# POSETS OF COPIES OF COUNTABLE SCATTERED LINEAR ORDERS 

Miloš S. Kurilić<br>Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia.<br>e-mail: milos@dmi.uns.ac.rs


#### Abstract

We show that the separative quotient of the poset $\langle\mathbb{P}(L), \subset\rangle$ of isomorphic suborders of a countable scattered linear order $L$ is $\sigma$-closed and atomless. So, under the CH , all these posets are forcing-equivalent (to $\left.(P(\omega) / \text { Fin })^{+}\right)$


## 1 Introduction

The posets of the form $\langle\mathbb{P}(\mathbb{X}), \subset\rangle$, where $\mathbb{X}$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures, were investigated in [4]. In particular, a classification of countable binary structures related to the order-theoretic and forcing-related properties of the posets of their copies is described in Diagram 11: for the structures from column $A$ (resp. $B ; D$ ) the corresponding posets are forcing equivalent to the trivial poset (resp. the Cohen forcing, $\left\langle{ }^{<\omega} 2\right.$, $\left.\supset\right\rangle$; an $\omega_{1}$-closed atomless poset) and, for the structures from the class $C_{4}$, the posets of copies are forcing equivalent to the posets of the form $(P(\omega) / \mathcal{I})^{+}$, for some co-analytic tall ideal $\mathcal{I}$. For example, all countable non-scattered linear orders are in the class $C_{4}$, moreover, as a consequence of the main result of [3] we have

Theorem 1.1 For each countable non-scattered linear order $L$ the poset $\langle\mathbb{P}(L), \subset\rangle$ is forcing equivalent to the two-step iteration $\mathbb{S} * \pi$, where $\mathbb{S}$ is the Sacks forcing and $1_{\mathbb{S}} \Vdash$ " $\pi$ is a $\sigma$-closed forcing". If the equality $\operatorname{sh}(\mathbb{S})=\aleph_{1}$ or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega) / \text { Fin })^{+}$of the Sacks extension.
The aim of this paper is to complete the picture of countable linear orders in this context and, having in mind Theorem 1.1 we concentrate our attention on countable scattered linear orders. In the simplest case, if $L$ is the ordinal $\omega$, then $\langle\mathbb{P}(L), \subset\rangle=\left\langle[\omega]^{\omega}, \subset\right\rangle$ is a homogeneous atomless partial order of size $\mathfrak{c}$ and its separative quotient, the poset $(P(\omega) / \mathrm{Fin})^{+}$, is $\sigma$-closed. We will show that the same holds for each countable scattered linear order. So the following theorem is our main result.

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Diagram 1: Binary relations on countable sets

Theorem 1.2 For each countable scattered linear order $L$ the poset $\langle\mathbb{P}(L), \subset\rangle$ is homogeneous, atomless, of size $\mathfrak{c}$ and its separative quotient is $\sigma$-closed.

Corollary 1.3 If $L$ is a countable linear order, then the poset $\langle\mathbb{P}(L), \subset\rangle$ is forcing equivalent to
$-\mathbb{S} * \pi$, where $1_{\mathbb{S}} \Vdash$ " $\pi$ is $\sigma$-closed", if $L$ is non-scattered [3];

- A $\sigma$-closed atomless forcing, if $L$ is scattered.

Under the CH , the poset $\langle\mathbb{P}(L), \subset\rangle$ is forcing equivalent to

- $\mathbb{S} * \pi$, where $1_{\mathbb{S}} \Vdash$ " $\pi=(P(\check{\omega}) / \text { Fin })^{+} "$, if $L$ is non-scattered [3];
- $(P(\omega) / \text { Fin })^{+}$, if $L$ is scattered.

The most difficult part of the proof of Theorem 1.2 is to show that the separative quotient of $\langle\mathbb{P}(L), \subset\rangle$ is $\sigma$-closed (this result is the best possible: " $\sigma$-closed" can not be replaced by " $\omega_{2}$-closed", see Example [.2]. Namely, it is easy to see that
there are copies of an $\omega$-sum $\sum_{\omega} L_{i}$ of linear orders $L_{i}$, which are not of the form $\bigcup_{i \in \omega} C_{i}$, where $C_{i} \in \mathbb{P}\left(L_{i}\right)$, so the Hausdorff hierarchy of scattered linear orders can not be used (easily) for an inductive proof. Instead of that hierarchy we use the result of Laver [7] that a countable scattered linear order is a finite sum of hereditarily indecomposable (ha) linear orders. So we first prove the statement for ha-orders, then for special blocks of ha-orders and, finally, for finite sums of blocks.

## 2 Preliminaries

A linear order $L$ is said to be scattered iff it does not contain a dense suborder or, equivalently, iff the rational line, $\mathbb{Q}$, does not embed in $L$. By $\mathcal{S}$ we denote the class of all countable scattered linear orders.

Fact 2.1 If $L$ is a linear order satisfying $L+L \hookrightarrow L$, then $L$ is not scattered (see [8], p. 180).

Proof. By the assumption, $L+(L+L) \hookrightarrow L+L \hookrightarrow L$. By recursion we construct the sequences $\left\langle L_{\varphi}: \varphi \in^{<\omega} 2\right\rangle$ and $\left\langle L_{\varphi}^{\prime}: \varphi \in^{<\omega} 2\right\rangle$ in $\mathbb{P}(L)$ and $\left\langle q_{\varphi}: \varphi \in^{<\omega} 2\right\rangle$ in $L$ such that (i) $L_{\emptyset}=L$, (ii) $L_{\varphi\urcorner 0}<L_{\varphi}^{\prime}<L_{\varphi\urcorner 1}$, (iii) $L_{\varphi\urcorner 0} \cup L_{\varphi}^{\prime} \cup L_{\varphi \neg 1} \subset L_{\varphi}$, (iv) $q_{\varphi} \in L_{\varphi}^{\prime}$. Then $\left\{q_{\varphi}: \varphi \in^{<\omega} 2\right\}$ is a copy of $\mathbb{Q}$ in $L$.

A linear order $L$ is said to be additively indecomposable (respectively left indecomposable; right indecomposable) iff for each decomposition $L=L_{0}+L_{1}$ we have $L \hookrightarrow L_{0}$ or $L \hookrightarrow L_{1}$ (respectively $L \hookrightarrow L_{0} ; L \hookrightarrow L_{1}$ ). The class $\mathcal{H}$ of hereditarily additively indecomposable (or ha-indecomposable) linear orders is the smallest class of order types of countable linear orders containing the one element order type, $\mathbf{1}$, and containing the $\omega$-sum, $\sum_{\omega} L_{i}$, and the $\omega^{*}$-sum, $\sum_{\omega^{*}} L_{i}$, for each sequence $\left\langle L_{i}: i \in \omega\right\rangle$ in $\mathcal{H}$ satisfying

$$
\begin{equation*}
\forall i \in \omega\left|\left\{j \in \omega: L_{i} \hookrightarrow L_{j}\right\}\right|=\aleph_{0} \tag{1}
\end{equation*}
$$

Fact 2.2 (a) $\mathcal{H} \subset \mathcal{S}$ (see [8], p. 196);
(b) If $L \in \mathcal{H}$ is an $\omega$-sum, then $L$ is right indecomposable (see [8], p. 196);
(c) If $L \in \mathcal{H}$ is an $\omega^{*}$-sum, then $L$ is left indecomposable (see [8], p. 196);
(d) If $L \in \mathcal{S}$ is additively indecomposable, then $L$ is left indecomposable or right indecomposable (see [8], p. 175);
(e) (Laver, [7]) If $L \in \mathcal{S}$, then $L \in \mathcal{H}$ iff $L$ is additively indecomposable (see [8], p. 201);
(f) (Laver, [7]) If $L \in \mathcal{S}$, then $L$ is a finite sum of elements of $\mathcal{H}$ (see [8], p. 201).

Let $\mathbb{P}=\langle P, \leq\rangle$ be a pre-order. Then $p \in P$ is an atom iff each $q, r \leq p$ are compatible (there is $s \leq q, r$ ). $\mathbb{P}$ is called: atomless iff it has no atoms; homogeneous iff it has the largest element and $\mathbb{P} \cong p \downarrow$, for each $p \in \mathbb{P}$. If $\kappa$ is a regular cardinal, $\mathbb{P}$ is called $\kappa$-closed iff for each $\gamma<\kappa$ each sequence $\left\langle p_{\alpha}: \alpha\langle\gamma\rangle\right.$ in $P$, such that $\alpha<\beta \Rightarrow p_{\beta} \leq p_{\alpha}$, has a lower bound in $P$. $\omega_{1}$-closed pre-orders are called $\sigma$-closed. Two pre-orders $\mathbb{P}$ and $\mathbb{Q}$ are called forcing equivalent iff they produce the same generic extensions.

Fact 2.3 If $\mathbb{P}_{i}, i \in I$, are $\kappa$-closed pre-orders, then $\prod_{i \in I} \mathbb{P}_{i}$ is $\kappa$-closed.
A partial order $\mathbb{P}=\langle P, \leq\rangle$ is called separative iff for each $p, q \in P$ satisfying $p \not \leq q$ there is $r \leq p$ such that $r \perp q$. The separative modification of $\mathbb{P}$ is the separative pre-order $\operatorname{sm}(\mathbb{P})=\left\langle P, \leq^{*}\right\rangle$, where

$$
\begin{equation*}
p \leq^{*} q \Leftrightarrow \forall r \leq p \exists s \leq r s \leq q . \tag{2}
\end{equation*}
$$

The separative quotient of $\mathbb{P}$ is the separative partial order $\operatorname{sq}(\mathbb{P})=\left\langle P /=^{*}, \unlhd\right\rangle$, where $p=^{*} q \Leftrightarrow p \leq^{*} q \wedge q \leq^{*} p$ and $[p] \unlhd[q] \Leftrightarrow p \leq^{*} q$.

Fact 2.4 Let $\mathbb{P}, \mathbb{Q}$ and $\mathbb{P}_{i}, i \in I$, be partial orderings. Then
(a) $\mathbb{P}, \operatorname{sm}(\mathbb{P})$ and $\mathrm{sq}(\mathbb{P})$ are forcing equivalent forcing notions;
(b) $\operatorname{sm}(\mathbb{P})$ is $\kappa$-closed iff $\operatorname{sq}(\mathbb{P})$ is $\kappa$-closed;
(c) If $p_{0}, p_{1}, \ldots p_{n} \in \mathbb{P}$, where $p_{n} \leq^{*} p_{n-1} \leq^{*} \ldots \leq^{*} p_{0}$, then there is $q \in \mathbb{P}$ such that $q \leq p_{k}$, for all $k \leq n$.
(d) $\mathbb{P} \cong \mathbb{Q}$ implies that $\mathrm{sm} \mathbb{P} \cong \mathrm{sm} \mathbb{Q}$ and $\mathrm{sq} \mathbb{P} \cong \mathrm{sq} \mathbb{Q}$;
(e) $\operatorname{sm}\left(\prod_{i \in I} \mathbb{P}_{i}\right)=\prod_{i \in I} \operatorname{sm} \mathbb{P}_{i}$ and $\mathrm{sq}\left(\prod_{i \in I} \mathbb{P}_{i}\right) \cong \prod_{i \in I} \mathrm{sq} \mathbb{P}_{i}$.
(f) If $X$ is an infinite set, $\mathcal{I} \subset P(X)$ an ideal containing $[X]^{<\omega}$ and $\mathcal{I}^{+}=$ $P(X) \backslash \mathcal{I}$ the corresponding family of $\mathcal{I}$-positive sets, then $\mathrm{sm}\left\langle\mathcal{I}^{+}, \subset\right\rangle=\left\langle\mathcal{I}^{+}, \subset_{\mathcal{I}}\right\rangle$, where $A \subset_{\mathcal{I}} B \Leftrightarrow A \backslash B \in \mathcal{I}$, for $A, B \in \mathcal{I}^{+}$. Also sq $\left\langle\mathcal{I}^{+}, \subset\right\rangle=(P(X) / \mathcal{I})^{+}$.

