Forcing lightface definable well-orders without the GCH

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Abstract

For any given uncountable cardinal κ with $\kappa^{<\kappa} = \kappa$, we present a forcing that is $<\kappa$-directed closed, has the κ^+-cc and introduces a lightface definable well-order of $H(\kappa^+)$. We use this to define a global iteration that adds such a well-order for all such κ simultaneously and is capable of preserving the existence of many large cardinals in the universe.

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1 Introduction

If κ is an infinite cardinal, a lightface definable wellorder of $H(\kappa^+)$ is a well-order of $H(\kappa^+)$ that is definable over $\langle H(\kappa^+), \in\rangle$ without parameters. In [2] and [3], Sy Friedman and the first author show that given any uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$ (note that this implies that κ is regular) and $2^\kappa = \kappa^+$, there is a $<\kappa$-directed closed, κ^+-cc partial order of size 2^κ which yields a lightface definable well-order of $H(\kappa^+)^{V[G]}$ whenever G is generic for that forcing. They use this to define a class sized iteration which, assuming the GCH, introduces a lightface definable well-order of $H(\kappa^+)$ for every uncountable cardinal κ, preserving the GCH and all cofinalities, and show that whenever κ is λ-supercompact for λ regular, then the λ-supercompactness of κ is preserved by the iteration. Moreover they show that introducing those well-orders by a variant of the above class sized iteration also allows for preserving many instances of n-hugeness.

We generalize those results to a non-GCH context as follows. First we show that even if $2^\kappa > \kappa^+$, there is a very nice forcing to introduce a lightface definable well-order of $H(\kappa^+)$. The key new ingredient will be a new coding forcing (that we call *Club Coding*) which will be introduced in Section 3.

Theorem 1.1. *Suppose κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. Then there is a partial order $\mathbb{Q}$ with the following properties.*

1. *$\mathbb{Q}$ has a $<\kappa$-directed closed dense subset.*
2. *$\mathbb{Q}$ has the κ^+-chain condition.*

3. $\mathcal{Q} \subseteq H(\kappa^+)$

4. *$\mathcal{Q}$ forces the existence of a lightface definable wellorder of $H(\kappa^+)$.*

Using the properties of those *single-step forcings*, it is straightforward (see Section 6) to iterate this for all uncountable cardinals κ that satisfy $\kappa^{<\kappa} = \kappa$ which are not *counterexamples to the* SCH and obtain the following.

Theorem 1.2. *There is a* ZFC*-preserving class forcing $\mathbb{P}$ so that*

- *$\mathbb{P}$ preserves cofinality κ whenever κ is not a* counterexample to the SCH, *which means there is no singular strong limit cardinal λ with $\lambda^+ < \kappa \leq 2^\lambda$.*
- *$\mathbb{P}$ preserves the value of 2^κ whenever neither κ nor 2^κ are counterexamples to the* SCH*.*
- *$\mathbb{P}$ introduces a lightface definable well-order of $H(\kappa^+)$ whenever $\kappa \geq \omega_1$ is such that $\kappa^{<\kappa} = \kappa$ and κ is not a counterexample to the* SCH*.*

Under the assumption of SCH, the above turns into the following, much nicer form.

Theorem 1.3. *Assume* SCH*. There is a* ZFC*-preserving class forcing $\mathbb{P}$ so that*

- *$\mathbb{P}$ preserves cofinalities and the continuum function (i.e. the value of 2^α for every α).*
- *$\mathbb{P}$ introduces a lightface definable well-order of $H(\kappa^+)$ whenever $\kappa \geq \omega_1$ is such that $\kappa^{<\kappa} = \kappa$.*

The role of the SCH in the above is very similar to the situation in [5]. We refer the reader to the first chapter of that paper (or also to [6]) for a more detailed discussion.

In Section 7, we will show that forcing with $\mathbb{P}$ allows for various forms of large cardinal preservation. Supercompactness preservation seems to be a difficult issue in a non-GCH setting and we only obtain a partial result (originating from [2]) that relies on instances of the GCH to hold. Using *sparser* iterations, one may use supercompactness preservation arguments for a non-GCH context that were developed in [6] and given a simplified presentation in a somewhat different context in [5]. We give a sample result of this in Section 8. Back in Section 7, we also present stronger results on large cardinal preservation for other types of large cardinals: hyperstrong and n-superstrong cardinals for $2 \leq n \leq \omega$.

2 More on related Results

In this short section, we want to comment on the results of this paper and their relationship with other recent results on introducing locally lightface definable well-orders by forcing. In [5], Sy Friedman together with the second and third author provides a class sized iteration that introduces a lightface definable well-order of $H(\kappa^+)$ whenever κ is inaccessible (see Section 8 of the present article for the exact statement of their theorem). It is fairly simple to introduce a lightface definable well-order of $H(\kappa^+)$ for a single inaccessible cardinal κ, so that paper is mainly concerned with finding a sufficiently uniform way of building a class sized forcing that does this for all inaccessibles κ and allows for large cardinal preservation. In the present article, we improve on this by providing a much more *well-behaved* forcing to introduce a lightface definable well-order of $H(\kappa^+)$ that works both for (suitable) successor cardinals and for inaccessibles. This actually allows us to give (as a sample result) a different proof (which we will only hint towards) of the main result of [5] in Section 8.

In [8], the second and third author show that given an uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$, $\lambda^{<\lambda} < \kappa$ for any $\lambda < \kappa$ and 2^κ regular, under an additional *anti-large cardinal hypothesis*[1] it is possible to introduce a Σ_1-definable well-order of $H(\kappa^+)$ that only uses κ as parameter (and is thus Σ_3-definable over $H(\kappa^+)$ without parameters) by $<\kappa$-closed forcing that preserves all cofinalities $\leq 2^\kappa$ and the value of 2^κ. In particular, this shows that it is consistent to have 2^κ large while having a lightface definable well-order of $H(\kappa^+)$. While the complexity of the well-orders introduced by the forcing provided in the present article is certainly higher than Σ_3, our forcings are cofinality-preserving, they work for a larger class of cardinals and they lend themselves well to large cardinal preservation (see Section 7). Most importantly however, we do not need to assume any kind of anti-large cardinal hypothesis to hold.

3 Club Coding (relative to a stationary set)

In this section we will introduce a coding forcing that could be seen as combining ideas from Solovay's *almost disjoint coding* ([10]) and the *canonical function coding* introduced by Sy Friedman and the first author in [2] and [3]. Although we will never explicitly make use of this coding forcing, it will be woven into our main forcing construction in Section 5 and later proofs will be variations of the arguments given in this section. Moreover the forcing itself might prove to be interesting (in fact it has already been made use of in [7] and [8]). We will call the coding we want to introduce *club coding (relative to a stationary set S)*. Given an uncountable cardinal κ that satisfies $\kappa^{<\kappa} = \kappa$, we will present a notion of forcing with nice properties that will allow us to make a subset of $H(\kappa^+)$ definable by a generically added subset of κ. Under the above assumptions on κ, both the almost disjoint coding forcing at κ and canonical function coding at κ are capable of making a subset of $H(\kappa^+)$ definable by a generically added subset of κ, however canonical function coding requires the additional assumption that $2^\kappa = \kappa^+$ and almost disjoint coding does not possess the crucial property (for our present purposes) that we will verify for club coding in Lemma 3.7 (see the paragraph following its proof).

Throughout this section we fix a regular uncountable cardinal κ with $\kappa = \kappa^{<\kappa}$, a stationary set $S \subseteq \kappa \cap \operatorname{cof}(\omega)$, and a non-empty subset A of ${}^\kappa\kappa$. We will first recall the definition of almost disjoint coding at κ (see [8] for a more detailed account and a collection of its basic properties).

Definition 3.1. Assume that $\vec{s} = \langle s_\alpha \mid \alpha < \kappa \rangle$ is an enumeration of ${}^{<\kappa}\kappa$ with the property that every element of ${}^{<\kappa}\kappa$ is enumerated κ-many times. We define a partial order $\mathbb{Q}(A)$ by the following clauses.

[1] Namely that fat stationary subsets of κ in $\mathbf{L}$ remain fat stationary in $\mathbf{V}$.

- A condition in $\mathbb{Q}(A)$ is a pair $p = \langle t_p, a_p\rangle$ with $t_p\colon \alpha_p \to 2$ for some $\alpha_p < \kappa$ and $a_p \in [A]^{<\kappa}$.
- We have $q \leq_{\mathbb{Q}(A)} p$ if and only if $t_p \subseteq t_q$, $a_p \subseteq a_q$ and

$$s_\beta \subseteq x \;\longrightarrow\; t(\beta) = 0$$

  for every $x \in a_p$ and $\alpha_p \leq \beta < \alpha_q$.

Now we want to introduce the definition of club coding (relative to a stationary set S). The important differences when compared to the almost disjoint coding forcing are that the enumeration of ${}^{<\kappa}\kappa$ is added generically and (and that's the main point) whenever $x \in A$, this is reflected correctly only on a club (relative to S) and not (as is the case with the almost disjoint coding) on a final segment of the generically added coding subset of κ (if G is generic for either the almost disjoint coding forcing $\mathbb{Q}(A)$ or the club coding forcing $\mathbb{Q}^*(A,S)$, this coding subset of κ is equal to $\bigcup_{p\in G} t_p$).

Definition 3.2. We define $\mathbb{Q}^*(A,S)$ to be the partial order whose conditions are tuples

$$p = \langle s_p, t_p, \langle c_x^p \mid x \in a_p\rangle\rangle$$

such that the following statements hold for some successor ordinal $\gamma_p < \kappa$.

- $s_p : \gamma_p \longrightarrow {}^{<\kappa}\kappa$, $t_p : \gamma_p \longrightarrow 2$ and $a_p \in [A]^{<\kappa}$.
- If $x \in a_p$, then c_x^p is a closed subset of γ_p and

$$s_p(\alpha) \subseteq x \;\longrightarrow\; t_p(\alpha) = 0$$

  for all $\alpha \in c_x^p \cap S$.

