SOME SUPPLEMENTS TO FEFERMAN-VAUGHT RELATED TO THE MODEL THEORY OF ADELES

JAMSHID DERAKHSHAN AND ANGUS MACINTYRE

ABSTRACT. We give foundational results for the model theory of \mathbb{A}^{fin}_K , the ring of finite adeles over a number field, construed as a restricted product of local fields. In contrast to Weispfenning we work in the language of ring theory, and various sortings interpretable therein. In particular we give a systematic treatment of the product valuation and the valuation monoid. Deeper results are given for the adelic version of Krasner's hyperfields, relating them to the Basarab-Kuhlmann formalism.

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1. Introduction

We have recently revisited Weispfenning's work [25] on the rings of adeles \mathbb{A}_K over number fields K. That work in turn depends on the classic paper of Feferman-Vaught [12] on generalized products. Our objective is to obtain the most refined analysis possible of definable sets in \mathbb{A}_K (paying special attention to uniformity in K). One intended application is to computation of measures and integrals over \mathbb{A}_K . A first paper [10] on this will soon be available. We think of our approach as rather more geometric, and less abstractly model theoretic, than the analysis in [25] and [12]. We prefer to work in the language of ring theory (or sometimes topological ring theory), without the Boolean or lattice-theoretic scaffolding from [25] and [12] (which has much more general applicability). We wish to stress that we add little to the foundations of the theory of generalized products. The treatment in [25] and [12] can hardly be improved. Rather, we work directly on the adeles \mathbb{A}_K as a ring. However, we depend on various quantifier eliminations for completions K_v and some of these are in many-sorted languages appropriate to Henselian fields, so we need a version of [12] for many-sorted structures. Moreover \mathbb{A}_K is a restricted product in the sense of [12] even in a language (like ring-theory) with function-symbols. Though it is implicit in [12] how to deal with function symbols and sorts, we prefer to prepare this short paper providing foundations appropriate to the adelic setting. Further motivation is provided by the model theory of the product valuation on \mathbb{A}_K . The image is a submonoid of a lattice ordered group, and so not literally itself a restricted product. But some simple technical work allows us to find a restricted product interpretation of the image of the valuation. So it is convenient to provide some foundational discussion appropriate to this case.

Much more interesting is our adelic version of the Basarab-Kuhlmann structures [1][20] on local fields. We relate this to Krasner's hyperrings [18] associated to local fields, and provide a natural quantifier-elimination for the adelic version.

Finally, we address the issues of stable embedding of the local fields in the adeles, stable interpretation of the value monoid of the adeles, and the property of not having the tree property of the second kind, NTP_2 .

All readers of [12] know the importance of enrichments of atomic Boolean algebras. A specially important case is (Powerset(I), Fin), where Powerset(I) is the powerset of I and Fin is the ideal of finite sets. In [9] we showed that there is good elimination theory for various refinements, e.g. by Even, where Even picks out the finite sets of even cardinality, or by predicates expressing congruence conditions on cardinality of finite sets. We hope that these refinements will find applications.

We are able to work internally, in the language of ring theory, because \mathbb{A}_K has lots of idempotents. It is not a von Neumann regular ring, so we are appealing to more than is used in the observations used by Kochen [17] and Serre [24, pp.389] that an ultraproduct of fields is canonically isomorphic to the product of the fields modulo a maximal ideal.

2. Generalities

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The data for Theorems of Feferman-Vaught type consists of:

- (i) A (possibly many-sorted) first-order language L, which has the equality symbol = of various sorts, and may have relation symbols and function-symbols. Convenient references for many-sorted logic are [19, 13, 23, 21]. One convention from [23] that we choose not to follow is that the sorts be disjoint. This is an unnecessary restriction, especially when the only well-formed equality statements in our formalism demand that the terms involved be of the same sort.
 - (ii) \mathcal{L}_0 , the usual language for Boolean algebra, with $\{0, 1, \wedge, \vee, \bar{}\}$,
 - (iii) \mathcal{L} , any extension of \mathcal{L}_0 ,
- (iv) I, an index set, with associated atomic Boolean algebra Powerset(I) (the powerset of I, which will be denoted by \mathbb{B} say),

 $\mathbb{B}_{\mathcal{L}}$ will be some \mathcal{L} -structure on \mathbb{B} where $\{0, 1, \wedge, \vee, \bar{}\}$ have their usual interpretations.

(v) A family $\{M_i : i \in I\}$ of L-structures with product $\Pi = \prod_{i \in I} M_i$. One first forms, for each sort σ , the product

$$\prod_{i\in I} Sort_{\sigma}(M_i),$$

where $Sort_{\sigma}(M_i)$, qua set, is just the σ -sort of M_i . This product is the σ -sort of the product Π . The elements are just functions f_{σ} on I with

$$f_{\sigma}(i) \in Sort_{\sigma}(M_i)$$

for all i.

We generally write

$$\bar{f}_{\sigma_1},\ldots,\bar{f}_{\sigma_i},\ldots$$

for tuples of elements of sorts $\sigma_1, \ldots, \sigma_j, \ldots$ respectively; and

$$\bar{x}_{\sigma_1},\ldots,\bar{x}_{\sigma_j},\ldots$$

for tuples of L-variables of sorts $\sigma_1, \ldots, \sigma_j, \ldots$ respectively. Suppose τ is a function-symbol of sort

$$\sigma_1 \times \cdots \times \sigma_r \to \sigma$$
.

Then the interpretation of τ in Π is given by

$$\tau^{(\Pi)}(\bar{f}_{\sigma_1},\ldots,\bar{f}_{\sigma_n})(i) = \tau^{(M_i)}(\bar{f}_{\sigma_1}(i),\ldots,\bar{f}_{\sigma_n}(i)).$$

For an L-formula $\Phi(\bar{w}_{\sigma_1},\ldots,\bar{w}_{\sigma_r})$ we define

$$[[\Phi(\bar{f}_{\sigma_1},\ldots,\bar{f}_{\sigma_r})]] = \{i: M_i \models \Phi(\bar{f}_{\sigma_1}(i),\ldots,\bar{f}_{\sigma_r}(i))\}.$$

The interpretation of a basic relation symbol R of sort

$$\sigma_1 \times \cdots \times \sigma_r$$

is given by

$$R^{(\Pi)}(\bar{f}_{\sigma_1},\ldots,\bar{f}_{\sigma_r}) \Leftrightarrow [[R(\bar{f}_{\sigma_1},\ldots,\bar{f}_{\sigma_r})]] = 1.$$

In this way we have defined the natural product L-structure on Π , agreeing with the usual 1-sorted version.

We usually write z_1, \ldots, z_j, \ldots for variables of the language \mathcal{L} .

Now we bring in $\mathbb{B}_{\mathcal{L}}$, by defining new relations

$$\Psi \circ < \Phi_1, \ldots, \Phi_m >$$

where $\Psi(z_1, \ldots, z_m)$ is an \mathcal{L} -formula, and Φ_1, \ldots, Φ_m are L-formulas in a common set of variables $\bar{x}_{\sigma_1}, \ldots, \bar{x}_{\sigma_s}$ of sorts $\sigma_1, \ldots, \sigma_s$ respectively, by:

$$\Pi \models \Psi \circ < \Phi_1, \dots, \Phi_m > (\bar{f}_{\sigma_1}, \dots, \bar{f}_{\sigma_s}) \Leftrightarrow \\
\mathbb{B}_{\mathcal{L}} \models \Psi([[\Phi_1(\bar{f}_{\sigma_1}, \dots, \bar{f}_{\sigma_s}), \dots, [[\Phi_m(\bar{f}_{\sigma_1}, \dots, \bar{f}_{\sigma_s})]]),$$

for $\bar{f}_{\sigma_1}, \ldots, \bar{f}_{\sigma_s} \in \Pi$.

We extend L by adding a new relation symbol, of appropriate arity, for each of the above. In this way we get $L(\mathbb{B}_{\mathcal{L}})$, and Π has been given an $L(\mathbb{B}_{\mathcal{L}})$ -structure.

The results of Feferman-Vaught [12] are proved for one-sorted languages with no function symbols, though it is pointed out that their basic theorems on generalized products readily adapt to the more general case. The following, adequate for our adelic purposes, is a special case of an even more general theorem in [12].

1. **Theorem.** Uniformly for all families $\{M_i : i \in I\}$ the product Π has constructive quantifier elimination in $L(\mathbb{B}_{\mathcal{L}})$.

Proof. The proof of Theorem 3.1 in [12, pp.65] goes through.

3. Restricted products I

This is a more delicate matter. To our knowledge, the construction has been studied only for 1-sorted situations. We quickly review the definitions in that case.

So L is assumed 1-sorted. Moreover, we assume \mathcal{L} has a 1-ary relation symbol Fin, interpreted in Powerset(I) as the set of finite subsets of I.

Let $\Phi(x)$ be a fixed L-formula in a single variable v, and $\{M_i : i \in I\}$ be L-structures subject only to the constraint that each $\Phi(M_i)$ is an L-substructure of M_i (here $\Phi(M_i)$ denotes the set defined by $\Phi(x)$ in M_i).

With the above assumptions we define the restricted product of the M_i , $(i \in I)$, relative to the formula $\Phi(x)$, denoted $\Pi^{(\Phi)}$ (or $\prod_{i \in I}^{(\Phi)} M_i$), as the L-substructure of $\Pi = \prod_{i \in I} M_i$ consisting of the f such that

$$Fin([[\neg \Phi(f)]])$$

holds. (Our preceding assumptions make it an *L-substructure* of Π).

- 1. **Example.** L is the language of group theory, with primitives $\{\cdot,^{-1}, 1\}$, and $\Phi(x)$ is the formula x = 1. The M_i are arbitrary. The restricted product is the direct sum.
- 2. **Example.** L is the language of fields with a valuation ring, with primitives $\{+,-,\cdot,0,1,V\}$, where where V(x) is a unary predicate for the valuation ring. $\Phi(x)$ is the formula V(x).

If $\{M_i : i \in I\}$ is the family of completions of an algebraic number field K with respect to the normalized valuations of K, the restricted product is the ring of adeles of K, denoted A_K (cf.[3]).

If $\{M_i : i \in I\}$ is the family of completions of K with respect to the non-archimedean normalized valuations of K, then restricted product is the ring of finite adeles of K, denoted \mathbb{A}_K^{fin} .

In fact $\Pi^{(\Phi)}$ is an $L(\mathbb{B}_{\mathcal{L}})$ -definable $L(\mathbb{B}_{\mathcal{L}})$ -substructure of Π (remember, Fin is in \mathcal{L}) since the basic $L(\mathbb{B}_{\mathcal{L}})$ -formulas not in \mathcal{L} are all relational).

2. **Theorem.** $\Pi^{(\Phi)}$ has quantifier-elimination as an $L(\mathbb{B}_{\mathcal{L}})$ -structure uniformly in $\{M_i : i \in I\}$ and $\Phi(x)$, and effectively.

Proof. For an $L(\mathbb{B}_{\mathcal{L}})$ -formula Ψ , let $\Psi^{\Pi^{(\Phi)}}$ denote the relativization of Ψ to $\Pi^{(\Phi)}$ which is defined inductively by replacing a quantified subformula

$$\exists y(\Psi(...,y,...))$$

(where y is a single variable) in Ψ by the quantified formula

$$\exists y (Fin([[\neg \Phi(y)]]) \land \Psi(...,y,...)).$$

Now, by Theorem 1, we have for all tuples \bar{f} from Π ,

$$\Pi \models \Psi^{\Pi^{(\Phi)}}(\bar{f}) \Leftrightarrow \Pi \models \Theta(\bar{f})$$

for a quantifier-free formula Θ from $L(\mathbb{B}_{\mathcal{L}})$. Finally, since $\Pi^{(\Phi)}$ is an $L(\mathbb{B}_{\mathcal{L}})$ -definable $L(\mathbb{B}_{\mathcal{L}})$ -substructure of Π , we get

$$\Pi^{(\Phi)} \models \Psi(\bar{f}) \Leftrightarrow \Pi \models \Psi^{\Pi^{(\Phi)}}(\bar{f})$$
$$\Leftrightarrow \Pi \models \Theta(\bar{f}) \Leftrightarrow \Pi^{(\Phi)} \models \Theta(\bar{f}).$$

4. Restricted products II

We have experimented with various notions of restricted product in the many-sorted case. The notion explained below is the only viable notion we found. In this section, we give a many-sorted version of the results in Section 3.

Assume L is many-sorted, perhaps with both function-symbols and relation-symbols. Let M, N be L-structures. We put $N_{\sigma} = Sort_{\sigma}(N)$ for every sort σ .

1. **Definition.** An L-morphism $F: N \to M$ is a collection of maps

$$F_{\sigma}: N_{\sigma} \to M_{\sigma},$$

where σ ranges over the sorts, such that for any relation symbol R of sort

$$\sigma_1 \times \cdots \times \sigma_k$$

we have,

$$N_{\sigma_1} \times \cdots \times N_{\sigma_k} \models R(\bar{f}_1, \dots, \bar{f}_k) \Leftrightarrow M_{\sigma_1} \times \cdots \times M_{\sigma_k} \models R(F_{\sigma_1}(\bar{f}_1), \dots, F_{\sigma_k}(\bar{f}_k)),$$

and for any function symbol G of sort

$$\sigma_1 \times \cdots \times \sigma_k \to \sigma$$

we have

$$G(F_{\sigma_1}(\bar{f}_1),\ldots,F_{\sigma_k}(\bar{f}_k)) = F_{\sigma}(G(\bar{f}_1,\ldots,\bar{f}_k)),$$

where $\bar{f}_1, \ldots, \bar{f}_k$ denote tuples of elements of sorts $\sigma_1, \ldots, \sigma_k$ respectively.

1. Note. Our convention that we have the usual equality as a binary relation on each sort forces each F_{σ} to be injective. There is certainly a case for relaxing this convention or replacing $\Leftrightarrow by \to$, but there is no gain for our present purposes.

If each $N_{\sigma} \subseteq M_{\sigma}$, and the identity maps constitute an L-morphism, we say N is an L-substructure of M.

