POSETS OF COPIES OF COUNTABLE SCATTERED LINEAR ORDERS

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Abstract

We show that the separative quotient of the poset $\langle \mathbb{P}(L), \subset \rangle$ of isomorphic suborders of a countable scattered linear order *L* is σ -closed and atomless. So, under the CH, all these posets are forcing-equivalent (to $(P(\omega)/\operatorname{Fin})^+$).¹

1 Introduction

The posets of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where \mathbb{X} is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures, were investigated in [4]. In particular, a classification of countable binary structures related to the order-theoretic and forcing-related properties of the posets of their copies is described in Diagram 1: for the structures from column A (resp. B; D) the corresponding posets are forcing equivalent to the trivial poset (resp. the Cohen forcing, $\langle {}^{<\omega}2, \supset \rangle$; an ω_1 -closed atomless poset) and, for the structures from the class C_4 , the posets of copies are forcing equivalent to the posets of the form $(P(\omega)/\mathcal{I})^+$, for some co-analytic tall ideal \mathcal{I} . For example, all countable non-scattered linear orders are in the class C_4 , moreover, as a consequence of the main result of [3] we have

Theorem 1.1 For each countable non-scattered linear order L the poset $\langle \mathbb{P}(L), \subset \rangle$ is forcing equivalent to the two-step iteration $\mathbb{S} * \pi$, where \mathbb{S} is the Sacks forcing and $\mathbb{1}_{\mathbb{S}} \Vdash ``\pi$ is a σ -closed forcing''. If the equality $\operatorname{sh}(\mathbb{S}) = \aleph_1$ or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega)/\operatorname{Fin})^+$ of the Sacks extension.

The aim of this paper is to complete the picture of countable linear orders in this context and, having in mind Theorem 1.1, we concentrate our attention on countable scattered linear orders. In the simplest case, if L is the ordinal ω , then $\langle \mathbb{P}(L), \subset \rangle = \langle [\omega]^{\omega}, \subset \rangle$ is a homogeneous atomless partial order of size \mathfrak{c} and its separative quotient, the poset $(P(\omega)/\operatorname{Fin})^+$, is σ -closed. We will show that the same holds for each countable scattered linear order. So the following theorem is our main result.

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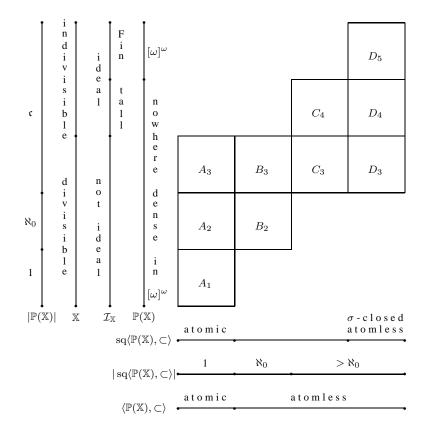


Diagram 1: Binary relations on countable sets

Theorem 1.2 For each countable scattered linear order *L* the poset $\langle \mathbb{P}(L), \subset \rangle$ is homogeneous, atomless, of size \mathfrak{c} and its separative quotient is σ -closed.

Corollary 1.3 If *L* is a countable linear order, then the poset $\langle \mathbb{P}(L), \subset \rangle$ is forcing equivalent to

- $\mathbb{S} * \pi$, where $1_{\mathbb{S}} \Vdash$ " π is σ -closed", if *L* is non-scattered [3];

- A σ -closed atomless forcing, if L is scattered.

Under the CH, the poset $\langle \mathbb{P}(L), \subset \rangle$ is forcing equivalent to

- $\mathbb{S} * \pi$, where $\hat{1}_{\mathbb{S}} \Vdash "\pi = (P(\check{\omega})/\operatorname{Fin})^+$ ", if L is non-scattered [3];

- $(P(\omega)/\operatorname{Fin})^+$, if L is scattered.

The most difficult part of the proof of Theorem 1.2 is to show that the separative quotient of $\langle \mathbb{P}(L), \subset \rangle$ is σ -closed (this result is the best possible: " σ -closed" can not be replaced by " ω_2 -closed", see Example 7.2). Namely, it is easy to see that

there are copies of an ω -sum $\sum_{\omega} L_i$ of linear orders L_i , which are not of the form $\bigcup_{i \in \omega} C_i$, where $C_i \in \mathbb{P}(L_i)$, so the Hausdorff hierarchy of scattered linear orders can not be used (easily) for an inductive proof. Instead of that hierarchy we use the result of Laver [7] that a countable scattered linear order is a finite sum of hereditarily indecomposable (ha) linear orders. So we first prove the statement for ha-orders, then for special blocks of ha-orders and, finally, for finite sums of blocks.

2 Preliminaries

A linear order L is said to be *scattered* iff it does not contain a dense suborder or, equivalently, iff the rational line, \mathbb{Q} , does not embed in L. By S we denote the class of all countable scattered linear orders.

Fact 2.1 If L is a linear order satisfying $L + L \hookrightarrow L$, then L is not scattered (see [8], p. 180).

Proof. By the assumption, $L + (L+L) \hookrightarrow L + L \hookrightarrow L$. By recursion we construct the sequences $\langle L_{\varphi} : \varphi \in {}^{<\omega}2 \rangle$ and $\langle L'_{\varphi} : \varphi \in {}^{<\omega}2 \rangle$ in $\mathbb{P}(L)$ and $\langle q_{\varphi} : \varphi \in {}^{<\omega}2 \rangle$ in L such that (i) $L_{\emptyset} = L$, (ii) $L_{\varphi \cap 0} < L'_{\varphi} < L_{\varphi \cap 1}$, (iii) $L_{\varphi \cap 0} \cup L'_{\varphi} \cup L_{\varphi \cap 1} \subset L_{\varphi}$, (iv) $q_{\varphi} \in L'_{\varphi}$. Then $\{q_{\varphi} : \varphi \in {}^{<\omega}2\}$ is a copy of \mathbb{Q} in L. \Box

A linear order L is said to be *additively indecomposable* (respectively *left indecomposable; right indecomposable*) iff for each decomposition $L = L_0 + L_1$ we have $L \hookrightarrow L_0$ or $L \hookrightarrow L_1$ (respectively $L \hookrightarrow L_0$; $L \hookrightarrow L_1$). The class \mathcal{H} of *hereditarily additively indecomposable* (or *ha-indecomposable*) linear orders is the smallest class of order types of countable linear orders containing the one element order type, **1**, and containing the ω -sum, $\sum_{\omega} L_i$, and the ω^* -sum, $\sum_{\omega^*} L_i$, for each sequence $\langle L_i : i \in \omega \rangle$ in \mathcal{H} satisfying

$$\forall i \in \omega \ |\{j \in \omega : L_i \hookrightarrow L_j\}| = \aleph_0. \tag{1}$$

Fact 2.2 (a) $\mathcal{H} \subset S$ (see [8], p. 196);

(b) If $L \in \mathcal{H}$ is an ω -sum, then L is right indecomposable (see [8], p. 196);

(c) If $L \in \mathcal{H}$ is an ω^* -sum, then L is left indecomposable (see [8], p. 196);

(d) If $L \in S$ is additively indecomposable, then L is left indecomposable or right indecomposable (see [8], p. 175);

(e) (Laver, [7]) If $L \in S$, then $L \in \mathcal{H}$ iff L is additively indecomposable (see [8], p. 201);

(f) (Laver, [7]) If $L \in S$, then L is a finite sum of elements of \mathcal{H} (see [8], p. 201).

Let $\mathbb{P} = \langle P, \leq \rangle$ be a pre-order. Then $p \in P$ is an *atom* iff each $q, r \leq p$ are compatible (there is $s \leq q, r$). \mathbb{P} is called: *atomless* iff it has no atoms; *homogeneous* iff it has the largest element and $\mathbb{P} \cong p \downarrow$, for each $p \in \mathbb{P}$. If κ is a regular cardinal, \mathbb{P} is called κ -closed iff for each $\gamma < \kappa$ each sequence $\langle p_{\alpha} : \alpha < \gamma \rangle$ in P, such that $\alpha < \beta \Rightarrow p_{\beta} \leq p_{\alpha}$, has a lower bound in P. ω_1 -closed pre-orders are called σ -closed. Two pre-orders \mathbb{P} and \mathbb{Q} are called *forcing equivalent* iff they produce the same generic extensions.

Fact 2.3 If \mathbb{P}_i , $i \in I$, are κ -closed pre-orders, then $\prod_{i \in I} \mathbb{P}_i$ is κ -closed.

A partial order $\mathbb{P} = \langle P, \leq \rangle$ is called *separative* iff for each $p, q \in P$ satisfying $p \not\leq q$ there is $r \leq p$ such that $r \perp q$. The *separative modification* of \mathbb{P} is the separative pre-order sm $(\mathbb{P}) = \langle P, \leq^* \rangle$, where

$$p \leq^* q \Leftrightarrow \forall r \leq p \; \exists s \leq r \; s \leq q. \tag{2}$$

The separative quotient of \mathbb{P} is the separative partial order $\operatorname{sq}(\mathbb{P}) = \langle P/=^*, \trianglelefteq \rangle$, where $p =^* q \Leftrightarrow p \leq^* q \land q \leq^* p$ and $[p] \trianglelefteq [q] \Leftrightarrow p \leq^* q$.

Fact 2.4 Let \mathbb{P}, \mathbb{Q} and $\mathbb{P}_i, i \in I$, be partial orderings. Then

(a) \mathbb{P} , sm(\mathbb{P}) and sq(\mathbb{P}) are forcing equivalent forcing notions;

(b) $\operatorname{sm}(\mathbb{P})$ is κ -closed iff $\operatorname{sq}(\mathbb{P})$ is κ -closed;

(c) If $p_0, p_1, \ldots, p_n \in \mathbb{P}$, where $p_n \leq^* p_{n-1} \leq^* \ldots \leq^* p_0$, then there is $q \in \mathbb{P}$ such that $q \leq p_k$, for all $k \leq n$.