We define $q \leq p$ to hold if $s_p = s_q \restriction \gamma_p$, $t_p = t_q \restriction \gamma_p$, $a_p \subseteq a_q$ and $c_x^p = c_x^q \cap \gamma_p$ for every $x \in a_p$.

Lemma 3.3. *The partial order $\mathbb{Q}^*(A,S)$ is $<\kappa$-closed, κ^+-Knaster and has cardinality at most 2^κ.*

Proof. Let $\lambda < \kappa$ and $\langle p_\alpha \mid \alpha < \lambda\rangle$ be a descending sequence in $\mathbb{Q}^*(A,S)$. If there is an $\bar{\alpha} < \lambda$ with $\gamma_{p_\alpha} = \gamma_{p_{\bar{\alpha}}}$ for all $\bar{\alpha} \leq \alpha < \lambda$, then

$$p \;=\; \left\langle s_{p_{\bar{\alpha}}}, t_{p_{\bar{\alpha}}}, \langle \bigcup_{x\in a_{p_\alpha}} c_x^{p_\alpha} \mid x \in \bigcup_{\alpha<\lambda} a_{p_\alpha}\rangle \right\rangle$$

is a condition in $\mathbb{Q}^*(A,S)$ with $p \leq p_\alpha$ for all $\alpha < \lambda$. Now define $\gamma = \sup_{\alpha<\lambda} \gamma_{p_\alpha}$ and assume that $\gamma > \gamma_{p_\alpha}$ for all $\alpha < \lambda$. Define

- $s \;=\; \{(\gamma, \emptyset)\} \;\cup\; \bigcup\{s_{p_\alpha} \mid \alpha < \lambda\}$.
- $t \;=\; \{(\gamma, 0)\} \;\cup\; \bigcup\{t_{p_\alpha} \mid \alpha < \lambda\}$.
- $a \;=\; \bigcup\{a_{p_\alpha} \mid \alpha < \lambda\}$.
- $c_x \;=\; \{\gamma\} \;\cup\; \bigcup\{c_x^{p_\alpha} \mid \alpha < \lambda,\ x \in a_{p_\alpha}\}$ for all $x \in a$.
- $p \;=\; \langle s, t, \langle c_x \mid x \in a\rangle\rangle$.

Then p is a condition in $\mathbb{Q}^*(A,S)$ with $p \leq p_\alpha$ for all $\alpha < \lambda$.

To show that $\mathbb{Q}^*(A,S)$ is κ^+-Knaster, let $\langle p_\alpha \mid \alpha < \kappa^+\rangle$ be an injective sequence of conditions in $\mathbb{Q}^*(A,S)$. Then there is an $X \in [\kappa^+]^{\kappa^+}$ and an $r \in [A]^{<\kappa}$ such that $s_{p_\alpha} = s_{p_{\bar{\alpha}}}$, $t_{p_\alpha} = t_{p_{\bar{\alpha}}}$, $r = a_{p_\alpha} \cap a_{p_{\bar{\alpha}}}$ and $c_x^{p_\alpha} = c_x^{p_{\bar{\alpha}}}$ for all $\alpha, \bar{\alpha} \in X$ with $\alpha \neq \bar{\alpha}$ and $x \in r$. Given $\alpha, \bar{\alpha} \in X$ the tuple

$$\langle s_{p_\alpha}, t_{p_\alpha}, \langle c_x^{p_\alpha} \mid x \in a_x^{p_\alpha}\rangle \ \cup \ \langle c_x^{p_{\bar{\alpha}}} \mid x \in a_x^{p_{\bar{\alpha}}}\rangle\rangle$$

is a condition in $\mathbb{Q}^*(A,S)$ that extends both p_α and $p_{\bar{\alpha}}$.

The last claim of the lemma follows from a simple counting argument. $\square$

It follows from Lemma 3.3 that forcing with $\mathbb{Q}^*(A,S)$ preserves cofinalities as well as the stationarity of S.

Proposition 3.4. *If $x \in A$ and $\alpha < \kappa$, then the set*

$$D_{x,\alpha} = \{p \in \mathbb{Q}^*(A,S) \mid x \in a_p, \ c_x^p \setminus \alpha \neq \emptyset\}$$

is dense in $\mathbb{Q}^*(A,S)$.

Proof. Pick a condition p in $\mathbb{Q}^*(A,S)$. We may assume $x \in a_p$, because otherwise we work with the condition

$$p' \ = \ \langle s_p, t_p, \langle c_y^p \mid y \in a_p\rangle \cup \{(x,\emptyset)\}\rangle$$

Pick $\gamma > \gamma_p$ and define

- $s \ = \ s_p \ \cup \ \{(\beta,\emptyset) \mid \gamma_p \leq \beta \leq \gamma\}$.
- $t \ = \ t_p \ \cup \ \{(\beta,0) \mid \gamma_p \leq \beta \leq \gamma\}$.
- $p_* \ = \ \langle s, t, \langle c_y^p \cup [\gamma_p, \gamma] \mid y \in a_p\rangle\rangle$.

Then p_* is a condition in $D_{x,\alpha}$ with $p_* \leq p$. $\square$

Let $\dot{s}$ and $\dot{t}$ denote the canonical $\mathbb{Q}^*(A,S)$-names such that

$$\dot{s}^G \ = \ \bigcup\{s_p \mid p \in G\}$$

and

$$\dot{t}^G \ = \ \bigcup\{t_p \mid p \in G\}$$

whenever G is $\mathbb{Q}^*(A,S)$-generic over $\mathbf{V}$.

Theorem 3.5. *If G is $\mathbb{Q}^*(A,S)$-generic over $\mathbf{V}$, then $\dot{s}^G : \kappa \longrightarrow {}^{<\kappa}\kappa$, $\dot{t}^G : \kappa \longrightarrow 2$ and A is equal to the set of all $x \in ({}^\kappa\kappa)^{\mathbf{V}[G]}$ with the property that*

$$\forall \alpha \in C \cap S \ [\dot{s}^G(\alpha) \subseteq x \ \longrightarrow \ \dot{t}^G(\alpha) = 0] \tag{1}$$

holds for some club subset C of κ in $\mathbf{V}[G]$.

Proof. The first two statements follow directly from the above proposition. Pick $x \in A$ and define $C = \bigcup\{c_x^p \mid p \in G,\ x \in a_p\}$. Then the definition of $\mathbb{Q}^*(A, S)$ implies that C is a closed subset of κ that satisfies (1) and the above proposition shows that C is unbounded in κ.

Now work in the ground model $\mathbf{V}$, pick a $\mathbb{Q}^*(A, S)$-name $\dot{y}$ for an element of ${}^\kappa\kappa$ and a $\mathbb{Q}^*(A, S)$-name $\dot{C}$ for a club subset of κ and assume, towards a contradiction, that there is a condition p_0 in $\mathbb{Q}^*(A, S)$ with

$$p_0 \Vdash \dot{y} \notin \check{A} \ \wedge\ \forall \alpha \in \dot{C} \cap S\ [\dot{s}(\alpha) \subseteq \dot{y} \ \longrightarrow\ \dot{t}(\alpha) = 0]. \tag{2}$$

Let N be a countable elementary substructure of some large enough $H(\theta)$ containing $\mathbb{Q}^*(A, S)$, $\dot{y}$, $\dot{C}$ and p_0 and such that $\gamma := N \cap \kappa \in S$. Let $\langle p_n \mid n < \omega\rangle$ be a descending $\langle N, \mathbb{Q}^*(A, S)\rangle$-generic sequence of conditions extending p_0. By the above proposition together with the genericity of $\langle p_n \mid n < \omega\rangle$,

(i) $\sup_n \gamma_{p_n} = \gamma$, and

(ii) there is some $u : \gamma \longrightarrow \kappa$ such that for every $n < \omega$ there is some $m \geq n$ such that p_m forces
  - $\dot{y}\restriction\gamma_{p_n} = u\restriction\gamma_{p_n}$ and such that
  - $x\restriction\gamma_{p_n} \neq u\restriction\gamma_{p_n}$ for all $x \in a_{p_n}$.

Now we define

- $s\ =\ \{(\gamma, u)\}\ \cup\ \bigcup\{s_{p_n} \mid n < \omega\}$.
- $t\ =\ \{(\gamma, 1)\}\ \cup\ \bigcup\{t_{p_n} \mid n < \omega\}$.
- $a\ =\ \bigcup\{a_{p_n} \mid n < \omega\}$.
- $c_x\ =\ \{\gamma\}\ \cup\ \bigcup\{c_x^{p_n} \mid n < \omega,\ x \in a_{p_n}\}$ for all $x \in a$.

Then the tuple $p = \langle s, t, \langle c_x \mid x \in a\rangle\rangle$ is a condition in $\mathbb{Q}^*(A, S)$, because $u \nsubseteq x$ for all $x \in a$. But $p \leq p_0$ and

$$p \Vdash \check{\gamma} \in \dot{C}\ \wedge\ \dot{s}(\check{\gamma}) \subseteq \dot{y}\ \wedge\ \dot{t}(\check{\gamma}) = 1,$$

contradicting (2). □

Remark 3.6. (i) The above theorem shows that the set A is definable over the structure $\langle H(\kappa^+)^{\mathbf{V}[G]}, \in\rangle$ by a Σ_1-formula with parameters S, $\dot{s}^G$ and $\dot{t}^G$ whenever G is $\mathbb{Q}^*(A, S)$-generic over the ground model $\mathbf{V}$.

(ii) A small variation of the above proof shows that this coding has nice persistence properties - the set A is still defined by the formula (1) after further forcing with a σ-strategically closed partial order that preserves the regularity of κ. The next lemma however provides the crucial, for our present purposes, property that club coding (relative to S) satisfies in contrast to the classical almost disjoint coding.