Suppose that for each sort σ we have a formula $\Phi_{\sigma}(x_{\sigma})$ in a single free variable x_{σ} of sort σ , and we make the assumption that for each σ for all i the sets

$$S_{\sigma,i} = \{x \in Sort_{\sigma}(M_i) : M_i \models \Phi_{\sigma}(x)\}$$

naturally constitute an L-substructure of M_i . Note that in particular, for any function symbol F of sort

$$\sigma \to \tau$$

and any $a \in S_{\sigma}(M_i)$ we have that

$$F(a) \in S_{\tau}(M_i),$$

for all i.

Then we define $\Pi^{(\Phi_{\sigma})}$ (also denoted $\prod_{i\in I}^{(\Phi_{\sigma})} M_i$), the restricted product with respect to the formulas $\Phi_{\sigma}(x)$, as the $L(\mathbb{B}_{\mathcal{L}})$ -substructure of Π consisting in sort σ of the

$$f_{\sigma} \in \prod_{i \in I} S_{\sigma}(M_i)$$

such that

$$Fin([[\neg \Phi_{\sigma}(f_{\sigma})]])$$

holds.

Note that $\Pi^{(\Phi_{\sigma})}$ is L-sorted: given σ , a sort of L, the σ -sort of $\Pi^{(\Phi_{\sigma})}$ is the set of all $f_{\sigma} \in \prod_{i \in I} S_{\sigma}(M_i)$ such that

$$Fin([[\neg \Phi_{\sigma}(f_{\sigma})]])$$

holds.

If F is a function symbol of sort

$$\sigma \to \tau$$
,

and a is in the σ -sort of $\Pi^{(\Phi_{\sigma})}$, then since the sets $S_{\sigma}(i)$ are L-substructures of M_i for all i, we deduce that

$$Fin([[\neg \Phi_{\tau}(F(f_{\sigma}))]])$$

holds. Hence F(a) lies in $Sort_{\tau}(\Pi^{(\Phi_{\sigma})})$. Thus $\Pi^{(\Phi_{\sigma})}$ is a substructure of Π . It is clearly $L(\mathbb{B}_{\mathcal{L}})$ -definable.

Now it is clear that the quantifier-elimination and effectivity of Theorem 2 goes through. It is convenient to refer to the general version as Theorem 2_{sort} .

Below we will give some important examples of restricted products associated to adeles, namely, the lattice-ordered value monoid and various hyperring structures connected to Basarab's formalism [1], which in turn connects to the much earlier work of Krasner [18].

5. Eliminating the Boolean superstructure and quantifier elimination for adele rings

Already in Kochen [17] (and surely in von Neumann's work) one sees that products $\prod_{i \in I} M_i$ of fields M_i are von Neumann regular rings, with idempotents corresponding to subsets S of I via the correspondence

$$S \longrightarrow e_S$$

where e_S is the idempotent defined by

$$e_S(i) = 1, i \in S,$$

$$e_S(i) = 0, i \notin S,$$

and conversely, an idempotent e corresponds to the set

$$S(e) = \{ i \in I : e(i) = 1 \}.$$

Given an element $a=(f_1,f_2,\dots)\in\prod_{i\in I}M_i$, let e_a be the idempotent corresponding to the support

$$supp(a)=\{i\in I: a(i)\neq 0\}.$$

Thus $e_a(i) = 0$ if a(i) = 0 and $e_a(i) = 1$ if $a(i) \neq 0$.

Note that a and e_a generate the same ideal. Indeed, it is clear that $a=(e_a)a$, thus a is in the ideal generated by e_a . Conversely, let $b \in \prod_{i \in I} M_i$ be defined by b(i) = 0 when a(i) = 0 and $b(i) = a(i)^{-1}$ when $a(i) \neq 0$. Then $e_a = ab$, showing that e_a is in the ideal generated by a. This shows von Neumann regularity of the products $\prod_{i \in I} M_i$ of fields M_i .

Kochen [17] observes that maximal ideals in $\prod_{i \in I} M_i$ correspond to ultrafilters on the Boolean algebra Powerset(I). The point is that in some cases of products one can code the Boolean algebra Powerset(I) purely algebraically in the product ring $\prod_{i \in I} M_i$. In this way one can sometimes reconstruct the external Boolean apparatus inside the product ring. Note that there are limits to this, for example, one cannot define Fin internally in any infinite product $\prod_{i \in I} M_i$. But, and this is crucial for us, we can define Fin internally in the adeles \mathbb{A}_K and in the finite adeles \mathbb{A}_K^{fin} as is shown in [10]. We remark that \mathbb{A}_K and \mathbb{A}_K^{fin} are not von Neumann regular (cf. [10]).

The main novelty of [10] over [25] is the internalization of [12] for the case of the adeles. In this way we get better quantifier-elimination. Below we briefly review the first-order definitions of the Boolean algebra, Boolean value, and the ideal Fin in the case of adeles from [10].

Let K be a number field and V_K (resp. V_K^f) denote the set of normalized valuations (resp. normalized non-archimedean (discrete) valuations) of K. Note the correspondence between subsets of V_K^f and idempotents in \mathbb{A}_K^{fin} given by

$$S \longrightarrow e_S$$
,

where $e_S(v) = 1$ if $v \in S \subseteq V_K^f$, and $e_S = 0$ if $v \notin S$. Clearly $e_S \in \mathbb{A}_K^{fin}$. Conversely, given an idempotent $e \in \mathbb{A}_K^{fin}$, let $S = \{v : e(v) = 1\}$. Then $e = e_S$.

There is a similar correspondence between subsets of V_K and idempotents in \mathbb{A}_K . Denote by \mathbb{B}_K^f the Boolean algebra of idempotents of \mathbb{A}_K^{fin} with the operations

$$e \wedge f = ef.$$

$$e \vee f = 1 - (1 - e)(1 - f) = e + f - ef.$$

$$\bar{e} = 1 - e.$$

 \mathbb{B}_K^f is quantifier-free definable in \mathbb{A}_K^{fin} in the language of rings. Note that minimal idempotents e correspond to normalized valuations v_e of K, and vice-versa, v corresponds to $e_{\{v\}}$ above. One has

$$\mathbb{A}_K^{fin}/(1-e)\mathbb{A}_K^{fin} \cong e\mathbb{A}_K^{fin} \cong K_{v_e}.$$

Note that the first of these structures is a definable quotient of \mathbb{A}_K^{fin} and the second is a definable subring of \mathbb{A}_K^{fin} with e as a unit.

Similarly, one can define the Boolean algebra of idempotents in \mathbb{A}_K and similar assertions hold.

Given a formula $\Phi(x_1, \ldots, x_n)$ of the ring language, define $Loc(\Phi)$ as the set of all

$$(e, a_1, \dots, a_n) \in \mathbb{A}_K^{n+1}$$

such that e is a minimal idempotent and

$$e\mathbb{A}_K^{fin} \models \Phi(ea_1, \dots, ea_n).$$

Here, $e\mathbb{A}_K^{fin}$ is a subring of \mathbb{A}_K^{fin} with e as its unit, is definable with the parameter e, and

$$e\mathbb{A}_K^{fin} \models \Phi(ea_1,\ldots,ea_n) \Leftrightarrow \mathbb{A}_K^{fin} \models \Phi(ea_1,\ldots,ea_n).$$

Note that $Loc(\Phi)$ is a definable subset of $(\mathbb{A}_K^{fin})^{n+1}$.

Let $a_1, \ldots, a_n \in \mathbb{A}_K^{fin}$. Define the Boolean value

$$[[\Phi(a_1,\ldots,a_n)]]$$

as the supremum of all the minimal idempotents e in \mathbb{B}^f_K such that

$$(e, a_1, \dots, a_n) \in Loc(\Phi).$$

This is a definition in the language of rings which is uniform for all number fields K. If Φ has a string of quantifiers $Q_1x_1 \dots Q_kx_k$, then the the formula defining $[[\Phi(a_1, \dots, a_n)]]$ has the string of quantifiers $\forall z Q_1x_1 \dots Q_kx_k$, where z is a variable distinct from the x_1, \dots, x_k .

The functions $(\mathbb{A}_K^{fin})^n \to \mathbb{A}_K^{fin}$ given by

$$(a_1,\ldots,a_n) \to [[\Phi(a_1,\ldots,a_n)]]$$

(for each formula Φ) are definable in the language of rings, uniformly for all K. The support of an element $a \in \mathbb{A}_K^{fin}$, denoted supp(a), is defined as $[[x \neq 0]]0$.

We remark that the concepts of Boolean value can be defined similarly for the adele ring \mathbb{A}_K and similar assertions hold.

We shall denote by \mathbb{F}_K^f (resp. \mathbb{F}_K) the ideal of idempotents in \mathbb{A}_K^{fin} (resp. \mathbb{A}_K) whose support is finite.

The sets \mathbb{F}_K^f (resp. \mathbb{F}_K) are definable in \mathbb{A}_K^{fin} (resp. \mathbb{A}_K) in the language of rings. The defining formulas are existential-universal-existential (Cf. [10]).

These results in conjunction with Theorem 2_{sort} yield the following quantifier elimination theorem for \mathbb{A}_K^{fin} in suitable extensions of the language of rings.

3. **Theorem.** [10] K be a number field. Let L be a one-sorted (resp. many-sorted) extension of the language of rings in which the non-archimedean completions K_v have uniform quantifier elimination (resp. uniform quantifier elimination in a sort σ). Let \mathcal{L} be any extension of the language of Boolean algebras containing a unary predicate Fin(x) for the ideal of finite sets and unary predicates $C_j(x)$, for all $j \geq 1$, stating the there are at least j distinct atoms below x. Let $\Phi(\bar{x})$ be an L-formula. Then there are L-formulas $\Psi_1(\bar{x}), \ldots, \Psi_m(\bar{x})$ which are quantifier-free (resp. quantifier-free in sort σ) and a quantifier-free \mathcal{L} -formula $\Theta(x_1, \ldots, x_m)$ such that $\Phi(\bar{x})$ is equivalent, modulo $Th(\mathbb{A}_K^{fin})$, to

$$\Theta([[\Psi_1(\bar{x})]],\ldots,[[\Psi_m(\bar{x})]]).$$

Note that by the definition of Fin in the language of rings, this is a quantifier elimination in L. Now using the quantifier elimination in the theory of infinite atomic Boolean algebras in the Boolean language enriched by the predicates Fin(x), $C_j(x)$, for all $j \geq 1$, (cf. [9]), we deduce the following.

- 4. Corollary. [10] A definable set $X \subseteq (\mathbb{A}_K^{fin})^m$ is a finite Boolean combination of sets of the form
 - $Fin([[\Psi(\bar{x})]]),$
 - $C_j([[\Xi(\bar{x})]]),$

where $\Psi(\bar{x})$ and $\Xi(\bar{x})$ are quantifier-free L-formulas (resp. quantifier-free L-formulas in sort σ), where L is as in the Theorem 3.

Note that the condition $[[\Psi(\bar{x})]] = 1$ is equivalent to $\neg C_1([[\neg \Psi(\bar{x})]])$.

These results imply similar results for the ring of adeles \mathbb{A}_K (cf. [10]).

We will give examples of languages which can be used in Theorem 3 and Corollary 4 in the next section.

6. Sortings of valued fields

In the fifty years of the history of model theory of valued fields, many formalisms and sortings has proved useful. Each of these will provide a formalism for rings of finite adeles \mathbb{A}_K^{fin} . We have, as of now, seen no need to make a systematic study of all the possibilities, but some have engaged our attention. Below are some of the standard ingredients (it is intended that each sort has = as a primitive). Given a local field K, v denotes the valuation, \mathcal{O}_K the valuation ring, \mathcal{M} the maximal ideal, U the unit group of \mathcal{O}_K , and Γ the value group.

- (1) The field sort K with primitives $\{+, -, ., 0, 1\}$,
- (2) The multiplicative group sort K^* with primitives $\{.,^{-1},1\}$
- (3) The valuation ring sort \mathcal{O}_K with primitives $\{+, -, ., 0, 1\}$,
- (4) The residue field sort k with primitives $\{+, -, ., 0, 1\}$,
- (5) The extended residue field sort $k \cup \{\infty\}$ with primitives $\{+, -, ., 0, 1, \infty\}$,
- (6) The value group sort Γ with primitives $\{+, -, 0, <\}$,
- (7) The extended value group sort $\Gamma \cup \{\infty\}$ with primitives $\{+, -, 0, <, \infty\}$,
- (8) The maximal ideal sort \mathcal{M} , with primitives $\{+, -, ., 0\}$,
- (9) The 3-sorted structure consisting of the sorts (1), (6), and (4) together with the connecting valuation and residue maps.
- (10) The Basarab-Kuhlmann sorts $(K, K^*/1 + \mathcal{M}^n, \mathcal{O}_K/\mathcal{M}^m)$, for all $n \geq 1$, with primitives $\{.,^{-1}, 1\}$ for the sort $K^*/1 + \mathcal{M}^n$ and $\{+, -, ., 0, 1\}$ for the sorts $\mathcal{O}_K/\mathcal{M}^m$, with the valuation maps and canonical projection maps from the field sort into the other sorts.
- (11) The many-sorted language $(K, K/1+\mathcal{M}^n)$, for all n, with primitives $\{., 1, 0, \Sigma\}$ with the valuation map, Σ (the hyperring of Krasner, cf. Section 10), and the connecting map $K \to K/1 + \mathcal{M}^n$.
- (12) The formalisms (10) and (11) with \mathcal{M}^n replaced by $\mathcal{M}_{n,K}$ defined in Section 13.
 - (13) Valuations between appropriate sorts.
 - (14) Residue maps to residue rings.
 - (15) The place map from (1) to (5),
 - (16) The cross-section from (6) to (1) or (2),
- (17) Angular component maps from K to residue fields. Addition of this to the formalism (9) gives the Denef-Pas language.

Thus there is a large stock of sortings relevant to the adeles. We will concentrate here on those connected to the value group sorts and the Basarab sorts.