Proof. All the statements are folklore except, maybe, (c). For a proof of (c), by recursion we define the sequence $\left\langle q_{k}: k \leq m\right\rangle$ such that (i) $q_{0}=p_{n}$ and (ii) $q_{k} \leq q_{k-1}, p_{n-k}$, for $0<k \leq n$. Then $q_{n} \leq p_{k}$, for all $k \leq n$.

Fact 2.5 (Folklore) Under the $\mathbf{C H}$, each atomless separative $\omega_{1}$-closed pre-order of size $\omega_{1}$ is forcing equivalent to $(P(\omega) / \mathrm{Fin})^{+}$.

We recall that the ideal Fin $\times$ Fin $\subset P(\omega \times \omega)$ is defined by:

$$
\text { Fin } \times \text { Fin }=\left\{A \subset \omega \times \omega:\left|\left\{i \in \omega:\left|A \cap L_{i}\right|=\omega\right\}\right|<\omega\right\},
$$

where $L_{i}=\{i\} \times \omega$, for $i \in \omega$. By $\mathfrak{h}(\mathbb{P})$ we denote the distributivity number of a poset $\mathbb{P}$. In particular, for $n \in \mathbb{N}$, let $\mathfrak{h}_{n}=\mathfrak{h}\left(\left((P(\omega) / \text { Fin })^{+}\right)^{n}\right)$; thus $\mathfrak{h}=\mathfrak{h}_{1}$.

Fact 2.6 (a) $\operatorname{sm}\left(\left\langle[\omega]^{\omega}, \subset\right\rangle^{n}\right)=\left\langle[\omega]^{\omega}, \subset^{*}\right\rangle^{n}$ and $\mathrm{sq}\left(\left\langle[\omega]^{\omega}, \subset\right\rangle^{n}\right)=\left((P(\omega) / \text { Fin })^{+}\right)^{n}$ are forcing equivalent, $\mathfrak{t}$-closed atomless pre-orders of size $\boldsymbol{c}$.
(b) (Shelah and Spinas [9]) $\operatorname{Con}\left(\mathfrak{h}_{n+1}<\mathfrak{h}_{n}\right)$, for each $n \in \mathbb{N}$.
(c) (Szymański and Zhou [10]) $(P(\omega \times \omega) /(\text { Fin } \times \text { Fin }))^{+}$is an $\omega_{1}$-closed, but not $\omega_{2}$-closed atomless poset.
(d) (Hernández-Hernández [2]) $\operatorname{Con}\left(\mathfrak{h}\left((P(\omega \times \omega) /(\text { Fin } \times \text { Fin }))^{+}\right)<\mathfrak{h}\right)$.

Now we prove the first part of Theorem 1.2
Proposition 2.7 For each countable scattered linear order $L$ the partial ordering $\langle\mathbb{P}(L), \subset\rangle$ is homogeneous, atomless and of size $\mathfrak{c}$.

Proof. The homogeneity of $\langle\mathbb{P}(L), \subset\rangle$ is evident. For a proof that it is atomless first we show

$$
\begin{equation*}
\forall L \in \mathcal{H} \quad(|L|=\omega \Rightarrow \exists X, Y \in \mathbb{P}(L) \quad X \cap Y=\emptyset) . \tag{3}
\end{equation*}
$$

If $L$ is an $\omega$-sum, that is $L=\sum_{\omega} L_{i}$, where $\left\langle L_{i}: i \in \omega\right\rangle$ is a sequence in $\mathcal{H}$ satisfying (1), by recursion we define the sequences $\left\langle k_{i}: i \in \omega\right\rangle$ and $\left\langle l_{i}: i \in \omega\right\rangle$ in $\omega$ such that for each $i$
(i) $k_{i}<l_{i}$,
(ii) $l_{i}<k_{i+1}$,
(iii) $L_{i} \hookrightarrow L_{k_{i}}, L_{l_{i}}$.

Using (11) we choose $k_{0}, l_{0} \in \omega$ such that $k_{0}<l_{0}$ and $L_{0} \hookrightarrow L_{k_{0}}, L_{l_{0}}$.
Let the sequences $k_{0}, \ldots, k_{i}$ and $l_{0}, \ldots, l_{i}$ satisfy (i)-(iii). Then $k_{0}<l_{0}<$ $\ldots<k_{i}<l_{i}$. Using (1) we choose $k_{i+1}, l_{i+1} \in \omega$ such that $l_{i}<k_{i+1}<l_{i+1}$ and $L_{i+1} \hookrightarrow L_{k_{i+1}}, L_{l_{i+1}}$. Thus, the recursion works.

By (iii) there are $X_{i}, Y_{i} \cong L_{i}$ such that $X_{i} \subset L_{k_{i}}$ and $Y_{i} \subset L_{l_{i}}$. Then $X=$ $\sum_{\omega} X_{i}, Y=\sum_{\omega} Y_{i} \cong L$ and, by (i) and (ii) we have $X \cap Y=\emptyset$.

If $L$ is an $\omega^{*}$-sum, we proceed in the same way. Thus (3) is proved.
By Fact 2.2 for $L \in \mathcal{S}$ there is $m \in \mathbb{N}$ such that $L=\sum_{i<m} L_{i}$, where $L_{i} \in \mathcal{H}$. Let $J=\left\{i<m:\left|L_{i}\right|=\omega\right\}$. By (3), for $i \in J$ there are $X_{i}, Y_{i} \in \mathbb{P}\left(L_{i}\right)$ such that $X_{i} \cap Y_{i}=\emptyset$. Let $X=\bigcup_{i \in J} X_{i} \cup \bigcup_{i \in m \backslash J} L_{i}$ and $Y=\bigcup_{i \in J} Y_{i} \cup$ $\bigcup_{i \in m \backslash J} L_{i}$. Then $X, Y \in \mathbb{P}(L)$ and $|X \cap Y|=\left|\bigcup_{i \in m \backslash J} L_{i}\right|<\omega$ and, hence, $X$ and $Y$ are incompatible elements of the poset $\langle\mathbb{P}(L), \subset\rangle$. So, since $\langle\mathbb{P}(L), \subset\rangle$ is a homogeneous partial order, it is atomless.

It is known (see [1], p. 170) that the equivalence classes corresponding to the relation $\sim$ on $L$, defined by $x \sim y$ iff $|[\min \{x, y\}, \max \{x, y\}]|<\omega$, are convex parts of $L$ which are finite or isomorphic to $\omega$, or $\omega^{*}$ or $\mathbb{Z}$. Since $|L|=\omega$ and two consecutive parts can not be finite, there is one infinite part, say $L^{\prime}$, and, clearly, it
has $\mathfrak{c}$-many copies. For each $C \in \mathbb{P}\left(L^{\prime}\right)$ we have $\left(L \backslash L^{\prime}\right) \cup C \in \mathbb{P}(L)$ and, hence, $|\mathbb{P}(L)|=\mathbf{c}$.

In the rest of the paper we prove that $\mathrm{sq}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed poset, for each countable scattered linear order $L$. By Fact 2.4(b), it is sufficient to show that the pre-order $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is $\sigma$-closed. In the sequel we use the following notation:

$$
\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle=\langle\mathbb{P}(L), \leq\rangle
$$

## 3 Elements of $\mathcal{H}$

Proposition 3.1 Let $L=\sum_{\omega} L_{i} \in \mathcal{H}$, where $\left\langle L_{i}: i \in \omega\right\rangle$ is a sequence in $\mathcal{H}$ satisfying (1). Then
(a) $A \subset L$ contains a copy of $L$ iff for each $i, m \in \omega$ there is finite $K \subset \omega \backslash m$ such that $L_{i} \hookrightarrow \bigcup_{j \in K} L_{j} \cap A$. So, each $A \in \mathbb{P}(L)$ intersects infinitely many $L_{i}$ 's.
(b) If $A, B \in \mathbb{P}(L)$, then $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ and each $i, m \in \omega$ there exists a finite $K \subset \omega \backslash m$ such that $L_{i} \hookrightarrow \bigcup_{j \in K} L_{j} \cap C \cap B$.
(c) $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed pre-order.

The same statement holds for the $\omega^{*}$-sum $\sum_{\omega^{*}} L_{i}$.
Proof. (a) $(\Rightarrow)$ Let $f: L \hookrightarrow L$ and $C=f[L] \subset A$. Then $C=\sum_{i \in \omega} f\left[L_{i}\right]$.
Claim 1. For each $i \in \omega$ there is a finite set $K \subset \omega$ such that $f\left[L_{i}\right] \subset \bigcup_{j \in K} L_{j}$.
Proof of Claim 1. Since $f$ is an embedding and $L_{i}<L_{i+1}$ we have $f\left[L_{i}\right]<$ $f\left[L_{i+1}\right]$. For $x \in L_{i+1}$ we have $f(x) \in f\left[L_{i+1}\right] \subset \bigcup_{j \in \omega} L_{j}$ and, hence, $f(x) \in$ $L_{j_{0}}$, for some $j_{0} \in \omega$. Now, by the monotonicity of $f$ we have $f\left[L_{i}\right]<\{f(x)\} \subset$ $L_{j_{0}}$, thus $f\left[L_{i}\right] \subset \bigcup_{j \leq j_{0}} L_{j}$, so we can take $K=j_{0}+1$ and Claim 1 is proved.

For $i \in \omega$ let $K_{i}=\left\{j \in \omega: f\left[L_{i}\right] \cap L_{j} \neq \emptyset\right\}$. By Claim 1 we have

$$
\begin{equation*}
K_{i} \in[\omega]^{<\omega} \text { and } f\left[L_{i}\right] \subset \bigcup_{j \in K_{i}} L_{j} . \tag{4}
\end{equation*}
$$

Claim 2. $K_{i} \leq K_{i+1}$, for each $i \in \omega$. Consequently, either $K_{i} \cap K_{i+1}=\emptyset$ or $K_{i} \cap K_{i+1}=\left\{\max K_{i}\right\}=\left\{\min K_{i+1}\right\}$.
Proof of Claim 2. Let $j^{\prime} \in K_{i}$ and $j^{\prime \prime} \in K_{i+1}$. Then there are $x \in L_{i}$ and $y \in L_{i+1}$ such that $f(x) \in L_{j^{\prime}}$ and $f(y) \in L_{j^{\prime \prime}}$ and, clearly, $x<y$. Now $j^{\prime \prime}<j^{\prime}$ would imply $f(y)<f(x)$, which is impossible. Thus $j^{\prime} \leq j^{\prime \prime}$. Claim 2 is proved.
Claim 3. $\bigcup_{i \in \omega} K_{i}$ is an infinite subset of $\omega$.
Proof of Claim 3. On the contrary, suppose that $j_{0}=\max \bigcup_{i \in \omega} K_{i}$. Let $i_{0}=$ $\min \left\{i \in \omega: j_{0} \in K_{i}\right\}$. Then $j_{0} \in K_{i_{0}} \leq\left\{j_{0}\right\}$ and, by Claim 2,

$$
\forall i>i_{0}\left(K_{i}=\left\{j_{0}\right\} \wedge f\left[L_{i}\right] \subset L_{j_{0}}\right) .
$$

By (11), there are $i_{1}, i_{2} \in \omega$ such that $i_{0}+1<i_{1}<i_{2}$ and $L_{j_{0}} \hookrightarrow L_{i_{1}}, L_{i_{2}}$, which implies $L_{j_{0}}+L_{j_{0}} \hookrightarrow L_{i_{1}}+L_{i_{2}} \hookrightarrow f\left[L_{i_{1}}\right]+f\left[L_{i_{2}}\right] \subset L_{j_{0}}$. But $L_{j_{0}}$ is a scattered linear order and, by Fact 2.1 $L_{j_{0}}+L_{j_{0}} \nrightarrow L_{j_{0}}$. A contradiction. Claim 3 is proved.

Let $i_{0}, m_{0} \in \omega$. By (1), the set $I_{i_{0}}=\left\{j \in \omega: L_{i_{0}} \hookrightarrow L_{j}\right\}$ is an infinite set.
Claim 4. There is $j_{0} \in I_{i_{0}}$ such that $K_{j_{0}} \cap m_{0}=\emptyset$.
Proof of Claim 4. On the contrary, suppose that $K_{j} \cap m_{0} \neq \emptyset$, for each $j \in I_{i_{0}}$. Then

$$
\begin{equation*}
\forall j \in I_{i_{0}} \quad \min K_{j}<m_{0} \tag{5}
\end{equation*}
$$

For $i \in \omega$ there is $j \in I_{i_{0}}$ such that $j>i+1$ and, by Claim $2, K_{i} \leq K_{i+1} \leq K_{j}$ and, by (5), $\max K_{i} \leq \min K_{i+1} \leq \min K_{j}<m_{0}$. Thus $K_{i} \subset m_{0}$, for all $i \in \omega$, which is impossible by Claim 3. Claim 4 is proved.