(d) $\mathbb{P} \cong \mathbb{Q}$ implies that $\operatorname{sm} \mathbb{P} \cong \operatorname{sm} \mathbb{Q}$ and $\operatorname{sq} \mathbb{P} \cong \operatorname{sq} \mathbb{Q}$;

(e) sm $(\prod_{i \in I} \mathbb{P}_i) = \prod_{i \in I} \operatorname{sm} \mathbb{P}_i$ and sq $(\prod_{i \in I} \mathbb{P}_i) \cong \prod_{i \in I} \operatorname{sq} \mathbb{P}_i$.

(f) If X is an infinite set, $\mathcal{I} \subset P(X)$ an ideal containing $[X]^{<\omega}$ and $\mathcal{I}^+ = P(X) \setminus \mathcal{I}$ the corresponding family of \mathcal{I} -positive sets, then $\operatorname{sm}\langle \mathcal{I}^+, \subset \rangle = \langle \mathcal{I}^+, \subset_{\mathcal{I}} \rangle$, where $A \subset_{\mathcal{I}} B \Leftrightarrow A \setminus B \in \mathcal{I}$, for $A, B \in \mathcal{I}^+$. Also $\operatorname{sq}\langle \mathcal{I}^+, \subset \rangle = (P(X)/\mathcal{I})^+$.

Proof. All the statements are folklore except, maybe, (c). For a proof of (c), by recursion we define the sequence $\langle q_k : k \leq m \rangle$ such that (i) $q_0 = p_n$ and (ii) $q_k \leq q_{k-1}, p_{n-k}$, for $0 < k \leq n$. Then $q_n \leq p_k$, for all $k \leq n$.

Fact 2.5 (Folklore) Under the CH, each atomless separative ω_1 -closed pre-order of size ω_1 is forcing equivalent to $(P(\omega)/\operatorname{Fin})^+$.

We recall that the ideal $\operatorname{Fin} \times \operatorname{Fin} \subset P(\omega \times \omega)$ is defined by:

$$\operatorname{Fin} \times \operatorname{Fin} = \{ A \subset \omega \times \omega : |\{i \in \omega : |A \cap L_i| = \omega\}| < \omega \},\$$

where $L_i = \{i\} \times \omega$, for $i \in \omega$. By $\mathfrak{h}(\mathbb{P})$ we denote the *distributivity number* of a poset \mathbb{P} . In particular, for $n \in \mathbb{N}$, let $\mathfrak{h}_n = \mathfrak{h}(((P(\omega)/\operatorname{Fin})^+)^n)$; thus $\mathfrak{h} = \mathfrak{h}_1$.

Fact 2.6 (a) sm $(\langle [\omega]^{\omega}, \subset \rangle^n) = \langle [\omega]^{\omega}, \subset^* \rangle^n$ and sq $(\langle [\omega]^{\omega}, \subset \rangle^n) = ((P(\omega)/\operatorname{Fin})^+)^n$ are forcing equivalent, t-closed atomless pre-orders of size \mathfrak{c} .

(b) (Shelah and Spinas [9]) $\operatorname{Con}(\mathfrak{h}_{n+1} < \mathfrak{h}_n)$, for each $n \in \mathbb{N}$.

(c) (Szymański and Zhou [10]) $(P(\omega \times \omega)/(\text{Fin} \times \text{Fin}))^+$ is an ω_1 -closed, but not ω_2 -closed atomless poset.

(d) (Hernández-Hernández [2]) $\operatorname{Con}(\mathfrak{h}((P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+) < \mathfrak{h}).$

Now we prove the first part of Theorem 1.2.

Proposition 2.7 For each countable scattered linear order *L* the partial ordering $\langle \mathbb{P}(L), \subset \rangle$ is homogeneous, atomless and of size c.

Proof. The homogeneity of $\langle \mathbb{P}(L), \subset \rangle$ is evident. For a proof that it is atomless first we show

$$\forall L \in \mathcal{H} \ (|L| = \omega \Rightarrow \exists X, Y \in \mathbb{P}(L) \ X \cap Y = \emptyset).$$
(3)

If L is an ω -sum, that is $L = \sum_{\omega} L_i$, where $\langle L_i : i \in \omega \rangle$ is a sequence in \mathcal{H} satisfying (1), by recursion we define the sequences $\langle k_i : i \in \omega \rangle$ and $\langle l_i : i \in \omega \rangle$ in ω such that for each i

(i)
$$k_i < l_i$$
,
(ii) $l_i < k_{i+1}$,
(iii) $L_i \hookrightarrow L_{k_i}, L_{l_i}$.

Using (1) we choose $k_0, l_0 \in \omega$ such that $k_0 < l_0$ and $L_0 \hookrightarrow L_{k_0}, L_{l_0}$.

Let the sequences k_0, \ldots, k_i and l_0, \ldots, l_i satisfy (i)-(iii). Then $k_0 < l_0 < \ldots < k_i < l_i$. Using (1) we choose $k_{i+1}, l_{i+1} \in \omega$ such that $l_i < k_{i+1} < l_{i+1}$ and $L_{i+1} \hookrightarrow L_{k_{i+1}}, L_{l_{i+1}}$. Thus, the recursion works.

By (iii) there are $X_i, Y_i \cong L_i$ such that $X_i \subset L_{k_i}$ and $Y_i \subset L_{l_i}$. Then $X = \sum_{\omega} X_i, Y = \sum_{\omega} Y_i \cong L$ and, by (i) and (ii) we have $X \cap Y = \emptyset$.

If L is an ω^* -sum, we proceed in the same way. Thus (3) is proved.

By Fact 2.2 for $L \in S$ there is $m \in \mathbb{N}$ such that $L = \sum_{i < m} L_i$, where $L_i \in \mathcal{H}$. Let $J = \{i < m : |L_i| = \omega\}$. By (3), for $i \in J$ there are $X_i, Y_i \in \mathbb{P}(L_i)$ such that $X_i \cap Y_i = \emptyset$. Let $X = \bigcup_{i \in J} X_i \cup \bigcup_{i \in m \setminus J} L_i$ and $Y = \bigcup_{i \in J} Y_i \cup \bigcup_{i \in m \setminus J} L_i$. Then $X, Y \in \mathbb{P}(L)$ and $|X \cap Y| = |\bigcup_{i \in m \setminus J} L_i| < \omega$ and, hence, X and Y are incompatible elements of the poset $\langle \mathbb{P}(L), \subset \rangle$. So, since $\langle \mathbb{P}(L), \subset \rangle$ is a homogeneous partial order, it is atomless.

It is known (see [1], p. 170) that the equivalence classes corresponding to the relation \sim on L, defined by $x \sim y$ iff $|[\min\{x, y\}, \max\{x, y\}]| < \omega$, are convex parts of L which are finite or isomorphic to ω , or ω^* or \mathbb{Z} . Since $|L| = \omega$ and two consecutive parts can not be finite, there is one infinite part, say L', and, clearly, it

has c-many copies. For each $C \in \mathbb{P}(L')$ we have $(L \setminus L') \cup C \in \mathbb{P}(L)$ and, hence, $|\mathbb{P}(L)| = \mathfrak{c}$.

In the rest of the paper we prove that $sq\langle \mathbb{P}(L), \subset \rangle$ is a σ -closed poset, for each countable scattered linear order L. By Fact 2.4(b), it is sufficient to show that the pre-order $sm\langle \mathbb{P}(L), \subset \rangle$ is σ -closed. In the sequel we use the following notation:

$$\operatorname{sm}\langle \mathbb{P}(L), \subset \rangle = \langle \mathbb{P}(L), \leq \rangle$$

3 Elements of \mathcal{H}

Proposition 3.1 Let $L = \sum_{\omega} L_i \in \mathcal{H}$, where $\langle L_i : i \in \omega \rangle$ is a sequence in \mathcal{H} satisfying (1). Then

(a) $A \subset L$ contains a copy of L iff for each $i, m \in \omega$ there is finite $K \subset \omega \setminus m$ such that $L_i \hookrightarrow \bigcup_{j \in K} L_j \cap A$. So, each $A \in \mathbb{P}(L)$ intersects infinitely many L_i 's.

(b) If A, B ∈ P(L), then A ≤ B iff for each C ∈ P(L) satisfying C ⊂ A and each i, m ∈ ω there exists a finite K ⊂ ω \ m such that L_i ⇔ ⋃_{j∈K} L_j ∩ C ∩ B.
(c) sm⟨P(L), ⊂⟩ is a σ-closed pre-order.

The same statement holds for the ω^* -sum $\sum_{\omega^*} L_i$.

Proof. (a) (\Rightarrow) Let $f : L \hookrightarrow L$ and $C = f[L] \subset A$. Then $C = \sum_{i \in \omega} f[L_i]$.

Claim 1. For each $i \in \omega$ there is a finite set $K \subset \omega$ such that $f[L_i] \subset \bigcup_{i \in K} L_i$.

Proof of Claim 1. Since f is an embedding and $L_i < L_{i+1}$ we have $f[L_i] < f[L_{i+1}]$. For $x \in L_{i+1}$ we have $f(x) \in f[L_{i+1}] \subset \bigcup_{j \in \omega} L_j$ and, hence, $f(x) \in L_{j_0}$, for some $j_0 \in \omega$. Now, by the monotonicity of f we have $f[L_i] < \{f(x)\} \subset L_{j_0}$, thus $f[L_i] \subset \bigcup_{j \leq j_0} L_j$, so we can take $K = j_0 + 1$ and Claim 1 is proved. For $i \in \omega$ let $K_i = \{j \in \omega : f[L_i] \cap L_j \neq \emptyset\}$. By Claim 1 we have

$$K_i \in [\omega]^{<\omega} \text{ and } f[L_i] \subset \bigcup_{j \in K_i} L_j.$$
 (4)

Claim 2. $K_i \leq K_{i+1}$, for each $i \in \omega$. Consequently, either $K_i \cap K_{i+1} = \emptyset$ or $K_i \cap K_{i+1} = \{\max K_i\} = \{\min K_{i+1}\}.$

Proof of Claim 2. Let $j' \in K_i$ and $j'' \in K_{i+1}$. Then there are $x \in L_i$ and $y \in L_{i+1}$ such that $f(x) \in L_{j'}$ and $f(y) \in L_{j''}$ and, clearly, x < y. Now j'' < j' would imply f(y) < f(x), which is impossible. Thus $j' \leq j''$. Claim 2 is proved.