There are natural extensions of the above languages in which the completions K_v , where $v \in V_K^f$, of a number field K have uniform quantifier elimination. These languages can be used in Theorem 3 and Corollary 4. A one-sorted example is the extension of the language of rings by the Macintyre predicates $P_n(x)$, for all $n \geq 1$, stating that x is an n-th power, and the solvability predicates $Sol_k(y_0, \ldots, y_n)$, $k \geq 1$, stating that $v(y_i) \geq 0$, for $0 \leq i \leq n$, and the reduction of the polynomial

$$x^n + y_1 x^{n-1} + \dots + y_n$$

modulo the maximal ideal \mathcal{M} has a root in the residue field. Belair [2] proved that the p-adic fields \mathbb{Q}_p , for all p, have uniform quantifier elimination in this language and his proof carries over to the case of all K_v . Many-sorted examples are the language of Basarab and Kuhlmann (cf. [1],[20] and Section 13 below) sated in (10)

above and the language of Denef-Pas [22] stated in (17) above. In these languages the K_v have uniform quantifier elimination in the field sort relative to the other sorts. These languages have been used in motivic integration (cf. [8],[6],[16]).

7. The value monoid: various options

In connection with the value group, two particular sortings stand out, closely connected. The objective is to work out the meaning of these two on the finite adeles \mathbb{A}_K^{fin} . We also consider a third, connected to the ideles.

First version:

We have two sorts, corresponding to K and $\Gamma \cup \{\infty\}$, and v connecting them. K has usual ring structure, $\Gamma \cup \{\infty\}$ has primitives $\{<,+,0,1\}$, and Γ is an ordered abelian group. For technical reasons we need to replace the total order < by lattice operations \land, \lor (which are respectively min and max).

The axioms regarding ∞ in the value group sort are:

$$\infty + \infty = \infty,$$

$$\forall g \in \Gamma(\infty + g = \infty = g + \infty = \infty - g),$$

and

$$\forall g \in \Gamma(g < \infty).$$

The most natural thing is to have a constant ∞ of the extended value group sort.

On the factors K_v , $v(0) = \infty$, and for $x \in K_v^*$, v(x) is the standard normalized valuation of x. This justifies the laws above. Note that $\infty - \infty$ is not defined, since 0/0 is not. We justify $\infty - g$ by the remark that $0.x^{-1} = 0$, $x \neq 0$.

Thus $\Gamma \cup \{\infty\}$ is a commutative monoid under the operation +, as is Γ (which is in fact a group, though $\Gamma \cup \infty$) is not). So, at the cost of changing the notion of *substructure* there is a case, which we accept, for taking the inverse - away from the basic formalism of the $\Gamma \cup \{\infty\}$ sort. $\Gamma \cup \{\infty\}$ is an ordered monoid, and Γ an ordered group.

Given a number field K with completion K_v at the normalized non-archimedean discrete valuation v, the product $\prod_{v \in V_K^f} K_v$ has a product valuation $\prod v$ to

$$\prod_{v \in V_K^f} (\Gamma_v \cup \{\infty\})$$

where Γ_v is $\mathbb Z$ for all v, and the Feferman-Vaught theory gives us a decidable model theory for this. Note that the product $\prod_{v \in V_K^f} (\Gamma \cup \{\infty\})$ is a lattice-ordered monoid. Note too that the image of $\mathbb A_K^{fin}$ is the set of g such that $g(v) \geq 0$ for all but finitely many v. We shall see a bit later how to mimic in the $\Gamma \cup \{\infty\}$ sorting what we did in the adelic setting, i.e. give an internal definition of the Boolean value $[[\Phi(\bar{x})]]$. That will involve a switch from < to the lattice operations \wedge, \vee .

Our goal is to represent the product valuation on \mathbb{A}_K^{fin} in terms of a restricted many-sorted formalism. Because of the substructure constraints in the general definition of restricted product, we proceed as follows. To the one-sorted formalism for the ring of adeles we add just one more sort, the value sort, which has as primitives $\{+, \wedge, \vee, 0, \infty\}$. For a valued field K these get their standard interpretation for + and 0 and ∞ , but \wedge and \vee are respectively min and max in the ordering.

Note that the following axioms are true in this value sort in the case of valued fields:

- i) Axioms for lattice-ordered commutative monoids with 0 as neutral element,
- ii) Axioms about the distinguished element ∞ , namely

$$\infty + \infty = \infty,$$
$$\forall g(\infty \land g = g),$$
$$\forall g(\infty \lor g = \infty).$$

These axioms are preserved under products. Note in contrast that the axiom special to the valued sort of a value field case, namely,

$$\forall x \forall y (x \land y = x) \lor (x \land y = y),$$

is *not* preserved under products.

We want to carry this sorting to the adeles. So now we consider valued fields as 2-sorted structures consisting of a sort for the valued field, a sort for the lattice-ordered monoid with ∞ , and a connecting map v. The product of these 2-sorted structures will have in its first sort a von Neumann regular ring (as product of the field sorts), and in its second sort a lattice-ordered commutative monoid with distinguished element ∞ satisfying the axioms we gave before and in addition the following version of the valuation axioms:

$$\forall f \forall g (v(f.g) = v(f) + v(g)),$$
$$\forall f \forall g (v(f+g) \ge v(f) \land v(g)).$$

We are mainly interested in this product valuation on the finite adeles \mathbb{A}_K^{fin} . As remarked above, the image of the finite adeles \mathbb{A}_K^{fin} under the product valuation is contained in the set of g in $\prod_{v \in V_K^f} (\Gamma_v \cup \{\infty\})$ such that

$$Fin([[\neg(g \land 0 = 0)]])$$

holds. In fact the set of such g is exactly the image of \mathbb{A}_K^{fin} under the product valuation. This is immediate by lifting such a g back to any f with v(f(v)) = g(v). Let us note that the pair

$$(v(x) \ge 0, y \land 0 = 0)$$

satisfies the assumption in Section 4 that allows us to define a restricted product. So we can now identify, inside the 2-sorted structure with K and $\Gamma \cup \{\infty\}$, and connecting valuation v, a natural restricted product, namely that with respect to the formulas $v(x) \geq 0$ in the K-sort, and the formula $y \wedge 0 = 0$ in the $\Gamma \cup \{\infty\}$

sort. This we call the structure of \mathbb{A}_K^{fin} with totally defined product valuation. By Theorem 2_{sort}, it has a Feferman-Vaught quantifier-elimination.

In Section 9 below we will go further, eliminating the Boolean scaffolding in the value group sort, in terms of the formalism of that sort.

Second version:

We have three sorts, corresponding to K, K^* , and Γ , and

$$v: K^* \to \Gamma, \ i: K^* \to K.$$

Again we will use \land , \lor on Γ . Now there is no need for ∞ . We take K^* with primitives $\{.,1\}$, but not with the operation of inverse $\{^{-1}\}$.

Obviously there is essentially no difference between the first and second versions in terms of expressive power. We could if needed make this precise in terms of bi-interpretability.

In the product we have $\prod_v v$, $\prod_v K_v^*$, and $\prod_v \Gamma_v$ and, now we get a restricted product using $v(x) \geq 0$ in the K-sort, $v(x) \geq 0$ in the K^* -sort, and $y \wedge 0 = 0$ in the Γ -sort.

Notice that the formula in the second sort actually involves the connecting map between the second and third sorts.

Now the restricted product that emerges consists of the finite adeles \mathbb{A}_K^{fin} with the submonoid of elements with no zero coordinate and the product valuation from this set to the restricted product of the Γ_v . Call this the ∞ -free restricted product for this version.

Third version:

Note however that another interesting possibility emerges if we take the formula of the middle sort to be v(x) = 0 and the formula of the last sort to be g = 0. Then the restricted product that emerges consists of the finite adeles with the finite ideles as a subgroup together with a valuation from it onto the direct sum of the value groups Γ_v (a group!). Call this the *idelic* restricted product for this version.

The difference between the ∞ -free and idelic restricted products for this version are:

- (i) The former has three sorts, namely \mathbb{A}_K^{fin} , the submonoid of elements with no zero coordinates, and the value monoid sort.
- (ii) The latter has three sorts, namely \mathbb{A}_K^{fin} , the ideles, and the submonoid of the value monoid from (i) consisting of elements which are zero at all but finitely many coordinates (i.e. a direct sum).

We show later that (i) defines $(Powerset(V_K^f), Fin)$, and we can "remove the Boolean scaffolding".

In (ii), it turns out that all $[[\Phi(f_1,\ldots,f_n)]]$, where f_j belong to the value monoid, are finite or cofinite, i.e. belong to the finite/cofinite subalgebra of $Powerset(V_K^f)$.

It turns out that we can *define Fin*, and then *interpret* the finite/cofinite algebra, and thereby "remove the Boolean scaffolding".

8. Interpreting the sorts in the field sort

In [5] it is shown that the valuation ring is uniformly definable in all K_v (by an existential-universal formula of the language of rings). From this it follows directly that all the sorts in Section 6 and the maps listed with them, together with the connecting maps between the sorts are uniformly interpretable in the field sort.

The angular component maps are known not to be interpretable, but have proved very useful, e.g. in motivic matters via the Pas language [8].

The "corpoid" or "hyperring" structure in (11) merits special attention. Fix n, and consider the group $K^*/1 + \mathcal{M}^n$ under multiplication. This is certainly interpretable. There is also a valuation v on $K^*/1 + \mathcal{M}^n$ to Γ , the value group sort, clearly interpretable. Of course, the quotient

$$\pi_n: K^* \to K^*/1 + \mathcal{M}^n$$

is interpretable. Finally, the relation Σ_n which is the image of the graph of addition intersected with $(K^*)^3$ is interpretable and gives an "approximation to addition".

Basarab [1] showed that one has quantifier elimination for the field sort in terms of essentially extra sorts involving higher residue ring sorts $\mathcal{O}_K/\mathcal{M}^n$ and the group sorts $K^*/1 + \mathcal{M}^n$.

- 2. Note. The Basarab construction works for general initial segments I of the value group, but there is now no functorial sort. I may not be interpretable.
 - 9. Removing the Boolean scaffolding in the value monoids of the \mathbb{A}^{fin}_K , for the totally defined, ∞ -free, and idelic restricted products

Recall that in the first and second versions discussed in Section 7 we dealt respectively with

- i) A restricted product involving two sorts, the usual valued field sort, and a value group sort which was a lattice-ordered monoid $\prod_{v \in V_K^f} (\Gamma_v \cup \{\infty\})$,
- ii) A restricted product involving three sorts, the valued field sort, the multiplicative group sort, and a lattice-ordered monoid sort $\prod_{v \in V_K^f} \Gamma_v$.

There are only minor differences between these versions, whereas the third version is somewhat different.

Versions 1 and 2:

The restricted product is relative to the formula $v(x) \geq 0$ in the field sort, and the formula $y \wedge 0 = 0$ in the lattice-ordered monoid sort for both of the restricted products from (i) and (ii). The restricted product is then the adeles with the (surjective)

product valuation to the lattice-ordered monoid

$$\{g: Fin(g \lor 0 \neq g)\}$$

for both versions.

Now note that for each version the lattice-ordered monoid is itself a restricted product with respect to the formula $g \vee 0 \neq g$ over the index set V_K^f . So the question arises as to whether we can eliminate the Boolean scaffolding for the restricted products. We do not have the machinery of idempotents which we exploited in \mathbb{A}_K^{fin} , so the problem is nontrivial. The following argument works for both versions as there is no reference to ∞ .

How to interpret the elements of V_K^f ? An atom of the lattice order is a minimal non-zero e > 0. Such e correspond exactly to the g in the restricted product so that g(v) = 0 except for a single v_0 , where $g(v_0) = 1$. The Boolean algebra $\mathbb{B} = Powerset(V_K^f)$ can be identified with the set of all e which are either 0 or a supremum of atoms. There is a largest such element which we call 1. The Boolean operations on \mathbb{B} are actually the lattice operations \wedge, \vee of the lattice-ordered monoid. Note that the complexity of definition is higher than in the adele case.

How to define the finite elements of \mathbb{B} ? Just note that b in \mathbb{B} is finite if and only if b is invertible in the restricted product monoid. (We write -b for the inverse). Note, of course, that we are now living a bit dangerously notation-wise: -b is the group-theoretic inverse (defined as the c with b+c=0), and has little to do with the -b in the Boolean ring. So again we see that the complexity of our definitions is greater than in the ring case. Thus we can interpret (\mathbb{B} , Fin). It remains to define $[\Phi(\bar{x})]$ and it suffices to define or interpret the stalk at an atom e, uniformly in e.

One should note how the monoid operation + relates to \mathbb{B} . The operation + is not a group operation on the restricted product, but a trace of the operation - survives on \mathbb{B} . Namely, if $e \in \mathbb{B}$, the Boolean complement f of e in \mathbb{B} is 1-e, i.e. e+f=1.

Let e be an atom, corresponding to a valuation v. The stalk at e is just the lattice-ordered monoid Γ_v (= $\mathbb{Z} \cup \{\infty\}$). We identify it with the substructure consisting of the h such that h(w) = 0 for all $w \neq v$. This we call the internal stalk at e, and denote it by $\hat{\Gamma}_e$. First suppose h is any element of the restricted product with $h \wedge e = e$ (i.e. $h \geq e$). Then $h \geq 0$ and $h(v) \geq 1$ (the stalks are discretely ordered). If for some w, $h(w) \neq 0$, then $h(w) \geq 1$.

Suppose $w_0 \neq v$ and $h(w_0) \geq 1$. Define j_1 and j_2 by

$$j_1(v) = 2h(v),$$

 $j_1(w_0) = h(w_0),$
 $j_2(v) = h(v),$
 $j_2(w_0) = 2h(w_0),$

and if $w \neq v, w_0$

$$j_1(w) = h(w), \ j_2(w) = h(w).$$

Now $e \leq j_1 \leq 2h$ and $e \leq j_2 \leq 2h$, but neither $j_1 \leq j_2$ nor $j_2 \leq j_1$. Thus the interval [e, 2h] is not linearly ordered.

Conversely, if h(w) = 0 for all $w \neq v$, then [e, 2h] is linearly ordered.

Next, suppose we have h with

$$h \wedge (-e) = h$$
.

Then $h(w) \leq 0$ for $w \neq v$, and $h(v) \leq -1$. Then the preceding argument, mutatis mutandis, shows that h(w) = 0 for all $w \neq v$ if and only if [2h, -e] is linearly ordered.