By Claim 4, $K_{j_{0}} \in\left[\omega \backslash m_{0}\right]^{<\omega}$. By (4) we have $f\left[L_{j_{0}}\right] \subset \bigcup_{j \in K_{j_{0}}} L_{j}$. Since $j_{0} \in I_{i_{0}}$ and $f\left[L_{j_{0}}\right] \subset C \subset A$ we have $L_{i_{0}} \hookrightarrow L_{j_{0}} \hookrightarrow f\left[L_{j_{0}}\right] \subset \bigcup_{j \in K_{j_{0}}} L_{j} \cap A$ and the proof of " $\Rightarrow$ " is finished.
$(\Leftarrow)$ Suppose that a set $A \subset L$ satisfies the given condition. By recursion we define the sequences $\left\langle K_{i}: i \in \omega\right\rangle$ and $\left\langle f_{i}: i \in \omega\right\rangle$ such that for each $i \in \omega$
(i) $K_{i} \in[\omega]^{<\omega}$,
(ii) $K_{0}<K_{1}<\ldots$,
(iii) $f_{i}: L_{i} \hookrightarrow \bigcup_{j \in K_{i}} L_{j} \cap A$.

By the assumption, for $i=m=0$ there are $K_{0} \in[\omega]^{<\omega}$ and $f_{0}: L_{0} \hookrightarrow$ $\bigcup_{j \in K_{0}} L_{j} \cap A$.

Let $K_{0}, \ldots, K_{i}$ and $f_{0}, \ldots, f_{i}$ satisfy (i)-(iii) and let $m=\max \left(\bigcup_{r \leq i} K_{r}\right)+1$. By the assumption for $i+1$ and $m$ there are $K_{i+1} \in[\omega \backslash m]^{<\omega}$ and $f_{i+1}: L_{i+1} \hookrightarrow$ $\bigcup_{j \in K_{i+1}} L_{j} \cap A$ and the recursion works.

Let $f=\bigcup_{i \in \omega} f_{i}$. By (ii) and (iii), $i_{1}<i_{2}$ implies $K_{i_{1}}<K_{i_{2}}$, which implies $f_{i_{1}}\left[L_{i_{1}}\right]<f_{i_{2}}\left[L_{i_{2}}\right]$ and, hence, $f: L \hookrightarrow A$. Thus $C=f[L] \in \mathbb{P}(L)$ and $C \subset A$.
(b) By (2), $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ the set $C \cap B$ contains a copy of $L$. Now we apply (a) to $C \cap B$.
(c) For $A_{n} \in \mathbb{P}(L), n \in \omega$, where $A_{0} \geq A_{1} \geq \ldots$ we will construct $A \in \mathbb{P}(L)$ such that $A \leq A_{n}$, for all $n \in \omega$. First, by Fact 2.4(c), there are $C_{i} \in \mathbb{P}(L), i \in \omega$, such that $C_{0}=A_{0}$ and

$$
\begin{equation*}
\forall i \in \omega C_{i} \subset A_{0} \cap \ldots \cap A_{i} . \tag{6}
\end{equation*}
$$

By recursion we define the sequences $\left\langle K_{i}: i \in \omega\right\rangle$ and $\left\langle f_{i}: i \in \omega\right\rangle$ such that for each $i \in \omega$
(i) $K_{i} \in[\omega]^{<\omega}$,
(ii) $K_{i}<K_{i+1}$,
(iii) $f_{i}: L_{i} \hookrightarrow \bigcup_{j \in K_{i}} L_{j} \cap C_{i}$.

Since $C_{0}=A_{0} \in \mathbb{P}(L)$, by (a), for $i=m=0$ there are $K_{0} \in[\omega]^{<\omega}$ and $f_{0}: L_{0} \hookrightarrow \bigcup_{j \in K_{0}} L_{j} \cap C_{0}$.

Let the sequences $K_{0}, \ldots, K_{i^{\prime}}$ and $f_{0}, \ldots, f_{i^{\prime}}$ satisfy (i)-(iii). Since $A_{i^{\prime}+1} \leq$ $A_{i^{\prime}}, C_{i^{\prime}+1} \in \mathbb{P}(L)$ and, by (6), $C_{i^{\prime}+1} \subset A_{i^{\prime}+1}$, according to (b), for $i^{\prime}+1$ and $m=\max \left(K_{0} \cup \ldots \cup K_{i^{\prime}}\right)+1$ there are

$$
\begin{gather*}
K_{i^{\prime}+1} \in\left[\omega \backslash\left(\max \left(K_{0} \cup \ldots \cup K_{i^{\prime}}\right)+1\right)\right]^{<\omega}  \tag{7}\\
f_{i^{\prime}+1}: L_{i^{\prime}+1} \hookrightarrow \bigcup_{j \in K_{i^{\prime}+1}} L_{j} \cap C_{i^{\prime}+1} \tag{8}
\end{gather*}
$$

(since, by (6)), $C_{i^{\prime}+1} \cap A_{i^{\prime}}=C_{i^{\prime}+1}$ ). By (7)) we have (i) and (ii) and (iii) follows from (8)). The recursion works.

Let $f=\bigcup_{i \in \omega} f_{i}$. By (ii) and (iii), $i_{1}<i_{2}$ implies $K_{i_{1}}<K_{i_{2}}$, which implies $f_{i_{1}}\left[L_{i_{1}}\right]<f_{i_{2}}\left[L_{i_{2}}\right]$ and, hence, $f: L \hookrightarrow L$. Thus

$$
\begin{equation*}
A=f[L]=\bigcup_{i \in \omega} f_{i}\left[L_{i}\right] \in \mathbb{P}(L) \tag{9}
\end{equation*}
$$

Using the characterization from (b), for $n^{*} \in \omega$ we show that $A \leq A_{n^{*}}$. So, for $C^{*} \in \mathbb{P}(L)$ such that $C^{*} \subset A$ and $i^{*}, m^{*} \in \omega$ we prove that

$$
\begin{equation*}
\exists K \in\left[\omega \backslash m^{*}\right]^{<\omega} L_{i^{*}} \hookrightarrow \bigcup_{j \in K} L_{j} \cap C^{*} \cap A_{n^{*}} \tag{10}
\end{equation*}
$$

By (ii), (iii) and (9) we have $A=\sum_{i \in \omega} \Lambda_{i} \cong L$, where $\Lambda_{i}=f_{i}\left[L_{i}\right] \cong L_{i}$, thus $A \in \mathcal{H}$. Since $C^{*} \cong L \cong A$ we have $C^{*} \in \mathbb{P}(A)$ so, applying (a) to the linear order $A$ instead of $L$ we obtain

$$
\begin{equation*}
\forall i, m \in \omega \quad \exists K \in[\omega \backslash m]^{<\omega} \quad f_{i}\left[L_{i}\right] \hookrightarrow \bigcup_{j \in K} f_{j}\left[L_{j}\right] \cap C^{*} . \tag{11}
\end{equation*}
$$

Let $m^{\prime}>m^{*}, n^{*}$. By (11), for $i^{*}$ and $m^{\prime}$ there is

$$
\begin{gather*}
K^{*} \in\left[\omega \backslash m^{\prime}\right]<\omega \text { such that }  \tag{12}\\
f_{i^{*}}\left[L_{i^{*}}\right] \hookrightarrow \bigcup_{j \in K^{*}} f_{j}\left[L_{j}\right] \cap C^{*} . \tag{13}
\end{gather*}
$$

By (12), for $j \in K^{*}$ we have $j>n^{*}$ and, by (6), $C_{j} \subset A_{n^{*}}$. Thus, by (iii) we have $f_{j}\left[L_{j}\right] \subset \bigcup_{s \in K_{j}} L_{s} \cap C_{j} \subset \bigcup_{s \in K_{j}} L_{s} \cap A_{n^{*}}$ which, together with (iii) and (13) gives $L_{i^{*}} \hookrightarrow f_{i^{*}}\left[L_{i^{*}}\right] \hookrightarrow \bigcup_{j \in K^{*}} f_{j}\left[L_{j}\right] \cap C^{*} \subset \bigcup_{j \in K^{*}} \bigcup_{s \in K_{j}} L_{s} \cap A_{n^{*}} \cap C^{*}=$ $\bigcup_{s \in \bigcup_{j \in K^{*}} K_{j}} L_{s} \cap C^{*} \cap A_{n^{*}}$.

In order to finish the proof of (10) we prove that $\bigcup_{j \in K^{*}} K_{j} \cap m^{*}=\emptyset$. By (12), for $j \in K^{*}$ we have $j>m^{*}$. By (ii) the sequence $\left\langle\min K_{i}: i \in \omega\right\rangle$ is increasing and, hence, $\min K_{j} \geq j>m^{*}$, which implies $K_{j} \cap m^{*}=\emptyset$ and (10) is proved.

## 4 Finite sums of $\omega$-sums. Finite sums of $\omega^{*}$-sums

Lemma 4.1 Let $L_{0}=\sum_{\omega} L_{i}^{0}, L_{1}=\sum_{\omega} L_{i}^{1} \in \mathcal{H}$, where $\left\langle L_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle L_{i}^{1}: i \in \omega\right\rangle$ are sequences in $\mathcal{H}$ satisfying (1). Then
(a) $\exists i \in \omega L_{0} \hookrightarrow L_{i}^{1} \Leftrightarrow \exists m \in \omega L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$;
(b) $L_{0}+L_{1} \notin \mathcal{H} \Rightarrow \neg \exists m \in \omega \quad L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$.
(c) If $L=L_{0}+L_{1} \notin \mathcal{H}$ and $f: L \hookrightarrow L$, then $f\left[L_{k}\right] \subset L_{k}$, for $k=0,1$.

Proof. (a) Suppose that $L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$ and let $i_{0}=\max \left\{i \leq m: f\left[L_{0}\right] \cap L_{i}^{1} \neq\right.$ $\emptyset\}$. Then $f\left[L_{0}\right] \cap L_{i_{0}}^{1}$ is a final part of the ordering $f\left[L_{0}\right] \cong L_{0}$ and, by Fact[2.2(a), contains a copy of $L_{0}$. Thus $L_{0} \hookrightarrow L_{i_{0}}^{1}$.
(b) If $L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$ then, by (a), there are $i_{0} \in \omega$ and $f: L_{0} \hookrightarrow L_{i_{0}}^{1}$. Then $\left\langle L_{0}, L_{0}^{1}, L_{1}^{1}, \ldots, L_{i_{0}}^{1}, \ldots\right\rangle$ is a sequence in $\mathcal{H}$ satisfying (1) and $L_{0}+L_{1}=$ $L_{0}+L_{0}^{1}+L_{1}^{1}+\ldots+L_{i_{0}}^{1}+\ldots \in \mathcal{H}$.
(c) Suppose that $f\left[L_{0}\right] \cap L_{1} \neq \emptyset$. Then $f\left[L_{0}\right] \cap L_{1}$ is a final part of the ordering $f\left[L_{0}\right] \cong L_{0}$ and, by Fact 2.2(a), contains a copy of $L_{0}$. Thus, by (b), $f\left[L_{0}\right] \cap L_{i}^{1} \neq \emptyset$, for infinitely many $i \in \omega$. But this is impossible because $f\left[L_{0}\right]<$ $f\left[L_{1}\right]$. Thus $f\left[L_{0}\right] \subset L_{0}$ and, hence, $f\left[L_{0}\right] \in \mathbb{P}\left(L_{0}\right)$. By Proposition 3.1 (a) we have $f\left[L_{0}\right] \cap L_{i}^{0} \neq \emptyset$, for infinitely many $i \in \omega$, which implies $f\left[L_{1}\right] \subset L_{1}$.