Claim 3. $\bigcup_{i \in \omega} K_i$ is an infinite subset of ω .

Proof of Claim 3. On the contrary, suppose that $j_0 = \max \bigcup_{i \in \omega} K_i$. Let $i_0 = \min\{i \in \omega : j_0 \in K_i\}$. Then $j_0 \in K_{i_0} \leq \{j_0\}$ and, by Claim 2,

$$\forall i > i_0 \ (K_i = \{j_0\} \land f[L_i] \subset L_{j_0}).$$

By (1), there are $i_1, i_2 \in \omega$ such that $i_0 + 1 < i_1 < i_2$ and $L_{j_0} \hookrightarrow L_{i_1}, L_{i_2}$, which implies $L_{j_0} + L_{j_0} \hookrightarrow L_{i_1} + L_{i_2} \hookrightarrow f[L_{i_1}] + f[L_{i_2}] \subset L_{j_0}$. But L_{j_0} is a scattered linear order and, by Fact 2.1, $L_{j_0} + L_{j_0} \nleftrightarrow L_{j_0}$. A contradiction. Claim 3 is proved.

Let $i_0, m_0 \in \omega$. By (1), the set $I_{i_0} = \{j \in \omega : L_{i_0} \hookrightarrow L_j\}$ is an infinite set.

Claim 4. There is $j_0 \in I_{i_0}$ such that $K_{j_0} \cap m_0 = \emptyset$.

Proof of Claim 4. On the contrary, suppose that $K_j \cap m_0 \neq \emptyset$, for each $j \in I_{i_0}$. Then

$$\forall j \in I_{i_0} \quad \min K_j < m_0. \tag{5}$$

For $i \in \omega$ there is $j \in I_{i_0}$ such that j > i + 1 and, by Claim 2, $K_i \leq K_{i+1} \leq K_j$ and, by (5), max $K_i \leq \min K_{i+1} \leq \min K_j < m_0$. Thus $K_i \subset m_0$, for all $i \in \omega$, which is impossible by Claim 3. Claim 4 is proved.

By Claim 4, $K_{j_0} \in [\omega \setminus m_0]^{<\omega}$. By (4) we have $f[L_{j_0}] \subset \bigcup_{j \in K_{j_0}} L_j$. Since $j_0 \in I_{i_0}$ and $f[L_{j_0}] \subset C \subset A$ we have $L_{i_0} \hookrightarrow L_{j_0} \hookrightarrow f[L_{j_0}] \subset \bigcup_{j \in K_{j_0}} L_j \cap A$ and the proof of " \Rightarrow " is finished.

(\Leftarrow) Suppose that a set $A \subset L$ satisfies the given condition. By recursion we define the sequences $\langle K_i : i \in \omega \rangle$ and $\langle f_i : i \in \omega \rangle$ such that for each $i \in \omega$

(i) $K_i \in [\omega]^{<\omega}$, (ii) $K_0 < K_1 < \dots$,

(iii) $f_i: L_i \hookrightarrow \bigcup_{j \in K_i} L_j \cap A.$

By the assumption, for i = m = 0 there are $K_0 \in [\omega]^{<\omega}$ and $f_0 : L_0 \hookrightarrow \bigcup_{i \in K_0} L_i \cap A$.

Let K_0, \ldots, K_i and f_0, \ldots, f_i satisfy (i)-(iii) and let $m = \max(\bigcup_{r \le i} K_r) + 1$. By the assumption for i + 1 and m there are $K_{i+1} \in [\omega \setminus m]^{<\omega}$ and $f_{i+1} : L_{i+1} \hookrightarrow \bigcup_{j \in K_{i+1}} L_j \cap A$ and the recursion works.

Let $f = \bigcup_{i \in \omega} f_i$. By (ii) and (iii), $i_1 < i_2$ implies $K_{i_1} < K_{i_2}$, which implies $f_{i_1}[L_{i_1}] < f_{i_2}[L_{i_2}]$ and, hence, $f : L \hookrightarrow A$. Thus $C = f[L] \in \mathbb{P}(L)$ and $C \subset A$.

(b) By (2), $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ the set $C \cap B$ contains a copy of L. Now we apply (a) to $C \cap B$.

(c) For $A_n \in \mathbb{P}(L)$, $n \in \omega$, where $A_0 \ge A_1 \ge \ldots$ we will construct $A \in \mathbb{P}(L)$ such that $A \le A_n$, for all $n \in \omega$. First, by Fact 2.4(c), there are $C_i \in \mathbb{P}(L)$, $i \in \omega$, such that $C_0 = A_0$ and

$$\forall i \in \omega \ C_i \subset A_0 \cap \ldots \cap A_i. \tag{6}$$

By recursion we define the sequences $\langle K_i : i \in \omega \rangle$ and $\langle f_i : i \in \omega \rangle$ such that for each $i \in \omega$

(i) $K_i \in [\omega]^{<\omega}$, (ii) $K_i < K_{i+1}$, (iii) $f_i : L_i \hookrightarrow \bigcup_{j \in K_i} L_j \cap C_i$. Since $C_0 = A_0 \in \mathbb{P}(L)$, by (a), for i = m = 0 there are $K_0 \in [\omega]^{<\omega}$ and $f_0: L_0 \hookrightarrow \bigcup_{j \in K_0} L_j \cap C_0$.

Let the sequences $K_0, \ldots, K_{i'}$ and $f_0, \ldots, f_{i'}$ satisfy (i)-(iii). Since $A_{i'+1} \leq A_{i'}, C_{i'+1} \in \mathbb{P}(L)$ and, by (6), $C_{i'+1} \subset A_{i'+1}$, according to (b), for i' + 1 and $m = \max(K_0 \cup \ldots \cup K_{i'}) + 1$ there are

$$K_{i'+1} \in [\omega \setminus (\max(K_0 \cup \ldots \cup K_{i'}) + 1)]^{<\omega}$$
(7)

$$f_{i'+1}: L_{i'+1} \hookrightarrow \bigcup_{j \in K_{i'+1}} L_j \cap C_{i'+1} \tag{8}$$

(since, by (6)), $C_{i'+1} \cap A_{i'} = C_{i'+1}$). By (7)) we have (i) and (ii) and (iii) follows from (8)). The recursion works.

Let $f = \bigcup_{i \in \omega} f_i$. By (ii) and (iii), $i_1 < i_2$ implies $K_{i_1} < K_{i_2}$, which implies $f_{i_1}[L_{i_1}] < f_{i_2}[L_{i_2}]$ and, hence, $f : L \hookrightarrow L$. Thus

$$A = f[L] = \bigcup_{i \in \omega} f_i[L_i] \in \mathbb{P}(L).$$
(9)

Using the characterization from (b), for $n^* \in \omega$ we show that $A \leq A_{n^*}$. So, for $C^* \in \mathbb{P}(L)$ such that $C^* \subset A$ and $i^*, m^* \in \omega$ we prove that

$$\exists K \in [\omega \setminus m^*]^{<\omega} \ L_{i^*} \hookrightarrow \bigcup_{j \in K} L_j \cap C^* \cap A_{n^*}.$$
(10)

By (ii), (iii) and (9) we have $A = \sum_{i \in \omega} \Lambda_i \cong L$, where $\Lambda_i = f_i[L_i] \cong L_i$, thus $A \in \mathcal{H}$. Since $C^* \cong L \cong A$ we have $C^* \in \mathbb{P}(A)$ so, applying (a) to the linear order A instead of L we obtain

$$\forall i, m \in \omega \; \exists K \in [\omega \setminus m]^{<\omega} \; f_i[L_i] \hookrightarrow \bigcup_{j \in K} f_j[L_j] \cap C^*.$$
(11)

Let $m' > m^*$, n^* . By (11), for i^* and m' there is

$$K^* \in [\omega \setminus m']^{<\omega}$$
 such that (12)

$$f_{i^*}[L_{i^*}] \hookrightarrow \bigcup_{j \in K^*} f_j[L_j] \cap C^*.$$
(13)

By (12), for $j \in K^*$ we have $j > n^*$ and, by (6), $C_j \subset A_{n^*}$. Thus, by (iii) we have $f_j[L_j] \subset \bigcup_{s \in K_j} L_s \cap C_j \subset \bigcup_{s \in K_j} L_s \cap A_{n^*}$ which, together with (iii) and (13) gives $L_{i^*} \hookrightarrow f_{i^*}[L_{i^*}] \hookrightarrow \bigcup_{j \in K^*} f_j[L_j] \cap C^* \subset \bigcup_{j \in K^*} \bigcup_{s \in K_j} L_s \cap A_{n^*} \cap C^* = \bigcup_{s \in \bigcup_{j \in K^*} K_j} L_s \cap C^* \cap A_{n^*}$.

In order to finish the proof of (10) we prove that $\bigcup_{j \in K^*} K_j \cap m^* = \emptyset$. By (12), for $j \in K^*$ we have $j > m^*$. By (ii) the sequence $\langle \min K_i : i \in \omega \rangle$ is increasing and, hence, $\min K_j \ge j > m^*$, which implies $K_j \cap m^* = \emptyset$ and (10) is proved. \Box

4 Finite sums of ω -sums. Finite sums of ω^* -sums

Lemma 4.1 Let $L_0 = \sum_{\omega} L_i^0, L_1 = \sum_{\omega} L_i^1 \in \mathcal{H}$, where $\langle L_i^0 : i \in \omega \rangle$ and $\langle L_i^1 : i \in \omega \rangle$ are sequences in \mathcal{H} satisfying (1). Then

(a) $\exists i \in \omega \ L_0 \hookrightarrow L_i^1 \Leftrightarrow \exists m \in \omega \ L_0 \hookrightarrow \sum_{i \leq m} L_i^1;$ (b) $L_0 + L_1 \notin \mathcal{H} \Rightarrow \neg \exists m \in \omega \ L_0 \hookrightarrow \sum_{i \leq m} L_i^1.$ (c) If $L = L_0 + L_1 \notin \mathcal{H}$ and $f : L \hookrightarrow L$, then $f[L_k] \subset L_k$, for k = 0, 1.