We conclude:

1. **Lemma.** h is in the stalk at e if and only if either h = 0 or $h \ge e$ and [e, 2h] is linearly ordered, or $h \le -e$ and [2h, -e] is linearly ordered.

Proof. Done.
$$\Box$$

This is however, not quite enough to get a definition of $[\Phi(\bar{x})]$ in the style of what we did for \mathbb{A}_K^{fin} . We need to define the natural map from the value monoid to the stalk at e. Our restricted product is a structure of functions on I (identified with set of atoms) and the stalk at $v \in I$ (which also call the *external stalk*) is the set of all f(v), for f in the restricted product. We now show how to interpret this. For this, we show the following. Let Γ_e (or $\Gamma_e \cup \{\infty\}$) denote the stalk at the atom e defined as the set of h in the product such that h(v) = 0 for all atoms $v \neq e$.

By Lemma 1 there is a definition, in the restricted product, for the internal stalk at e, $\hat{\Gamma}_e$, where e is an atom. Define the relation \equiv_e on the restricted product by

$$f \equiv_e g \Leftrightarrow f(e) = g(e).$$

This is a congruence for $\land, \lor, +, 0, 1$. If $f, g \ge 0$, then it is clear that

$$(9.0.1) f(e) = g(e) \Leftrightarrow \forall h \in \hat{\Gamma}_e \ (h \ge f \Leftrightarrow h \ge g),$$

(i.e. $h \wedge f = f \Leftrightarrow h \wedge g = g$ holds in the restricted product).

So we can define

$$f \equiv_e g \Leftrightarrow \exists f^+, f^-, g^+, g^- \ (f^+ \ge 0 \land g^+ \ge 0 \land f^- \le 0 \land g^- \le 0$$

$$\wedge f = f^+ + f^- \wedge g = g^+ + g^- \wedge f^+(e) = g^+(e) \wedge f^-(e) = g^-(e)$$

This follows from applying 9.0.1 to the f^+ and g^+ and to f^- and g^- with the order reversed. Thus:

2. **Lemma.** The stalk Γ_e at the atom e is interpretable uniformly in e in the restricted product.

Proof. Done.
$$\Box$$

So we identify the stalk at e with the set of congruence classes modulo \equiv_e , thereby giving a definable meaning to the condition that the stalk at e satisfies

$$\Phi(f_1(e),\ldots,f_k(e)),$$

and so we define $[[\Phi(\bar{f})]]$ as the set of e's where this holds, and have completed the removal of the Boolean scaffolding.

Version 3:

Now the valuation is defined on the group of finite ideles, and the value monoid is the direct sum of the Γ_v , $v \in V_K^f$. We define Boolean operations as in previous versions, but in this case we do not get a Boolean algebra, just a lattice because all elements are *finite*, there is no top element, and no element is complemented (though we have relative complements).

Note that some of the discussion of Case 1 goes through, namely that giving the interpretation of the stalk at e and the natural projection to the stalk. Thus we can define

$$e \in [[\Phi(\bar{f})]].$$

Now we show that we can define Fin in the restricted product. Given a formula $\Phi(\bar{x})$, we can define $Fin([[\Phi(\bar{f})]]$ by

$$Fin([\Phi(\bar{f})]) \Leftrightarrow \exists f \forall e (e \in [\Phi(\bar{f})]) \to e \leq f).$$

Let $\mathbb{B}_{fin/cofin}$ denote the Boolean algebra of finite and cofinite subsets of $Powerset(V_K^f)$.

3. **Lemma.** For any formula $\Phi(\bar{x})$, and \bar{f} from the direct sum $\bigoplus_{v \in V_K^f} \Gamma_v$, the Boolean value $[\Phi(\bar{f})]$ belongs to $\mathbb{B}_{fin/cofin}$.

Proof. For almost all atoms e, we have $\bar{f}(e) = 0$. Hence for almost all atoms e, the formula $\Phi(\bar{f}(e))$ is $\Phi(0)$, hence is either true or false in \mathbb{Z} .

So we have in effect defined Fin, but no Boolean algebra. However, we can interpret the finite-cofinite algebra $\mathbb{B}_{fin/cofin}$ in the direct sum of the Γ_v .

4. Lemma. The Boolean algebra $\mathbb{B}_{fin/cofin}$ is interpretable in $\bigoplus_{v \in V_K^f} \Gamma_v$.

Proof. We interpret a Boolean algebra \mathbb{B} in $\mathbb{B}_0 := \bigoplus_{v \in V_K^f} \Gamma_v$ as follows. Choose an element $\beta \in \mathbb{B}_0 \setminus \{0\}$ and let

$$\mathbb{B}_{\beta} = \mathbb{B}_0 \times \{0\} \cup \mathbb{B}_0 \times \{\beta\}.$$

usual on \mathbb{B}_0 , and on $\mathbb{B}_0 \times \{\beta\}$ define

$$(x,\beta) \wedge (y,\beta) := (x \vee y,\beta),$$

$$(x,\beta) \lor (y,\beta) := (x \land y,\beta).$$

and

$$(x,0) \wedge (y,\beta) = (x \wedge \overline{y},0),$$

where $x \wedge \overline{y}$ is defined as the supremum of atoms γ such that $\gamma \leq x$ and $\gamma \nleq y$. Put

$$\overline{(x,0)} = (x,\beta),$$

$$\overline{(x,\beta)} = (x,0),$$

and

$$(x,0) \lor (x,\beta) = \neg(\overline{(x,0)} \land \overline{(y,\beta)}) = \overline{((x,\beta) \land (y,0))}.$$

Thus \mathbb{B}_{β} is a Boolean algebra. Clearly, different choices of β give isomorphic Boolean algebras.

Given
$$(x, 0) \in \mathbb{B}_0 \times \{0\}$$
, define $Fin((x, 0)) \Leftrightarrow Fin(x)$, and given $(x, \beta) \in \mathbb{B}_0 \times \{\beta\}$, define $Fin((x, \beta)) \Leftrightarrow \neg Fin(x)$.

In any case, we have shown that the Boolean scaffolding can be removed, up to interpretation, but probably not up to definition, and the lattice-ordered monoid is decidable and has a quantifier-elimination in all the cases.

1. Remark. In the language of Boolean algebras $\mathbb{B}_{fin/cofin}$ is an elementary substructure of Powerset(V_K^f), but not in the Boolean language with a predicate Fin for finite subsets.

Proof. This follows from the quantifier elimination theorem for infinite atomic Boolean algebras in the Boolean language enriched by unary predicates $C_j(x)$ stating the there are at least j distinct atoms below x (cf. [9]) since the Boolean algebras $\mathbb{B}_{fin/cofin}$ and $Powerset(V_K^f)$ have the same atoms.

10. The Basarab sorts and hyperrings

This notion of hyperring was defined by Krasner [18] and used by Connes-Consani [7]. We recall this notion. A set H is called a *canonical hypergroup* (cf. [7]) if there is multivalued addition

$$+: H \rightarrow Powerset(H)$$

(where the variables x, y, z range over elements in H) satisfying the following axioms:

- $(1) \ \forall x \forall y (x + y = y + x),$
- $(2) \forall x \forall y ((x+y) + z = x + (y+z)),$
- (3) $\forall x(0+x=x+0=x),$
- (4) $\forall x \exists ! y (0 \in x + y)$ (y is written as -x),
- (5) $\forall x \forall y \forall z (x \in y + z \Rightarrow z \in x y) (=x + (-y)).$

The operation + is called hyperaddition. The hyperring axioms require in addition that multiplication gives a monoid with multiplicative identity, and we have

$$\forall r \forall s \forall t (r(s+t) = rs + rt),$$
$$\forall r \forall s \forall t (s+t)r = sr + tr,$$
$$0 \neq 1.$$

A hyperfield H is a hyperring such that it's nonzero elements form a group under multiplication.

Let K denote a local field. In [18], Krasner defined a hyperring associated to K. This definition can be slightly generalized as follows.

Let Δ be a subset of Γ with $0 \in \Gamma$, and closed downwards in the sense that $g \leq h$ and $h \in \Delta$ imply $g \in \Delta$. Such a Δ is called here *convex*. Note that if $-g \in \Delta$ then

 $\Delta + g$ is also convex. We denote $\mathcal{M}_{\Delta} = \{x : v(x) > \Delta\}$. This is an ideal in \mathcal{O}_K (since $\infty > \Gamma$). Clearly $1 + \mathcal{M}_{\Delta}$ is a subgroup of U.

Let G_{Δ} be the group $K^*/1 + \mathcal{M}_{\Delta}$ and R_{Δ} the ring $\mathcal{O}/\mathcal{M}_{\Delta}$. Let H_{Δ} be the monoid $K/1 + \mathcal{M}_{\Delta}$ (of orbits for the action of $1 + \mathcal{M}_{\Delta}$ on K). Note that the valuation v is 0 on $1 + \mathcal{M}_{\Delta}$, and so induces "valuations" v from G_{Δ} to Γ , and H_{Δ} to $\Gamma \cup \{\infty\}$. Let

$$P_{\Delta} = \{ x \in H_{\Delta} : v(x) \ge 0 \},$$

and

$$U_{\Delta} = \{ x \in H_{\Delta} : v(x) = 0 \}.$$

Note that $0 \in P_{\Delta}$.

The set H_{Δ} carries the structure of a hyperfield. More generally, by the construction of Krasner [18] (cf. also [7]), given a commutative unital ring R and a subgroup G of its multiplicative group, the set of all orbits of R under G, denoted by R/G carries the structure of a hyperring defined as follows:

- Hyperaddition: $xG + yG = \{(xG + yG)/G\}$ (a a subset of R/G),
- Multiplication:: xG.yG = (xy)G.

In the above we use the standard notations $A + B = \{a + b : a \in A, b \in B\}$, called the sumset of A and B; and $A/G = \{aG : a \in A\}$ for a subset $A \subseteq R$. We are using + for the hyperaddition by slight abuse of language since we use the same notation for the sumset of the G-orbits, but it will hopefully be clear from the context. Hyperaddition is a multi-valued addition.

The axioms for canonical hypergroup are all satisfied in Krasner's construction R/G, with 0 = 0G and -(xG) = (-x)G. For uniqueness in Axiom (4), note that if 0 = a + b, where $a \in xG$, $b \in yG$, then

$$b = -a.a$$

for some $g \in G$, so $b \in (-x)G$, so yG = (-x)G.

The other axioms are verified in [18], with 1 = 1G, provided $0 \neq 1$ in the ring R. Another useful way to think of the hyperaddition (following Krasner [18]) is as follows. Given xG and yG, the sumset xG + yG is a union of cosets and the hyper sum xG + yG is the set of these cosets. So

$$xG+yG=\{zG\in R/G:zG\subseteq xG+yG\},$$

where, by slight abuse of language, the sum on the right hand side is sumset, and on the left hand side is hyperaddition.

Model-theoretically, it is more natural to replace the "hyperoperation" + by Σ , the graph of that operation, namely

$$H \models \Sigma(x, y, z) \Leftrightarrow z \in x + y,$$

and we will often use this version.

We define the language of hyperrings to be the language with a predicate for multiplication, and predicate for Σ , and constants for 0, 1. This is a natural language for hyperrings.

We do not take the time to write out the hyperring axioms in terms of the primitives $\{., 1, 0, \Sigma\}$. This is easily done, and will often be used.

In the model theory of Henselian valued fields K, some important work of Basarab [1] and Kuhlmann [20] is closely related to the construction above. We take R = K, and G to be $1 + \mathcal{M}_{\Delta}$, where Δ is an initial segment of the value group, $0 \in \Delta$, and \mathcal{M}_{Δ} is the ideal of elements of the valuation ring consisting of the x with $v(x) > \Delta$. (We make no further restriction on Δ).

The hyperring

$$(K/1 + \mathcal{M}_{\Delta}, ., 1, 0, \Sigma)$$

is called (by us) the Krasner-Basarab hyperring associated to Δ , and denoted $Kras(\Delta)$. It has some extra structure coming from the valuation on K. Note that $1 + \mathcal{M}_{\Delta}$ is a subgroup of the units of \mathcal{O}_K , and the action of $1 + \mathcal{M}_{\Delta}$ preserves the valuation. Thus the valuation induces a map

$$v: K/1 + \mathcal{M}_{\Delta} \longrightarrow \Gamma \cup \{\infty\}$$

satisfying v(xy) = v(x) + v(y) with usual conventions about $v(x) + \infty$ and $\infty + \infty$. Inside $K/1 + \mathcal{M}_{\Delta}$ we consider $\mathcal{O}_K/1 + \mathcal{M}_{\Delta}$, a hyperring by the same construction. One checks easily that $K/1 + \mathcal{M}_{\Delta}$ is a hyperring extension of $\mathcal{O}_K/1 + \mathcal{M}_{\Delta}$, in the sense of [7], and $K/1 + \mathcal{M}_{\Delta}$ is a hyperfield. We denote by

$$\pi_{\Delta}: K \to K/1 + \mathcal{M}_{\Delta}$$

the canonical projection map.

The surjection $\mathcal{O}_K \to \mathcal{O}_K/1 + \mathcal{M}_\Delta$ clearly respects division. Since

$$v(x) = v(y)$$

holds in \mathcal{O}_K if and only if x and y divide each other, we can define unambiguously $v(x(1+\mathcal{M}_{\Delta}))$ as v(x). Then the relation

$$v(x) \le v(y)$$

on $\mathcal{O}_K/1 + \mathcal{M}_{\Delta}$ is definable by x|y (which denotes x divides y). Also every non-zero element in $K/1 + \mathcal{M}_{\Delta}$ is of the form ab^{-1} , with $a, b \in \mathcal{O}_K/1 + \mathcal{M}_{\Delta}$. We have to check how v relates to the hyperaddition +. In fact it is easily checked that

$$\Sigma(x,y,t) \Rightarrow v(t) \geq \min\{v(x),v(y)\}.$$

In [1] and [20], Basarab and Kuhlmann work with $K^*/1 + \mathcal{M}_{\Delta}$, i.e. a multiplicative group. This is part of the hyperring (in fact hyperfield) H_{Δ} (namely the multiplicative group of its nonzero elements), and is quantifier-free definable in H_{Δ} (since we have a constant for 0).