Proposition 4.2 (Finite sums of $\omega$-sums) Let $L=\sum_{i \leq n} L_{i}$, where $L_{i} \in \mathcal{H}$ are $\omega$-sums of sequences in $\mathcal{H}$ satisfying (1) and $L_{i}+L_{i+1} \notin \mathcal{H}$, for $i<n$. Then
(a) If $f: L \hookrightarrow L$, then $f\left[L_{i}\right] \subset L_{i}$, for each $i \leq n$;
(b) $\mathbb{P}(L)=\left\{\bigcup_{i \leq n} C_{i}: \forall i \leq n \quad C_{i} \in \mathbb{P}\left(L_{i}\right)\right\}$;
(c) $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed pre-order.

Proof. (a) For $n=1$ this is (c) of Lemma 4.1 Assuming that the statement is true for $n-1$ we prove that it is true for $n$. Suppose that $f\left[L_{0}\right] \not \subset L_{0}$. Then, since $f\left[L_{n}\right] \subset \bigcup_{i \leq n} L_{i}$, for $i^{*}=\max \left\{i \leq n: f\left[L_{i}\right] \not \subset \bigcup_{j \leq i} L_{j}\right\}$ we have $0 \leq i^{*}<n, f\left[L_{i^{*}}^{-}\right] \not \subset \bigcup_{j \leq i^{*}} L_{j}$ and $f\left[L_{i^{*}+1}\right] \subset \bigcup_{j \leq i^{*}} L_{j} \cup L_{i^{*}+1}$. Since $f\left[L_{i^{*}}\right]<f\left[L_{i^{*}+1}\right]$ we have $\bar{f}\left[L_{i^{*}+1}\right] \subset L_{i^{*}+1}$ so $f\left[L_{i^{*}}\right]^{-} \cap L_{i^{*}+1}$ is a final part of $f\left[L_{i^{*}}\right] \cong L_{i^{*}}$ and, by Fact[2.2(a), contains a copy of $L_{i^{*}}$. This copy is contained in the union of finitely many summands of $L_{i^{*}+1}$. But, since $L_{i^{*}}+L_{i^{*}+1} \notin$ $\mathcal{H}$, this is impossible by Lemma 4.11b). Thus $f\left[L_{0}\right] \subset L_{0}$ and, by Proposition 3.1 a), the set $f\left[L_{0}\right]$ intersects infinitely many summands of $L_{0}$, which implies $f\left[L_{1} \cup \ldots \cup L_{n}\right] \subset L_{1} \cup \ldots \cup L_{n}$. Thus, by the induction hypothesis, $f\left[L_{i}\right] \subset L_{i}$, for each $i \in\{1, \ldots, n\}$.
(b) The inclusion " $\supset$ " is evident and we prove " $\subset$ ". If $C \in \mathbb{P}(L)$ and $f: L \hookrightarrow$ $L$, where $C=f[L]$, then by (a), $C_{i}=f\left[L_{i}\right] \subset L_{i}$ and, hence, $C_{i} \in \mathbb{P}\left(L_{i}\right)$ and, clearly, $C=\bigcup_{i \leq n} C_{i}$.
(c) By the statement (b) and, since the sets $L_{i}, i \leq n$, are disjoint, the mapping $F: \prod_{i \leq n}\left\langle\mathbb{P}\left(L_{i}\right), \subset\right\rangle \rightarrow\langle\mathbb{P}(L), \subset\rangle$ given by $F\left(\left\langle C_{0}, \ldots, C_{n}\right\rangle\right)=C_{0} \cup \ldots \cup$ $C_{n}$ is an isomorphism and, by Fact $2.4 \operatorname{sm}\langle\mathbb{P}(L), \subset\rangle \cong \operatorname{sm}\left(\prod_{i \leq n}\left\langle\mathbb{P}\left(L_{i}\right), \subset\right\rangle\right) \cong$ $\prod_{i \leq n} \operatorname{sm}\left\langle\mathbb{P}\left(L_{i}\right), \subset\right\rangle$. By Proposition 3.1 (c), the pre-orders $\operatorname{sm}\left\langle\mathbb{P}\left(L_{i}\right), \subset\right\rangle, i \leq n$, are $\sigma$-closed, and, by Fact 2.3 the same holds for their direct product and, hence, for $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ as well.

The following dual statements can be proved in the same way.

Lemma 4.3 Let $L_{0}=\sum_{\omega^{*}} L_{i}^{0}, L_{1}=\sum_{\omega^{*}} L_{i}^{1} \in \mathcal{H}$, where $\left\langle L_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle L_{i}^{1}: i \in \omega\right\rangle$ are sequences in $\mathcal{H}$ satisfying (11). Then
(a) $\exists i \in \omega L_{1} \hookrightarrow L_{i}^{0} \Leftrightarrow \exists m \in \omega L_{1} \hookrightarrow L_{m}^{0}+\ldots+L_{0}^{0}$;
(b) $L_{0}+L_{1} \notin \mathcal{H} \Rightarrow \neg \exists m \in \omega L_{1} \hookrightarrow L_{m}^{0}+\ldots+L_{0}^{0}$.
(c) If $L=L_{0}+L_{1} \notin \mathcal{H}$ and $f: L \hookrightarrow L$, then $f\left[L_{k}\right] \subset L_{k}$, for $k=0,1$.

Proposition 4.4 (Finite sums of $\omega^{*}$-sums) Let $L=\sum_{i<n} L_{i}$, where $L_{i} \in \mathcal{H}$ are $\omega^{*}$-sums and $L_{i}+L_{i+1} \notin \mathcal{H}$, for $i<n-1$. Then
(a) If $f: L \hookrightarrow L$, then $f\left[L_{i}\right] \subset L_{i}$, for each $i<n$;
(b) $\mathbb{P}(L)=\left\{\bigcup_{i<n} C_{i}: \forall i<n \quad C_{i} \in \mathbb{P}\left(L_{i}\right)\right\}$;
(c) $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed pre-order.

## $5 \quad \omega^{*}$-sum plus $\omega$-sum

Lemma 5.1 Let $L=L_{0}+L_{1}$, where $L_{0}=\sum_{\omega^{*}} L_{i}^{0}, L_{1}=\sum_{\omega} L_{i}^{1} \in \mathcal{H}$ and $\left\langle L_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle L_{i}^{1}: i \in \omega\right\rangle$ are sequences in $\mathcal{H}$ satisfying (1). Then
(a) $\exists i \in \omega L_{0} \hookrightarrow L_{i}^{1} \Leftrightarrow \exists m \in \omega L_{0} \hookrightarrow L_{0}^{1}+\ldots+L_{m}^{1}$;
(b) $\exists i \in \omega L_{1} \hookrightarrow L_{i}^{0} \Leftrightarrow \exists m \in \omega L_{1} \hookrightarrow L_{m}^{0}+\ldots+L_{0}^{0}$;
(c) If $L_{0}+L_{1} \notin \mathcal{H}$, then

$$
\begin{equation*}
\forall m \in \omega\left(L_{0} \nrightarrow L_{0}^{1}+\ldots+L_{m}^{1} \wedge L_{1} \nrightarrow L_{m}^{0}+\ldots+L_{0}^{0}\right) . \tag{14}
\end{equation*}
$$

Proof. (a) If $f: L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$ and $i_{0}=\min \left\{i \leq m: f\left[L_{0}\right] \cap L_{i}^{1} \neq \emptyset\right\}$, then $f\left[L_{0}\right] \cap L_{i_{0}}^{1}$ is a initial part of the ordering $f\left[L_{0}\right] \cong L_{0}$ and, by Fact[2.2 (c), contains a copy of $L_{0}$. Thus $L_{0} \hookrightarrow L_{i_{0}}^{1}$. The proof of (b) is dual.
(c) If $L_{0} \hookrightarrow \sum_{i \leq m} L_{i}^{1}$ then, by (a), there are $i_{0} \in \omega$ and $f: L_{0} \hookrightarrow L_{i_{0}}^{1}$. Then $\left\langle L_{0}, L_{0}^{1}, L_{1}^{1}, \ldots, L_{i_{0}}^{1}, \ldots\right\rangle$ is a sequence in $\mathcal{H}$ satisfying (1) and, hence, $L_{0}+L_{1}=$ $L_{0}+L_{0}^{1}+L_{1}^{1}+\ldots+L_{i_{0}}^{1}+\ldots \in \mathcal{H}$. If $L_{1} \hookrightarrow L_{m}^{0}+\ldots+L_{0}^{0}$, we prove $L_{0}+L_{1} \in \mathcal{H}$ in a similar way.

Proposition 5.2 Let $L=L_{0}+L_{1} \notin \mathcal{H}$, where $L_{0}=\sum_{\omega^{*}} L_{i}^{0}, L_{1}=\sum_{\omega} L_{i}^{1} \in \mathcal{H}$ and $\left\langle L_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle L_{i}^{1}: i \in \omega\right\rangle$ are sequences in $\mathcal{H}$ satisfying (1). Then
(a) $A \subset L$ contains a copy of $L$ iff for each $i, m \in \omega$ there is a finite $K \subset \omega \backslash m$ such that $L_{i}^{0} \hookrightarrow \bigcup_{j \in K} L_{j}^{0} \cap A$ and $L_{i}^{1} \hookrightarrow \bigcup_{j \in K} L_{j}^{1} \cap A$.
(b) If $A, B \in \mathbb{P}(L)$, then $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ and each $i, m \in \omega$ there is a finite $K \subset \omega \backslash m$ such that $L_{i}^{0} \hookrightarrow \bigcup_{j \in K} L_{j}^{0} \cap C \cap B$ and $L_{i}^{1} \hookrightarrow \bigcup_{j \in K} L_{j}^{1} \cap C \cap B$.
(c) $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed pre-order.

Proof. (a) $(\Rightarrow)$ Let $C \in \mathbb{P}(L), C \subset A, f: L \hookrightarrow L$ and $C=f[L]$. First we prove

$$
\begin{equation*}
\exists C_{0} \in \mathbb{P}\left(L_{0}\right) \quad \exists C_{1} \in \mathbb{P}\left(L_{1}\right) \quad C_{0} \cup C_{1} \subset A . \tag{15}
\end{equation*}
$$

Suppose that $f\left[L_{0}\right] \subset L_{1}$. Then, by Lemma[5.1 (c), $f\left[L_{0}\right] \cap L_{i}^{1} \neq \emptyset$, for infinitely many $i \in \omega$. But this is impossible since $f\left[L_{0}\right]<f\left[L_{1}\right]$. Thus $f\left[L_{0}\right] \cap L_{0} \neq \emptyset$, this set is an initial part of the order $f\left[L_{0}\right] \cong L_{0}$ and, by Fact 2.2(c), there is $C_{0} \in \mathbb{P}\left(L_{0}\right)$ such that $C_{0} \subset f\left[L_{0}\right] \cap L_{0} \subset C \subset A$. Similarly, there is $C_{1} \in \mathbb{P}\left(L_{1}\right)$ such that $C_{1} \subset f\left[L_{1}\right] \cap L_{1} \subset C \subset A$ and (15) is proved.