Proof. (a) Suppose that $L_0 \hookrightarrow \sum_{i \le m} L_i^1$ and let $i_0 = \max\{i \le m : f[L_0] \cap L_i^1 \ne \emptyset\}$. Then $f[L_0] \cap L_{i_0}^1$ is a final part of the ordering $f[L_0] \cong L_0$ and, by Fact 2.2(a), contains a copy of L_0 . Thus $L_0 \hookrightarrow L_{i_0}^1$.

(b) If $L_0 \hookrightarrow \sum_{i \le m} L_i^1$ then, by (a), there are $i_0 \in \omega$ and $f : L_0 \hookrightarrow L_{i_0}^1$. Then $\langle L_0, L_0^1, L_1^1, \ldots, L_{i_0}^1, \ldots \rangle$ is a sequence in \mathcal{H} satisfying (1) and $L_0 + L_1 = L_0 + L_0^1 + L_1^1 + \ldots + L_{i_0}^1 + \ldots \in \mathcal{H}$.

(c) Suppose that $f[L_0] \cap L_1 \neq \emptyset$. Then $f[L_0] \cap L_1$ is a final part of the ordering $f[L_0] \cong L_0$ and, by Fact 2.2(a), contains a copy of L_0 . Thus, by (b), $f[L_0] \cap L_i^1 \neq \emptyset$, for infinitely many $i \in \omega$. But this is impossible because $f[L_0] < f[L_1]$. Thus $f[L_0] \subset L_0$ and, hence, $f[L_0] \in \mathbb{P}(L_0)$. By Proposition 3.1(a) we have $f[L_0] \cap L_i^0 \neq \emptyset$, for infinitely many $i \in \omega$, which implies $f[L_1] \subset L_1$. \Box

Proposition 4.2 (Finite sums of ω -sums) Let $L = \sum_{i \leq n} L_i$, where $L_i \in \mathcal{H}$ are ω -sums of sequences in \mathcal{H} satisfying (1) and $L_i + L_{i+1} \notin \mathcal{H}$, for i < n. Then

(a) If $f: L \hookrightarrow L$, then $f[L_i] \subset L_i$, for each $i \le n$; (b) $\mathbb{P}(L) = \{\bigcup_{i \le n} C_i : \forall i \le n \ C_i \in \mathbb{P}(L_i)\};$

(c) sm $\langle \mathbb{P}(L), \subset \rangle$ is a σ -closed pre-order.

Proof. (a) For n = 1 this is (c) of Lemma 4.1. Assuming that the statement is true for n - 1 we prove that it is true for n. Suppose that $f[L_0] \not\subset L_0$. Then, since $f[L_n] \subset \bigcup_{i \leq n} L_i$, for $i^* = \max\{i \leq n : f[L_i] \not\subset \bigcup_{j \leq i} L_j\}$ we have $0 \leq i^* < n$, $f[L_{i^*}] \not\subset \bigcup_{j \leq i^*} L_j$ and $f[L_{i^*+1}] \subset \bigcup_{j \leq i^*} L_j \cup L_{i^*+1}$. Since $f[L_{i^*}] < f[L_{i^*+1}]$ we have $f[L_{i^*+1}] \subset L_{i^*+1}$ so $f[L_{i^*}] \cap L_{i^*+1}$ is a final part of $f[L_{i^*}] \cong L_{i^*}$ and, by Fact 2.2(a), contains a copy of L_{i^*} . This copy is contained in the union of finitely many summands of L_{i^*+1} . But, since $L_{i^*} + L_{i^*+1} \not\in \mathcal{H}$, this is impossible by Lemma 4.1(b). Thus $f[L_0] \subset L_0$ and, by Proposition 3.1(a), the set $f[L_0]$ intersects infinitely many summands of L_0 , which implies $f[L_1 \cup \ldots \cup L_n] \subset L_1 \cup \ldots \cup L_n$. Thus, by the induction hypothesis, $f[L_i] \subset L_i$, for each $i \in \{1, \ldots, n\}$.

(b) The inclusion " \supset " is evident and we prove " \subset ". If $C \in \mathbb{P}(L)$ and $f : L \hookrightarrow L$, where C = f[L], then by (a), $C_i = f[L_i] \subset L_i$ and, hence, $C_i \in \mathbb{P}(L_i)$ and, clearly, $C = \bigcup_{i \le n} C_i$.

(c) By the statement (b) and, since the sets L_i , $i \leq n$, are disjoint, the mapping $F : \prod_{i \leq n} \langle \mathbb{P}(L_i), \subset \rangle \rightarrow \langle \mathbb{P}(L), \subset \rangle$ given by $F(\langle C_0, \ldots, C_n \rangle) = C_0 \cup \ldots \cup C_n$ is an isomorphism and, by Fact 2.4, $\operatorname{sm}\langle \mathbb{P}(L), \subset \rangle \cong \operatorname{sm}(\prod_{i \leq n} \langle \mathbb{P}(L_i), \subset \rangle) \cong \prod_{i \leq n} \operatorname{sm}\langle \mathbb{P}(L_i), \subset \rangle$. By Proposition 3.1(c), the pre-orders $\operatorname{sm}\langle \mathbb{P}(L_i), \subset \rangle$, $i \leq n$, are σ -closed, and, by Fact 2.3 the same holds for their direct product and, hence, for $\operatorname{sm}\langle \mathbb{P}(L), \subset \rangle$ as well. \Box

The following dual statements can be proved in the same way.

Lemma 4.3 Let $L_0 = \sum_{\omega^*} L_i^0, L_1 = \sum_{\omega^*} L_i^1 \in \mathcal{H}$, where $\langle L_i^0 : i \in \omega \rangle$ and $\langle L_i^1 : i \in \omega \rangle$ are sequences in \mathcal{H} satisfying (1). Then (a) $\exists i \in \omega \ L_1 \hookrightarrow L_i^0 \Leftrightarrow \exists m \in \omega \ L_1 \hookrightarrow L_m^0 + \ldots + L_0^0$; (b) $L_0 + L_1 \notin \mathcal{H} \Rightarrow \neg \exists m \in \omega \ L_1 \hookrightarrow L_m^0 + \ldots + L_0^0$. (c) If $L = L_0 + L_1 \notin \mathcal{H}$ and $f : L \hookrightarrow L$, then $f[L_k] \subset L_k$, for k = 0, 1.

Proposition 4.4 (Finite sums of ω^* -sums) Let $L = \sum_{i < n} L_i$, where $L_i \in \mathcal{H}$ are ω^* -sums and $L_i + L_{i+1} \notin \mathcal{H}$, for i < n - 1. Then (a) If $f : L \hookrightarrow L$, then $f[L_i] \subset L_i$, for each i < n; (b) $\mathbb{P}(L) = \{\bigcup_{i < n} C_i : \forall i < n \ C_i \in \mathbb{P}(L_i)\};$ (c) sm $\langle \mathbb{P}(L), \zeta \rangle$ is a σ -closed pre-order.

5 ω^* -sum plus ω -sum

Lemma 5.1 Let $L = L_0 + L_1$, where $L_0 = \sum_{\omega^*} L_i^0, L_1 = \sum_{\omega} L_i^1 \in \mathcal{H}$ and $\langle L_i^0 : i \in \omega \rangle$ and $\langle L_i^1 : i \in \omega \rangle$ are sequences in \mathcal{H} satisfying (1). Then (a) $\exists i \in \omega \ L_0 \hookrightarrow L_i^1 \Leftrightarrow \exists m \in \omega \ L_0 \hookrightarrow L_0^1 + \ldots + L_m^1$; (b) $\exists i \in \omega \ L_1 \hookrightarrow L_i^0 \Leftrightarrow \exists m \in \omega \ L_1 \hookrightarrow L_m^0 + \ldots + L_0^0$; (c) If $L_0 + L_1 \notin \mathcal{H}$, then

$$\forall m \in \omega \ (L_0 \not\hookrightarrow L_0^1 + \ldots + L_m^1 \ \land \ L_1 \not\hookrightarrow L_m^0 + \ldots + L_0^0).$$
(14)

Proof. (a) If $f: L_0 \hookrightarrow \sum_{i \le m} L_i^1$ and $i_0 = \min\{i \le m : f[L_0] \cap L_i^1 \ne \emptyset\}$, then $f[L_0] \cap L_{i_0}^1$ is a initial part of the ordering $f[L_0] \cong L_0$ and, by Fact 2.2(c), contains a copy of L_0 . Thus $L_0 \hookrightarrow L_{i_0}^1$. The proof of (b) is dual.

(c) If $L_0 \hookrightarrow \sum_{i \le m} L_i^1$ then, by (a), there are $i_0 \in \omega$ and $f : L_0 \hookrightarrow L_{i_0}^1$. Then $\langle L_0, L_0^1, L_1^1, \ldots, L_{i_0}^1, \ldots \rangle$ is a sequence in \mathcal{H} satisfying (1) and, hence, $L_0 + L_1 = L_0 + L_0^1 + L_1^1 + \ldots + L_{i_0}^1 + \ldots \in \mathcal{H}$. If $L_1 \hookrightarrow L_m^0 + \ldots + L_0^0$, we prove $L_0 + L_1 \in \mathcal{H}$ in a similar way.

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Proposition 5.2 Let $L = L_0 + L_1 \notin \mathcal{H}$, where $L_0 = \sum_{\omega^*} L_i^0, L_1 = \sum_{\omega} L_i^1 \in \mathcal{H}$ and $\langle L_i^0 : i \in \omega \rangle$ and $\langle L_i^1 : i \in \omega \rangle$ are sequences in \mathcal{H} satisfying (1). Then

(a) $A \subset L$ contains a copy of L iff for each $i, m \in \omega$ there is a finite $K \subset \omega \setminus m$ such that $L_i^0 \hookrightarrow \bigcup_{j \in K} L_j^0 \cap A$ and $L_i^1 \hookrightarrow \bigcup_{j \in K} L_j^1 \cap A$.

(b) If $A, B \in \mathbb{P}(L)$, then $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ and each $i, m \in \omega$ there is a finite $K \subset \omega \setminus m$ such that $L_i^0 \hookrightarrow \bigcup_{j \in K} L_j^0 \cap C \cap B$ and $L_i^1 \hookrightarrow \bigcup_{j \in K} L_j^1 \cap C \cap B$.