Lemma 3.7. *If $A_0 \subseteq A$, then $\mathbb{Q}^*(A_0, S)$ is a complete subforcing of $\mathbb{Q}^*(A, S)$.*

Proof. By the definition of $\mathbb{Q}^*(A,S)$, it suffices to show that every maximal antichain $\mathcal{A}$ in $\mathbb{Q}^*(A_0,S)$ is predense in $\mathbb{Q}^*(A,S)$. Then $\mathcal{A}$ is a a maximal antichain in $\mathbb{Q}^*(A,S)$. Pick a condition p in $\mathbb{Q}^*(A,S)$ and define

$$\bar{p} \;=\; \langle s_p, t_p, \langle c_x^p \mid x \in a_p \cap A_0 \rangle\rangle.$$

Then $\bar{p}$ is a condition in $\mathbb{Q}^*(A_0,S)$ and there are conditions q and r in $\mathbb{Q}^*(A_0,S)$ such that $q \in \mathcal{A}$ and r is a common extension of $\bar{p}$ and q in $\mathbb{Q}^*(A_0,S)$. Define

$$p_* \;=\; \langle s_r, t_r, \langle c_x^r \mid x \in a_r \rangle \cup \langle c_x^p \mid x \in a_p \setminus A_0 \rangle\rangle.$$

Then p_* is a condition in $\mathbb{Q}^*(A,S)$ and it is a common extension of p and q in $\mathbb{Q}^*(A,S)$. □

Remark 3.8. It is easy to see that the almost disjoint coding forcing does not possess the property stated in the above lemma: Assume, towards a contradiction, that $\mathbb{Q}(A)$ is a complete subforcing of $\mathbb{Q}({}^\kappa\kappa)$ for every $A \subseteq {}^\kappa\kappa$ and let G be $\mathbb{Q}({}^\kappa\kappa)$-generic over $\mathbf{V}$. For each $A \subseteq {}^\kappa\kappa$, the generic filter in $\mathbb{Q}(A)$ induced by G yields a function $t_A \in {}^\kappa 2$ coding A. It is easy to see that the resulting function $[A \mapsto t_A]$ is an injection of $\mathcal{P}({}^\kappa\kappa)^{\mathbf{V}}$ into $({}^\kappa 2)^{\mathbf{V}[G]}$ that is definable in $\mathbf{V}[G]$. Since forcing with $\mathbb{Q}(A)$ preserves cardinalities and the value of 2^κ, this yields a contradiction. This consideration does not apply if we work with $\mathbb{Q}^*(A,S)$ instead: Suppose $A_0 \subseteq A$, G is generic for $\mathbb{Q}^*(A,S)$, and G' is the restriction of G to $\mathbb{Q}^*(A_0,S)$. Then G' adds $t_{A_0} \in {}^\kappa 2$ coding A_0 in $\mathbf{V}[G']$. However, the same code t_{A_0} will code A in $\mathbf{V}[G]$. The reason is that in moving from $\mathbf{V}[G']$ to $\mathbf{V}[G]$ we are adding new club subsets of κ that ensure this to be the case.

4 More Preliminaries

Let $\prec\cdot,\cdot\succ\colon \mathrm{Ord}\times\mathrm{Ord} \to \mathrm{Ord}$ denote Gödel's pairing function.[2] We also let $\prec\cdot,\cdot,\cdot\succ\colon \mathrm{Ord}^3 \to \mathrm{Ord}$ be $\prec\cdot,\prec\cdot,\cdot\succ\succ$.

It will be convenient to define the following notion of rank of an ordinal with respect to a set of ordinals and the corresponding notion of perfect ordinal (see for example [2] or [3]).

Definition 4.1. Let X be a set of ordinals and let η, μ be ordinals. We define the relation $\mathrm{rank}_X(\eta) \geq \mu$ by recursion as follows:

- $\mathrm{rank}_X(\eta) > 0$ if and only if there is a nonempty $X' \subseteq X$ such that $\sup(X') = \eta$.
- If $\mu > 0$, then $\mathrm{rank}_X(\eta) > \mu$ if and only if η is a limit of ordinals ξ such that $\mathrm{rank}_X(\xi) \geq \mu$.

We say that an ordinal η is *perfect* if and only if $\mathrm{rank}_\eta(\eta) = \eta$.

Note that the first nonzero perfect ordinal is $\epsilon_0 = sup\{\omega, \omega^\omega, \omega^{(\omega^\omega)}, \ldots\}$. Note also that $\mathrm{rank}_\delta(\delta) \leq \delta$ for every ordinal δ and that, given any uncountable cardinal λ, the set of perfect ordinals below λ forms a club subset of λ of order

[2] That is, $\prec\alpha,\beta\succ$ is the order type of $\{\langle\gamma,\delta\rangle \in \mathrm{Ord}\times\mathrm{Ord} \mid \langle\gamma,\delta\rangle < \langle\alpha,\beta\rangle\}$, where $\langle\gamma,\delta\rangle < \langle\alpha,\beta\rangle$ if and only if either $max\{\gamma,\delta\} < max\{\alpha,\beta\}$, or $max\{\gamma,\delta\} = max\{\alpha,\beta\}$ and $\gamma < \alpha$, or $max\{\gamma,\delta\} = max\{\alpha,\beta\}$, $\gamma = \alpha$ and $\delta < \beta$ (see for example [9], p. 30).

type λ. Let $(\eta_\xi)_{\xi\in\mathrm{Ord}}$ be the strictly increasing enumeration of all nonzero perfect ordinals of cofinality ω.

The notions defined in the following two paragraphs appear in [2] and [3].

Given two sets of ordinals X and Y, let $X \cap^* Y$ be the collection of all $\delta \in X \cap Y$ such that δ is not a limit point of X.[3] A sequence $\vec{C} = \langle C_\delta \mid \delta \in \mathrm{dom}(\vec{C})\rangle$ is a club-sequence if $\mathrm{dom}(\vec{C})$ is a set of ordinals and C_δ is a club subset of δ for each $\delta \in \mathrm{dom}(\vec{C})$. We will say that $\vec{C}$ is *coherent* if there is a club-sequence $\vec{D} = \langle D_\delta \mid \delta \in \mathrm{dom}(\vec{D})\rangle$ such that

- $\vec{C} \subseteq \vec{D}$ and
- for every $\delta \in \mathrm{dom}(\vec{D})$ and every limit point γ of D_δ, $\gamma \in \mathrm{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$.

If $\vec{C} = \langle C_\delta \mid \delta \in \mathrm{dom}(\vec{C})\rangle$ is a club-sequence, we denote $\bigcup_{\delta\in\mathrm{dom}(\vec{C})} C_\delta$ by $\mathrm{range}(\vec{C})$. Also, we will say that an ordinal τ is the *height of* $\vec{C}$, and will write $\mathrm{ht}(\vec{C}) = \tau$, if $\mathrm{ot}(C_\delta) = \tau$ for all $\delta \in \mathrm{dom}(\vec{C})$.[4] A club-sequence is called a ladder system if it has height ω. We will say that a club-sequence $\vec{C} = \langle C_\delta \mid \delta \in \mathrm{dom}(\vec{C})\rangle$ with stationary domain such that $\sup(\mathrm{dom}(\vec{C})) = \chi$ is *strongly type-guessing* if for every club subset $C \subseteq \chi$ there is a club $D \subseteq \chi$ such that $\mathrm{ot}(C_\delta \cap^* C) = \mathrm{ot}(C_\delta)$ for every $\delta \in \mathrm{dom}(\vec{C}) \cap D$.

The following related form of club-guessing will also be used:[5] A ladder system $\langle C_\delta \mid \delta \in S\rangle$, where S is a stationary subset of some κ, is strongly guessing if for every club $C \subseteq \kappa$ there is a club $D \subseteq \kappa$ such that $C_\delta \setminus C$ is bounded in δ for every $\delta \in D \cap S$.

The following notion of closure for partial orders will be useful: Let $\mathbb{P}$ be a partial order, κ a cardinal, $\beta_\mathbb{P} : \mathbb{P} \longrightarrow \mathrm{Ord}$ a function, and S a set of ordinals. We will say that a partial order $\mathbb{P}$ is *uniformly $<\kappa$-closed relative to $\beta_\mathbb{P}$ outside S* if for every cardinal $\theta > |\mathbb{P}|$, $\theta > \sup(\mathrm{range}(\beta_\mathbb{P}))$, and every well-order Δ of $H(\theta)$ there is a function $F : {}^{<\kappa}\mathbb{P} \longrightarrow \mathbb{P}$, F definable over the structure $\langle H(\theta), \in, \Delta, \mathbb{P}, \beta_\mathbb{P}\rangle$ without parameters, such that for every $\lambda < \kappa$ and every decreasing sequence $\langle p_i \mid i < \lambda\rangle$ of conditions in $\mathbb{P}$, if $\sup(\beta_\mathbb{P}``\{p_i \mid i < \lambda\}) \notin S$, then $F(\langle p_i \mid i < \lambda\rangle)$ is a condition in $\mathbb{P}$ extending all p_i. If $S = \emptyset$, then we will simply say that $\mathbb{P}$ is *uniformly $<\kappa$-closed*.

Finally, it will also be convenient to define the following notion of hereditary internal approachability (see [2]). Let θ be an infinite cardinal and Δ a well-order of $H(\theta)$. Given $x \in H(\theta)$ we define, by recursion on the cardinals less than θ, the notion of being a *hereditarily internally approachable* (HIA) elementary substructure of $\langle H(\theta), \in, \Delta\rangle$ containing x as follows: A structure $N \prec \langle H(\theta), \in, \Delta\rangle$ such that $x \in N$ is HIA if $N = \bigcup_{i<\mathsf{cf}(|N|)} N_i$ for a $\subseteq$-continuous $\in$-chain $\langle N_i \mid i < \mathsf{cf}(|N|)\rangle$ of sets of size less than $|N|$ such that N_i is an HIA elementary substructure of $\langle H(\theta), \in, \Delta\rangle$ containing x whenever N_i is infinite and i is not a nonzero limit ordinal. It is easy to see that the set of HIA elementary substructures of $\langle H(\theta), \in, \Delta\rangle$ containing x of size μ is a stationary subset of $[H(\theta)]^\mu$ whenever $\mu \leq |H(\theta)|$ is an infinite cardinal.