Let $i, m \in \omega$. By (15) we have $C_{0} \subset A \cap L_{0} \subset L_{0}$ and $C_{1} \subset A \cap L_{1} \subset L_{1}$, so, by Proposition 3.1 a), there are finite sets $K_{0}, K_{1} \subset \omega \backslash m$ such that $L_{i}^{0} \hookrightarrow$ $\bigcup_{j \in K_{0}} L_{j}^{0} \cap A \cap L_{0}$ and $L_{i}^{1} \hookrightarrow \bigcup_{j \in K_{1}} L_{j}^{1} \cap A \cap L_{1}$. Clearly, $K=K_{0} \cup K_{1}$ is a finite subset of $\omega \backslash m$ and $L_{i}^{0} \hookrightarrow \bigcup_{j \in K} L_{j}^{0} \cap A$ and $L_{i}^{1} \hookrightarrow \bigcup_{j \in K} L_{j}^{1} \cap A$.
$(\Leftarrow)$ Suppose that the given condition is satisfied by $A$. Then, by Proposition 3.1 a), there are $C_{0} \in \mathbb{P}\left(L_{0}\right)$ and $C_{1} \in \mathbb{P}\left(L_{1}\right)$ such that $C_{0} \subset A \cap L_{0}$ and $C_{1} \subset$ $A \cap L_{1}$. Now $\mathbb{P}(L) \ni C_{0} \cup C_{1} \subset A$.
(b) By (2), $A \leq B$ iff for each $C \in \mathbb{P}(L)$ satisfying $C \subset A$ the set $C \cap B$ contains a copy of $L$. Now we apply (a) to $C \cap B$.
(c) For $A_{n} \in \mathbb{P}(L), n \in \omega$, where $A_{0} \geq A_{1} \geq \ldots$ we will construct $A \in \mathbb{P}(L)$ such that $A \leq A_{n}$, for all $n \in \omega$. First, by Fact 2.4 (c), there are $C_{i} \in \mathbb{P}(L), i \in \omega$, such that $C_{0}=A_{0}$ and

$$
\begin{equation*}
\forall i \in \omega C_{i} \subset A_{0} \cap \ldots \cap A_{i} \tag{16}
\end{equation*}
$$

By recursion we define the sequences $\left\langle K_{i}: i \in \omega\right\rangle,\left\langle f_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle f_{i}^{1}: i \in \omega\right\rangle$ such that for each $i \in \omega$
(i) $K_{i} \in[\omega]^{<\omega}$,
(ii) $K_{i}<K_{i+1}$,
(iii) $f_{i}^{0}: L_{i}^{0} \hookrightarrow \bigcup_{j \in K_{i}} L_{j}^{0} \cap C_{i}$,
(iv) $f_{i}^{1}: L_{i}^{1} \hookrightarrow \bigcup_{j \in K_{i}}^{j} L_{j}^{1} \cap C_{i}$.

Since $C_{0}=A_{0} \in \mathbb{P}(L)$, by (a) (for $i=m=0$ ), there exist $K_{0} \in[\omega]^{<\omega}$, $f_{0}^{0}: L_{0}^{0} \hookrightarrow \bigcup_{j \in K_{0}} L_{j}^{0} \cap C_{0}$ and $f_{0}^{1}: L_{0}^{1} \hookrightarrow \bigcup_{j \in K_{0}} L_{j}^{1} \cap C_{0}$.

Let the sequences $K_{0}, \ldots, K_{i^{\prime}}, f_{0}^{0}, \ldots, f_{i^{\prime}}^{0}$ and $f_{0}^{1}, \ldots, f_{i^{\prime}}^{1}$ satisfy (i)-(iv). Since $A_{i^{\prime}+1} \leq A_{i^{\prime}}, C_{i^{\prime}+1} \in \mathbb{P}(L)$ and, by (16), $C_{i^{\prime}+1} \subset A_{i^{\prime}+1}$, according to (b), for $i^{\prime}+1$ and $m=\max \left(K_{0} \cup \ldots \cup K_{i^{\prime}}\right)+1$ there are

$$
\begin{gather*}
K_{i^{\prime}+1} \in\left[\omega \backslash\left(\max \left(K_{0} \cup \ldots \cup K_{i^{\prime}}\right)+1\right)\right]^{<\omega}  \tag{17}\\
f_{i^{\prime}+1}^{0}: L_{i^{\prime}+1}^{0} \hookrightarrow \bigcup_{j \in K_{i^{\prime}+1}} L_{j}^{0} \cap C_{i^{\prime}+1}  \tag{18}\\
f_{i^{\prime}+1}^{1}: L_{i^{\prime}+1}^{1} \hookrightarrow \bigcup_{j \in K_{i^{\prime}+1}} L_{j}^{1} \cap C_{i^{\prime}+1} \tag{19}
\end{gather*}
$$

(since, by (16)), $C_{i^{\prime}+1} \cap A_{i^{\prime}}=C_{i^{\prime}+1}$ ). By (17)) we have (i) and (ii). (iii) and (iv) follow from (18) and (19). The recursion works.

Let $f=\bigcup_{i \in \omega} f_{i}^{0} \cup \bigcup_{i \in \omega} f_{i}^{1}$. By (ii) and (iii), $i_{1}<i_{2}$ implies $K_{i_{1}}<K_{i_{2}}$, which implies $f_{i_{1}}^{0}\left[L_{i_{1}}^{0}\right]>f_{i_{2}}^{0}\left[L_{i_{2}}^{0}\right]$ and $f_{i_{1}}^{1}\left[L_{i_{1}}^{1}\right]<f_{i_{2}}^{1}\left[L_{i_{2}}^{1}\right]$ and, hence, $f: L \hookrightarrow L$. Thus

$$
\begin{equation*}
A=f[L]=\bigcup_{i \in \omega} f_{i}^{0}\left[L_{i}^{0}\right] \cup \bigcup_{i \in \omega} f_{i}^{1}\left[L_{i}^{1}\right] \in \mathbb{P}(L) \tag{20}
\end{equation*}
$$

Using the characterization from (b), for $n^{*} \in \omega$ we show that $A \leq A_{n^{*}}$. So, for $C^{*} \in \mathbb{P}(L)$ such that $C^{*} \subset A$ and $i^{*}, m^{*} \in \omega$ we prove that
$\exists K \in\left[\omega \backslash m^{*}\right]^{<\omega}\left(L_{i^{*}}^{0} \hookrightarrow \bigcup_{j \in K} L_{j}^{0} \cap C^{*} \cap A_{n^{*}} \wedge L_{i^{*}}^{1} \hookrightarrow \bigcup_{j \in K} L_{j}^{1} \cap C^{*} \cap A_{n^{*}}\right)$.
By (ii)-(iv) and (20) we have $A=\sum_{\omega^{*}} \Lambda_{i}^{0}+\sum_{\omega} \Lambda_{i}^{1} \cong L$, where $\Lambda_{i}^{0}=f_{i}^{0}\left[L_{i}^{0}\right] \cong$ $L_{i}^{0}$ and $\Lambda_{i}^{1}=f_{i}^{1}\left[L_{i}^{1}\right] \cong L_{i}^{1}$, so $A$ is a sum of an $\omega^{*}$-sum, $\Lambda_{0}=\sum_{\omega^{*}} \Lambda_{i}^{0} \cong L_{0}$ and an $\omega$-sum, $\Lambda_{1}=\sum_{\omega} \Lambda_{i}^{1} \cong L_{1}$. In addition, $L_{0}+L_{1} \notin \mathcal{H}$ implies $\Lambda_{0}+\Lambda_{1} \notin \mathcal{H}$.

Since $C^{*} \cong L \cong A$ and $C^{*} \subset A$ we have $C^{*} \in \mathbb{P}(A)$ so, applying (a) to the linear order $A$ instead of $L$ we obtain

$$
\begin{equation*}
\forall i, m \in \omega \exists K \in[\omega \backslash m]^{<\omega}\left(\Lambda_{i}^{0} \hookrightarrow \bigcup_{j \in K} \Lambda_{j}^{0} \cap C^{*} \wedge \Lambda_{i}^{1} \hookrightarrow \bigcup_{j \in K} \Lambda_{j}^{1} \cap C^{*}\right) . \tag{22}
\end{equation*}
$$

Let $m^{\prime}>m^{*}, n^{*}$. By (22), for $i^{*}$ and $m^{\prime}$ there is

$$
\begin{gather*}
K^{*} \in\left[\omega \backslash m^{\prime}\right]^{<\omega} \text { such that }  \tag{23}\\
\Lambda_{i^{*}}^{0} \hookrightarrow \bigcup_{j \in K^{*}} \Lambda_{j}^{0} \cap C^{*} \wedge \Lambda_{i^{*}}^{1} \hookrightarrow \bigcup_{j \in K^{*}} \Lambda_{j}^{1} \cap C^{*} \tag{24}
\end{gather*}
$$

By (23), for $j \in K^{*}$ we have $j>n^{*}$ and, by (16), $C_{j} \subset A_{n^{*}}$. Thus, by (iii) and (iv) we have $\Lambda_{j}^{0} \subset \bigcup_{s \in K_{j}} L_{s}^{0} \cap C_{j} \subset \bigcup_{s \in K_{j}} L_{s}^{0} \cap A_{n^{*}}$ and $\Lambda_{j}^{1} \subset \bigcup_{s \in K_{j}} L_{s}^{1} \cap C_{j} \subset$ $\bigcup_{s \in K_{j}} L_{s}^{1} \cap A_{n^{*}}$ which, together with (iii),(iv) and (24) gives $L_{i^{*}}^{0} \hookrightarrow \Lambda_{i^{*}}^{0} \hookrightarrow$ $\bigcup_{j \in K^{*}} f_{j}\left[\Lambda_{j}^{0}\right] \cap C^{*} \subset \bigcup_{j \in K^{*}} \bigcup_{s \in K_{j}} L_{s}^{0} \cap A_{n^{*}} \cap C^{*}=\bigcup_{s \in \bigcup_{j \in K^{*}} K_{j}} L_{s}^{0} \cap C^{*} \cap A_{n^{*}}$. Similarly we prove that $L_{i^{*}}^{0} \hookrightarrow \bigcup_{s \in \bigcup_{j \in K^{*}}} L_{j}^{0} \cap C^{*} \cap A_{n^{*}}$.

In order to finish the proof of (21) we show that $\bigcup_{j \in K^{*}} K_{j} \cap m^{*}=\emptyset$. By (23), for $j \in K^{*}$ we have $j>m^{*}$. By (ii) the sequence $\left\langle\min K_{i}: i \in \omega\right\rangle$ is increasing and, hence, $\min K_{j} \geq j>m^{*}$, which implies $K_{j} \cap m^{*}=\emptyset$ and (21) is proved.

## 6 The general case

For $L \in \mathcal{S}$, let $m(L)=\min \{n \in \omega: L$ is a sum of $n$ elements of $\mathcal{H}\}$. For $m \in \mathbb{N}$, let $\mathcal{S}_{m}=\{L \in \mathcal{S}: m(L)=m\}$.

Lemma 6.1 (a) There is no $L \in \mathcal{H}$ such that $L=\sum_{\omega^{*}} L_{i}^{0}$ and $L=\sum_{\omega} L_{i}^{1}$, where $\left\langle L_{i}^{0}: i \in \omega\right\rangle$ and $\left\langle L_{i}^{1}: i \in \omega\right\rangle$ are sequences in $\mathcal{H}$ satisfying (1).
(b) Let $L \in \mathcal{S}_{m}$ and $L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$. Then

$$
\begin{gather*}
\forall i<m\left(\left|L_{i}\right|=1 \underline{\vee} L_{i} \text { is an } \omega \text {-sum } \underline{\vee} L_{i} \text { is an } \omega^{*} \text {-sum }\right)  \tag{25}\\
\forall i<m-1 L_{i}+L_{i+1} \notin \mathcal{H} .  \tag{26}\\
\left|L_{i}\right|=1 \Rightarrow\left(L_{i+1} \text { is not an } \omega \text {-sum } \wedge L_{i-1} \text { is not an } \omega^{*} \text {-sum }\right) . \tag{27}
\end{gather*}
$$

Proof. (a) On the contrary, by Fact [2.2, $L$ would be both left and right indecomposable and, for a partition $L=L^{\prime}+L^{\prime \prime}$ there would be $C^{\prime}, C^{\prime \prime} \cong L$ such that $C^{\prime} \subset L^{\prime}$ and $C^{\prime \prime} \subset L^{\prime \prime}$, which would imply $L+L \hookrightarrow L$. But this is impossible by Fact 2.1
(b) The first statement follows from (a), the second from the minimality of $m$ and the third from the second statement ( $1+\omega$-sum is an $\omega$-sum satisfying (11).

Lemma 6.2 If $m \in \mathbb{N}, L \in \mathcal{S}_{m}, L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$, and $f: L \hookrightarrow L$, then for each $i<m$ there is $C_{i} \in \mathbb{P}\left(L_{i}\right)$ such that $C_{i} \subset f\left[L_{i}\right]$.

Proof. We use induction. For $m=1$ the statement is trivially true.
Suppose that the statement holds for all $k \leq m$. Let $L \in \mathcal{S}_{m+1}, L_{0}, \ldots, L_{m} \in$ $\mathcal{H}, L=L_{0}+L_{1}+\ldots+L_{m}$ and $f: L \hookrightarrow L$. Let $L^{\prime}=L_{1}+\ldots+L_{m}$.