(c) $\sin \langle \mathbb{P}(L), \subset \rangle$ is a σ -closed pre-order.

Proof. (a) (\Rightarrow) Let $C \in \mathbb{P}(L)$, $C \subset A$, $f : L \hookrightarrow L$ and C = f[L]. First we prove

$$\exists C_0 \in \mathbb{P}(L_0) \ \exists C_1 \in \mathbb{P}(L_1) \ C_0 \cup C_1 \subset A.$$
(15)

Suppose that $f[L_0] \subset L_1$. Then, by Lemma 5.1(c), $f[L_0] \cap L_i^1 \neq \emptyset$, for infinitely many $i \in \omega$. But this is impossible since $f[L_0] < f[L_1]$. Thus $f[L_0] \cap L_0 \neq \emptyset$, this set is an initial part of the order $f[L_0] \cong L_0$ and, by Fact 2.2(c), there is $C_0 \in \mathbb{P}(L_0)$ such that $C_0 \subset f[L_0] \cap L_0 \subset C \subset A$. Similarly, there is $C_1 \in \mathbb{P}(L_1)$ such that $C_1 \subset f[L_1] \cap L_1 \subset C \subset A$ and (15) is proved.

Let $i, m \in \omega$. By (15) we have $C_0 \subset A \cap L_0 \subset L_0$ and $C_1 \subset A \cap L_1 \subset L_1$, so, by Proposition 3.1(a), there are finite sets $K_0, K_1 \subset \omega \setminus m$ such that $L_i^0 \hookrightarrow \bigcup_{j \in K_0} L_j^0 \cap A \cap L_0$ and $L_i^1 \hookrightarrow \bigcup_{j \in K_1} L_j^1 \cap A \cap L_1$. Clearly, $K = K_0 \cup K_1$ is a finite subset of $\omega \setminus m$ and $L_i^0 \hookrightarrow \bigcup_{j \in K} L_j^0 \cap A$ and $L_i^1 \hookrightarrow \bigcup_{j \in K} L_j^1 \cap A$.

(\Leftarrow) Suppose that the given condition is satisfied by A. Then, by Proposition 3.1(a), there are $C_0 \in \mathbb{P}(L_0)$ and $C_1 \in \mathbb{P}(L_1)$ such that $C_0 \subset A \cap L_0$ and $C_1 \subset A \cap L_1$. Now $\mathbb{P}(L) \ni C_0 \cup C_1 \subset A$.

(b) By (2), $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ the set $C \cap B$ contains a copy of L. Now we apply (a) to $C \cap B$.

(c) For $A_n \in \mathbb{P}(L)$, $n \in \omega$, where $A_0 \ge A_1 \ge \ldots$ we will construct $A \in \mathbb{P}(L)$ such that $A \le A_n$, for all $n \in \omega$. First, by Fact 2.4(c), there are $C_i \in \mathbb{P}(L)$, $i \in \omega$, such that $C_0 = A_0$ and

$$\forall i \in \omega \ C_i \subset A_0 \cap \ldots \cap A_i. \tag{16}$$

By recursion we define the sequences $\langle K_i : i \in \omega \rangle$, $\langle f_i^0 : i \in \omega \rangle$ and $\langle f_i^1 : i \in \omega \rangle$ such that for each $i \in \omega$

(i) $K_i \in [\omega]^{<\omega}$, (ii) $K_i < K_{i+1}$, (iii) $f_i^0 : L_i^0 \hookrightarrow \bigcup_{j \in K_i} L_j^0 \cap C_i$, (iv) $f_i^1 : L_i^1 \hookrightarrow \bigcup_{j \in K_i} L_j^1 \cap C_i$. Since $C_0 = A_0 \in \mathbb{P}(L)$, by (a) (for i = m = 0), there exist $K_0 \in [\omega]^{<\omega}$, $f_0^0 : L_0^0 \hookrightarrow \bigcup_{j \in K_0} L_j^0 \cap C_0$ and $f_0^1 : L_0^1 \hookrightarrow \bigcup_{j \in K_0} L_j^1 \cap C_0$. Let the sequences $K_0, \ldots, K_{i'}, f_0^0, \ldots, f_{i'}^0$ and $f_0^1, \ldots, f_{i'}^1$ satisfy (i)-(iv). Since $A_{i'+1} \leq A_{i'}, C_{i'+1} \in \mathbb{P}(L)$ and, by (16), $C_{i'+1} \subset A_{i'+1}$, according to (b), for i'+1 and $m = \max(K_0 \cup \ldots \cup K_{i'}) + 1$ there are

$$K_{i'+1} \in [\omega \setminus (\max(K_0 \cup \ldots \cup K_{i'}) + 1)]^{<\omega}$$
(17)

$$f_{i'+1}^{0}: L_{i'+1}^{0} \hookrightarrow \bigcup_{j \in K_{i'+1}} L_{j}^{0} \cap C_{i'+1}$$
(18)

$$f_{i'+1}^{1}: L_{i'+1}^{1} \hookrightarrow \bigcup_{j \in K_{i'+1}} L_{j}^{1} \cap C_{i'+1}$$
(19)

(since, by (16)), $C_{i'+1} \cap A_{i'} = C_{i'+1}$). By (17)) we have (i) and (ii). (iii) and (iv) follow from (18) and (19). The recursion works.

Let $f = \bigcup_{i \in \omega} f_i^0 \cup \bigcup_{i \in \omega} f_i^1$. By (ii) and (iii), $i_1 < i_2$ implies $K_{i_1} < K_{i_2}$, which implies $f_{i_1}^0[L_{i_1}^0] > f_{i_2}^0[L_{i_2}^0]$ and $f_{i_1}^1[L_{i_1}^1] < f_{i_2}^1[L_{i_2}^1]$ and, hence, $f : L \hookrightarrow L$. Thus

$$A = f[L] = \bigcup_{i \in \omega} f_i^0[L_i^0] \cup \bigcup_{i \in \omega} f_i^1[L_i^1] \in \mathbb{P}(L).$$
⁽²⁰⁾

Using the characterization from (b), for $n^* \in \omega$ we show that $A \leq A_{n^*}$. So, for $C^* \in \mathbb{P}(L)$ such that $C^* \subset A$ and $i^*, m^* \in \omega$ we prove that

$$\exists K \in [\omega \setminus m^*]^{<\omega} \ (L^0_{i^*} \hookrightarrow \bigcup_{j \in K} L^0_j \cap C^* \cap A_{n^*} \wedge L^1_{i^*} \hookrightarrow \bigcup_{j \in K} L^1_j \cap C^* \cap A_{n^*}).$$

$$(21)$$

By (ii)-(iv) and (20) we have $A = \sum_{\omega^*} \Lambda_i^0 + \sum_{\omega} \Lambda_i^1 \cong L$, where $\Lambda_i^0 = f_i^0[L_i^0] \cong L_i^0$ and $\Lambda_i^1 = f_i^1[L_i^1] \cong L_i^1$, so A is a sum of an ω^* -sum, $\Lambda_0 = \sum_{\omega^*} \Lambda_i^0 \cong L_0$ and an ω -sum, $\Lambda_1 = \sum_{\omega} \Lambda_i^1 \cong L_1$. In addition, $L_0 + L_1 \notin \mathcal{H}$ implies $\Lambda_0 + \Lambda_1 \notin \mathcal{H}$.

Since $C^* \cong L \cong A$ and $C^* \subset A$ we have $C^* \in \mathbb{P}(A)$ so, applying (a) to the linear order A instead of L we obtain

$$\forall i, m \in \omega \; \exists K \in [\omega \setminus m]^{<\omega} \; (\Lambda_i^0 \hookrightarrow \bigcup_{j \in K} \Lambda_j^0 \cap C^* \land \Lambda_i^1 \hookrightarrow \bigcup_{j \in K} \Lambda_j^1 \cap C^*).$$
(22)

Let $m' > m^*$, n^* . By (22), for i^* and m' there is

1

$$K^* \in [\omega \setminus m']^{<\omega}$$
 such that (23)

$$\Lambda_{i^*}^0 \hookrightarrow \bigcup_{j \in K^*} \Lambda_j^0 \cap C^* \wedge \Lambda_{i^*}^1 \hookrightarrow \bigcup_{j \in K^*} \Lambda_j^1 \cap C^*$$
(24)

By (23), for $j \in K^*$ we have $j > n^*$ and, by (16), $C_j \subset A_{n^*}$. Thus, by (iii) and (iv) we have $\Lambda_j^0 \subset \bigcup_{s \in K_j} L_s^0 \cap C_j \subset \bigcup_{s \in K_j} L_s^0 \cap A_{n^*}$ and $\Lambda_j^1 \subset \bigcup_{s \in K_j} L_s^1 \cap C_j \subset \bigcup_{s \in K_j} L_s^1 \cap A_{n^*}$ which, together with (iii),(iv) and (24) gives $L_{i^*}^0 \hookrightarrow \Lambda_{i^*}^0 \hookrightarrow \bigcup_{j \in K^*} f_j[\Lambda_j^0] \cap C^* \subset \bigcup_{j \in K^*} \bigcup_{s \in K_j} L_s^0 \cap A_{n^*} \cap C^* = \bigcup_{s \in \bigcup_{j \in K^*} K_j} L_s^0 \cap C^* \cap A_{n^*}$. Similarly we prove that $L_{i^*}^0 \hookrightarrow \bigcup_{s \in \bigcup_{j \in K^*} K_j} L_s^0 \cap C^* \cap A_{n^*}$.

In order to finish the proof of (21) we show that $\bigcup_{j \in K^*} K_j \cap m^* = \emptyset$. By (23), for $j \in K^*$ we have $j > m^*$. By (ii) the sequence $\langle \min K_i : i \in \omega \rangle$ is increasing and, hence, $\min K_j \ge j > m^*$, which implies $K_j \cap m^* = \emptyset$ and (21) is proved. \Box

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6 The general case

For $L \in S$, let $m(L) = \min\{n \in \omega : L \text{ is a sum of } n \text{ elements of } \mathcal{H}\}$. For $m \in \mathbb{N}$, let $S_m = \{L \in S : m(L) = m\}$.