The following lemma is easy.

[3] Note that $\cap^*$ is not commutative. For example, $\{\omega\}\cap^*(\omega+1) = \{\omega\}$ but $(\omega+1)\cap^*\{\omega\} = \emptyset$.

[4] The height of a club-sequence may of course not be defined.

[5] This notion is rather standard; see for example [1].

Lemma 4.2. *Let κ be a cardinal and let $\mathbb{P}$ be a partial order which is uniformly $<\kappa$-closed. If $\theta > |\mathbb{P}|$ is a cardinal, Δ is a well-order of $H(\theta)$ and N is an* HIA *elementary substructure of $\langle H(\theta), \in, \Delta\rangle$ containing $\mathbb{P}$ and of size at most κ with $p \in \mathbb{P} \cap N$, then there is an $\langle N, \mathbb{P}\rangle$-generic sequence of conditions extending p.*

We will also need the following technical lemma in the next section.

Lemma 4.3. *Assume κ is regular and uncountable, $S \subseteq \kappa \cap \operatorname{cof}\omega$ is stationary, $(\operatorname{cof}\omega \cap \kappa) \setminus S$ is stationary and θ is large enough and regular. For every countable $X \subseteq H(\theta)$, there is an $\in$-chain $\langle N_n \mid n < \omega\rangle$ of countable elementary substructures of $\langle H(\theta), \in\rangle$ such that $X \subseteq N_0$, $\gamma_n := \sup(N_n) \cap \kappa \notin S$ for any $n < \omega$ and $\sup_{n<\omega} \gamma_n \in S$.*

Proof. Let $\langle M_i \mid i < \kappa\rangle$ be a continuous $\in$-chain of elementary substructures of $H(\theta)$ of size less than κ such that $X \subseteq M_0$ and such that $\sup(M_i \cap \kappa) \in (\operatorname{cof}\omega \cap \kappa) \setminus S$ whenever i is a successor ordinal. Let $Lim^2 := \lim(\lim(\kappa))$. $\{\sup(M_i) \cap \kappa \mid i \in Lim^2\}$ is a club subset of κ. Choose $i \in Lim^2$ least possible such that $\sup(M_i \cap \kappa) \in S$. Note that $\operatorname{cof}(i) = \omega$. Pick a sequence $\langle i_n \mid n < \omega\rangle$ with supremum i, consisting only of limit ordinals of cofinality ω. This is possible for $i \in Lim^2$.

First assume that there is $n < \omega$ such that $\sup(M_{i_n} \cap \kappa) \in S$. By minimality of i, $i_n \notin Lim^2$. But this means that $i_n = k + \omega$ for some limit ordinal k. For $n < \omega$, let $N^*_n := M_{k+n+1}$.

Now assume $\sup(M_{i_n} \cap \kappa) \notin S$ for any $n < \omega$ and let $N^*_n := M_{i_n}$ for any $n < \omega$ in this case. In both cases, we have that $\sup(N^*_n \cap \kappa) \in (\operatorname{cof}\omega \cap \kappa) \setminus S$ for any $n < \omega$. Now we inductively construct $\langle N_n \mid n < \omega\rangle$ as follows. Inside of N^*_1, let N_0 be a countable elementary substructure of N^*_0 with $\sup(N_0 \cap \kappa) = \sup(N^*_0 \cap \kappa)$ and $X \subseteq N_0$. Given N_i for $i < \omega$, work inside of N^*_{i+2} to choose a countable elementary substructure N_{i+1} of N^*_{i+1} with $N_i \in N_{i+1}$ and $\sup(N_{i+1} \cap \kappa) = \sup(N^*_{i+1} \cap \kappa)$. Then $\sup_{n<\omega}(\sup(N_n) \cap \kappa) \in S$, as desired. $\square$

5 The single-step forcing

In this section we prove Theorem 1.1.

Proof of Theorem 1.1. Let us fix a regular uncountable cardinal κ such that $\kappa^{<\kappa} = \kappa$. $\mathcal{Q}$ will consist of tuples of the form $\langle p, q\rangle$ where $p \in \vec{S}$ and $p \Vdash_{\vec{S}} \check{q} \in \dot{\mathbb{P}}$, for $\vec{S}$ a notion of forcing in $\mathbf{V}$ described below and $\dot{\mathbb{P}}$ an $\vec{S}$-name for a notion of forcing $\mathbb{P}$ in $\mathbf{V}^{\vec{S}}$ described below, such that $\mathbf{1} \Vdash_{\vec{S}} \dot{\mathbb{P}} \subseteq \mathbf{V}$.[6]

For any ordinal β, $\beta^{\prec + \succ}$ denotes the least ordinal greater than β that is closed under Gödel pairing. Let $C^{\prec + \succ}$ denote the closed unbounded subset of κ consisting of 0 and all limit ordinals closed under Gödel pairing.

Conditions in $\vec{S}$ will be pairs $\langle s, \sigma\rangle$ such that $s\colon \gamma \to 2$ for some ordinal $\gamma < \kappa$, $\{\delta < \gamma \mid s(\delta) = 1\} \subseteq C^{\prec + \succ} \cap \operatorname{cof}(\omega)$ and, letting $\tilde{s} = \{\gamma \mid s(\gamma) = 1\}$, σ is a function $\sigma\colon \tilde{s} \to {}^{<\kappa}\kappa$ (in a slight abuse of notation, we will sometimes identify s and $\tilde{s}$ later on). A condition $\langle s_1, \sigma_1\rangle$ extends a condition $\langle s_0, \sigma_0\rangle$ in $\vec{S}$

[6]What we basically want to do here is to let $\mathcal{Q}$ be the two-step iteration of $\vec{S} * \dot{\mathbb{P}}$. However for technical reasons, we choose it to be a dense subset of this two-step iteration. Since conditions in $\vec{S}$ will be elements of $H(\kappa^+)$ and $\mathbf{1}_{\vec{S}}$ forces conditions in $\dot{\mathbb{P}}$ to be elements of $H(\kappa^+)^{\mathbf{V}}$, the above will in particular help us to obtain that $\mathcal{Q} \subseteq H(\kappa^+)$.

if $s_0 \subseteq s_1$ and $\sigma_0 \subseteq \sigma_1$. Forcing with $\vec{S}$ adds a stationary subset S of $\kappa \cap \mathrm{cof}(\omega)$ such that $(\kappa \cap \mathrm{cof}(\omega)) \setminus S$ is also stationary, and adds a generic enumeration $\vec{s}$ of ${}^{<\kappa}\kappa$ with domain S and with the property that every element of ${}^{<\kappa}\kappa$ is enumerated stationarily often. Let $\dot{S}$ and $\dot{s}$ be the canonical $\vec{S}$-names for S and $\vec{s}$ respectively.

Let $\lambda := 2^\kappa$. Let $\bar{\mathcal{W}}$ be a well-order of ${}^\kappa\kappa$ of order-type λ with smallest element $\vec{0}$. We want to use $\bar{\mathcal{W}}$ to construct a very specific well-order $\mathcal{W}$ of ${}^\kappa\kappa$ of order-type $\lambda+1$. If $x \in {}^{<\omega}\kappa$ and $y \in {}^\kappa\kappa$, we let $x^\frown y$ denote the concatenation of x and y, i.e. if $x = \langle x_i \mid i < n\rangle$ we let $(x^\frown y)(i) = x_i$ if $i < n$ and we let $(x^\frown y)(n+\alpha) = y(\alpha)$ for $\alpha < \kappa$. $\mathcal{W}$ will be made up of λ-many κ-blocks with $\vec{0}$ atop of them. Assuming that $x, y \in {}^\kappa\kappa$ are both not equal to $\vec{0}$, $x = \langle\alpha\rangle^\frown\bar{x}$ and $y = \langle\beta\rangle^\frown\bar{y}$, we set

$$x\mathcal{W}y \leftrightarrow [(\bar{x} = \bar{y} \wedge \alpha < \beta) \vee \bar{x}\bar{\mathcal{W}}\bar{y}].$$

We will need this well-order $\mathcal{W}$ in our coding construction in order for every $\bar{x} \in {}^\kappa\kappa$ to be canonically connected to a κ-block of $\mathcal{W}$-consecutive elements. Having $\vec{0}$ as its largest element will just be notationally convenient.

Let $\bar{F}\colon \lambda \to H(\kappa^+)$ be a bookkeeping function for $H(\kappa^+)$ (i.e., for every $x \in H(\kappa^+)$, $\bar{F}^{-1}(x)$ is unbounded in λ) and let $F\colon {}^\kappa\kappa \setminus \{\vec{0}\} \to H(\kappa^+)$ be defined by $F(x) = \bar{F}(\mathrm{ot}\{y \mid y\mathcal{W}x\})$.

Work in an $\vec{S}$-generic extension $\mathbf{W}$ of $\mathbf{V}$ until further notice and let G_0 denote the $\vec{S}$-generic filter. We want to construct by recursion along $\mathcal{W}$ a collection of partial orders P_x for $x \in ({}^\kappa\kappa)^{\mathbf{V}}$ and set $\mathbb{P} = P_{\vec{0}}$. $\mathbb{P}$ and the P_x will depend on S and $\vec{s}$ and we write $\mathbb{P}(S, \vec{s})$ instead of $\mathbb{P}$ when we want to emphasize this fact. Each P_x will have a canonical $\vec{S}$-name in $\mathbf{V}$, denoted by $\dot{P}_x$. Conditions in P_x will be of the form

$$p = \left\langle t, \vec{e}, \langle\langle\vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\right\rangle.$$

We will set $t^p = t$ and similarly for any other object appearing within p as above. Suppose now that p is a tuple as above such that

(1) $\beta \in C^{\prec + \succ}$,

(2) $t \in {}^{\beta+1}2$,

(3) $\vec{e}$ is a ladder system on $S \cap (\beta+1)$,

(4) for $i < \beta$, $\vec{C}^i$ and $\vec{D}^i$ are club-sequences with domains included in $\beta+1$,

(5) $a \in [({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}]^{<\kappa}$ and

(6) for every $\bar{x} \in a$, $c_{\bar{x}}$ is a subset of $\beta+1$.

