neg\varepsilon^n x \vee \varepsilon^n x = 1, \tag{3.20}$$

that we shall denote by $\mathbb{BISRL}_n$.

Clearly $\mathbb{BISRL}_n \subseteq \mathbb{BISRL}_{n+1}$ for each $n \geq 1$. The next example shows that the inclusions are proper.

Example 3.4. For each $n \geq 2$ define the negation $\neg_n$ over the real segment $[0, 1]$ by the prescriptions:

$$\neg_n(x) = \begin{cases} 1, & \text{if } x = 0, \\ \frac{n-k}{n}, & \text{if } \frac{k-1}{n} < x \leq \frac{k}{n},\ \text{for } 1 \leq k \leq n-1, \\ 0, & \text{if} \frac{n-1}{n} < x \leq 1. \end{cases}$$

Let $\boldsymbol{A}_n$ denote the standard WNM-algebra determined by $\neg_n$, equipped with an involution $\sim$ such that $\sim x > 1 - x$. For instance, given $\frac{1}{2} < z < 1$, one can take

$$\sim x = \begin{cases} (1 - \frac{1}{z})x + 1, & \text{if } 0 \leq x \leq z, \\ \frac{z}{z-1}x - \frac{z}{z-1}, & \text{if } z \leq x \leq 1. \end{cases}$$

It is easy to check that $\boldsymbol{A}_{n+1} \in \mathbb{BISRL}_{n+1}$ and $\boldsymbol{A}_{n+1} \notin \mathbb{BISRL}_n$, for $n \geq 1$. □

Let $\boldsymbol{A} \in \mathbb{NSRL}$. From items (iii) in Theorem 1.7, (iv) in Lemma 2.5 and (iv) in Theorem 2.9 it follows that ε is an interior operator such that $K(\boldsymbol{A}) = B(\boldsymbol{A})$ if and only if $\boldsymbol{A}$ satisfies the Stone equation (1.11). Therefore $\mathbb{BISRL}_1$ *coincides with the variety* $\mathbb{NSSRL}$ *of normal symmetric stonean residuated lattices.*

Proposition 3.5. *The following equations hold in every $\boldsymbol{A} \in \mathbb{BISRL}_n$, for each $n \geq 1$:*

(i) $\varepsilon^n(x * y) = \varepsilon^n x * \varepsilon^n y$,
(ii) $(\varepsilon^n x \to \varepsilon^n y) \to \varepsilon^n(x \to y) = 1$.

Proof. Since $\varepsilon^n x \in B(\boldsymbol{A})$, by (iii) in Lemma 1.3 we have that $\varepsilon^n x * \varepsilon^n y = \varepsilon^n x \wedge \varepsilon^n y \in B(\boldsymbol{A}) = K(\boldsymbol{A})$. Therefore $\varepsilon^n x * \varepsilon^n y = \varepsilon^n(\varepsilon^n x * \varepsilon^n y) \leq \varepsilon^n(x * y) \leq \varepsilon^n x \wedge \varepsilon^n y = \varepsilon^n x * \varepsilon^n y$, and (i) is proved. To prove (ii), note that by (i) $\varepsilon^n x * \varepsilon^n(x \to y) = \varepsilon^n(x * (x \to y)) \leq \varepsilon^n y$, and apply the residuation condition (1.1). □

Let $\boldsymbol{A} \in \mathbb{BISRL}_n$, with $n \geq 1$, and let F be an i-filter of $\boldsymbol{A}$. Clearly for each $x \in A$, $\varepsilon x \in F$ if and only if $\varepsilon^n x \in F$, and since $\varepsilon^n x$ is an interior operator on $\boldsymbol{A}$ such that $\varepsilon^n(A) = B(\boldsymbol{A})$, we have that the ε-filters of $\boldsymbol{A}$ coincide with the stonean filters. Hence by (i) of Theorem 2.2 we have that for each congruence ϑ of $\boldsymbol{A}$, $F(\vartheta)$ is a stonean filter, and $\vartheta = \theta(F(\vartheta))$. On the other hand, it follows from Lemma 1.4 that each stonean filter of $\boldsymbol{A}$ is an implicative filter, and it follows that the congruence $\theta(F)$ preserves the involution $\sim$. Indeed, suppose that $(x, y) \in \theta(F)$. By the same lemma we know that there is $z \in F \cap B(\boldsymbol{A})$ such that $x \wedge z = y \wedge z$, and since by (iv) in Lemma 2.5, $\sim z = \neg z$, taking into account M_1, M_3, (iii) of Lemma 1.3 and (1.5), we have $z \wedge \sim x = z * (\sim x \vee \sim z) = z * (\sim y \vee \sim z) = z \wedge \sim y$. Summing up we have:

Theorem 3.6. *For each integer $n \geq 1$ and each $\boldsymbol{A} \in \mathbb{BISRL}_n$ the correspondence $F \mapsto \theta(F)$ establishes a one–one inclusion preserving correspondence between the stonean filters and the congruence relations of $\boldsymbol{A}$. The inverse mapping is given by the correspondence $\theta \mapsto 1/\theta$.* □

Recall that an algebra $\boldsymbol{A}$ is *simple* if the only congruences of $\boldsymbol{A}$ are the identity and $A \times A$, and that an algebra is called *semisimple* if it is a subdirect product of simple algebras [4].

Corollary 3.7. *The following are equivalent conditions for each $\boldsymbol{A} \in \mathbb{BISRL}_n$, $n \geq 1$:*

(i) *$\boldsymbol{A}$ is simple,*
(ii) $B(\boldsymbol{A}) = \{0, 1\}$,
(iii) $\varepsilon x = \begin{cases} 1, & \text{if } x = 1, \\ 0, & \text{if } x \neq 1. \end{cases}$

Proof. Suppose there is $z \in B(\boldsymbol{A})$ such that $0 < z < 1$. Then $F_z = \{x \in A : z \leq x\}$ is a stonean filter, and $\{1\} \subsetneq F_z \subsetneq A$. This shows that (i) implies (ii). On the other hand, (ii) implies that the only stonean filters of $\boldsymbol{A}$ are $\{1\}$ and A. Hence (i) and (ii) are equivalent. The equivalence between (ii) and (iii) follows from the fact that $B(\boldsymbol{A}) = K(\boldsymbol{A}) = \varepsilon^n(A)$. □

A *maximal stonean filter* of a residuated lattice $\boldsymbol{A}$ is a stonean filter F of $\boldsymbol{A}$ such that $F \neq A$ and for each stonean filter G of $\boldsymbol{A}$, $F \subsetneq G$ implies $G = A$. It follows from Theorem 3.6 that *maximal stonean filters are in one–one correspondence with maximal congruences of* $\boldsymbol{A}$. Therefore, for each maximal stonean filter of $\boldsymbol{A}$, the quotient $\boldsymbol{A}/F := \boldsymbol{A}/\theta(F)$ is simple (see, for instance, [4]).

It is easy to check that a stonean filter F of $\boldsymbol{A}$ is maximal if and only if $F \cap B(\boldsymbol{A})$ is a maximal filter of the boolean algebra $\boldsymbol{B}(\boldsymbol{A})$), and that the correspondence

$$U \mapsto \langle U \rangle := \{x \in A : x \geq u \text{ for some } u \in U\} \tag{3.21}$$

defines a bijection between the maximal filters of the boolean algebra $\boldsymbol{B}(\boldsymbol{A})$ and the maximal stonean filters of $\boldsymbol{A}$.

For each bounded residuated lattice $\boldsymbol{A}$, the *stonean radical* of $\boldsymbol{A}$, $\mathrm{StRad}(\boldsymbol{A})$, is the intersection of all maximal stonean filters of $\boldsymbol{A}$.