Lemma 6.1 (a) There is no $L \in \mathcal{H}$ such that $L = \sum_{\omega^*} L_i^0$ and $L = \sum_{\omega} L_i^1$, where $\langle L_i^0 : i \in \omega \rangle$ and $\langle L_i^1 : i \in \omega \rangle$ are sequences in \mathcal{H} satisfying (1).

(b) Let $L \in S_m$ and $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$. Then

 $\forall i < m \ (|L_i| = 1 \ \ u \ L_i \text{ is an } \omega \text{-sum } \ u \ L_i \text{ is an } \omega^* \text{-sum})$ (25)

$$\forall i < m-1 \ L_i + L_{i+1} \notin \mathcal{H}. \tag{26}$$

$$|L_i| = 1 \Rightarrow (L_{i+1} \text{ is not an } \omega \text{-sum } \land L_{i-1} \text{ is not an } \omega^* \text{-sum }).$$
 (27)

Proof. (a) On the contrary, by Fact 2.2, L would be both left and right indecomposable and, for a partition L = L' + L'' there would be $C', C'' \cong L$ such that $C' \subset L'$ and $C'' \subset L''$, which would imply $L + L \hookrightarrow L$. But this is impossible by Fact 2.1.

(b) The first statement follows from (a), the second from the minimality of m and the third from the second statement $(1 + \omega$ -sum is an ω -sum satisfying (1)). \Box

Lemma 6.2 If $m \in \mathbb{N}$, $L \in S_m$, $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$, and $f : L \hookrightarrow L$, then for each i < m there is $C_i \in \mathbb{P}(L_i)$ such that $C_i \subset f[L_i]$.

Proof. We use induction. For m = 1 the statement is trivially true.

Suppose that the statement holds for all $k \leq m$. Let $L \in S_{m+1}, L_0, \ldots, L_m \in \mathcal{H}, L = L_0 + L_1 + \ldots + L_m$ and $f : L \hookrightarrow L$. Let $L' = L_1 + \ldots + L_m$.

Claim 1. $f[L_1] \cap L_0$ does not contain a copy of L_1 .

Proof of Claim 1. On the contrary suppose that $L_1 \cong C_1 \subset f[L_1] \cap L_0$.

First we show that L_0 is an ω^* -sum. Namely, $|L_0| = 1$ would imply $C_1 = L_0 = f[L_1]$, which is impossible because $f[L_0] < f[L_1]$. Suppose that L_0 is an ω -sum, $L_0 = \sum_{\omega} \Lambda_i$. Then, since $f[L_0] < f[L_1] \cap L_0$, $L_0 \hookrightarrow \sum_{i \le m} \Lambda_i$, for some $m \in \omega$, which is impossible by Proposition 3.1(a).

Thus L_0 is an ω^* -sum, $L_0 = \sum_{\omega^*} L_i^0$ and, by (25) and (27), L_1 is either an ω -sum or an ω^* -sum. Since $f[L_0] < f[L_1] \cap L_0 \leftrightarrow L_1$, there is $m \in \omega$ such that $L_1 \hookrightarrow L_m^0 + \ldots + L_0^0$. By (26) we have $L_0 + L_1 \notin \mathcal{H}$ and this is impossible by Lemma 5.1(c) in the first case and Lemma 4.3(b) in the second. A contradiction. Claim 1 is proved.

By (25), regarding the summand L_1 we have the following three cases.

Case 1: $|L_1| = 1$. Then, by Claim 1, $f[L_1] \cap L_0 = \emptyset$, which implies that $f \upharpoonright L' : L' \hookrightarrow L'$. Clearly $m(L') \le m$ and m(L') < m is impossible, because of the minimality of m(L). Thus m(L') = m and, by the induction hypothesis,

$$\forall i \in \{1, \dots m\} \ \exists C_i \in \mathbb{P}(L_i) \ C_i \subset (f \upharpoonright L')[L_i] = f[L_i].$$

$$(28)$$

Since $|L_1| = 1$ we have $C_1 = L_1 = f[L_1] > f[L_0]$, for $C_0 = f[L_0]$ we have $C_0 \in \mathbb{P}(L_0)$ and the proof is over.

Case 2: L_1 is an ω^* -sum. By Fact 2.2(c), $f[L_1] \cap L_0 \neq \emptyset$ would imply that $f[L_1] \cap L_0$ contains a copy of L_1 , which is impossible by Claim 1. Thus $f[L_1] \cap L_0 = \emptyset$ and, as in Case 1, we have (28). In particular, $\mathbb{P}(L_1) \ni C_1 \subset f[L_1]$ and, by Proposition 3.1(a) (for ω^* -sums), $f[L_1]$ intersects infinitely many summands of L_1 , which implies $f[L_0] \subset L_0$. Again, for $C_0 = f[L_0]$ we have $C_0 \in \mathbb{P}(L_0)$ and the proof is over.

Case 3: L_1 is an ω -sum. By (25) and (27), regarding the summand L_0 we have the following two subcases.

Subcase 3.1: L_0 is an ω -sum. $f[L_1] \cap L_0 \neq \emptyset$ would imply that L_0 is embeddable in an initial part of L_0 , which is impossible by Proposition 3.1(a). Thus $f[L_1] \cap L_0 = \emptyset$ and, as in Case 1, we have (28). Since $C_1 \subset f[L_1] \cap L_1$ we have $f[L_0] \subset L_0 \cup L_1$. Suppose that $f[L_0] \cap L_1 \neq \emptyset$. Then $f[L_0] \cap L_1$ is contained in finitely many summands of L_1 and, by Fact 2.2(a), contains a copy of L_0 , which is impossible by (26) and Lemma 4.1(b). Thus $f[L_0] \subset L_0$ and, for $C_0 = f[L_0]$ we have $C_0 \in \mathbb{P}(L_0)$ which, together with (28), finishes the proof.

Subcase 3.2: L_0 is an ω^* -sum. Let $L_0 = \sum_{\omega^*} A_i$ and $L_1 = \sum_{\omega} B_i$. By Claim 1, there is $x \in L_1$ such that $L_0 < \{f(x)\}$. By Fact 2.2(b), there is $L'_1 \cong L_1$ such that $L'_1 \subset [x, \infty)_{L_1}$. Let $\varphi : L_1 + L_2 + \ldots + L_m \to L'_1 + L_2 + \ldots + L_m$ be an isomorphism, where $\varphi \upharpoonright L_i = id_{L_i}$, for $i \in \{2, 3, \ldots, m\}$. Then $f \circ \varphi : L' \hookrightarrow L'$ and, by the induction hypothesis, there are $C_i \in \mathbb{P}(L_i), i \in \{1, \ldots, m\}$, satisfying $C_i \subset f[\varphi[L_i]]$. Since $C_1 \subset f[\varphi[L_1]] = f[L'_1]$ we have

$$C_1 \subset f[L'_1] \cap L_1 \subset f[L_1] \cap L_1.$$
(29)

$$\forall i \in \{2, \dots m\} \ (C_i \in \mathbb{P}(L_i) \land C_i \subset f[\varphi[L_i]] = f[L_i]).$$
(30)

By (29) we have $f[L_0] \subset L_0 \cup L_1$. Suppose that $f[L_0] \subset L_1$. Then, by (29), $f[L_0]$ is contained in the union of finitely many summands of L_1 , which is impossible by (26) and Lemma 5.1(c). Thus $f[L_0] \cap L_0 \neq \emptyset$ is an initial part of $f[L_0] \cong L_0$ and, by Fact 2.2(c), there is $C_0 \cong L_0$ such that $C_0 \subset f[L_0] \cap L_0$. By (29) and (30) the proof is over.

Let $L \in S_m$ and $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$. Then we have (25), (26) and (27) and we divide L into *blocks*, groups of consecutive summands L_i , in the following way:

- first we glue each two consecutive summands such that the first is an ω^* -sum and the second an ω -sum (blocks of the type D),

- then we divide the rest into the groups of consecutive (in L) L_i 's of the same form: groups of singletons (blocks of the type A), groups of ω -sums (blocks of the type B) and groups of ω^* -sums (blocks of the type C).

For example $111|\omega^*\omega^*|\omega^*\omega|\omega|11|\omega^*\omega|\omega\omega\omega\omega|\omega^*\omega^*$. More formally, we define a *block* of *L* as a sum of consecutive summands $B = L_i + L_{i+1} + \ldots + L_{i+k}$, where $k \ge 0$ and satisfying one of the following conditions.

- (A) $|L_j| = 1$, for all $j \in \{i, ..., i + k\}$ and (i) $i = 0 \lor |L_{i-1}| = \omega$ and (ii) $i + k = m - 1 \lor |L_{i+k+1}| = \omega$;
- (B) L_j is an ω -sum, for all $j \in \{i, \ldots, i+k\}$ and (iii) $i = 0 \lor (L_{i-1} \text{ is an } \omega \text{-sum} \land L_{i-2} \text{ is an } \omega^*\text{-sum})$ and (iv) $i + k = m - 1 \lor L_{i+k+1}$ is not an ω -sum;
- (C) L_j is an ω^* -sum, for all $j \in \{i, \ldots, i+k\}$ and (v) $i = 0 \lor L_{i-1}$ is not an ω^* -sum and (vi) $i + k = m - 1 \lor (L_{i+k+1} \text{ is an } \omega^* \text{-sum} \land L_{i+k+2} \text{ is an } \omega\text{-sum});$
- (D) k = 1 and L_i is an ω^* -sum and L_{i+1} is an ω -sum.

By Block(L) we will denote the set of blocks.

Lemma 6.3 Blocks determine a partition of the set $\{L_0, \ldots, L_{m-1}\}$ and a partition of L into convex parts.

Proof. Let $L \in S_m$ and $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$. First we show that each summand L_j is contained in some block. We have the following three cases

Case 1: $|L_j| = 1$. Let $L_i, L_{i+1}, \ldots, L_j, \ldots, L_{i+k}$ be the maximal sequence of consecutive summands of size 1, including L_j . Then conditions (i) and (ii) are satisfied and, hence, L_j belongs to a block of the type (A).

Case 2: L_j is an ω -sum.