Note that any such tuple is an element of $\mathbf{V}$ for $\vec{S}$ is $<\kappa$-closed. We want to associate to p a certain set $C(p) \subseteq ({}^\kappa\kappa)^{\mathbf{V}}$ which canonically codes p.[7] Given $x, y \in {}^{\leq\kappa}\kappa$, let $|x, y| \in {}^\kappa\kappa$ be defined by setting $|x, y|(\alpha) = \beta$ iff $\alpha = 2 \cdot \bar{\alpha}$ and

[7] In fact we will not be able to read off from $C(p)$ whether $\bar{x} \in a$ in case $c_{\bar{x}} = \emptyset$. So one may rather say that $C(p)$ only codes partial information about p. This minor point will however be irrelevant.

$x(\bar\alpha) = \beta$ or $\alpha = 2 \cdot \bar\alpha + 1$ and $y(\bar\alpha) = \beta$. We set $|x, y|(\alpha) = 0$ whenever it is not given a value by the above.

We code t, $\vec{e}$ and $\langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle$ by $b \in ({}^\kappa\kappa)^{\mathbf{V}}$ as follows. For $\gamma < \kappa$, let $b(2\cdot\gamma) = 1$ iff $t(\gamma) = 1$, let $b(6\cdot\gamma+1) = 1$ iff $\gamma_0 \in E_{\gamma_1}$ where $\gamma = \prec\gamma_0, \gamma_1\succ$ and $\vec{e} = \langle E_\xi \mid \xi \in S \cap (\beta+1)\rangle$, $b(6\cdot\gamma+3) = 1$ iff $\gamma_0 \in C^i_{\gamma_1}$ where $\gamma = \prec\gamma_0, \gamma_1, i\succ$,[8] and let $b(6\cdot\gamma+5) = 1$ iff $\gamma_0 \in D^i_{\gamma_1}$, where again $\gamma = \prec\gamma_0, \gamma_1, i\succ$.

Now we want to define $C(p) \subseteq ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ coding b and $\langle c_{\bar x} \mid \bar x \in a\rangle$. For $x \in {}^\kappa\kappa$, we let $x \in C(p)$ iff one of the following holds.

- There is $\alpha < \kappa$ such that $x = \langle 1+\alpha\rangle^\frown\vec{0}$ and $\alpha \in b$.
- There is $\alpha < \kappa$ and $\bar x \in a$ such that $x = \langle \alpha, 1\rangle^\frown\bar x$ and $\alpha \in c_{\bar x}$.

We code F by $F^* \subseteq ({}^\kappa\kappa)^{\mathbf{V}}$ as follows. Set $z \in F^*$ iff there is (x, y) such that $x \in ({}^\kappa\kappa)^{\mathbf{V}}$, $y \subseteq \kappa$ codes $y^* \in H(\kappa^+)^{\mathbf{V}}$,[9] $F(x) = y^*$ and $z = |x, y|$, where we identify y with its characteristic function in the latter. Let $\mathcal{W}^* \subseteq ({}^\kappa\kappa)^{\mathbf{V}}$ code $\mathcal{W}$ by letting $z \in \mathcal{W}^*$ iff there is $(x, y) \in \mathcal{W}$ such that $z = |x, y|$.

Now we define $A^p \subseteq ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ coding $C(p)$, F^* and $\mathcal{W}^*$ by letting, for every $x \in {}^\kappa\kappa$, $x \in A^p$ iff one of the following holds.

- $x \in C(p)$.
- There is $\bar x \in {}^\kappa\kappa$ such that $x = \langle 0, 2\rangle^\frown\bar x$ and $\bar x \in F^*$.
- There is $\bar x \in {}^\kappa\kappa$ such that $x = \langle 0, 3\rangle^\frown\bar x$ and $\bar x \in \mathcal{W}^*$.

Let $s^* \in ({}^\kappa\kappa)^{\mathbf{W}}$ be a canonical code for $\vec{s}$, say if $\kappa > \alpha = \prec\beta, \gamma\succ$ we set

$$s^*(\alpha) = \begin{cases} 0 & \text{if } \vec{s}(\beta)(\gamma) = 0 \\ 1 & \text{if } \vec{s}(\beta)(\gamma) = 1 \\ 2 & \text{if } \beta \notin \operatorname{dom}(\vec{s}) \vee \gamma \geq \operatorname{dom}(\vec{s}(\beta)) \end{cases}.$$

If $x \in {}^\kappa\kappa$, let x^- be defined by $x^-(\alpha) = x(1+\alpha)$ for every $\alpha < \kappa$.[10] Let $\mathscr{C}$ be the set of all $x \in ({}^\kappa\kappa)^{\mathbf{V}}$ such that either $x = \vec{0}$ or whenever $y\mathcal{W}x$ then both $y^-\mathcal{W}x$ and for every $\alpha < \kappa$, $\langle\alpha\rangle^\frown y\mathcal{W}x$.[11] Since $\operatorname{cof}(\lambda) > \kappa$, $\mathscr{C} \cap \{y \mid y\mathcal{W}\vec{0}\}$ is a closed and unbounded (w.r.t. $\mathcal{W}$) subset of $\{y \mid y\mathcal{W}\vec{0}\}$. P_x will be defined iff $x \in \mathscr{C}$.

For $x \in ({}^\kappa\kappa)^{\mathbf{V}}$ and $p = \left\langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar x} \mid \bar x \in a\rangle \right\rangle$ as above, let

$$p\restriction x = \left\langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar x} \mid \bar x \in a \cap \{y \mid y\mathcal{W}x\}\rangle \right\rangle.$$

The next claim follows by the properties of $\mathscr{C}$.

Claim 5.1. *Let p be a tuple for which A^p is defined and let $x \in \mathscr{C}$. Then for every $y\mathcal{W}x$, $y \in A^p$ iff $y \in A^{p\restriction x}$.*

[8] If $\vec{C}^i$ is a club-sequence and $j \in \operatorname{dom}(\vec{C}^i)$, we write C^i_j to abbreviate $\vec{C}^i(j)$.

[9] In the usual way of subsets of κ coding elements of $H(\kappa^+)$.

[10] Thus when passing from x to x^-, we just *throw away* the first component of x.

[11] Note that in particular $\langle 1\rangle^\frown\vec{0}$, the $\mathcal{W}$-least element of $({}^\kappa\kappa)^{\mathbf{V}}$, is an element of $\mathscr{C}$.

Proof. Assume p is such that A^p is defined. Note that $A^{p\restriction x} \subseteq A^p$ for any $x \in ({}^\kappa\kappa)^{\mathbf{V}}$. Now assume $x \in \mathscr{C}$, $y\mathcal{W}x$ and $y \in A^p$. We want to show that $y \in A^{p\restriction x}$. Clearly the only nontrivial case is when y is of the form $\langle\alpha, 1\rangle^\frown\bar{y}$, i.e. y codes the property that $\alpha \in c_{\bar{y}}$ for some $\alpha < \kappa$ and $\bar{y} \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$. Using the fact that $x \in \mathscr{C}$, it follows that $\bar{y}\mathcal{W}x$. But this obviously implies that $y \in A^{p\restriction x}$. □

Given $x \in \mathscr{C}$ and assuming that P_y has been defined for all $y\mathcal{W}x$ with $y \in \mathscr{C}$, conditions in P_x are tuples of the form

$$p = \left\langle t, \vec{e}, \langle\langle\vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\right\rangle$$

satisfying the following properties (which imply properties (1)-(6) above).

(i) $\beta \in C^{\prec + \succ}$.

(ii) $t \in {}^{\beta+1}2$

(iii) $\vec{e} = (E_\delta \mid \delta \in S \cap (\beta + 1))$ is a ladder system.

(iv) a is a subset of $W(\beta^{\prec + \succ}) \cap \{y \mid y\mathcal{W}x\}$ of size less than κ, where for $\xi < \kappa$,

$$W(\xi) = \left[\langle 1\rangle^\frown\vec{0}, \langle\xi\rangle^\frown\vec{0}\right) \cup \bigcup_{\bar{x}\in({}^\kappa\kappa)^{\mathbf{V}}\setminus\{\vec{0}\}} \left[\langle 0\rangle^\frown\bar{x}, \langle\xi\rangle^\frown\bar{x}\right)^{\mathcal{W}}$$

and for any $x_0, x_1 \in ({}^\kappa\kappa)^{\mathbf{V}}$, $[x_0, x_1)^{\mathcal{W}}$ denotes the interval $[x_0, x_1)$ w.r.t. $\mathcal{W}$, i.e. $[x_0, x_1)^{\mathcal{W}} = \{z \in {}^\kappa\kappa \mid z = x_0 \vee x_0\mathcal{W}z\mathcal{W}x_1\}$.

(v) For each $\bar{x} \in a$, $c_{\bar{x}}$ is a closed subset of $\beta + 1$.