Claim 1. $f\left[L_{1}\right] \cap L_{0}$ does not contain a copy of $L_{1}$.
Proof of Claim 1. On the contrary suppose that $L_{1} \cong C_{1} \subset f\left[L_{1}\right] \cap L_{0}$.
First we show that $L_{0}$ is an $\omega^{*}$-sum. Namely, $\left|L_{0}\right|=1$ would imply $C_{1}=$ $L_{0}=f\left[L_{1}\right]$, which is impossible because $f\left[L_{0}\right]<f\left[L_{1}\right]$. Suppose that $L_{0}$ is an $\omega$-sum, $L_{0}=\sum_{\omega} \Lambda_{i}$. Then, since $f\left[L_{0}\right]<f\left[L_{1}\right] \cap L_{0}, L_{0} \hookrightarrow \sum_{i \leq m} \Lambda_{i}$, for some $m \in \omega$, which is impossible by Proposition 3.1(a).

Thus $L_{0}$ is an $\omega^{*}$-sum, $L_{0}=\sum_{\omega^{*}} L_{i}^{0}$ and, by (25) and (27), $L_{1}$ is either an $\omega$-sum or an $\omega^{*}$-sum. Since $f\left[L_{0}\right]<f\left[L_{1}\right] \cap L_{0} \hookleftarrow L_{1}$, there is $m \in \omega$ such that $L_{1} \hookrightarrow L_{m}^{0}+\ldots+L_{0}^{0}$. By (26) we have $L_{0}+L_{1} \notin \mathcal{H}$ and this is impossible by Lemma 5.1 (c) in the first case and Lemma 4.3(b) in the second. A contradiction. Claim 1 is proved.

By (25), regarding the summand $L_{1}$ we have the following three cases.

Case 1: $\left|L_{1}\right|=1$. Then, by Claim 1, $f\left[L_{1}\right] \cap L_{0}=\emptyset$, which implies that $f \upharpoonright$ $L^{\prime}: L^{\prime} \hookrightarrow L^{\prime}$. Clearly $m\left(L^{\prime}\right) \leq m$ and $m\left(L^{\prime}\right)<m$ is impossible, because of the minimality of $m(L)$. Thus $m\left(L^{\prime}\right)=m$ and, by the induction hypothesis,

$$
\begin{equation*}
\forall i \in\{1, \ldots m\} \exists C_{i} \in \mathbb{P}\left(L_{i}\right) C_{i} \subset\left(f \upharpoonright L^{\prime}\right)\left[L_{i}\right]=f\left[L_{i}\right] . \tag{28}
\end{equation*}
$$

Since $\left|L_{1}\right|=1$ we have $C_{1}=L_{1}=f\left[L_{1}\right]>f\left[L_{0}\right]$, for $C_{0}=f\left[L_{0}\right]$ we have $C_{0} \in \mathbb{P}\left(L_{0}\right)$ and the proof is over.
Case 2: $L_{1}$ is an $\omega^{*}$-sum. By Fact $2.2(\mathrm{c}), f\left[L_{1}\right] \cap L_{0} \neq \emptyset$ would imply that $f\left[L_{1}\right] \cap L_{0}$ contains a copy of $L_{1}$, which is impossible by Claim 1. Thus $f\left[L_{1}\right] \cap$ $L_{0}=\emptyset$ and, as in Case 1, we have (28). In particular, $\mathbb{P}\left(L_{1}\right) \ni C_{1} \subset f\left[L_{1}\right]$ and, by Proposition 3.1 a) (for $\omega^{*}$-sums), $f\left[L_{1}\right]$ intersects infinitely many summands of $L_{1}$, which implies $f\left[L_{0}\right] \subset L_{0}$. Again, for $C_{0}=f\left[L_{0}\right]$ we have $C_{0} \in \mathbb{P}\left(L_{0}\right)$ and the proof is over.

Case 3: $L_{1}$ is an $\omega$-sum. By (25) and (27), regarding the summand $L_{0}$ we have the following two subcases.

Subcase 3.1: $L_{0}$ is an $\omega$-sum. $f\left[L_{1}\right] \cap L_{0} \neq \emptyset$ would imply that $L_{0}$ is embeddable in an initial part of $L_{0}$, which is impossible by Proposition 3.1(a). Thus $f\left[L_{1}\right] \cap L_{0}=$ $\emptyset$ and, as in Case 1, we have (28). Since $C_{1} \subset f\left[L_{1}\right] \cap L_{1}$ we have $f\left[L_{0}\right] \subset L_{0} \cup L_{1}$. Suppose that $f\left[L_{0}\right] \cap L_{1} \neq \emptyset$. Then $f\left[L_{0}\right] \cap L_{1}$ is contained in finitely many summands of $L_{1}$ and, by Fact 2.2(a), contains a copy of $L_{0}$, which is impossible by (26) and Lemma 4.1 b). Thus $f\left[L_{0}\right] \subset L_{0}$ and, for $C_{0}=f\left[L_{0}\right]$ we have $C_{0} \in \mathbb{P}\left(L_{0}\right)$ which, together with (28), finishes the proof.

Subcase 3.2: $L_{0}$ is an $\omega^{*}$-sum. Let $L_{0}=\sum_{\omega^{*}} A_{i}$ and $L_{1}=\sum_{\omega} B_{i}$. By Claim 1 , there is $x \in L_{1}$ such that $L_{0}<\{f(x)\}$. By Fact 2.2 b), there is $L_{1}^{\prime} \cong L_{1}$ such that $L_{1}^{\prime} \subset[x, \infty)_{L_{1}}$. Let $\varphi: L_{1}+L_{2}+\ldots+L_{m} \rightarrow L_{1}^{\prime}+L_{2}+\ldots+L_{m}$ be an isomorphism, where $\varphi \upharpoonright L_{i}=i d_{L_{i}}$, for $i \in\{2,3, \ldots, m\}$. Then $f \circ \varphi: L^{\prime} \hookrightarrow L^{\prime}$ and, by the induction hypothesis, there are $C_{i} \in \mathbb{P}\left(L_{i}\right), i \in\{1, \ldots m\}$, satisfying $C_{i} \subset f\left[\varphi\left[L_{i}\right]\right]$. Since $C_{1} \subset f\left[\varphi\left[L_{1}\right]\right]=f\left[L_{1}^{\prime}\right]$ we have

$$
\begin{align*}
& C_{1} \subset f\left[L_{1}^{\prime}\right] \cap L_{1} \subset f\left[L_{1}\right] \cap L_{1} .  \tag{29}\\
& \forall i \in\{2, \ldots m\}\left(C_{i} \in \mathbb{P}\left(L_{i}\right) \wedge C_{i} \subset f\left[\varphi\left[L_{i}\right]\right]=f\left[L_{i}\right]\right) . \tag{30}
\end{align*}
$$

By (29) we have $f\left[L_{0}\right] \subset L_{0} \cup L_{1}$. Suppose that $f\left[L_{0}\right] \subset L_{1}$. Then, by (29), $f\left[L_{0}\right]$ is contained in the union of finitely many summands of $L_{1}$, which is impossible by (26) and Lemma5.1(c). Thus $f\left[L_{0}\right] \cap L_{0} \neq \emptyset$ is an initial part of $f\left[L_{0}\right] \cong L_{0}$ and, by Fact 2.2(c), there is $C_{0} \cong L_{0}$ such that $C_{0} \subset f\left[L_{0}\right] \cap L_{0}$. By (29) and (30) the proof is over.

Let $L \in \mathcal{S}_{m}$ and $L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$. Then we have (25), (26) and (27) and we divide $L$ into blocks, groups of consecutive
summands $L_{i}$, in the following way:

- first we glue each two consecutive summands such that the first is an $\omega^{*}$-sum and the second an $\omega$-sum (blocks of the type D ),
- then we divide the rest into the groups of consecutive (in $L$ ) $L_{i}$ 's of the same form: groups of singletons (blocks of the type A), groups of $\omega$-sums (blocks of the type B) and groups of $\omega^{*}$-sums (blocks of the type C).

For example $111\left|\omega^{*} \omega^{*}\right| \omega^{*} \omega|\omega| 11\left|\omega^{*} \omega\right| \omega \omega \omega \omega \mid \omega^{*} \omega^{*}$. More formally, we define a block of $L$ as a sum of consecutive summands $B=L_{i}+L_{i+1}+\ldots+L_{i+k}$, where $k \geq 0$ and satisfying one of the following conditions.
(A) $\left|L_{j}\right|=1$, for all $j \in\{i, \ldots, i+k\}$ and
(i) $i=0 \vee\left|L_{i-1}\right|=\omega$ and
(ii) $i+k=m-1 \vee\left|L_{i+k+1}\right|=\omega$;
(B) $L_{j}$ is an $\omega$-sum, for all $j \in\{i, \ldots, i+k\}$ and
(iii) $i=0 \vee\left(L_{i-1}\right.$ is an $\omega$-sum $\wedge L_{i-2}$ is an $\omega^{*}$-sum $)$ and
(iv) $i+k=m-1 \vee L_{i+k+1}$ is not an $\omega$-sum;
(C) $L_{j}$ is an $\omega^{*}$-sum, for all $j \in\{i, \ldots, i+k\}$ and
(v) $i=0 \vee L_{i-1}$ is not an $\omega^{*}$-sum and
(vi) $i+k=m-1 \vee\left(L_{i+k+1}\right.$ is an $\omega^{*}$-sum $\wedge L_{i+k+2}$ is an $\omega$-sum $)$;
(D) $k=1$ and $L_{i}$ is an $\omega^{*}$-sum and $L_{i+1}$ is an $\omega$-sum.

By $\operatorname{Block}(L)$ we will denote the set of blocks.
Lemma 6.3 Blocks determine a partition of the set $\left\{L_{0}, \ldots, L_{m-1}\right\}$ and a partition of $L$ into convex parts.

Proof. Let $L \in \mathcal{S}_{m}$ and $L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$. First we show that each summand $L_{j}$ is contained in some block. We have the following three cases

Case 1: $\left|L_{j}\right|=1$. Let $L_{i}, L_{i+1}, \ldots, L_{j}, \ldots L_{i+k}$ be the maximal sequence of consecutive summands of size 1 , including $L_{j}$. Then conditions (i) and (ii) are satisfied and, hence, $L_{j}$ belongs to a block of the type (A).
Case 2: $L_{j}$ is an $\omega$-sum.
Subcase 2.1: $j=0$. Let $L_{0}, \ldots, L_{k}$ be a maximal sequence of consecutive $\omega$ sums. Then $k=m-1$ or $L_{k+1}$ is not an $\omega$-sum so, conditions (iii) and (iv) are satisfied and $L_{j}$ belongs to a block of the type (B).

Subcase 2.2: $j>0$ and $L_{j-1}$ is an $\omega^{*}$-sum. Then $L_{j-1}+L_{j}$ is a block of the type (D) containing $L_{j}$.

Subcase 2.3: $j>0$ and $L_{j-1}$ is not an $\omega^{*}$-sum. Then, by (27), $\left|L_{j-1}\right| \neq 1$, so, by (25), $L_{j-1}$ is an $\omega$-sum. Let $L_{i}, L_{i+1}, \ldots, L_{j-1}, L_{j}, \ldots, L_{i+k}$ be the maximal sequence of consecutive $\omega$-sums containing $L_{j}$. Then (iv) is true.