Subcase 2.1: j = 0. Let L_0, \ldots, L_k be a maximal sequence of consecutive ω -sums. Then k = m - 1 or L_{k+1} is not an ω -sum so, conditions (iii) and (iv) are satisfied and L_j belongs to a block of the type (B).

Subcase 2.2: j > 0 and L_{j-1} is an ω^* -sum. Then $L_{j-1} + L_j$ is a block of the type (D) containing L_j .

Subcase 2.3: j > 0 and L_{j-1} is not an ω^* -sum. Then, by (27), $|L_{j-1}| \neq 1$, so, by (25), L_{j-1} is an ω -sum. Let $L_i, L_{i+1}, \ldots, L_{j-1}, L_j, \ldots, L_{i+k}$ be the maximal sequence of consecutive ω -sums containing L_j . Then (iv) is true.

If i = 0, then (iii) is true and L_j belongs to a block of the type (B).

If i > 0, then, by the maximality of the sequence and (27) and (25), L_{i-1} is an ω^* -sum. Now $L_{i+1}, \ldots, L_{j-1}, L_j, \ldots, L_{i+k}$ satisfies (iii) and (iv), so it is a block of the type (B) containing L_j (since, clearly, $i + 1 \le j$).

Case 3: L_j is an ω^* -sum.

Subcase 3.1: j = m - 1. Let L_i, \ldots, L_j be a maximal sequence of consecutive ω^* -sums. Then i = 0 or L_{i-1} is not an ω^* -sum so, conditions (v) and (vi) are satisfied and L_j belongs to a block of the type (C).

Subcase 3.2: j < m - 1 and L_{j+1} is an ω -sum. Then $L_j + L_{j+1}$ is a block of the type (D) containing L_j .

Subcase 3.3: j < m - 1 and L_{j+1} is not an ω -sum. Since, by (27), $|L_{j+1}| \neq 1$ by (25) we have that L_{j+1} is an ω^* -sum. Let $L_i, L_{i+1}, \ldots, L_j, L_{j+1}, \ldots, L_{i+k}$ be the maximal sequence of consecutive ω^* -sums containing L_j . Then (v) is true.

If i + k = m - 1, then (vi) is true and L_j belongs to a block of the type (C).

If i + k < m - 1, then, by the maximality of the sequence and (27) and (25), L_{i+k+1} is an ω -sum. Now $L_i, \ldots, L_{j-1}, L_j, \ldots, L_{i+k-1}$ satisfies (v) and (vi), so it is a block of the type (C) containing L_j (since, clearly, $j \le i + k - 1$).

Now we prove that different blocks are disjoint. Suppose that $B', B'' \in \text{Block}(L)$ and $x \in B' \cap B''$. Then $x \in L_j$ for some L_j contained in $B' \cap B''$. By (25) we have the following three cases:

Case 1: $|L_j| = 1$. Then B' and B'' are blocks of the type (A). Since $L_j \subset B' \cap B''$, by (i) and (ii) we have B' = B''.

Case 2: L_j is an ω -sum. Then, by Lemma 6.1(a), the blocks are of the type (B) or (D).

Subcase 2.1: B' and B'' are of the type (D). Then, since $L_j \subset B' \cap B''$ is an ω -sum, by Lemma 6.1(a) we have B' = B''.

Subcase 2.2: B' and B'' are of the type (B). Then, since $L_j \subset B' \cap B''$, from (iii) and (iv) it follows that in L the blocks have the same beginning and the same end. Thus, B' = B''.

Subcase 2.2: B' is of the type (B) and B'' of the type (D). Then, by Lemma 6.1(a), L_j is the second summand of B'' and, hence, $B'' = L_{j-1} + L_j$ and $B' = L_j + \dots + L_k$. But this is impossible by (iii)

Posets of copies of countable scattered linear orders

Case 3: L_i is an ω^* -sum. This case is dual to Case 2.

Lemma 6.4 If $m \in \mathbb{N}$, $L \in S_m$, $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$ and $\operatorname{Block}(L) = \{B_0, \ldots, B_r\}$, then $\operatorname{Block}(L \setminus B_0) = \operatorname{Block}(L) \setminus \{B_0\}$.

Proof. Let $L = L_0 + ... + L_{n-1} + L_n + ... + L_{m-1}$, where $B_0 = L_0 + ... + L_{n-1}$, $L' = L \setminus B_0 = L_n + ... + L_{m-1}$ and 0 < n < m. First we show that

$$\operatorname{Block}(L') \subset \operatorname{Block}(L).$$
 (31)

Let $B = L_i + \ldots + L_{i+k} \in \operatorname{Block}(L')$. Clearly, if B is of the type (D) in L', then the same holds in L and $B \in \operatorname{Block}(L)$. If B is of the type (A) (resp. (B), (C)), then it satisfies (ii) (resp. (iv), (vi)) in L' and, clearly, in L. If i > n, then, in addition, B satisfies (i) (resp. (iii), (v)) in L' and, again, in L; thus $B \in \operatorname{Block}(L)$. So it remains to be proved that B satisfies (i) (resp. (iii), (v)) in L, when i = n.

Case 1: *B* is of the type (A). Then $|L_{n-1}| = 1$ would imply that B_0 is not a block in *L*. Thus $|L_{n-1}| = \omega$ and *B* satisfies (i) in *L*.

Case 2: *B* is of the type (B). Then L_n is an ω -sum and, by (27), $|L_{n-1}| = \omega$. By (iv) and (vi), B_0 is not of the type (B) or (C). Thus, B_0 is of the type (D) and, hence, *B* satisfies (iii) in *L*.

Case 3: *B* is of the type (C). Then L_n is an ω^* -sum. Suppose that L_{n-1} is an ω^* -sum. Then B_0 must be of the type (C) and, by (vi) for B_0 in *L*, L_{n+1} is an ω -sum. But then *B* should be a block of the type (D) in *L'*, which is not true. Thus L_{n-1} is not an ω^* -sum and, hence, *B* satisfies (v) in *L*.

So (31) is proved, which implies $\operatorname{Block}(L') \subset \operatorname{Block}(L) \setminus \{B_0\} = \{B_1, \ldots, B_r\}$. By Lemma 6.3 we have $\bigcup \operatorname{Block}(L') = L' = B_1 \cup \ldots \cup B_r$, which gives the another inclusion.

Lemma 6.5 If $m \in \mathbb{N}$, $L \in S_m$, $L_0, \ldots, L_{m-1} \in \mathcal{H}$, where $L = L_0 + \ldots + L_{m-1}$, and $f : L \hookrightarrow L$, then for each $B \in \text{Block}(L)$ we have $f[B] \subset B$.

Proof. We prove the statement by induction. For m = 1 it is trivially true.

Suppose that it is true for all k < m. Let $L = L_0 + \ldots + L_{m-1}$ and $Block(L) = \{B_0, \ldots, B_r\}$. If r = 0, we are done. Otherwise we have

$$L = B_0 + L_{i+1} + \ldots + L_{m-1}, \tag{32}$$

where $B_0 = L_0 + \ldots + L_i$. Let $L' = L_{i+1} + \ldots + L_{m-1}$. By Lemma 6.2,

$$\forall j \in \{0, \dots, m-1\} \ \exists C_j \in \mathbb{P}(L_j) \ C_j \subset f[L_j] \cap L_j.$$
(33)

Regarding the type of B_0 we have the following cases.

Case 1: B_0 is of the type (A). Then, by (25), (27) and (ii), L_{i+1} is an ω^* -sum. By (33) and Proposition 3.1(a) (for ω^* -sums), C_{i+1} intersects infinitely many summands of L_{i+1} and, since B_0 is finite and $f[B_0] < f[L_{i+1}]$, we have $f[B_0] = B_0$. Hence $f \upharpoonright L' : L' \hookrightarrow L'$ and m(L') = m - i - 1. By Lemma 6.4 we have

$$Block(L') = Block(L) \setminus \{B_0\} = \{B_1, \dots, B_r\}$$
(34)

and, by the induction hypothesis, $f[B_j] = (f \upharpoonright L')[B_j] \subset B_j$, for j > 0.

Case 2: B_0 is of the type (B). By Proposition 3.1(a) C_i intersects infinitely many summands of L_i , which implies that $f \upharpoonright L' : L' \hookrightarrow L'$.

If $|L_{i+1}| = 1$, then $f[L_{i+1}] = L_{i+1}$ and, hence, $f[B_0] \subset B_0$. By (34) and the induction hypothesis $f[B_i] \subset B_i$, for j > 0.

If L_{i+1} is an ω^* -sum, then C_{i+1} intersects infinitely many summands of L_{i+1} and, hence, $f[B_0] \subset B_0$. Also, C_i intersects infinitely many summands of L_i , which implies that $f[L'] \subset L'$. By (34) and the induction hypothesis $f[B_j] \subset B_j$, for j > 0 again.

Case 3: B_0 is of the type (C). Then by (vi), L_{i+1} is an ω^* -sum. By (33) we have $C_{i+1} \subset f[L_{i+1}] \cap L_{i+1}$ and, by Proposition 3.1, $f[L_{i+1}]$ intersects infinitely many summands of L_{i+1} , which implies $f[B_0] \subset B_0$. Suppose that $f[L_{i+1}] \cap L_i \neq \emptyset$. By (33), $C_i \subset f[L_i] \cap L_i$, which implies that $f[L_{i+1}] \cap L_i$ is an initial part of $f[L_{i+1}]$ contained in an final part of L_i . By Fact 2.2(c) $f[L_{i+1}] \cap L_i$ contains a copy of L_{i+1} , which is impossible by Lemma 4.3(b) and (26). Thus $f[L_{i+1}] \cap L_i = \emptyset$, which implies $f[L'] \subset L'$ and again, by (34) and the induction hypothesis $f[B_j] \subset B_j$, for j > 0.

Case 4: B_0 is of the type (D). Then $B_0 = L_0 + L_1$ and, by (33) and Proposition 3.1, $f[L_1]$ intersects infinitely many summands of L_1 , which implies

$$f[L'] \subset L'. \tag{35}$$

By (33) there is C_2 such that

$$C_2 \in \mathbb{P}(L_2) \land C_2 \subset f[L_2] \cap L_2.$$
(36)

Regarding the form of L_2 we distinguish the following three subcases.