(vi) $\forall\bar{x} \in a \cap A^p\, \forall\alpha \in c_{\bar{x}} \cap S\ [s_\alpha \subseteq \bar{x} \to t(\alpha) = 0]$

(vii) For every $i < \beta$, $\vec{C}^i$ and $\vec{D}^i$ are club-sequences with domain included in $\beta+1$, $\mathrm{ht}(\vec{C}^i)$ is defined and is a perfect ordinal of countable cofinality, and $\vec{C}^i$ is coherent as witnessed by $\vec{D}^i$. Moreover, for every $\xi < \beta$,

   (a) if $\xi = 2 \cdot \bar{\xi}$, then

   $$\bar{\xi} \in S \leftrightarrow \exists i < \beta\ \mathrm{ht}(\vec{C}^i) = \eta_\xi.$$

   (b) if $\xi = 4 \cdot \bar{\xi} + 1$ and $\bar{\xi} = \prec\xi_0, \xi_1\succ$, then

   $$(\xi_0, \xi_1) \in s^* \leftrightarrow \exists i < \beta\ \mathrm{ht}(\vec{C}^i) = \eta_\xi.$$

   (c) if $\xi = 4 \cdot \bar{\xi} + 3$ and $s_{\bar{\xi}} \neq s_\zeta$ for all $\zeta < \bar{\xi}$, then

   $$s_{\bar{\xi}} \subseteq t \leftrightarrow \exists i < \beta\ \mathrm{ht}(\vec{C}^i) = \eta_\xi.$$

(viii) For every $i < \beta$,

   (a) every successor point of every member of the range of $\vec{C}^i$ has countable cofinality,

(b) $\mathrm{dom}(\vec{D}^i) \cap (i+1) = \emptyset$,

(c) $(\mathrm{dom}(\vec{D}^i) \cup \mathrm{range}(\vec{D}^i)) \cap S = \emptyset$, and

(d) $\mathrm{dom}(\vec{C}^i) \cap \mathrm{range}(\vec{C}^j) = \emptyset$ for all $j < \beta$,

(e) $\mathrm{dom}(\vec{D}^i) \cap \mathrm{dom}(\vec{D}^j) = \emptyset$ for all $j \neq i$.

(ix) Let $\bar{x} \in a$ be given and suppose there is a $\mathcal{W}$-least $z\mathcal{W}\bar{x}$ with $z \in \mathscr{C}$ such that $F(\bar{x})$ is a $\mathcal{Q} \cap (\vec{S} * \dot{P}_z)$-name in $\mathbf{V}$ for a club subset of κ, let $F(\bar{x})^{G_0}$ denote its partial evaluation by the $\vec{S}$-generic filter G_0.[12] Then $p\restriction z$ is a condition in P_z and for every $\nu < \max(c_{\bar{x}})$, $p\restriction z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c_{\bar{x}})$ such that $p\restriction z \Vdash_{P_z} \nu \in F(\bar{x})^{G_0}$. Then

(a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}} \cap S$, and

(b) $\mathrm{ot}(C^i_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^i)$ for every $i < \beta$ and $\delta \in c_{\bar{x}} \cap \mathrm{dom}(\vec{C}^i)$.

Given conditions $p_\epsilon = \langle t^\epsilon, \vec{e}^\epsilon, \langle\langle \vec{C}^{i,\epsilon}, \vec{D}^{i,\epsilon}\rangle \mid i < \beta^\epsilon\rangle, \langle c^\epsilon_{\bar{x}} \mid \bar{x} \in a^\epsilon\rangle\rangle$ for $\epsilon \in \{0,1\}$, we order P_x by setting $p_1 \leq_x p_0$ iff

(i) $\beta^0 \leq \beta^1$, $t^0 \subseteq t^1$, $\vec{e}^0 \subseteq \vec{e}^1$, $a^0 \subseteq a^1$,

(ii) for all $\bar{x} \in a^0$, $c^0_{\bar{x}} = c^1_{\bar{x}} \cap (\beta^0 + 1)$, and

(iii) $\vec{C}^{i,0} = \vec{C}^{i,1}\restriction(\beta^0+1)$ and $\vec{D}^{i,0} = \vec{D}^{i,1}\restriction(\beta^0+1)$ for all $i < \beta^0$.

Note that $p_1 \leq_x p_0$ implies that $C(p_1) \supseteq C(p_0)$ and therefore $A^{p_1} \supseteq A^{p_0}$. We go down to $\mathbf{V}$ for a moment to observe that our definitions yield the following.

Lemma 5.2. $\mathcal{Q} \subseteq H(\kappa^+)$. $\square$

Back in $\mathbf{W}$, note that if $p_1 \leq p_0$ are conditions in $\mathbb{P}$, then $A^{p_1} \supseteq A^{p_0}$. The following is immediate by Claim 5.1 and noting (for the proof that (ix) holds for $p\restriction z$) that if $y\mathcal{W}z\mathcal{W}x$ and $p \in P_x$, then $p\restriction y = (p\restriction z)\restriction y$.

Claim 5.3. *If $x \in \mathscr{C}$, $p \in P_x$ and $z\mathcal{W}x$ with $z \in \mathscr{C}$, then $p\restriction z \in P_z$. If p, q are both in P_x and $q \leq p$, then $q\restriction z \leq p\restriction z$.* $\square$

It is immediate (using Claim 5.1) that if $z\mathcal{W}x$ and $z, x \in \mathscr{C}$, then $P_z \subseteq P_x$. In fact, the following holds.

Claim 5.4. *If $z\mathcal{W}x$ and $z, x \in \mathscr{C}$, then P_z is a complete suborder of P_x.*

Proof. First note that if $p \perp_z q$, then $p \perp_x q$. To see this, assume $r \leq_x p, q$. Then $r\restriction z \in P_z$ and $r\restriction z \leq_z p, q$ by Claim 5.3. To see that P_z is a complete suborder of P_x, let B be a maximal antichain of P_z. Let $q \in P_x$. There is $b \in B$ which is compatible to $q\restriction z$. Let $p \in P_z$ be stronger than both b and $q\restriction z$. Let $c^*_{\bar{x}}$ be $c^p_{\bar{x}}$ if $\bar{x} \in a^p$ and let it be $c^q_{\bar{x}}$ if $\bar{x} \in a^q \setminus a^p$. Let

$$q^* = \langle t^p, \vec{e}^p, \langle\langle \vec{C}^{i,p}, \vec{D}^{i,p}\rangle \mid i < \beta^p\rangle, \langle c^*_{\bar{x}} \mid \bar{x} \in a^p \cup a^q\rangle\rangle.$$

It suffices to show that q^* is a condition in P_x extending q and p. Given the former, the latter will be obvious considering the nature of the extension relation

[12] $F(\bar{x})^{G_0}$ will be a P_z-name in $\mathbf{W}$ for the same club subset of κ.

of P_x (which is end-extension). We will show, by induction on $t\mathcal{W}x$, that $q^*\restriction t$ is a condition in P_t whenever $t \in \mathscr{C}$. For simplicity of notation, let us assume that $t = x$ and that $q^*\restriction z$ is a condition in P_z for $z\mathcal{W}x$ whenever $z \in \mathscr{C}$. We want to show that q^* is a condition in P_x by showing that it satisfies conditions (i)-(ix) above. Conditions (i)-(v), (vii) and (viii) in the definition of P_x are immediate. For (vi), note that $A^{q^*} = A^q \cup A^p$. We thus have to show that

$$\forall \bar{x} \in (a^p \cup a^q) \cap (A^p \cup A^q)\ \forall \alpha \in c^*_{\bar{x}} \cap S\ [s_\alpha \subseteq \bar{x} \to t^p(\alpha) = 0].$$

If $\bar{x}\mathcal{W}z$, then $\bar{x} \in a^p \cap A^p$ and the above follows for $\bar{x}$ from (vi) for p. Otherwise $\bar{x} \in a^q \cap A^q$ and the above follows for $\bar{x}$ from (vi) for q.

We still need to verify (ix) - let $\bar{x} \in a^p \cup a^q$ be given. If $\bar{x} \in a^p$, then (ix) follows from (ix) for p as $c^*_{\bar{x}} = c^p_{\bar{x}}$, $q^*\restriction z \in P_z$ by induction hypothesis, and $q^*\restriction z \leq p$. So assume that $\bar{x} \in a^q \setminus a^p$. Then $\bar{x} \in a^q \setminus \{y \mid y\mathcal{W}z\}$. Suppose there is a $\mathcal{W}$-least $y\mathcal{W}\bar{x}$ with $y \in \mathscr{C}$ such that $F(\bar{x})^{G_0}$ is a P_y-name for a club subset of κ. As, by induction hypothesis, $q^*\restriction y$ is a condition in P_y stronger than $q\restriction y$, and as $c^*_{\bar{x}} = c^q_{\bar{x}}$, it follows that for every $\nu < \max c^*_{\bar{x}}$, $q^*\restriction y$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c^*_{\bar{x}})$ such that $q^*\restriction y \Vdash \nu \in F(\bar{x})^{G_0}$, which of course coincides with the set of $\nu < \max(c^q_{\bar{x}})$ such that $q\restriction y \Vdash \nu \in F(\bar{x})^{G_0}$. We have to show that

(a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c^q_{\bar{x}} \cap S$, and

(b) $\mathrm{ot}(C^{i,p}_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^{i,p})$ for every $i < \beta$ and $\delta \in c^*_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{i,p})$.

Condition (a) follows immediately from (ix) for q. For (b) fix some $i < \beta^p$ and $\delta \in c^*_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{i,p}) = c^q_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{i,p})$. It follows that $i < \delta \leq \beta^q$, as $\mathrm{dom}(\vec{C}^{i,p}) \cap (i+1) = \emptyset$ by condition (viii). Therefore $C^{i,p}_\delta = C^{i,q}_\delta$ and thus $\mathrm{ot}(C^{i,p}_\delta \cap^* C_{\bar{x}}) = \mathrm{ot}(C^{i,q}_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^{i,q}) = \mathrm{ht}(\vec{C}^{i,p})$. □

Next we show that $\mathbb{P}$ has the κ^+-chain condition. In fact we show that $\mathbb{P}$ is κ^+-Knaster where, for a cardinal θ, a poset $\mathbb{Q}$ is θ-Knaster if for every $\{q_\xi \mid \xi < \theta\} \subseteq \mathbb{Q}$ there is $I \subseteq \theta$ of size θ such that q_ξ and $q_{\xi'}$ are compatible conditions for all ξ, ξ' in I. We first need the following.