Corollary 3.8. *Every $\boldsymbol{A} \in \mathbb{BISRL}_n$ is semisimple.*

Proof. Since maximal congruences of $\boldsymbol{A}$ are in one–one correspondence with maximal stonean filters, it is sufficient to prove that $\mathrm{StRad}(\boldsymbol{A}) = \{1\}$ (see, for instance, [4]). Let $x \in A$, $x \neq 1$. Then $1 \neq \varepsilon^n x \in B(\boldsymbol{A})$, and there is a maximal filter U of $\boldsymbol{B}(\boldsymbol{A})$ such that $\varepsilon^n x \notin U$. Hence $x \notin \langle U \rangle$, and this shows that $x \notin \mathrm{Rad}(\boldsymbol{A})$. □

Observe that applying the construction indicated after Remark 1.1 to a normal symmetric residuated lattice, and extending the involution defining $\sim \top = \bot$ and $\sim \bot = \top$, we obtain a simple normal symmetric stonean residuated lattice. This shows that the structure of the variety $\mathbb{NSSRL} = \mathbb{BISRL}_1$ is rather complex. In particular, since it contains simple algebras of arbitrary large cardinality, *there are no injective algebras in* $\mathbb{BISRL}_n$ for every $n \geq 1$ (see [15]).

By a *boolean n-interior symmetric residuated chain*, or *bi$_n$-chain* for short, we mean a symmetric residuated chain that satisfies equations (3.19) and (3.20). Observe that *bi$_1$-chains are precisely the normal symmetric pseudocomplemented residuated chains*.

Our next aim is to characterize the subvarieties of the varieties $\mathbb{BISRL}_n$ generated by the bi$_n$-chains.

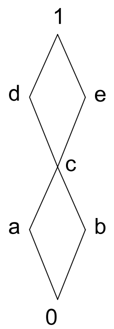
Observe that since ε is order preserving, each symmetric residuated chain satisfies the equation:

$$\varepsilon(x \vee y) = \varepsilon x \vee \varepsilon y. \tag{3.22}$$

Therefore each variety generated by normal symmetric residuated chains must satisfy (3.22).

Lemma 3.9. *Eq. (3.22) holds in a normal symmetric pseudocomplemented residuated lattice $\boldsymbol{A}$ if and only if $\boldsymbol{A}$ is stonean.*

Proof. If $\boldsymbol{A}$ is stonean, then by M_2 and Lemma 1.6 $\varepsilon(x \vee y) = \neg(\sim x \wedge \sim y) = \varepsilon x \vee \varepsilon y$. Suppose now that $\boldsymbol{A} \in NSPRL$ and that (3.22) holds in $\boldsymbol{A}$. Then by M_1 and M_3 we have that $\neg x \vee \neg y = \varepsilon \sim x \vee \varepsilon \sim y = \varepsilon(\sim x \vee \sim y) = \neg(x \wedge y)$. Hence Eq. (1.12) holds in $\boldsymbol{A}$, and by Lemma 1.6, $\boldsymbol{A}$ is stonean. □



x	$\neg x$	$\sim x$
0	1	1
a	b	e
b	a	d
c	0	c
d	0	b
e	0	a
1	0	0

Fig. 1. $\boldsymbol{A} \notin \mathbb{BISRL}_2^{\vee}$.

We denote by $\mathbb{BISRL}_n^{\vee}$ the subvariety of $\mathbb{BISRL}_n$ determined by Eq. (3.22).

It follows from the above lemma that $\mathbb{BISRL}_1^{\vee} = \mathbb{BISRL}_1$. On the other hand, the finite non-stonean Heyting algebra $\boldsymbol{A} \in \mathbb{BISRL}_2$ depicted in Fig. 1 shows that $\mathbb{BISRL}_n^{\vee}$ is a *proper* subvariety of $\mathbb{BISRL}_n$ for $n \geq 2$.

By a $\triangle$-*residuated lattice* we mean a bounded residuated lattice $\boldsymbol{A}$ equipped with a unary operation $\triangle$ satisfying the following properties (cf [1], [18, Definition 2.4.6]):

(i) $\triangle 1 = 1$,
(ii) $\triangle x \vee \neg\triangle x = 1$,
(iii) $\triangle(x \vee y) = \triangle x \vee \triangle y$,
(iv) $\triangle x \leq x$,
(v) $\triangle\triangle x = \triangle x$,
(vi) $\triangle(x * y) = \triangle x * \triangle y$.

Taking into account Proposition 3.5, we see that *each* $\boldsymbol{A} \in \mathbb{BISRL}_n^{\vee}$ *becomes a* $\triangle$*-residuated lattice if we define* $\triangle x = \varepsilon^n x$ *for each* $x \in A$.