If $i=0$, then (iii) is true and $L_{j}$ belongs to a block of the type (B).
If $i>0$, then, by the maximality of the sequence and (27) and (25), $L_{i-1}$ is an $\omega^{*}$-sum. Now $L_{i+1}, \ldots, L_{j-1}, L_{j}, \ldots, L_{i+k}$ satisfies (iii) and (iv), so it is a block of the type (B) containing $L_{j}$ (since, clearly, $i+1 \leq j$ ).
Case 3: $L_{j}$ is an $\omega^{*}$-sum.
Subcase 3.1: $j=m-1$. Let $L_{i}, \ldots, L_{j}$ be a maximal sequence of consecutive $\omega^{*}$-sums. Then $i=0$ or $L_{i-1}$ is not an $\omega^{*}$-sum so, conditions (v) and (vi) are satisfied and $L_{j}$ belongs to a block of the type (C).
Subcase 3.2: $j<m-1$ and $L_{j+1}$ is an $\omega$-sum. Then $L_{j}+L_{j+1}$ is a block of the type (D) containing $L_{j}$.
Subcase 3.3: $j<m-1$ and $L_{j+1}$ is not an $\omega$-sum. Since, by (27), $\left|L_{j+1}\right| \neq 1$ by (25) we have that $L_{j+1}$ is an $\omega^{*}$-sum. Let $L_{i}, L_{i+1}, \ldots, L_{j}, L_{j+1}, \ldots, L_{i+k}$ be the maximal sequence of consecutive $\omega^{*}$-sums containing $L_{j}$. Then (v) is true.

If $i+k=m-1$, then (vi) is true and $L_{j}$ belongs to a block of the type (C).
If $i+k<m-1$, then, by the maximality of the sequence and (27) and (25), $L_{i+k+1}$ is an $\omega$-sum. Now $L_{i}, \ldots, L_{j-1}, L_{j}, \ldots, L_{i+k-1}$ satisfies (v) and (vi), so it is a block of the type (C) containing $L_{j}$ (since, clearly, $j \leq i+k-1$ ).

Now we prove that different blocks are disjoint. Suppose that $B^{\prime}, B^{\prime \prime} \in \operatorname{Block}(L)$ and $x \in B^{\prime} \cap B^{\prime \prime}$. Then $x \in L_{j}$ for some $L_{j}$ contained in $B^{\prime} \cap B^{\prime \prime}$. By (25) we have the following three cases:
Case 1: $\left|L_{j}\right|=1$. Then $B^{\prime}$ and $B^{\prime \prime}$ are blocks of the type (A). Since $L_{j} \subset B^{\prime} \cap B^{\prime \prime}$, by (i) and (ii) we have $B^{\prime}=B^{\prime \prime}$.
Case 2: $L_{j}$ is an $\omega$-sum. Then, by Lemma6.1 (a), the blocks are of the type (B) or (D).

Subcase 2.1: $B^{\prime}$ and $B^{\prime \prime}$ are of the type (D). Then, since $L_{j} \subset B^{\prime} \cap B^{\prime \prime}$ is an $\omega$-sum, by Lemma6.1(a) we have $B^{\prime}=B^{\prime \prime}$.
Subcase 2.2: $B^{\prime}$ and $B^{\prime \prime}$ are of the type (B). Then, since $L_{j} \subset B^{\prime} \cap B^{\prime \prime}$, from (iii) and (iv) it follows that in $L$ the blocks have the same beginning and the same end. Thus, $B^{\prime}=B^{\prime \prime}$.
Subcase 2.2: $B^{\prime}$ is of the type (B) and $B^{\prime \prime}$ of the type (D). Then, by Lemma6.1 (a), $L_{j}$ is the second summand of $B^{\prime \prime}$ and, hence, $B^{\prime \prime}=L_{j-1}+L_{j}$ and $B^{\prime}=L_{j}+$ $\ldots+L_{k}$. But this is impossible by (iii)

Case 3: $L_{j}$ is an $\omega^{*}$-sum. This case is dual to Case 2.

Lemma 6.4 If $m \in \mathbb{N}, L \in \mathcal{S}_{m}, L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$ and $\operatorname{Block}(L)=\left\{B_{0}, \ldots B_{r}\right\}$, then $\operatorname{Block}\left(L \backslash B_{0}\right)=\operatorname{Block}(L) \backslash\left\{B_{0}\right\}$.

Proof. Let $L=L_{0}+\ldots+L_{n-1}+L_{n}+\ldots+L_{m-1}$, where $B_{0}=L_{0}+\ldots+L_{n-1}$, $L^{\prime}=L \backslash B_{0}=L_{n}+\ldots+L_{m-1}$ and $0<n<m$. First we show that

$$
\begin{equation*}
\operatorname{Block}\left(L^{\prime}\right) \subset \operatorname{Block}(L) \tag{31}
\end{equation*}
$$

Let $B=L_{i}+\ldots+L_{i+k} \in \operatorname{Block}\left(L^{\prime}\right)$. Clearly, if $B$ is of the type (D) in $L^{\prime}$, then the same holds in $L$ and $B \in \operatorname{Block}(L)$. If $B$ is of the type (A) (resp. (B), (C)), then it satisfies (ii) (resp. (iv), (vi)) in $L^{\prime}$ and, clearly, in $L$. If $i>n$, then, in addition, $B$ satisfies (i) (resp. (iii), (v)) in $L^{\prime}$ and, again, in $L$; thus $B \in \operatorname{Block}(L)$. So it remains to be proved that $B$ satisfies (i) (resp. (iii), (v)) in $L$, when $i=n$.
Case 1: $B$ is of the type (A). Then $\left|L_{n-1}\right|=1$ would imply that $B_{0}$ is not a block in $L$. Thus $\left|L_{n-1}\right|=\omega$ and $B$ satisfies (i) in $L$.
Case 2: $B$ is of the type (B). Then $L_{n}$ is an $\omega$-sum and, by (27), $\left|L_{n-1}\right|=\omega$. By (iv) and (vi), $B_{0}$ is not of the type (B) or (C). Thus, $B_{0}$ is of the type (D) and, hence, $B$ satisfies (iii) in $L$.

Case 3: $B$ is of the type (C). Then $L_{n}$ is an $\omega^{*}$-sum. Suppose that $L_{n-1}$ is an $\omega^{*}$-sum. Then $B_{0}$ must be of the type (C) and, by (vi) for $B_{0}$ in $L, L_{n+1}$ is an $\omega$-sum. But then $B$ should be a block of the type (D) in $L^{\prime}$, which is not true. Thus $L_{n-1}$ is not an $\omega^{*}$-sum and, hence, $B$ satisfies (v) in $L$.

So (31) is proved, which implies $\operatorname{Block}\left(L^{\prime}\right) \subset \operatorname{Block}(L) \backslash\left\{B_{0}\right\}=\left\{B_{1}, \ldots B_{r}\right\}$. By Lemma 6.3 we have $\bigcup \operatorname{Block}\left(L^{\prime}\right)=L^{\prime}=B_{1} \cup \ldots \cup B_{r}$, which gives the another inclusion.

Lemma 6.5 If $m \in \mathbb{N}, L \in \mathcal{S}_{m}, L_{0}, \ldots, L_{m-1} \in \mathcal{H}$, where $L=L_{0}+\ldots+L_{m-1}$, and $f: L \hookrightarrow L$, then for each $B \in \operatorname{Block}(L)$ we have $f[B] \subset B$.

Proof. We prove the statement by induction. For $m=1$ it is trivially true.
Suppose that it is true for all $k<m$. Let $L=L_{0}+\ldots+L_{m-1}$ and $\operatorname{Block}(L)=$ $\left\{B_{0}, \ldots B_{r}\right\}$. If $r=0$, we are done. Otherwise we have

$$
\begin{equation*}
L=B_{0}+L_{i+1}+\ldots+L_{m-1}, \tag{32}
\end{equation*}
$$

where $B_{0}=L_{0}+\ldots+L_{i}$. Let $L^{\prime}=L_{i+1}+\ldots+L_{m-1}$. By Lemma6.2,

$$
\begin{equation*}
\forall j \in\{0, \ldots, m-1\} \quad \exists C_{j} \in \mathbb{P}\left(L_{j}\right) C_{j} \subset f\left[L_{j}\right] \cap L_{j} . \tag{33}
\end{equation*}
$$

Regarding the type of $B_{0}$ we have the following cases.
Case 1: $B_{0}$ is of the type (A). Then, by (25), (27) and (ii), $L_{i+1}$ is an $\omega^{*}$-sum. By (33) and Proposition 3.1 a) (for $\omega^{*}$-sums), $C_{i+1}$ intersects infinitely many summands of $L_{i+1}$ and, since $B_{0}$ is finite and $f\left[B_{0}\right]<f\left[L_{i+1}\right]$, we have $f\left[B_{0}\right]=B_{0}$. Hence $f \upharpoonright L^{\prime}: L^{\prime} \hookrightarrow L^{\prime}$ and $m\left(L^{\prime}\right)=m-i-1$. By Lemma 6.4 we have

$$
\begin{equation*}
\operatorname{Block}\left(L^{\prime}\right)=\operatorname{Block}(L) \backslash\left\{B_{0}\right\}=\left\{B_{1}, \ldots, B_{r}\right\} \tag{34}
\end{equation*}
$$

and, by the induction hypothesis, $f\left[B_{j}\right]=\left(f \upharpoonright L^{\prime}\right)\left[B_{j}\right] \subset B_{j}$, for $j>0$.
Case 2: $B_{0}$ is of the type (B). By Proposition 3.1 (a) $C_{i}$ intersects infinitely many summands of $L_{i}$, which implies that $f \upharpoonright L^{\prime}: L^{\prime} \hookrightarrow L^{\prime}$.

If $\left|L_{i+1}\right|=1$, then $f\left[L_{i+1}\right]=L_{i+1}$ and, hence, $f\left[B_{0}\right] \subset B_{0}$. By (34) and the induction hypothesis $f\left[B_{j}\right] \subset B_{j}$, for $j>0$.

If $L_{i+1}$ is an $\omega^{*}$-sum, then $C_{i+1}$ intersects infinitely many summands of $L_{i+1}$ and, hence, $f\left[B_{0}\right] \subset B_{0}$. Also, $C_{i}$ intersects infinitely many summands of $L_{i}$, which implies that $f\left[L^{\prime}\right] \subset L^{\prime}$. By (34) and the induction hypothesis $f\left[B_{j}\right] \subset B_{j}$, for $j>0$ again.
Case 3: $B_{0}$ is of the type (C). Then by (vi), $L_{i+1}$ is an $\omega^{*}$-sum. By (33) we have $C_{i+1} \subset f\left[L_{i+1}\right] \cap L_{i+1}$ and, by Proposition 3.1] $f\left[L_{i+1}\right]$ intersects infinitely many summands of $L_{i+1}$, which implies $f\left[B_{0}\right] \subset B_{0}$. Suppose that $f\left[L_{i+1}\right] \cap L_{i} \neq \emptyset$. By (33), $C_{i} \subset f\left[L_{i}\right] \cap L_{i}$, which implies that $f\left[L_{i+1}\right] \cap L_{i}$ is an initial part of $f\left[L_{i+1}\right]$ contained in an final part of $L_{i}$. By Fact[2.2 (c) $f\left[L_{i+1}\right] \cap L_{i}$ contains a copy of $L_{i+1}$, which is impossible by Lemma 4.3 (b) and (26). Thus $f\left[L_{i+1}\right] \cap L_{i}=\emptyset$, which implies $f\left[L^{\prime}\right] \subset L^{\prime}$ and again, by (34) and the induction hypothesis $f\left[B_{j}\right] \subset B_{j}$, for $j>0$.
Case 4: $B_{0}$ is of the type (D). Then $B_{0}=L_{0}+L_{1}$ and, by (33) and Proposition 3.1. $f\left[L_{1}\right]$ intersects infinitely many summands of $L_{1}$, which implies

$$
\begin{equation*}
f\left[L^{\prime}\right] \subset L^{\prime} . \tag{35}
\end{equation*}
$$

By (33) there is $C_{2}$ such that

$$
\begin{equation*}
C_{2} \in \mathbb{P}\left(L_{2}\right) \wedge C_{2} \subset f\left[L_{2}\right] \cap L_{2} . \tag{36}
\end{equation*}
$$

Regarding the form of $L_{2}$ we distinguish the following three subcases.
$\left|L_{2}\right|=1$. Then, by (33), $f\left[L_{1}\right]=L_{1}$ and, hence, $f\left[B_{0}\right] \subset B_{0}$ and we use (35), (34) and the induction hypothesis.
$L_{2}$ is an $\omega^{*}$-sum. By (36) $f\left[L_{2}\right]$ intersects infinitely many summands of $L_{2}$ and, hence, $f\left[B_{0}\right] \subset B_{0}$ and we use (35), (34) and the induction hypothesis.
$L_{2}$ is an $\omega$-sum. By (36) we have $f\left[L_{1}\right] \subset L_{0} \cup L_{1} \cup L_{2} . f\left[L_{1}\right] \cap L_{2} \neq \emptyset$ is impossible by Lemma4.1(b), thus $f\left[B_{0}\right] \subset B_{0}$ and we continue as above.