Claim 5.5. *If $x \in \mathscr{C}$ and $p \in P_x$, then $C(p) \subseteq W((\beta^p)^{\prec + \succ}) \cap \{z \mid z\mathcal{W}x\}$ and is of size less than κ.*

Proof. Assume $y \in C(p)$. We will only treat the case when y is of the form $y = \langle \alpha, 1\rangle^\frown \bar{y}$ for some $\alpha < \kappa$ and $\bar{y} \in ({}^\kappa\kappa)^{\mathbf{V}}$, i.e. y codes the fact that $\alpha \in c^p_{\bar{y}}$. But the latter implies that $\alpha \leq \beta^p$ and thus $y \in W(\beta^p + 2) \subseteq W((\beta^p)^{\prec + \succ})$, and it implies that $\bar{y}\mathcal{W}x$ and hence by the closure properties of elements of $\mathscr{C}$, this implies that $y\mathcal{W}x$. The case that y is of the form $y = \langle 1+\alpha\rangle^\frown\vec{0}$ is similar. That $C(p)$ is of size less than κ is obvious from its definition and the definition of conditions in P_x. □

Lemma 5.6. *$\mathbb{P}$ is κ^+-Knaster.*

Proof. Let $\{p_\epsilon \mid \epsilon < \kappa^+\}$ be a set of conditions in $\mathbb{P}$. We want to show that there is $B \subseteq \kappa^+$ of size κ^+ such that p^ϵ and $p^{\epsilon'}$ are compatible whenever both ϵ and ϵ' are in B. Let

$$p_\epsilon = \langle t^\epsilon, \vec{e}^\epsilon, \langle\langle \vec{C}^{\epsilon,i}, \vec{D}^{\epsilon,i}\rangle \mid i < \beta^\epsilon\rangle, \langle c^\epsilon_{\bar{x}} \mid \bar{x} \in a^\epsilon\rangle\rangle.$$

By possibly strengthening the p^ϵ, we may assume that $a^\epsilon \supseteq C(p^\epsilon)$ for every $\epsilon < \kappa^+$, using Claim 5.5. This implies that if $\epsilon \neq \epsilon'$ then $A^{p^{\epsilon'}} \setminus A^{p^\epsilon} \subseteq a^{\epsilon'}$ and hence $(a^\epsilon \setminus a^{\epsilon'}) \cap (A^{p^{\epsilon'}} \setminus A^{p^\epsilon}) = \emptyset$. By a Δ-system argument using $2^{<\kappa} = \kappa$, there are β, t, $\vec{e} = \langle E_\delta \mid \delta \in S \cap (\beta+1)\rangle$, a, $\langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle$ and $\langle c_{\bar{x}} \mid \bar{x} \in a\rangle$ such that we may assume that for all distinct $\epsilon, \epsilon' < \kappa^+$,

(i) $t^\epsilon = t$, $\vec{e}^\epsilon = \vec{e}$, $\langle\langle \vec{C}^{\epsilon,i}, \vec{D}^{\epsilon,i}\rangle \mid i < \beta^\epsilon\rangle = \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle$,

(ii) $a^\epsilon \cap a^{\epsilon'} = a$, and

(iii) $c^\epsilon_{\bar{x}} = c_{\bar{x}}$ for all $\bar{x} \in a$.

We claim that any two such conditions are compatible, as

$$p_{\epsilon,\epsilon'} = \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c^{\bar{\epsilon}}_{\bar{x}} \mid \bar{\epsilon} \in \{\epsilon, \epsilon'\}, \bar{x} \in a^{\bar{\epsilon}}\rangle\rangle$$

is a condition in $\mathbb{P}$ stronger than both p_ϵ and $p_{\epsilon'}$: It suffices to show, by induction along $\mathcal{W}$, that $p_{\epsilon,\epsilon'}\restriction x$ is a condition in P_x whenever $x \in \mathscr{C}$. Thus assume that $x \in \mathscr{C}$ and inductively that $p_{\epsilon,\epsilon'}\restriction z$ is a condition in P_z whenever $z\mathcal{W}x$ and $z \in \mathscr{C}$. We want to show that $p_{\epsilon,\epsilon'}\restriction x$ is a condition in P_x. As in the proof of Claim 5.4, conditions (i)-(v), (vii) and (viii) are immediate. For (vi), by symmetry it suffices to show that

$$\forall \bar{x} \in a^\epsilon \cap (A^{p^\epsilon} \cup A^{p^{\epsilon'}})\, \forall \alpha \in c^\epsilon_{\bar{x}} \cap S\ [s_\alpha \subseteq \bar{x} \to t(\alpha) = 0].$$

Now this follows from (vi) for p^ϵ in case $\bar{x} \in a^\epsilon \cap A^{p^\epsilon}$ or from (vi) for $p^{\epsilon'}$ if $\bar{x} \in a^{\epsilon'} \cap A^{p^{\epsilon'}}$ and thus we may assume that $\bar{x} \in (a^\epsilon \setminus a^{\epsilon'}) \cap (A^{p^{\epsilon'}} \setminus A^{p^\epsilon})$. But the latter set is empty by our above assumption.

We are left with proving that (ix) holds for $p_{\epsilon,\epsilon'}\restriction x$. Given $\bar{x} \in a^\epsilon \cup a^{\epsilon'}$, we may assume (by symmetry) that $\bar{x} \in a^\epsilon$. Suppose there is a $\mathcal{W}$-least $z\mathcal{W}\bar{x}$ such that $z \in \mathscr{C}$ and $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ. As, by induction, $p_{\epsilon,\epsilon'}\restriction z$ is a condition in P_z stronger than $p_\epsilon\restriction z$, it follows that for every $\nu < \max(c^\epsilon_{\bar{x}})$, $p_{\epsilon,\epsilon'}\restriction z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c^\epsilon_{\bar{x}})$ such that $p_{\epsilon,\epsilon'}\restriction z \Vdash \nu \in F(\bar{x})^{G_0}$. We have to show that

(a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c^\epsilon_{\bar{x}} \cap S$, and

(b) $\mathrm{ot}(C^{i,p}_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^{i,p})$ for every $i < \beta$ and $\delta \in c^\epsilon_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{i,p})$.

But this is immediate from (ix) for p_ϵ. □

Let $\beta_{\mathbb{P}}$ be the function with domain $\mathbb{P}$ mapping a condition p to β^p.

Lemma 5.7. *$\mathbb{P}$ is uniformly $<\kappa$-closed relative to $\beta_{\mathbb{P}}$ outside S.*

Proof. Given $\gamma < \kappa$ and a decreasing sequence of conditions $\langle p^k \mid k < \gamma\rangle$ in $\mathbb{P}$ with

$$p^k = \langle t^k, \vec{e}^k, \langle\langle \vec{C}^{k,i}, \vec{D}^{k,i}\rangle \mid i < \beta^k\rangle, \langle c^k_{\bar{x}} \mid \bar{x} \in a^k\rangle\rangle,$$

let $\beta := \bigcup_{k<\gamma} \beta^k$, $t = \bigcup_{k<\gamma} t^k$, $\vec{e} = \bigcup_{k<\gamma} \vec{e}^k$, $\vec{C}^i = \bigcup_{k<\gamma} \vec{C}^{k,i}$ and $\vec{D}^i = \bigcup_{k<\gamma} \vec{D}^{k,i}$ for every $i < \beta$, $a = \bigcup_{k<\gamma} a^k$ and $c_{\bar{x}} = \bigcup\{c^k_{\bar{x}} \mid k < \gamma,\ \bar{c} \in a^k\}$ for every $\bar{x} \in a$. If there is $\bar{\gamma} < \gamma$ such that β^k is the same for all $k \geq \bar{\gamma}$, then $\langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle$ is a condition stronger than each p^k.

Otherwise, we may assume that $\beta \notin S$ and let p be defined by setting $\beta^p = \beta$, $t^p = t \cup (\beta, 0)$, $\vec{e}^p = \vec{e}$, $\vec{C}^{i,p} = \vec{C}^i$ and $\vec{D}^{i,p} = \vec{D}^i$ for every $i < \beta$, $a^p = a$ and $c^p_{\bar{x}} = c_{\bar{x}} \cup \{\sup(c_{\bar{x}})\}$ for every $\bar{x} \in a$.

We claim that p is a condition in $\mathbb{P}$. We will show by induction along $\mathcal{W}$ that for every $x \in \mathscr{C}$, $p\restriction x \in P_x$. By the particular specification of p, this will show that $\mathbb{P}$ is uniformly $<\kappa$-closed relative to $\beta_{\mathbb{P}}$ outside S. Thus assume $x \in \mathscr{C}$ and for every $y\mathcal{W}x$ with $y \in \mathscr{C}$, $p\restriction y \in P_y$. We want to check that conditions (i)-(ix) in the definition of P_x hold for $p\restriction x$ and thus $p\restriction x \in P_x$. Conditions (i), (ii), (v), (vii) and (viii) are immediate. Condition (iii) holds since $\beta \notin S$. Using that (iv) holds for the p^k and that $\beta^p > (\beta^k)^{\prec+\succ}$ for every $k < \gamma$, we obtain

$$(*) \quad a^p \subseteq W(\beta^p)$$

and thus (iv) holds for p. For (vi), we have to check that

$$\forall \bar{x} \in a^p \cap A^p \, \forall \alpha \in c^p_{\bar{x}} \cap S \, [s_\alpha \subseteq \bar{x} \to t^p(\alpha) = 0.]$$

If $\bar{x} \in \bigcup_{k<\gamma} A^{p^k}$, this is immediate from (vi) for p^k for some sufficiently large $k < \gamma$ if $\alpha < \beta^p$ and, if $\alpha = \beta^p$, because we set $t^p(\beta^p) = 0$. If $\bar{x} \in A^p \setminus \bigcup_{k<\gamma} A^{p^k}$, it is easily checked using the definition of $C(p)$ that $f(\bar{x})$ is of the form $\kappa \cdot \delta + \xi$ for some $\delta < \lambda$ and $\xi \geq \beta^p$. But by $(*)$ this means that $\bar{x} \notin a^p$ and therefore this case is vacuous.