They also use the (higher residue) rings $\mathcal{O}_K/\mathcal{M}_{\Delta}$, and we show in Section 8 that this is actually interpretable in $K/1 + \mathcal{M}_{\Delta}$, using the primitives $\{., \Sigma \text{ and } P_{\Delta}\}$ for all valued fields (and without P_{Δ} for all Henselian valued fields with finite or pseudofinite residue field). The definitions are uniform across all the stated fields and all Δ .

Note that on the sort $K^*/1 + \mathcal{M}_{\Delta}$, with $v : K^*/1 + \mathcal{M}_{\Delta} \to \Gamma$, the extra structure of hyperring on $K^*/1 + \mathcal{M}_{\Delta}$ is given by the 3-place relation which is the image of

the graph of addition on $(K^*)^3$. Taking K_v to be the family of completions K_v of a number field F under a non-archimedean absolute value v, we have the maps of products

$$\prod_{v \in V_F^f} K_v^* \to \prod_{v \in V_K^f} K_v^* / 1 + \mathcal{M}_\Delta \to \prod_{v \in V_F^f} \Gamma,$$

giving rise to several restricted products of fields and hyperfields, where the value monoid in the restricted product is what we considered in Sections 7 and 9, which will be studied in Section 14.

11. Uniform definition of valuation on the hyperrings

In this section, we will show that P_{Δ} is definable in H_{Δ} uniformly for all Henselian valued fields K with finite or pseudofinite residue field, for any convex subset Δ of Γ containing 0. The definition is an adaptation to the hyperfield situation of the definition given in [5] of \mathcal{O}_K in K uniformly for all K satisfying the above conditions. We use the notation of [5].

Let $P_2^{AS}(x)$ be the formula $\exists y(x=y^2+y)$. Let $T^+(x)$ be the formula

$$x \neq 0 \land \neg P_2^{AS}(x) \land \neg P_2^{AS}(x^{-1}).$$

Let $P_2^{AS,Kras}(x)$ be the "hyperversion" of $P_2^{AS}(x)$, namely,

$$\exists y \ \Sigma(y^2, y, x).$$

Let $T^{+,Kras}(x)$ be

$$x \neq 0 \land \neg P_2^{AS,Kras}(x) \land \neg P_2^{AS,Kras}(x^{-1}).$$

We need to review the use of T^+ in giving a uniform definition of \mathcal{O}_K in K, for K Henselian with k finite or pseudofinite (an assumption we now make, certainly true in all nonarchimedean completions of number fields).

We consider $T^+(K)$ and $T^+(k)$ the sets defined in K, resp. k, by the formula $T^+(x)$.

5. **Lemma.** • $T^+(K)$ is a subset of the units \mathcal{O}_K^{\times} , • If $v(\alpha) = 0$ and $res(\alpha) \in T^+(k)$, then $\alpha \in T^+(K)$.

Proof. Follows from [5, Lemmas 2 and 3].

We note the for k pseudofinite, $T^+(k)$ is infinite (cf. [5]). Much deeper is the following.

5. **Theorem.** There exists an integer N > 0 such that if k has cardinal at least N, then

$$\mathcal{O}_K = \{a + b + cd : a, b, c, d \in T^+(K)\}.$$

This is used to obtain the following comprehensive result:

6. **Theorem.** There exists an integer l > 0 such that for all K as above

$$\mathcal{O}_K = \{0, 1\} + \{a + b + cd : a, b, c, d \in T^+(K)\}$$

$$\cup \{x : \exists y (T^+(y) \land T^+(x^l - 1 + y))\},\$$

where A + B denotes the sumset of two sets A and B.

Proof. The result follows from the proof of [5, Theorem 2].

Now we take this definition and find a "hyperversion".

6. Lemma. Let $x \in K$. Suppose $T^{+,Kras}(x(1+\mathcal{M}))$ holds in H_{Δ} . Then $x \in T^{+}(K)$.

Proof. Obviously $x \neq 0$. If $P_2^{AS}(x)$ holds in K, then for some y in K

$$x = y^2 + y.$$

But then, taking $w = y(1 + \mathcal{M})$

$$H_{\Delta} \models \Sigma(w^2, w, x(1 + \mathcal{M}))$$

contradicting

$$H_{\Delta} \models T^{+,Kras}(x(1+\mathcal{M})).$$

So

$$K \models \neg P_2^{AS}(x).$$

Similarly

$$K \models \neg P_2^{AS}(x^{-1}).$$

7. **Lemma.** Let K be a valued field with residue characteristic different from 2. Let $x \in K$ be an element of value 0. Then x is a square in K if and only $x(1 + \mathcal{M}_{\Delta})$ is a square in $K/1 + \mathcal{M}_{\Delta}$.

Proof. We only have to show the right to left direction. Suppose that $x(1 + \mathcal{M}_{\Delta})$ is a square in $K/1 + \mathcal{M}_{\Delta}$. Then for some $y \in K$,

$$x(1+\mathcal{M}_{\Delta})=y^2(1+\mathcal{M}_{\Delta}).$$

Hence $x - y^2 \in \mathcal{M}_{\Delta}$. Let $f(y) := x - y^2$. Then f'(y) = 2y. Note that v(y) = 0 (since $v(x - y^2) > 0$ and v(x) = 0). Thus

$$v(f'(y)) = v(2y) = 0.$$

Applying Hensel's Lemma we deduce that x is a square in K.

8. Lemma. Suppose $x \in K$ and $x \in T^+(K)$. Then $T^{+,Kras}(x(1+\mathcal{M}))$ holds in H_{Δ} .

Proof. The argument is divided into two cases of whether the residue characteristic is 2 or not.

Case 1: k has characteristic 2.

By Lemma 5,
$$v(x) = 0$$
. So $x(1 + \mathcal{M}_{\Delta}) \neq 0$. Suppose

$$P_2^{AS,Kras}(x(1+\mathcal{M}_{\Delta}))$$

holds in H_{Δ} . Then for some y,

(11.0.2)
$$\Sigma(y^2(1+\mathcal{M}_{\Delta}), y(1+\mathcal{M}_{\Delta}), x(1+\mathcal{M}_{\Delta}))$$

holds in H_{Δ} . Then $y \neq 0$, and for some σ, τ in K with

$$y^{2}(1 + \mathcal{M}_{\Delta}) = \sigma(1 + \mathcal{M}_{\Delta}),$$
$$y(1 + \mathcal{M}_{\Delta}) = \tau(1 + \mathcal{M}_{\Delta}),$$

we have

$$(\sigma + \tau)(1 + \mathcal{M}_{\Delta}) = x(1 + \mathcal{M}_{\Delta}).$$

Note that if one of σ , τ has negative valuation, then so has y and then

$$v(\sigma) \neq v(\tau)$$

and v(x) < 0, a contradiction. So each of y, σ , and τ has non-negative valuation. But if one has positive valuation, then all have, so

$$v(\sigma + \tau) > 0$$

while v(x) = 0. So we conclude that

$$v(y) = v(\sigma) = v(\tau) = 0.$$

From 11.0.2 we have that

$$x \in x(1 + \mathcal{M}_{\Delta}) \subseteq y^2(1 + \mathcal{M}_{\Delta}) + y(1 + \mathcal{M}_{\Delta}),$$

hence

$$x = y^2 + y^2\lambda + y + y\rho,$$

for elements $\lambda, \ \rho \in \mathcal{M}_{\Delta}$. Thus

$$v(y^2 + y - x) > \Delta.$$

Let $f(y) := y^2 + y - x$. So $f(y) \in \mathcal{M}$. But

$$v(f'(y)) = v(2y+1),$$

and $2y \in \mathcal{M}$, hence $2y + 1 \notin \mathcal{M}$ and v(f'(y)) = 0. By Hensel's Lemma, we get $P_2^{AS}(x)$. But $x \in T^+(K)$, contradiction. So

$$\neg P_2^{AS,Kras}(x(1+\mathcal{M}_\Delta))$$

holds in H_{Δ} . Similarly

$$\neg P_2^{AS,Kras}(x^{-1}(1+\mathcal{M}_\Delta))$$

holds in H_{Δ} . So

$$T^{+,Kras}(x(1+\mathcal{M}_{\Delta}))$$

holds in H_{Δ} . This completes the proof in Case 1.

Case 2: k has characteristic different from 2.

In this case it is easy to see that the condition $P_2^{AS}(x)$ is equivalent to the condition $P_2(1+4x)$ in both K and in $K/1 + \mathcal{M}_{\Delta}$.

As in Case 1 we know that v(x) = 0, $x(1 + \mathcal{M}) \neq 0$, and we assume that

$$P_2^{AS,Kras}(x(1+\mathcal{M}))$$

holds in H_{Δ} . Thus K satisfies

$$P_2(1+4(x(1+\mathcal{M}_{\Delta}))).$$

Applying Lemma 7 we deduce that $P_2(1+4x)$ holds in K. Hence $P_2^{AS}(x)$ holds in K. The proof is now completed as in Case 1.

Note that Lemmas 6 and 8 show for $x \in K$ that

$$K \models T^+(x) \Leftrightarrow H_{\Delta} \models T^{+,Kras}(x(1+\mathcal{M}_{\Delta})).$$

To complete our work, it is convenient to introduce in the hyperrings the definable predicate $\Sigma_3(x, y, z, t)$, defined as

$$\exists w (\Sigma(x, y, w) \land \Sigma(w, z, t)).$$

Now fix l as in Theorem 6. Define $\Theta_1(X)$ as

$$\exists A, B, C, D[T^{+,Kras}(A) \wedge T^{+,Kras}(B) \wedge T^{+,Kras}(C) \\ \wedge T^{+,Kras}(D) \wedge \Sigma_3(A, B, CD, X)]$$

and $\Theta_2(X)$ as

$$\exists Y \exists W (T^{+,Kras}(Y) \wedge T^{+,Kras}(W) \wedge \Sigma_3(X^l, -1, Y, W)).$$

Now define $\Theta^{Kras}(X)$ as

$$\Theta_1(X) \vee \Theta_2(X) \vee \exists S(\Theta_2(S) \wedge \Sigma_3(X, -1, S)).$$

Then we have:

7. **Theorem.** Uniformly for all Henselian valued fields K with finite or pseudofinite residue field we have,

$$X \in P_{\Delta} \Leftrightarrow H_{\Delta} \models \Theta^{Kras}(X).$$

Proof. Suppose first $X \in P_{\Delta}$, and let $X = x(1 + \mathcal{M}_{\Delta})$. Then $v(x) \geq 0$. So by Theorem 6

$$K \models \exists a, b, c, d \in T^{+}(K)(x = a + b + cd) \lor \exists y (T^{+}(y) \land T^{+}(x^{l} - 1 + y)) \lor \exists z (x = z + 1 \land \exists w (T^{+}(w) \land T^{+}(z^{l} - 1 + w)).$$

So,

$$H_{\Delta} \models \Theta^{Kras}(x(1 + \mathcal{M}_{\Delta})).$$

Conversely, suppose

$$H_{\Delta} \models \Theta^{Kras}(x(1 + \mathcal{M}_{\Delta}).$$

This condition is a disjunction of three clauses and we examine each separately.

1. Claim. $H_{\Delta} \models \Theta_1(X) \Rightarrow X \in P_{\Delta}$.

Proof. Assume that

$$H_{\Delta} \models \exists A, B, C, D[T^{+,Kras}(A) \land T^{+,Kras}(B) \land T^{+,Kras}(C) \land T^{+,Kras}(D) \land \Sigma_3(A, B, CD, X)].$$

Choose

$$A = a(1 + \mathcal{M}_{\Delta}),$$

$$B = b(1 + \mathcal{M}_{\Delta}),$$

$$C = c(1 + \mathcal{M}_{\Delta}),$$

and

$$D = d(1 + \mathcal{M}_{\Delta}),$$

where $a, b, c, d \in K$, to witness the quantifiers. By Lemma 6

$$K \models T^+(a) \wedge T^+(b) \wedge T^+(c) \wedge T^+(d).$$

The meaning of $\Sigma_3(A, B, CD, X)$ is that there are $\alpha, \beta, \lambda, \lambda, x, \mu$ in K such that

$$a(1 + \mathcal{M}_{\Delta}) = \alpha(1 + \mathcal{M}_{\Delta}),$$

$$b(1 + \mathcal{M}_{\Delta}) = \beta(1 + \mathcal{M}_{\Delta}),$$

$$(\alpha + \beta)(1 + \mathcal{M}_{\Delta}) = \lambda(1 + \mathcal{M}_{\Delta}) = \lambda'(1 + \mathcal{M}_{\Delta}),$$

$$(cd)(1 + \mathcal{M}_{\Delta}) = \mu(1 + \mathcal{M}_{\Delta}),$$

$$(\lambda' + \mu)(1 + \mathcal{M}_{\Delta}) = x(1 + \mathcal{M}_{\Delta}).$$

Since

$$v(a) = v(b) = v(c) = v(d) = v(\mu) = 0,$$

(by Lemma 5), also

$$v(\alpha) = v(\beta) = v(cd) = v(\mu) = 0,$$

and

$$v(\lambda) = v(\alpha + \beta) \ge 0,$$

and

$$v(\lambda') \ge 0$$
,

so

$$v(x) = v(\lambda' + \mu) \ge 0.$$

So
$$X \in P_{\Delta}$$
.