Lemma 3.10. *If* $\boldsymbol{A} \in \mathbb{BISRL}_n^{\vee}$, $n \geq 1$, *then each maximal stonean filter of* $\boldsymbol{A}$ *is prime.*

Proof. Let F be a maximal stonean filter of $\boldsymbol{A}$, and suppose $x \vee y \in F$. Then $\varepsilon x \vee \varepsilon y = \varepsilon(x \vee y) \in F \cap B(\boldsymbol{A})$, and since $F \cap B(\boldsymbol{A})$ is a prime filter of the boolean algebra $\boldsymbol{B}(\boldsymbol{A})$, we have that $\varepsilon x \in F \cap B(\boldsymbol{A})$ or $\varepsilon y \in F \cap B(\boldsymbol{A})$. Therefore $x \in F$ or $y \in F$. $\square$

We denote by $\mathbb{LBISRL}_n$ the subvariety of $\mathbb{BISRL}_n^{\vee}$ determined by the prelinearity equation (1.9).

Theorem 3.11. *A symmetric residuated lattice* $\boldsymbol{A}$ *belongs to* $\mathbb{LBISRL}_n$ *if and only if it is a subdirect product of* bi_n*-chains, for each integer* $n \geq 1$.

Proof. Clearly each subdirect product of normal symmetric residuated chains satisfies (3.22) and (1.9). Therefore all subdirect products of bi_n-chains are in $\mathbb{LBISRL}_n$. Suppose now that $\boldsymbol{A} \in \mathbb{LBISRL}_n$. By Lemma 3.10, each maximal stonean filter of $\boldsymbol{A}$ is a prime i-filter, hence prelinearity implies that the quotient $\boldsymbol{A}/F$ is a bi_n-chain. The result follows from the fact that $\mathrm{StRad}(\boldsymbol{A}) = \{1\}$, as shown in the proof of Corollary 3.8. $\square$

Remark 3.12. Since the symmetric residuated chains considered in Example 3.4 are bi_n-chains, it follows that $\mathbb{LBISRL}_n \subsetneqq \mathbb{LBISRL}_{n+1}$ for each $n \geq 1$. $\square$

In [12,10] pseudocomplemented BL-algebras are called *strict BL-algebras*, or *SBL-algebras* for short. In [10], the $\mathrm{SBL}_\sim$-algebras introduced in [12] are characterized as a SBL-algebras equipped with an order reversing involution $\sim$ such that Eqs. (1.11) and (2.15) are satisfied. Since strict BL-algebras are pseudocomplemented MTL-algebras, by Remark 1.9 they satisfy Eq. (1.11)). Consequently $\mathit{SBL}_\sim$*-algebras coincide with normal symmetric BL-algebras*. It will follow from the next corollary that $\mathit{BL}_\sim$*-algebras also coincide with the symmetric BL-algebras that satisfy the Kleene equation* (2.16).

A. Monteiro [25, Chap. V, Section 4] showed that a symmetric Gödel algebra $\boldsymbol{A}$ is normal if and only if it satisfies the Kleene equation (2.16) if and only if it is a subdirect product of symmetric Heyting chains. On the other hand, it was shown in [12, Theorem 5] that every $\mathrm{SBL}_\sim$-algebra is a subdirect product of symmetric BL-chains. The next corollary generalizes both the results.

Corollary 3.13. *The following are equivalent conditions for each symmetric pseudocomplemented MTL-algebra* $\boldsymbol{A}$*:*

(i) $\boldsymbol{A}$ *is a subdirect product of symmetric pseudocomplemented residuated chains.*
(ii) $\boldsymbol{A}$ *satisfies the Kleene equation* (2.16).
(iii) $\boldsymbol{A}$ *is normal.*

Proof. That (i) implies (ii) and (ii) implies (iii) follow from Corollary 2.4. By Corollary 2.11, (iii) implies that $\boldsymbol{A}$ is stonean. Therefore $\boldsymbol{A} \in \mathbb{LBISRL}_1$, and by Theorem 3.11 we obtain (i). $\square$

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