Theorem 6.6 For each $L \in \mathcal{S}, \operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is a $\sigma$-closed pre-order.
Proof. Let $L \in \mathcal{S}_{m}, L=\sum_{i<r} B_{i}$, where $\operatorname{Block}(L)=\left\{B_{i}: i<r\right\}$. First we prove

$$
\begin{equation*}
\mathbb{P}(L)=\left\{\bigcup_{i<r} C_{i}: \forall i<r \quad C_{i} \in \mathbb{P}\left(B_{i}\right)\right\} . \tag{37}
\end{equation*}
$$

The inclusion " $\supset$ " is evident. If $C \in \mathbb{P}(L), f: L \hookrightarrow L$ and $C=f[L]$, then, by Lemma 6.5, for $C_{i}=f\left[B_{i}\right], i<r$, we have $C_{i} \subset B_{i}, C_{i} \in \mathbb{P}\left(B_{i}\right)$ and $C=\bigcup_{i<r} C_{i}$ and " $\subset$ " holds as well.

Clearly, the mapping $F: \prod_{i<r}\left\langle\mathbb{P}\left(B_{i}\right), \subset\right\rangle \rightarrow\langle\mathbb{P}(L), \subset\rangle$ defined by

$$
f\left(\left\langle C_{0}, \ldots, C_{r-1}\right\rangle\right)=\bigcup_{i<r} C_{i}
$$

is an isomorphism and, by Fact [2.4 dd),(e)

$$
\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle \cong \operatorname{sm} \prod_{i<r}\left\langle\mathbb{P}\left(B_{i}\right), \subset\right\rangle=\prod_{i<r} \operatorname{sm}\left\langle\mathbb{P}\left(B_{i}\right), \subset\right\rangle
$$

By Propositions 4.2, 4.4 and $5.2 \mathrm{sm}\left\langle\mathbb{P}\left(B_{i}\right), \subset\right\rangle, i<r$, are $\sigma$-closed partial orders and, by Fact 2.3 their product as well as the poset $\operatorname{sm}\langle\mathbb{P}(L), \subset\rangle$ is $\sigma$-closed.

## 7 Forcing by copies of countable scattered linear orders

The position of countable linear orders in Diagram 1 is presented in Diagram 2 ,
By Theorem 1.2 and Fact 2.5 CH implies that all posets of the form $\langle\mathbb{P}(L), \subset\rangle$, where $L$ is a scattered countable linear order, are forcing equivalent to $(P(\omega) / \text { Fin })^{+}$. The following examples show that this is not true in general and that the result of Theorem 1.2 is the best possible: " $\sigma$-closed" can not be replaced by " $\omega_{2}$-closed".

Example 7.1 It is consistent that the poset $\langle\mathbb{P}(\omega+\omega), \subset\rangle$ is not $\mathfrak{h}$-distributive and, hence, not forcing equivalent to $(P(\omega) / \text { Fin })^{+}$.

By Proposition 4.2 for $L=\omega+\omega$ the partial order $\langle\mathbb{P}(L), \subset\rangle$ is isomorphic to the product $\left\langle[\omega]^{\omega}, \subset\right\rangle \times\left\langle[\omega]^{\omega}, \subset\right\rangle$ and, by Fact [2.6(a), sq $\langle\mathbb{P}(\omega+\omega), \subset\rangle \cong$ $(P(\omega) / \text { Fin })^{+} \times(P(\omega) / \text { Fin })^{+}$. Now, by the result of Shelah and Spinas (Fact 2.6(b)), we have $\operatorname{Con}\left(\mathfrak{h}_{2}<\mathfrak{h}\right)$.

Example 7.2 The poset $\mathrm{sq}\langle\mathbb{P}(\omega \cdot \omega), \subset\rangle$ is not $\omega_{2}$-closed and it is consistent that $\operatorname{sq}\langle\mathbb{P}(\omega \cdot \omega), \subset\rangle$ is not $\mathfrak{h}$-distributive. Clearly $\omega \cdot \omega \cong\langle L,<\rangle$, where $L=\omega \times \omega$ and $\left\langle i_{0}, j_{0}\right\rangle<\left\langle i_{1}, j_{1}\right\rangle \Leftrightarrow i_{0}<i_{1} \vee\left(i_{0}=i_{1} \wedge j_{0}<j_{1}\right)$. Now $L=\sum_{i \in \omega} L_{i}$,


Diagram 2: Countable linear orders
where $L_{i}=\{i\} \times \omega$ and first we show that $\mathbb{P}(L)=(\text { Fin } \times \text { Fin })^{+}$. By Proposition 3.1(a), if $A \in \mathbb{P}(L)$, then for each $m \in \omega$ there is a finite set $K \subset \omega \backslash m$ such that $\omega \hookrightarrow \bigcup_{i \in K} A \cap L_{i}$ and, hence, there is $i \geq m$ satisfying $\left|A \cap L_{i}\right|=\omega$. Thus $A \notin$ Fin $\times$ Fin. Conversely, if $A \notin$ Fin $\times$ Fin and $\left\{i \in \omega:\left|A \cap L_{i}\right|=\omega\right\}=\left\{n_{j}:\right.$ $j \in \omega\}$, where $n_{0}<n_{1}<\ldots$, then $A=\bigcup_{j \in \omega} \Lambda_{j}$, where $\Lambda_{0}=\bigcup_{i \leq n_{0}}\left(A \cap L_{i}\right)$ and $\Lambda_{j}=\bigcup_{n_{j-1}<i \leq n_{j}}\left(A \cap L_{i}\right)$, for $j>0$. Clearly we have $\Lambda_{j} \cong \omega$ and, hence, $A \in \mathbb{P}(L)$. So, $\langle\mathbb{P}(L), \subset\rangle=\left\langle(\text { Fin } \times \text { Fin })^{+}, \subset\right\rangle$ and, by Fact $2.4(\mathrm{f}), \mathrm{sq}\langle\mathbb{P}(\omega \cdot \omega), \subset$ $\rangle \cong(P(\omega \times \omega) /(\text { Fin } \times \text { Fin }))^{+}$. Now we apply the results of Szymański and Zhou and of Hernández-Hernández (Fact 2.6(c) and (d)).

Some forcing-related properties of the posets $\mathrm{sq}\langle\mathbb{P}(L), \subset\rangle$ are described in the following table.

| $L$ | $\mathrm{sq}\{\mathbb{P}(L), \subset\rangle$ is <br> isomorphic to | $\mathrm{sq}\{\mathbb{P}(L), \subset\rangle$ is | $\mathrm{ZFC} \vdash \mathrm{sq}\{\mathbb{P}(L), \subset\rangle$ <br> is $\mathfrak{h}$-distributive |
| :---: | :---: | :---: | :---: |
| $\omega$ | $(P(\omega) / \text { Fin })^{+}$ | t -closed | yes |
| $\omega+\omega$ | $(P(\omega) / \text { Fin })^{+} \times(P(\omega) / \text { Fin })^{+}$ | t-closed | no |
| $\omega \cdot \omega$ | $(P(\omega \times \omega) /(\text { Fin } \times \text { Fin }))^{+}$ | $\omega_{1}$ but not $\omega_{2}$-closed | no |

Remark 7.3 Concerning Theorem 1.2 we note that for countable ordinals we have more information. Namely, by [6], if $\alpha=\omega^{\gamma_{n}+r_{n}} s_{n}+\ldots+\omega^{\gamma_{0}+r_{0}} s_{0}+k$ is a
countable ordinal presented in the Cantor normal form, where $k \in \omega, r_{i} \in \omega$, $s_{i} \in \mathbb{N}, \gamma_{i} \in \operatorname{Lim} \cup\{1\}$ and $\gamma_{n}+r_{n}>\ldots>\gamma_{0}+r_{0}$, then

$$
\begin{equation*}
\mathrm{sq}\langle\mathbb{P}(\alpha), \subset\rangle \cong \prod_{i=0}^{n}\left(\left(\mathrm{rp}^{r_{i}}\left(P\left(\omega^{\gamma_{i}}\right) / \mathcal{I}_{\omega^{\gamma_{i}}}\right)\right)^{+}\right)^{s_{i}} \tag{38}
\end{equation*}
$$

where, for an ordinal $\beta, \mathcal{I}_{\beta}=\{C \subset \beta: \beta \nrightarrow C\}$ and, for a poset $\mathbb{P}, \operatorname{rp}(\mathbb{P})$ denotes the reduced power $\mathbb{P}^{\omega} / \equiv_{\text {Fin }}$ and $\mathrm{rp}^{k+1}(\mathbb{P})=\operatorname{rp}\left(\mathrm{rp}^{k}(\mathbb{P})\right)$. In particular, for $\omega \leq \alpha<\omega^{\omega}$ we have

$$
\begin{equation*}
\operatorname{sq}\left(\mathbb{P}\left(\sum_{i=n}^{0} \omega^{1+r_{i}} s_{i}\right), \subset\right) \cong \prod_{i=0}^{n}\left(\left(\operatorname{rp}^{r_{i}}(P(\omega) / \text { Fin })\right)^{+}\right)^{s_{i}} . \tag{39}
\end{equation*}
$$

Remark 7.4 By [5], all countable equivalence relations, disconnected ultrahomogeneous graphs and disjoint unions of ordinals $\leq \omega$ are in column $D$ of Diagram 1 as well. In addition, the corresponding posets of copies are forcing equivalent to one of the following posets:
$\left((P(\omega) / \text { Fin })^{+}\right)^{n}$, for some $n \in \mathbb{N}$,
$(P(\omega \times \omega) /(\text { Fin } \times \text { Fin }))^{+}$,
$\left(P(\Delta) / \mathcal{E} \mathcal{D}_{\mathrm{fin}}\right)^{+} \times\left((P(\omega) / \text { Fin })^{+}\right)^{n}$, for some $n \in \omega$,
where $\Delta=\{\langle m, n\rangle \in \mathbb{N} \times \mathbb{N}: n \leq m\}$ and the ideal $\mathcal{E} \mathcal{D}_{\text {fin }} \subset P(\Delta)$ is defined by:

$$
\mathcal{E D}_{\text {fin }}=\{S \subset \Delta: \exists r \in \mathbb{N} \forall m \in \mathbb{N}|S \cap(\{m\} \times\{1,2, \ldots, m\})| \leq r\}
$$

## References

[1] R. Fraïssé, Theory of relations, Revised edition with an appendix by Norbert Sauer, Studies in Logic and the Foundations of Mathematics, 145, North-Holland Publishing Co., Amsterdam, 2000.
[2] F. Hernández-Hernández, Distributivity of quotients of countable products of Boolean algebras, Rend. Istit. Mat. Univ. Trieste 41 (2009) 27-33 (2010).
[3] M. S. Kurilić, S. Todorčević, Forcing by non-scattered sets, Ann. Pure Appl. Logic 163 (2012) 1299-1308.
[4] M. S. Kurilić, From $A_{1}$ to $D_{5}$ : Towards a forcing-related classification of relational structures, submitted.
[5] M. S. Kurilić, Maximally embeddable structures, submitted.
[6] M. S. Kurilić, Forcing with copies of countable ordinals, submitted.
[7] R. Laver, On Fraïssé's order type conjecture, Ann. of Math. 93,2 (1971) 89-111.
[8] J. G. Rosenstein, Linear orderings, Pure and Applied Mathematics, 98, Academic Press, Inc., Harcourt Brace Jovanovich Publishers, New York-London, 1982.
[9] S. Shelah, O. Spinas, The distributivity numbers of finite products of $P(\omega) /$ fin, Fund. Math. 158,1 (1998) 81-93.
[10] A. Szymański, Zhou Hao Xua, The behaviour of $\omega^{2^{*}}$ under some consequences of Martin's axiom, General topology and its relations to modern analysis and algebra, V (Prague, 1981), 577-584, Sigma Ser. Pure Math., 3, Heldermann, Berlin, 1983.


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