2. Claim. $H_{\Delta} \models \Theta_2(X) \Rightarrow X \in P_{\Delta}$.

Proof. Assume that

$$H_{\Delta} \models (\exists Y)(\exists W)(T^{+,Kras}(Y) \land T^{+,Kras}(W) \land \Sigma_3(X^l, -1, Y, W)).$$

Choose $Y = y(1 + \mathcal{M}_{\Delta})$ and $W = w(1 + \mathcal{M}_{\Delta})$, (where $y, w \in K$), to witness the quantifiers. By Lemma 6,

$$K \models T^+(y) \land T^+(w).$$

The meaning of $\Sigma_3(X^l, -1, Y, W)$ is that there are $x', \theta, y', \rho, \rho'$ in K such that

$$x'(1 + \mathcal{M}_{\Delta}) = x^{l}(1 + \mathcal{M}_{\Delta}),$$

$$\theta(1 + \mathcal{M}_{\Delta}) = (-1)(1 + \mathcal{M}_{\Delta}),$$

$$(x' + \theta)(1 + \mathcal{M}_{\Delta}) = \rho(1 + \mathcal{M}_{\Delta}) = \rho'(1 + \mathcal{M}_{\Delta}),$$

$$y'(1 + \mathcal{M}_{\Delta}) = y(1 + \mathcal{M}_{\Delta}),$$

$$(\rho' + y')(1 + \mathcal{M}_{\Delta}) = w(1 + \mathcal{M}_{\Delta}).$$

Thus (by Lemma 5)

$$v(y) = v(y') = v(w) = 0.$$

Obviously $v(\theta) = 0$. So

$$v(\rho' + \theta) = 0,$$

Hence $v(\rho') \geq 0$. Thus $v(\rho') \geq 0$. So

$$v(x' + \theta) \ge 0.$$

So $v(x') \ge 0$. Since

$$v(x') = lv(x),$$

we deduce that $v(x) \geq 0$. Therefore $X \in P_{\Delta}$, completing the proof of the claim. \square

To prove the theorem, suppose that

$$H_{\Lambda} \models \Theta^{Kras}(X).$$

Then either $\Theta_1(X)$ or $\Theta_2(X)$ holds, in which case we deduce from Claims 1 and 2 that $X \in P_{\Delta}$; or there exists S such that both $\Theta_2(S)$ and $\Sigma_3(X, -1, S)$ hold. Choose $s \in K$ with

$$S = s(1 + \mathcal{M}_{\Delta}).$$

By Claim 2, $v(s) \ge 0$.

Since $\Sigma_3(X, -1, S)$, there is $e, f \in K$ such that

$$e(1 + \mathcal{M}_{\Delta}) = x(1 + \mathcal{M}_{\Delta}),$$

$$f(1 + \mathcal{M}_{\Delta}) = (-1)(1 + \mathcal{M}_{\Delta}),$$

and

$$(e+f)(1+\mathcal{M}_{\Delta})=s(1+\mathcal{M}_{\Delta}).$$

Thus v(f) = 0 and thus $v(e) \ge 0$. But v(e) = v(x), and we deduce $X \in P_{\Delta}$. This proves the Theorem.

12. Uniform interpretation of the higher residue rings

In this section we show that the higher residue ring R_{Δ} is interpretable in the hyperring H_{Δ} uniformly for all valued fields and all Δ if we have a predicate for the valuation ring of H_{Δ} . We deduce that for all valued fields with finite or pseudofinite residue field, R_{Δ} is uniformly interpretable in H_{Δ} , uniformly in Δ .

We start with the following.

9. **Lemma.** There is a well-defined surjective set map $\Psi_{\Delta}: P_{\Delta} \to R_{\Delta}$ with

$$\Psi_{\Delta}(a(1+\mathcal{M}_{\Delta})) = a + \mathcal{M}_{\Delta},$$

for every $a \in \mathcal{O}_K$.

Proof. If

$$a(1 + \mathcal{M}_{\Delta}) = b(1 + \mathcal{M}_{\Delta}),$$

then

$$ab^{-1} \in 1 + \mathcal{M}_{\Delta},$$

so since $v(b) \geq 0$, we have

$$a \in b + \mathcal{M}_{\Delta}$$
.

3. Note. Ψ_{Δ} sends $0 \in P_{\Delta}$ (which is $0(1 + \mathcal{M}_{\Delta})$) to \mathcal{M}_{Δ} .

We need to understand the fibers $\Psi_{\Delta}^{-1}(a + \mathcal{M}_{\Delta})$, where $a \in \mathcal{O}_K$.

10. **Lemma.** Suppose $v(a) \in \Delta$ and $a \in \mathcal{O}_K$. Then the fiber $\Psi_{\Delta}^{-1}(a + \mathcal{M}_{\Delta})$ is naturally isomorphic to

$$\{a\} \times (1 + \mathcal{M}_{I-v(a)})/1 + \mathcal{M}_{\Delta}.$$

In particular, it has cardinal 1 if and only if I - v(a) = I which is true if v(a) = 0, i.e. $a \in U$ and $a(1 + \mathcal{M}_{\Delta}) \in U_{\Delta}$.

 ${\it Proof.} \ {\rm Suppose}$

$$b(1 + \mathcal{M}_{\Delta}) \in \Psi_{\Delta}^{-1}(a + \mathcal{M}_{\Delta}).$$

Then

$$b-a \in \mathcal{M}_{\Delta}$$
,

so v(a) = v(b), so

$$b = \theta a, \ v(\theta) = 0.$$

Also, $a(1-\theta) \in \mathcal{M}_{\Delta}$, so

$$v(1-\theta) + v(a) > \Delta,$$

SO

$$v(1-\theta) > \Delta - v(a)$$
.

Note that $0 \in \Delta - v(a)$. We have

$$\theta \in 1 + \mathcal{M}_{\Delta - v(a)}$$
.

Now

$$b(1+\mathcal{M}_{\Delta})=a(1+\mathcal{M}_{\Delta})$$

if and only if

$$\theta \in 1 + \mathcal{M}_{\Delta}$$
.

The proof is complete.

11. **Lemma.** Suppose $v(a) \notin \Delta$, i.e. $v(a) > \Delta$, and $a \in \mathcal{O}_K$. Then

$$\Psi_{\Delta}^{-1}(a) = \Psi_{\Delta}^{-1}(0) = \{ \gamma : \gamma > \Delta \} \times U/1 + \mathcal{M}_{\Delta},$$

which is infinite if $\Delta \neq \Gamma$.

Proof. We have

$$a + \mathcal{M}_{\Delta} = 0 \in R_{\Delta}.$$

Now suppose that

$$\Psi_{\Delta}(b(1+\mathcal{M}_{\Delta}))=0.$$

Then $v(b) > \Delta$. So

$$\Psi_{\Lambda}^{-1}(0) = \{ g \in K/1 + \mathcal{M}_K : v(g) > \Delta \}.$$

Suppose $v(a), v(b) > \Delta$ and

$$ab^{-1} \in 1 + \mathcal{M}_{\Delta},$$

then v(a) = v(b), as in Lemma 10,

$$b = \theta a$$

with $v(\theta) = 0$. Now

$$\theta \in 1 + \mathcal{M}_{\Lambda}$$
.

and we are done.

12. **Lemma.** Let A be the graph of addition on R_{Δ} . Then

$$P_{\Delta}^3 \cap \Sigma = \Psi_{\Delta}^{-1}(A).$$

Proof. Consider elements $a + \mathcal{M}_{\Delta}$, $b + \mathcal{M}_{\Delta} \in R_{\Delta}$. Then by Lemma 10 we have

$$\Psi_{\Delta}^{-1}(a+\mathcal{M}_{\Delta})=(a+\epsilon)(1+\mathcal{M}_{\Delta}),$$

$$\Psi_{\Delta}^{-1}(b+\mathcal{M}_{\Delta})=(b+\tau)(1+\mathcal{M}_{\Delta}),$$

where

$$v(\epsilon) > \Delta - v(a),$$

and

$$v(\tau) > \Delta - v(b)$$
.

Denoting by + the hyperaddition in the Krasner construction, we have that

$$(a+\epsilon)(1+\mathcal{M}_{\Lambda})+(b+\tau)(1+\mathcal{M}_{\Lambda})=$$

$$\{((a+\epsilon)(1+m_{\Delta})+(b+\tau)(1+m'_{\Delta})): m_{\Delta} \in \mathcal{M}_{\Delta}, m'_{\Delta} \in \mathcal{M}_{\Delta}\}.$$

Now it is immediate that

$$\Psi_{\Delta}(((a+\epsilon)(1+m_{\Delta})+(b+\tau)(1+m_{\Delta}'))(1+\mathcal{M}_{\Delta}))=a+b+\mathcal{M}_{\Delta},$$

which completes the proof of the lemma.

8. **Theorem.** R_{Δ} is interpretable in $(H_{\Delta}, ., 0, 1, P_{\Delta})$ (in any language where we have a predicate for P_{Δ}) uniformly for all valued fields and all Δ .

Proof. Define an equivalence relation E on H_{Δ} by E(g,h) if and only if

$$\Psi_{\Delta}(g) = \Psi_{\Delta}(h),$$

where $g, h \in P_{\Delta}$, with one extra class for $H_{\Delta} \setminus P_{\Delta}$. We first show that E is definable in the group H_{Δ} .

3. Claim. $\Psi_{\Delta}^{-1}(0)$ is definable.

Proof. If $g \in \Psi_{\Delta}^{-1}(0)$, then

$$g = \hat{g}(1 + \mathcal{M}_{\Delta}),$$

where $\hat{g} \in \mathcal{M}_{\Delta}$. We claim that

$$H_{\Delta} \models \Sigma(1, g, 1).$$

Indeed, this holds if and only if there are $\alpha, \beta \in K$ with

$$\alpha(1 + \mathcal{M}_{\Delta}) = 1(1 + \mathcal{M}_{\Delta}),$$

$$\beta(1+\mathcal{M}_{\Delta})=\hat{g}(1+\mathcal{M}_{\Delta}),$$

and

$$(\alpha + \beta)(1 + \mathcal{M}_{\Delta}) = 1(1 + \mathcal{M}_{\Delta}).$$

To satisfy this we take $\alpha = 1$, $\beta = \hat{g}$, and we are done since

$$(\alpha + \beta)(1 + \mathcal{M}_{\Delta}) = (1 + \hat{g})(1 + \mathcal{M}_{\Delta}) = 1(1 + \mathcal{M}_{\Delta}).$$

Conversely, suppose that

$$H_{\Delta} \models \Sigma(1, g, 1).$$

Then choosing $g = \hat{g}(1 + \mathcal{M}_K)$ where $\hat{g} \in K$, we have

$$1(1+\mathcal{M}_K) \subseteq \hat{g}(1+\mathcal{M}_K) + 1(1+\mathcal{M}_K).$$

Thus for any $\rho \in \mathcal{M}_K$ there are $\lambda, \tau \in \mathcal{M}_K$ such that

$$1 + \rho = 1 + \tau + \hat{g} + \hat{g}\lambda.$$

Choose such a ρ and get such λ and τ . We deduce that

$$\hat{g}(1-\lambda) = \rho - \tau \in \mathcal{M}_K.$$

Hence, $\hat{g} \in \mathcal{M}_K$, and

$$\Psi_{\Delta}(g) = 0.$$

4. Claim. For any $g, h \in H_{\Delta}$ we have

$$E(g,h) \Leftrightarrow H_{\Delta} \models \Sigma(g,\Psi_{\Delta}^{-1}(0),h).$$

Proof. Clear.

Now we interpret R_{Δ} setwise as the equivalence classes for E, and we have also given an interpretation of 0 and 1.

Denote the E-class of an element g by g_E . We define addition and multiplication on the classes by

$$g_E + h_E = j_E$$

where j is such that

$$H_{\Delta} \models \Sigma(q, h, j),$$

and

$$g_E.h_E = j_E,$$

where

$$j = qh$$
.

It is easy to see that addition is well-defined.

To show that multiplication is well-defined consider

$$a + \mathcal{M}_{\Delta} \in R_{\Delta}$$
,

and

$$b + \mathcal{M}_{\Delta} \in R_{\Delta}$$
.

We may assume that $v(a) \in \Delta$ and $v(b) \in \Delta$ otherwise we get the zero element after multiplying. Consider arbitrary elements

$$a\theta(1+\mathcal{M}_{\Delta})\in\Psi^{-1}(a+\mathcal{M}_{\Delta}),$$

and

$$b\psi(1+\mathcal{M}_{\Delta})\in\Psi^{-1}(b+\mathcal{M}_{\Delta}),$$

where θ and ψ have value zero (cf. Lemma 10). Since

$$\Psi_{\Delta}(a\theta(1+\mathcal{M}_{\Delta})b\psi(1+\mathcal{M}_{\Delta})) = ab + \mathcal{M}_{\Delta},$$

we need to show that

$$ab\theta\psi + \mathcal{M}_{\Lambda} = ab + \mathcal{M}_{\Lambda}.$$

By proof of Lemma 10,

$$\theta = 1 + \epsilon$$
,

where $v(\epsilon) > \Delta - v(a)$, and

$$\psi = 1 + \delta$$
.

where $v(\delta) > \Delta - v(b)$. So

$$\theta\psi = 1 + \epsilon + \delta + \epsilon\delta$$

where $v(\epsilon\delta) > \Delta - (v(a) + v(b))$. Thus

$$ab(1-\theta\psi)\in\mathcal{M}_{\Delta}$$

hence

$$1 - \theta \psi \in \mathcal{M}_{\Delta - (v(a) + v(b))},$$

as required.

We have thus defined a ring structure on H_{Δ}/E isomorphic, under the map Ψ_{Δ} , to R_{Δ} .

9. **Theorem.** The higher residue ring R_{Δ} is interpretable in the hyperring H_{Δ} uniformly for all Henselian valued fields with finite or pseudofinite residue field and uniformly in Δ

Proof. Use Theorem 7 together with Theorem 8.

13. A VARIANT OF THE BASARAB-KUHLMANN QUANTIFIER ELIMINATION

In the preceding discussion, for a general K, there may be many Δ to consider. There is, however, one family Δ_n of particular significance, defined for all K.

To define Δ_n , let p be the *characteristic exponent* of K, so p is the characteristic of the residue field k if k has prime characteristic, and 1 if k has characteristic 0.

Take $\Delta_{p,n}$ as $\{g: g \leq nv(p)\}$, where $n \geq 0$ and p is the characteristic exponent of K. Clearly $\Delta_{p,n}$ is convex. Note that $\mathcal{M}_{\Delta_{p,0}}$ is the maximal ideal of \mathcal{O}_K . If K has residue characteristic zero, we put $\Delta_n = \Delta_{1,n}$.

13. **Lemma.** If K has characteristic exponent 1, $\Delta_n = \{g : g \leq 0\} = \Delta_0$, for all n. *Proof.* Trivial.