 $|L_2| = 1$. Then, by (33), $f[L_1] = L_1$ and, hence, $f[B_0] \subset B_0$ and we use (35), (34) and the induction hypothesis.

 L_2 is an ω^* -sum. By (36) $f[L_2]$ intersects infinitely many summands of L_2 and, hence, $f[B_0] \subset B_0$ and we use (35), (34) and the induction hypothesis.

 L_2 is an ω -sum. By (36) we have $f[L_1] \subset L_0 \cup L_1 \cup L_2$. $f[L_1] \cap L_2 \neq \emptyset$ is impossible by Lemma 4.1(b), thus $f[B_0] \subset B_0$ and we continue as above. \Box

Theorem 6.6 For each $L \in S$, $\operatorname{sm} \langle \mathbb{P}(L), \subset \rangle$ is a σ -closed pre-order.

Proof. Let $L \in S_m$, $L = \sum_{i < r} B_i$, where $Block(L) = \{B_i : i < r\}$. First we prove

$$\mathbb{P}(L) = \{\bigcup_{i < r} C_i : \forall i < r \ C_i \in \mathbb{P}(B_i)\}.$$
(37)

The inclusion " \supset " is evident. If $C \in \mathbb{P}(L)$, $f : L \hookrightarrow L$ and C = f[L], then, by Lemma 6.5, for $C_i = f[B_i]$, i < r, we have $C_i \subset B_i$, $C_i \in \mathbb{P}(B_i)$ and $C = \bigcup_{i < r} C_i$ and " \subset " holds as well.

Clearly, the mapping $F: \prod_{i < r} \langle \mathbb{P}(B_i), \subset \rangle \to \langle \mathbb{P}(L), \subset \rangle$ defined by

$$f(\langle C_0, \dots, C_{r-1} \rangle) = \bigcup_{i < r} C_i$$

is an isomorphism and, by Fact 2.4(d),(e)

$$\operatorname{sm}(\mathbb{P}(L), \subset) \cong \operatorname{sm}\prod_{i < r} \langle \mathbb{P}(B_i), \subset \rangle = \prod_{i < r} \operatorname{sm}(\mathbb{P}(B_i), \subset).$$

By Propositions 4.2, 4.4 and 5.2 sm $\langle \mathbb{P}(B_i), \subset \rangle$, i < r, are σ -closed partial orders and, by Fact 2.3 their product as well as the poset sm $\langle \mathbb{P}(L), \subset \rangle$ is σ -closed. \Box

7 Forcing by copies of countable scattered linear orders

The position of countable linear orders in Diagram 1 is presented in Diagram 2.

By Theorem 1.2 and Fact 2.5, CH implies that all posets of the form $\langle \mathbb{P}(L), \subset \rangle$, where *L* is a scattered countable linear order, are forcing equivalent to $(P(\omega)/\operatorname{Fin})^+$. The following examples show that this is not true in general and that the result of Theorem 1.2 is the best possible: " σ -closed" can not be replaced by " ω_2 -closed".

Example 7.1 It is consistent that the poset $\langle \mathbb{P}(\omega + \omega), \subset \rangle$ is not \mathfrak{h} -distributive and, hence, not forcing equivalent to $(P(\omega)/\operatorname{Fin})^+$.

By Proposition 4.2, for $L = \omega + \omega$ the partial order $\langle \mathbb{P}(L), \subset \rangle$ is isomorphic to the product $\langle [\omega]^{\omega}, \subset \rangle \times \langle [\omega]^{\omega}, \subset \rangle$ and, by Fact 2.6(a), $\operatorname{sq}\langle \mathbb{P}(\omega + \omega), \subset \rangle \cong (P(\omega)/\operatorname{Fin})^+ \times (P(\omega)/\operatorname{Fin})^+$. Now, by the result of Shelah and Spinas (Fact 2.6(b)), we have $\operatorname{Con}(\mathfrak{h}_2 < \mathfrak{h})$.

Example 7.2 The poset $\operatorname{sq} \langle \mathbb{P}(\omega \cdot \omega), \subset \rangle$ is not ω_2 -closed and it is consistent that $\operatorname{sq} \langle \mathbb{P}(\omega \cdot \omega), \subset \rangle$ is not \mathfrak{h} -distributive. Clearly $\omega \cdot \omega \cong \langle L, < \rangle$, where $L = \omega \times \omega$ and $\langle i_0, j_0 \rangle < \langle i_1, j_1 \rangle \Leftrightarrow i_0 < i_1 \lor (i_0 = i_1 \land j_0 < j_1)$. Now $L = \sum_{i \in \omega} L_i$,

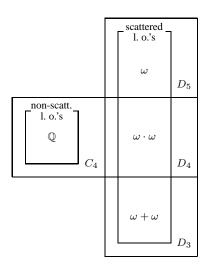


Diagram 2: Countable linear orders

where $L_i = \{i\} \times \omega$ and first we show that $\mathbb{P}(L) = (\operatorname{Fin} \times \operatorname{Fin})^+$. By Proposition 3.1(a), if $A \in \mathbb{P}(L)$, then for each $m \in \omega$ there is a finite set $K \subset \omega \setminus m$ such that $\omega \hookrightarrow \bigcup_{i \in K} A \cap L_i$ and, hence, there is $i \geq m$ satisfying $|A \cap L_i| = \omega$. Thus $A \notin \operatorname{Fin} \times \operatorname{Fin}$. Conversely, if $A \notin \operatorname{Fin} \times \operatorname{Fin}$ and $\{i \in \omega : |A \cap L_i| = \omega\} = \{n_j : j \in \omega\}$, where $n_0 < n_1 < \ldots$, then $A = \bigcup_{j \in \omega} \Lambda_j$, where $\Lambda_0 = \bigcup_{i \leq n_0} (A \cap L_i)$ and $\Lambda_j = \bigcup_{n_{j-1} < i \leq n_j} (A \cap L_i)$, for j > 0. Clearly we have $\Lambda_j \cong \omega$ and, hence, $A \in \mathbb{P}(L)$. So, $\langle \mathbb{P}(L), \mathbb{C} \rangle = \langle (\operatorname{Fin} \times \operatorname{Fin})^+, \mathbb{C} \rangle$ and, by Fact 2.4(f), $\operatorname{sq}(\mathbb{P}(\omega \cdot \omega), \mathbb{C})$ $\rangle \cong (P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+$. Now we apply the results of Szymański and Zhou and of Hernández-Hernández (Fact 2.6(c) and (d)).

Some forcing-related properties of the posets $sq \langle \mathbb{P}(L), \subset \rangle$ are described in the following table.

L	$\operatorname{sq} \langle \mathbb{P}(L), \subset \rangle$ is isomorphic to	$\mathrm{sq}\langle \mathbb{P}(L),\subset angle$ is	$\operatorname{ZFC} \vdash \operatorname{sq} \langle \mathbb{P}(L), \subset \rangle$ is \mathfrak{h} -distributive
ω	$(P(\omega)/\operatorname{Fin})^+$	t-closed	yes
$\omega+\omega$	$(P(\omega)/\operatorname{Fin})^+ \times (P(\omega)/\operatorname{Fin})^+$	t-closed	no
$\omega \cdot \omega$	$(P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^+$	ω_1 but not ω_2 -closed	no

Remark 7.3 Concerning Theorem 1.2 we note that for countable ordinals we have more information. Namely, by [6], if $\alpha = \omega^{\gamma_n + r_n} s_n + \ldots + \omega^{\gamma_0 + r_0} s_0 + k$ is a

countable ordinal presented in the Cantor normal form, where $k \in \omega$, $r_i \in \omega$, $s_i \in \mathbb{N}$, $\gamma_i \in \text{Lim} \cup \{1\}$ and $\gamma_n + r_n > \ldots > \gamma_0 + r_0$, then

$$\operatorname{sq}\langle \mathbb{P}(\alpha), \subset \rangle \cong \prod_{i=0}^{n} \left(\left(\operatorname{rp}^{r_{i}} \left(P(\omega^{\gamma_{i}}) / \mathcal{I}_{\omega^{\gamma_{i}}} \right) \right)^{+} \right)^{s_{i}}, \tag{38}$$

where, for an ordinal β , $\mathcal{I}_{\beta} = \{C \subset \beta : \beta \not\leftrightarrow C\}$ and, for a poset \mathbb{P} , $\operatorname{rp}(\mathbb{P})$ denotes the reduced power $\mathbb{P}^{\omega} / \equiv_{\operatorname{Fin}}$ and $\operatorname{rp}^{k+1}(\mathbb{P}) = \operatorname{rp}(\operatorname{rp}^{k}(\mathbb{P}))$. In particular, for $\omega \leq \alpha < \omega^{\omega}$ we have

$$\operatorname{sq}\left(\mathbb{P}\left(\sum_{i=n}^{0}\omega^{1+r_{i}}s_{i}\right),\subset\right)\cong\prod_{i=0}^{n}\left(\left(\operatorname{rp}^{r_{i}}\left(P(\omega)/\operatorname{Fin}\right)\right)^{+}\right)^{s_{i}}.$$
(39)

Remark 7.4 By [5], all countable equivalence relations, disconnected ultrahomogeneous graphs and disjoint unions of ordinals $\leq \omega$ are in column *D* of Diagram 1 as well. In addition, the corresponding posets of copies are forcing equivalent to one of the following posets:

$$((P(\omega)/\operatorname{Fin})^{+})^{n}$$
, for some $n \in \mathbb{N}$,
 $(P(\omega \times \omega)/(\operatorname{Fin} \times \operatorname{Fin}))^{+}$.

$$(P(\Delta)/\mathcal{ED}_{\text{fin}})^+ \times ((P(\omega)/\text{Fin})^+)^n$$
, for some $n \in \omega$,

where $\Delta = \{ \langle m, n \rangle \in \mathbb{N} \times \mathbb{N} : n \leq m \}$ and the ideal $\mathcal{ED}_{\text{fin}} \subset P(\Delta)$ is defined by:

$$\mathcal{ED}_{\text{fin}} = \{ S \subset \Delta : \exists r \in \mathbb{N} \ \forall m \in \mathbb{N} \ |S \cap (\{m\} \times \{1, 2, \dots, m\})| \le r \}.$$

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