Now we define the principal Krasner-Basarab sort as the 2-sorted structure

$$(K, H_{\Delta_0}, \pi_0),$$

where the sort K has the language of rings, the sort H_{Δ_0} has the structure of hyperrings, (cf. Section 10), and there is a symbol for the natural connecting map

$$\pi_0: K \to H_{\Delta_0}.$$

We denote this structure by $Kras_{\Delta_0}^B(K)$.

If K has residue characteristic 0, this is the only sorting we need. However, if K has residue characteristic p > 0 we need to consider the other convex sets $\Delta_{p,n}$, for $n \geq 0, p \geq 1$. In this case, we define the Krasner-Basarab p-sorting as the structure

$$(K, H_{\Delta_{p,n}}, \pi_{p,n})$$

with the language of rings for the field sort K, the language of hyperrings for the sorts $H_{\Delta_{p,n}}$, and symbols for the canonical maps

$$\pi_{n,p}:K\to H_{\Delta_{p,n}}.$$

We denote this structure by $Kras_{\Delta_{p,n}}^{B}(K)$.

Note that $\Delta_{p,m} \subseteq \Delta_{p,m+1}$, and we have natural (commuting) maps

$$K \longrightarrow H_{\Delta_{p,m+1}} \longrightarrow H_{\Delta_{p,m}}.$$

Note that $Kras_{\Delta_{1,n}}^{B} = Kras_{\Delta_{0}}^{B}$, (the principal sort).

We define the Krasner-Basarab language to be the many-sorted language consisting of the language of rings for the field sort, the language of hyperrings for the sorts $H_{\Delta_{p,n}}$, and function symbols for the connecting maps $\pi_{p,n}$ between the two sorts, for all $n \geq 0, p \geq 1$.

The theorem to be stated below, combining results of Basarab [1] and Kuhlmann [20] is one of the most comprehensive and important in the model theory of Henselian fields. We do not present the most general version.

Let K be a valued field K. Given $n \geq 0$, we denote

$$G_K^n = K^*/1 + \mathcal{M}_{K,n},$$

and

$$\mathcal{O}_K^n = \mathcal{O}_K/\mathcal{M}_{K,n},$$

where

$$\mathcal{M}_{K,n} = \{ a \in \mathcal{O}_K : v(a) > nv(p) \}.$$

We denote the corresponding canonical projection maps by

$$\pi_n: \mathcal{O}_K \to \mathcal{O}_K^n$$
,

and

$$\pi_n^*: K^* \to G_K^n.$$

Let $\Theta_n \subseteq \mathcal{O}_K^n \times \mathcal{G}_K^n$ be the relation defined by

$$\Theta_n(x,y) \Leftrightarrow \exists z \in \mathcal{O}_K(\pi_n(z) = x \land \pi_n^*(z) = y).$$

Given a valued K, the Basarab-Kuhlmann language for K is the many-sorted language with sorts:

$$\mathcal{K}_n := (K, \mathcal{O}_K^n, G_K^n, \pi_n, \pi_n^*, \Theta_n),$$

for all $n \geq 0$, with the language of rings for the field sort K and the higher residue ring sorts \mathcal{O}_K^n , and the language of groups for the sorts G_K^n . This language was defined by Kuhlmann [20] based on the language of Basarab [1].

Given a structure S with many sorts

$$(\sigma_0(S), \sigma_1(S), \sigma_2(S), \ldots),$$

where $\sigma_0(S) = S$ the home sort, and a formula Ψ of the many-sorted language for S (with free variables from the different sorts), we write

$$(S, \sigma_1, \sigma_2, \dots) \models \Psi$$

to indicate that the subformulas in Ψ from the sort σ_j hold in $\sigma_j(S)$, for all $j \geq 0$.

10. **Theorem.** Given a formula $\Phi(\bar{x})$ of the Basarab-Kuhlmann language, there is an integer $\beta(\Phi(\bar{x}))$, a formula $\Psi(\bar{x}, \bar{y})$ (in extra variables from the sorts other than the field), and integers $\gamma(p)$ for every prime $p \leq \beta(\Phi(\bar{x}))$, such that for every \bar{a} from K,

$$K \models \Phi(\bar{a}) \Leftrightarrow \mathcal{K}_0 \models \Psi(\bar{a}, \pi_0(\bar{a}), \pi^*(\bar{a})),$$

if the residue characteristic of K is greater than $\beta(\Phi(\bar{x}))$, and

$$K \models \Phi(\bar{a}) \Leftrightarrow \mathcal{K}_{\gamma(p)} \models \Psi(\bar{a}, \pi_{\gamma(p)}(\bar{a}), \pi^*_{\gamma(p)}(\bar{a})),$$

if the residue characteristic of K is p, for every prime $p \leq \beta(\Phi(\bar{x}))$.

Proof. By Theorem 2.4 in [20] there is a formula $\Psi_1(\bar{x}, \bar{y})$ involving the sorts K, \mathcal{O}_K^0 and G_K^0 (with the appropriate maps) such that if K has residue characteristic zero, then for every \bar{a} from K

$$K \models \Phi(\bar{a}) \Leftrightarrow \mathcal{K}_0 \models \Psi_1(\bar{a}, \pi(\bar{a}), \pi^*(\bar{a})).$$

Using a compactness argument, this holds for K of residue characteristic larger than some positive integer $\beta(\Phi(\bar{x}))$ depending only on $\Phi(\bar{x})$. The case of K with residue characteristic $p \leq \beta(\Phi(\bar{x}))$ follows from Theorem B in [1].

We do not know if there is a uniform quantifier elimination for Henselian fields of residue characteristic p. Theorem 10 does not apply to this case because of the dependency of the ideals $\mathcal{M}_{K,n}$ on v(p).

Theorem 10 holds in a very general setting. We can start with a notion of first-order formula of the language of valued fields which can be many-sorted (e.g. the standard 3-sorted (9) from Section 6, with the field sort most prominent). The only restriction on the other sorts is that they be interpretable in the 3-sorted case, and that the value and residue sorts be interpretable in them. This restriction excludes angular components and cross-sections, but it is easy to prove a version of the theorem taking account of those.

We can formulate the Basarab-Kuhlmann Theorem in the Krasner-Basarab language of hyperrings.

- 11. **Theorem.** There is a computable map, defined on first-order formulas of the language of valued fields, assigning to each $\Phi(\bar{x})$
 - i) an integer $\beta(\Phi(\bar{x}))$,
- ii) a formula $\Phi_0(\bar{x}, \bar{y})$ from the 2-sorted Krasner-Basarab language with field sort and principal sort, and having no bound variables of field sort,
 - iii) for each prime $p \leq \beta(\Phi(\bar{x}))$ an integer $\gamma_p(\Phi(\bar{x}))$ and formulas

$$\Phi_{p,1}(\bar{x},\bar{y}),\ldots,\Phi_{p,r_p}(\bar{x},\bar{y})$$

from the p-sorting, with no bound variables of field sort, and no sorts (p, n) for $n > \gamma_p(\Phi(\bar{x}))$,

such that for all Henselian valued fields K of characteristic 0, if the residue characteristic of K is not a prime at most $\beta(\Phi(\bar{x}))$ then for every \bar{a} from K

$$K \models \Phi(\bar{a}) \Leftrightarrow Kras_{\Delta_0}^B \models \Phi_0(\bar{a}, \pi_0(\bar{a})),$$

and if the residue characteristic of K is a prime $p \leq \beta(\Phi(\bar{x}))$, then for some r for all \bar{a} from K

$$K \models \Phi(\bar{a}) \Leftrightarrow Kras_{\Delta p,r}^B \models \Phi_{p,r}(\bar{a}, \pi_r(\bar{a})).$$

Proof. Combine Theorem 10 with the interpretation of the higher residue rings in Theorem 9. \Box

4. Note. Theorem 11 remains true without the condition on the residue field provided we add a predicate for the valuation ring of the hyperring to the (field sort of the) language of Krasner-Basarab.

- 5. Note. If the prime p is infinitely ramified, our understanding of the $H_{\Delta_{n,p}}$ is very limited. This stands in the way of proving decidability of the class of all finite extensions of \mathbb{Q}_p .
 - The preceding theorem does give much insight into definability in adele rings. It now leads us to look at adelic versions of the Krasner-Basarab hyperfields.

14. RESTRICTED PRODUCTS OF THE HYPERFIELDS

In this section we consider adelic versions of the Krasner-Basarab structures. We need to slightly extend our notations.

As before, K will be a number field, V_K^f the set of normalized non-archimedean valuations of K, and K_v the completion of K at $v \in V_K^f$. We will let S denote a finite subset of V_K^f .

Given K, we can consider several many-sorted restricted products constructed from the Krasner-Basarab structures on $Kras_{\Delta_{p,n}}(K)$ associated to K, for $p \geq 1, n \geq 0$. Given K_v , where $v \in V_K^f$, we will consider $Kras_{\Delta_{p(v),n}}(K_v)$, where p(v) is the residue characteristic of K_v . We will denote

$$\pi_{n,p(v)}: K \to Kras_{\Delta_{n(v),n}}(K_v)$$

the projection map.

The Krasner-Basarab language will give natural languages for these restricted products in the formalism of Section 2. We will be concerned here mainly with the 2-sorted structure consisting of the the restricted product of the K_v with the ring structure in the first sort, the restricted product of the $H_{\Delta_0,v}$ with the hyperring structure in the second sort, and the connecting map between the two sorts, for all $v \in V_K^f \setminus S$. This restricted product will be called the S-adelic principal Krasner-Basarab structure associated to K, and denoted $Kras_{S,0}^{\mathbb{A}}(K)$. Note that in particular, S can be empty, in which case we have the principal adelic Krasner-Basarab structure. In this case, since the connecting map between the sorts is surjective, the finite adeles map onto the f in $\prod_v H_{\Delta_0,v}$ with $[[\neg P_{\Delta}(f)]]$ finite, and the image of \mathbb{A}_K^{fin} is the restricted product of the $H_{\Delta_0,v}$ with respect to P_{Δ} . Note that $H_{\Delta_0,v}$ is relational, except for the monoid operation.

Note that P_{Δ} uniformly defines the valuation on the hyperrings $H_{\Delta_{p(v),n}}$, for all $v \in V_K^f$, and all $n \geq 1$, by Theorem 7. Thus adelic Krasner-Basarab structures can be construed as man-sorted restricted products in the formalism of Section 2. One takes for the first sort a ring formula which uniformly defines the valuation rings of the K_v , and for the second sort a formula of the language of hyperrings which uniformly defines the valuation of the hyperrings associated to K_v , for all $v \in V_K^f$.

One can also include other sorts $H_{\Delta_{n,p}}$ for all the $v \in S$ (where p is the residue characteristic of K_v). We call the resulting restricted product the *adelic-(p, n)* Krasner-Basarab structure. There are certainly several other possibilities of restricted products constructed from the family of K_v and $Kras_{\Delta_{p(v),n}}(K_v)$ for varying or fixed n or p. Note that given $p, n, \Delta_{p,n}$ is uniformly definable for all local fields K

(and even in much more generality, cf. Section 11). For residue characteristic zero, $\Delta_{p,n} = \Delta_0$.

For both the S-principal and the (p,n)-adelic Krasner-Basarab structures, it is easy to interpret the Boolean algebra $\mathbb B$ with Fin as follows. $\mathbb B$ is just the set of idempotents, with order \leq defined using multiplication. The atoms are defined as usual. The stalk at an atom e is naturally identified using the idempotents, (e.g. in the case of S-adelic principal Krasner-Basarab structures the stalk is $eKras_{S,0}^{\mathbb A}(K)$). This allows us to define the Boolean value $[[\Phi(\bar x)]]$, for a formula $\Phi(\bar x)$, as in the basic adelic situation. Finally Fin is defined using the Boolean value [[...]] and the valuation.

So for the adelic Krasner-Basarab structures we have a Feferman-Vaught Theorem, and we can eliminate the Boolean scaffolding with Fin.

12. **Theorem.** Let K be a number field. Given a formula $\Phi(\bar{x})$ of the language of rings (resp. the language of Basarab-Kuhlmann), there is an effectively computable finite set $S = \{v_1, \ldots, v_s\}$ of normalized non-Archimedean absolute values of K such that, for any \bar{a} from \mathbb{A}_K^{fin} , the condition

$$\mathbb{A}_K^{fin} \models \Phi(\bar{a}),$$

is equivalent to a Boolean combination of the following conditions:

• (type I)

$$Kras^{\mathbb{A}}_{S,0}(K) \models Fin([[\Psi(\bar{a}, \pi_{0,v}(\bar{a}))]]),$$

• (type II)

$$Kras_{S,0}^{\mathbb{A}}(K) \models C_k([[\Psi'(\bar{a}, \pi_{0,v}(\bar{a}))]]),$$

• (type III)

$$Kras^B_{\Delta_{p(v_j),m(j)}}(K_{v_j}) \models C_k([[\Psi_j(\bar{a},\pi_{m(j),p(v_j)}(\bar{a}))]]),$$

where $\Psi(\bar{x}, \bar{y}), \Psi'(\bar{x}, \bar{y}), \Psi_1(\bar{x}, \bar{y}), \ldots, \Psi_s(\bar{x}, \bar{y})$ are formulas of the 2-sorted Krasner-Basarab language which are quantifier free in the field sort, $k \geq 1$, $j \in \{1, \ldots, s\}$, $m(j) \geq 1$ (a positive integer depending on v_j), and $p(v_j)$ is the residue characteristic of K_{v_j} . The Boolean operations, Fin, C_k , and the Boolean value are expressible in the language of Basarab-Kuhlmann in each of the sorts.

Proof. By Theorem 2_{sort} , for every \bar{a} from \mathbb{A}_K^{fin} , the condition

$$\mathbb{A}_K^{fin} \models \Phi(\bar{a})$$

is equivalent to a Boolean combination of conditions of the form

$$\mathbb{A}_K^{fin} \models \Theta([[\Xi(\bar{a})]])$$

where Θ is either Fin or C_k , and $\Xi(x)$ is a formula of the language of rings (resp. language of Basarab-Kuhlmann). By Theorem 11, there is a finite set

$$S = \{v_1, \dots, v_s\}$$

of normalized non-Archimedean absolute values of K, positive integers m(j) for $j \in \{1, ..., s\}$, and formulas

$$\Psi(\bar{x},\bar{y}),\Psi_1(\bar{x},\bar{y}),\ldots,\Psi_s(\bar{x},\bar{y})$$

from the 2-sorted Krasner-Basarab language with no quantifiers of the field sort, and involving extra variables \bar{y} from the sorts other than the field sort, such that for every $v \notin S$, and every \bar{a} from K_v

$$K_v \models \Xi(\bar{a}) \Leftrightarrow Kras^B_{\Delta_{0,v}}(K_v) \models \Psi(\bar{a}, \pi_{0,v}(\bar{a})),$$

and for all $j \leq s$, and every \bar{a} from K_{v_i}

$$K_{v_j} \models \Xi(\bar{a}) \Leftrightarrow Kras^B_{\Delta_{p(v_j),m(j)}}(K_{v_j}) \models \Psi_j(\bar{a}, \pi_{m(j),v_j}(\bar{a})).$$

Here $p(v_j)$ is the residue characteristic of K_{v_j} . Note that

$$Fin(\{v : K_v \models \Xi(\bar{x})\}) \Leftrightarrow Fin(\{v \notin S : K_v \models \Xi(\bar{x})\})$$

$$\Leftrightarrow Fin(\{v : Kras_{\Delta_{0,v}}^B(K_v) \models \Psi(\bar{x}, \pi_{0,v}(\bar{x}))\}),$$

since S is finite,

Thus if Θ is Fin, then condition 14.0.3 is equivalent to

$$Kras_{S,0}^{\mathbb{A}} \models Fin([[\Psi(\bar{a}, \pi_{0,v}(\bar{a}))]]).$$

If Θ is C_k , for some $k \geq 1$, then condition 14.0.3 is equivalent to

$$Kras_{S,0}^{\mathbb{A}} \models C_{k}([[\Psi(\bar{a}, \pi_{0,v}(\bar{a}))]])) \lor (Kras_{S,0}^{\mathbb{A}} \models C_{k-1}([[\Psi(\bar{a}, \pi_{0,v}(\bar{a}))]]) \land \\ \bigvee_{1 \leq j \leq s} (Kras_{\Delta_{p(v_{j}),m(j)}}(K_{v_{j}}) \models \Psi_{j}(\bar{a}, \pi_{m(j),p(v_{j})}(\bar{a}))) \lor \\ \vdots \\ \lor (Kras_{S,0}^{\mathbb{A}} \models C_{k-t}([[\Psi(\bar{a}, \pi_{0,v}(\bar{a}))]]) \land \\ \bigvee_{(j_{1},...,j_{t}) \in S^{t}} (Kras_{\Delta_{p(v_{j_{1}}),m(j_{1})}}(K_{v_{j_{1}}}) \models \Psi_{j_{1}}(\bar{a}, \pi_{m(j_{1}),p(v_{j_{1}})}(\bar{a})) \\ \land \cdots \land Kras_{\Delta_{p(v_{j_{t}}),m(j_{t})}}(K_{v_{j_{t}}}) \models \Psi_{j_{t}}(\bar{a}, \pi_{m(j_{t}),p(v_{j_{t}})}(\bar{a}))) \lor \\ \vdots \\ \lor \bigvee_{(j_{1},...,j_{k}) \in S^{k}} (Kras_{\Delta_{p(v_{j_{1}}),m(j_{1})}}(K_{v_{j_{1}}}) \models \Psi_{j_{1}}(\bar{a}, \pi_{m(j_{1}),p(v_{j_{1}})}(\bar{a})) \land \cdots \land \\ Kras_{\Delta_{p(v_{j_{t}}),m(j_{k})}}(K_{v_{j_{k}}}) \models \Psi_{j_{k}}(\bar{a}, \pi_{m(j_{k}),p(v_{j_{k}})}(\bar{a})))$$

This yields (II). Taking negations, we deduce (III) for the case Θ is $\neg C_k$, for some k. The last statement concerning the definitions of Fin, Boolean structure, and Boolean value in each sort of the Krasner-Basarab language follow from the remarks before the theorem.

15. Stable embedding

It has been interesting and important in recent years, in the general area of valued fields, to analyze *stable embeddings and interpretations* (cf. [15]). For example for Henselian fields coming under the Ax-Kochen-Ershov analysis, one knows that parametrically definable subsets of the value group, using the three sorted language in (9) of Section 6, are already parametrically definable in the value group (cf. [15]).

We show now that the local fields K_v are stably embedded in the adeles \mathbb{A}_K (via the identification of K_v with $e_{\{v\}}K_v$).

13. **Theorem.** Let X be a definable subset of \mathbb{A}^n_K , where $n \geq 1$. Let e be a minimal idempotent. Then $X \cap (e\mathbb{A}_K)^n$ is definable in $e\mathbb{A}_K$.

Proof. By Corollary 4, X is defined by a Boolean combination of formulas of the type:

$$Fin([[\Psi(x_1,...,x_n,\bar{a})]]),$$

 $C_i([[\Phi(x_1,...,x_n,\bar{a})]]),$

where $j \geq 1$, $\bar{a} \in \mathbb{A}_K^m$, and Ψ and Φ are from the ring language. Recall that

$$[[\Phi(x_1,\ldots,x_n,\bar{a})]]=1$$

is equivalent to $\neg C_1([[\neg \Phi(x_1, \dots, x_n, \bar{a})]]).$

Let $b_1, \ldots, b_n \in e\mathbb{A}_K$. Let $v \in V_K$ be the normalized valuation that corresponds to e. Then $[[\Phi(b_1, \ldots, b_n, \bar{a})]]$ is equal to

$$\{w \neq v : K_w \models \Phi(0, \dots, 0, \bar{a}(w))\} \cup \{v\}$$

if

$$K_v \models \Phi(b_1(v), \dots, b_n(v), \bar{a}(v)),$$

and to

$$\{w \neq v : K_w \models \Phi(0, \dots, 0, \bar{a}(w))\}$$

if

$$K_v \models \neg \Phi(b_1, \dots, b_n, \bar{a}(v))\}.$$

Now let

$$f_1, \ldots, f_n \in X \cap (e\mathbb{A}_K)^n$$
.

We have two cases to consider:

The case of Fin:

In this case we have

$$\mathbb{A}_K \models Fin([[\Phi(f_1,\ldots,f_n,\bar{a})]]) \Leftrightarrow \mathbb{A}_K \models Fin(\Phi(0,\ldots,0,\bar{a})]).$$

There are two sub-cases.

(i)
$$\mathbb{A}_K \models Fin([[\Phi(0,\ldots,0,\bar{a})]])$$
. In this case

$$X \cap (e\mathbb{A}_K)^n = (e\mathbb{A}_K)^n$$

(ii)
$$\mathbb{A}_K \models \neg Fin([[\Phi(0,\ldots,0,\bar{a})]])$$
. In this case, $X \cap (e\mathbb{A}_K)^n = \emptyset$.

The case of C_i :

Note that

$$\mathbb{A}_K \models C_i([[\Phi(f_1,\ldots,f_n,\bar{a})]])$$

if and only if either

$$\mathbb{A}_K \models (C_{j-1}([[\Phi(f_1,\ldots,f_n,\bar{a})]]) \land \neg C_j([[\Phi(0,\ldots,0,\bar{a})]])$$

and

$$e\mathbb{A}_K \models \Phi(f_1(v), \dots, f_n(v), \bar{a}(v)),$$

or

$$\mathbb{A}_K \models C_i([[\Phi(0,\ldots,0,\bar{a})]])$$

(and in this case there is no condition on $e\mathbb{A}_K$).

In the first case

$$X \cap (e\mathbb{A}_K)^n = \{(g_1, \dots, g_n) \in (e\mathbb{A}_K)^n : \Phi(g_1, \dots, g_n, \bar{a})\},\$$

and in the second case

$$X \cap (e\mathbb{A}_K)^n = (e\mathbb{A}_K)^n.$$

In all the cases, it is thus clear that $X \cap (e\mathbb{A}_K)^n$ is definable with parameters from $e\mathbb{A}_K$.

6. Note. Although we do not take the time to state a general result here, it is clear that one has for generalized products a very general stable embedding theorem for factors.

Recall from Section 7 that we have the product valuation $\prod v$ from \mathbb{A}_K^{fin} to the restricted product Γ of the lattice-ordered monoids $\mathbb{Z} \cup \{\infty\}$. Evidently Γ is interpretable in the ring \mathbb{A}_K^{fin} .

14. **Theorem.** The value monoid Γ of $\mathbb{A}^{fin}_{\mathbb{Q}}$ is not stably interpreted (via the valuation map).

Proof. We can define in $\mathbb{A}_{\mathbb{Q}}$ the set X of idempotents whose support is the set of minimal idempotents e whose corresponding prime p is congruent to 1 modulo 4. Indeed, let Ψ be a sentence that holds in \mathbb{Q}_p for exactly the primes p with $p \equiv 1 \pmod{4}$, and let Ψ' be a sentence that holds in all non-Archimedean local fields and fails in all the Archimedean local fields. Then

$$X = \{x \in \mathbb{A}_{\mathbb{Q}} : supp(x) = [[\Psi \wedge \Psi']]\}.$$

The image of X under the product valuation $\prod v$ is the set Y of all g in $\prod_p(\mathbb{Z} \cup \{\infty\})$ which are 0 at p and ∞ elsewhere. It is an easy exercise using the Feferman-Vaught Theorem (or Theorem 2_{sort}) applied to Γ to show that this set is not definable

in the value monoid, using an appropriate modification of the Pressburger elimination in the factors. \Box

16. A REMARK ABOUT NTP_2

The property of not having the tree property of the second kind NTP_2 is a generalization of the properties of simple and NIP (the negation of the independence property). It is known that ultraproducts of \mathbb{Q}_p and certain valued difference fields have NTP_2 (cf. [4]).

Fix a number field K. The theory of \mathbb{A}_K has the independence property in two different ways. The first is via the residue fields (appealing to Duret [11], also cf. [14]), and the other comes from the definable Boolean algebra \mathbb{B}_K . Now the problem with the residue fields does not extend to the property NTP_2 . However, it turns out that we still have the following.

15. **Theorem.** The theory of finite adeles \mathbb{A}_K^{fin} and the theory of adeles \mathbb{A}_K do not have the property NTP_2 .

Proof. To show the negation of NTP_2 we have to produce a formula $\Psi(x,y)$, and an array

$$a_{11}, a_{12}, \dots$$

 a_{21}, a_{22}, \dots
:

so that for fixed j,

$$\{\Psi(x, a_{jk}) : k \ge 1\}$$

is inconsistent, but for each $f: \mathbb{N} \longrightarrow \mathbb{N}$,

$$\{\Psi(x, a_{if(j)}) : j \ge 1\}$$

is consistent. (x and y can be tuples).

Firstly, put $a_{jk} = a_{j1}^k$, $k \ge 1$. Secondly, for each j pick a minimal idempotent e_j so that $e_j \mathbb{A}_K^{fin}$ is the K_v , $v \in V_K^f$, which has residue field of characteristic p_j , the jth prime. Finally, pick a_{j1} to be an atom such that the coordinate at which it is nonzero lies in the maximal ideal of the valuation ring of $e_j \mathbb{A}_K$.

Now take the formula $\Psi(x,y)$ to be

$$C_1([[y \neq 0]]) \land \neg C_2([[y \neq 0]]) \land [[y \neq 0]] = [[\rho(y)]] \leq [[\sigma(x,y)]],$$

where $\rho(y)$ is a formula of the language of rings which is equivalent in all the K_v to the statement that y has positive valuation, and $\sigma(x,y)$ is a formula of the language of rings which is equivalent in all the K_v to the statement that x and y have the same valuation. Note that such formulas exist by the results in [5] on uniform definability of the valuation for all K_v in the ring language. Here v ranges over all non-archimedean valuations of K. Note that the first two conjuncts from the left state that $[[y \neq 0]]$ is a minimal idempotent, and the formula states that the support of y is minimal and the nonzero coordinate of y lies in the maximal ideal of the valuation ring, and x and y have the same valuation at that coordinate.

Now it is clear that

$$\{\Psi(x, a_{ik}) : k \ge 1\}$$

is inconsistent, since for $k_1 \neq k_2$

$$v(e_j a_{jk_1}) \neq v(e_j(a_{jk_2}))$$

for v the (normalized) valuation of $e_j \mathbb{A}_K^{fin}$ (since $k_1 v(e_j(a_{j1})) \neq k_2 v(e_j(a_{j1}))$). However, for any $f: \mathbb{N} \longrightarrow \mathbb{N}$

$$\{\Psi(x, a_{j,f(j)}) : j \ge 1\}$$

is consistent by choosing a finite adele $A \in \mathbb{A}_K^{fin}$ such that its coordinate A(j) in $e_j \mathbb{A}_K^{fin}$ satisfies

$$v(A(j)) = f(j)v(a_{j,1}),$$

for all j. Note that there is such an element in \mathbb{A}_K^{fin} .

This proves that \mathbb{A}_K^{fin} does not have the property NTP_2 . To deduce that the adeles \mathbb{A}_K does not have NTP_2 , it suffices to show that \mathbb{A}_K^{fin} is definable in \mathbb{A}_K (in the language of rings). To see this, take a sentence Θ which holds in all archimedean completions K_v of K but is not true in all the non-archimedean completions. Then \mathbb{A}_K^{fin} can be defined as the set of all $f \in \mathbb{A}_K$ such that f(v) = 0 for all $v \in [[\Theta]]$. \square

7. Note. The formula uniformly defining the valuation of all the local fields from [5] is existential-universal (this is shown in [5] to be optimal, i.e. there is no uniform universal or existential definition). It follows that the formula $\Psi(x,y)$ in the proof of Theorem 15 is universal-existential-universal.

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University of Oxford, Mathematical Institute, 24-29 St Giles', Oxford OX1 3LB, UK

E-mail address: derakhsh@maths.ox.ac.uk

QUEEN MARY, UNIVERSITY OF LONDON, SCHOOL OF MATHEMATICAL SCIENCES, QUEEN MARY, UNIVERSITY OF LONDON, MILE END ROAD, LONDON E1 4NS, UK

E-mail address: angus@eecs.qmul.ac.uk