It remains to show that (ix) holds for $p\restriction x$. Let $\bar{x} \in a^p$ be given and suppose there is a $\mathcal{W}$-least $z\mathcal{W}\bar{x}$ with $z \in \mathscr{C}$ such that $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ. As $p\restriction z$ is a condition in P_z by induction hypothesis and $p\restriction z \leq p^k\restriction z$ for every $k < \gamma$, we have that for every $\nu < \max(c^p_{\bar{x}})$, $p\restriction z$ either forces $\nu \in F(\bar{x})^{G_0}$ or forces $\nu \notin F(\bar{x})^{G_0}$. Let $C_{\bar{x}}$ be the set of all $\nu < \max(c^p_{\bar{x}})$ such that $p\restriction z \Vdash \nu \in F(\bar{x})^{G_0}$. It remains to show that

(a) $E_\delta \setminus C_{\bar{x}}$ is finite for every $\delta \in c_{\bar{x}} \cap S$, and

(b) $\mathrm{ot}(C^i_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^i)$ for every $i < \beta$ and $\delta \in c_{\bar{x}} \cap \mathrm{dom}(\vec{C}^i)$.

Condition (a) holds since, as $\beta \notin S$, every $\delta \in c_{\bar{x}} \cap S$ is such that $\delta \in c^k_{\bar{x}}$ for some k. For condition (b), fix some $i < \beta^p$ and $\delta \in c_{\bar{x}} \cap \mathrm{dom}(\vec{C}^i)$. Let $k < \gamma$ be such that $\bar{x} \in a^k$, $i < \beta^k$ and $\delta \in c^k_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{k,i})$. But then $\mathrm{ot}(C^i_\delta \cap^* C_{\bar{x}}) = \mathrm{ot}(C^{k,i}_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^i) = \mathrm{ht}(\vec{C}^{i,p})$, where the middle equation holds as p^k is a condition in $\mathbb{P}$. □

Lemma 5.8. *Let $p = \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle$ be a condition in $\mathbb{P}$. Then $\forall \beta' < \kappa \, \exists p' \leq p \, \left[\beta^{p'} > \beta' \text{ and } \langle c^{p'}_{\bar{x}} \mid \bar{x} \in a^{p'}\rangle = \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\right]$.*

Proof. Pick $\beta^* \in C^{\prec+\succ}$ such that $\beta^* > \beta', \beta$, $\beta^* = \eta_{\beta^*}$ and such that for every $\gamma < \beta^*$, $\beta^* \setminus S$ contains a closed subset of order-type $\gamma + 1$. The latter is easily possible for S was added generically. We construct

$$p' = \langle t^*, \vec{e}^*, \langle\langle \vec{C}^i_*, \vec{D}^i_*\rangle \mid i < \beta^*\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle \leq p$$

as follows. First we choose t^* of length $\beta^* + 1$ extending t and such that $t^*\restriction[\beta + 1, \beta^*] = \vec{0}$. Let $\vec{e}^* = \langle E^*_\xi \mid \xi \in S \cap (\beta^* + 1)\rangle$ be any ladder system on $(\beta^* + 1) \cap S$ extending $\vec{e}$. Let $\langle \vec{C}^i_*, \vec{D}^i_*\rangle = \langle \vec{C}^i, \vec{D}^i\rangle$ if $i < \beta$ and for $i \in [\beta, \beta^*)$, let $\vec{C}^i_* = \{\langle \sup(X_i), X_i\rangle\}$ and $\vec{D}^i_* = \{\langle \gamma, X_i \cap \gamma\rangle \mid \gamma$ a limit point of $X_i\}$, where $X_i \subseteq \beta^* \setminus (\beta^{\prec + \succ} \cup (i+1))$ has order-type η_{ρ_i}, has all its non-accumulation points of cofinality ω, and is closed in $\sup(X_i)$, for $\langle \rho_i \mid i \in [\beta, \beta^*)\rangle$ as determined by S, s^* and t^* (up to permutation of the indices) via condition (vii) in the definition of $\mathbb{P}$. We also make sure that for $i \neq i'$, $(X_i \cup \{\sup(X_i)\}) \cap (X_{i'} \cup \{\sup(X_{i'})\}) = \emptyset$ and $(X_i \cup \{\sup(X_i)\}) \cap S = \emptyset$.[13] We have to check that p' is a condition in $\mathbb{P}$. It is then obvious that p' is as desired. Conditions (i)-(v) in the definition of $\mathbb{P}$ are immediate, (vii) and (viii) are ensured by our above choice of $\vec{C}^i_*$ and $\vec{D}^i_*$ for $i < \beta^*$, and (ix) is shown as usual.

Finally, condition (vi) in the definition of $\mathbb{P}$ follows if we can show that $a \cap A^{p'} = a \cap A^p$, since (vi) holds for p. To show this is the case, assume $x \in a \cap A^{p'}$. Now if $x \in F^*$ or $x \in \mathcal{W}^*$ then trivially $x \in A^p$. Thus assume $x \in C(p')$. Assume first that there is $\alpha < \kappa$ such that $x = \langle 1 + \alpha\rangle^\frown \vec{0}$. We distinguish several cases.

- If $\alpha = 2 \cdot \gamma$, $x \in A^{p'}$ codes the fact that $t^*(\gamma) = 1$. But having set $t^* \restriction [\beta + 1, \beta^*] = \vec{0}$, this means that $\gamma \leq \beta$ and thus $x \in A^p$.

- If $\alpha = 6 \cdot \gamma + 1$ and $\gamma = \prec\gamma_0, \gamma_1\succ$, $x \in A^{p'}$ codes the fact that $\gamma_0 \in E^*_{\gamma_1}$. If $\gamma_1 \leq \beta$, $x \in A^p$ as in the preceding case. $\gamma_1 > \beta$ implies that $\gamma_1 \geq \beta^{\prec + \succ}$ and thus $\alpha > \beta^{\prec + \succ}$. But then by condition (iv) for p, x could not have been an element of a.

- If $\alpha = 6 \cdot \gamma + 3$ and $\gamma = \prec\gamma_0, \gamma_1, i\succ$, $x \in A^{p'}$ codes the fact that $\gamma_0 \in C^{i,\gamma_1}_*$. If $i < \beta$, $x \in A^p$ as in the preceding cases. If $i \geq \beta$, by our choice of C^i_* it follows that γ_0 and γ_1 are both $\geq \beta^{\prec + \succ}$, which in turn implies that $\alpha > \beta^{\prec + \succ}$ and again by condition (iv) for p, x thus could not have been an element of a.

- The case that $\alpha = 6 \cdot \gamma + 5$ is handled just like the previous case.

Now assume that there is $\alpha < \kappa$ and $\bar{x} \in {}^\kappa\kappa$ such that $x = \langle \alpha, 1\rangle^\frown \bar{x}$. Then $x \in A^{p'}$ codes the fact that $\alpha \in c_{\bar{x}}$. But of course then $x \in A^p$. □

Let us go down to $\mathbf{V}$ for a moment. The following lemma follows from the proof of Lemma 5.7 and from Lemma 5.8.

Lemma 5.9. *$\mathcal{Q}$ has a dense subset which is uniformly $<\kappa$-closed and $<\kappa$-directed closed in $\mathbf{V}$.*

Proof. We will only verify the first part of the statement of the lemma; its second part follows by essentially the same argument and will only be needed for large cardinal preservation arguments in Section 7. Let $\bar{\mathcal{Q}} = \{\langle \langle s, \sigma\rangle, p\rangle \in \mathcal{Q} \mid \mathrm{dom}(s) = \beta^p + 1\}$. It is straightforward to see from the definition of $\mathbb{P}$ that whenever $\langle \langle s, \sigma\rangle, p\rangle \in \mathcal{Q}$ and $\mathrm{dom}(s) > \beta^p + 1$, then in fact $\langle \langle s \restriction (\beta^p + 1), \sigma \restriction (\beta^p + 1)\rangle, p\rangle \in \mathcal{Q}$. This together with Lemma 5.8 implies that $\bar{\mathcal{Q}}$ is a dense subset of $\mathcal{Q}$. Now assume $\langle p^k \mid k < \gamma\rangle$ is a decreasing sequence of conditions in $\bar{\mathcal{Q}}$ for some $\gamma < \kappa$ with

$$p^k = \left\langle \langle s^k, \sigma^k\rangle, \langle t^k, \vec{e}^k, \langle\langle \vec{C}^{k,i}, \vec{D}^{k,i}\rangle \mid i < \beta^k\rangle, \langle c^k_{\bar{x}} \mid \bar{x} \in a^k\rangle\rangle \right\rangle.$$

[13]This is easy to arrange by our requirements on β^*.

If $\langle \beta^{p^k} \mid k < \gamma\rangle$ is eventually constant we can (uniformly) obtain a lower bound of $\langle p^k \mid k < \gamma\rangle$ as in the first part of the proof of Lemma 5.7. Otherwise, let $\beta := \bigcup_{k<\gamma} \beta^k$, let $s := \{(\beta, 0)\} \cup \bigcup_{k<\gamma} s^k$ and let $\sigma := \bigcup_{k<\gamma} \sigma^k$. Define t, $\vec{e}$, $\langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle$, a and $\langle c_{\bar{x}} \mid \bar{x} \in a\rangle$ as in the proof of Lemma 5.7. Since $\langle s, \sigma\rangle$ forces (in $\vec{S}$) that $\beta \notin \dot{S}$,

$$p = \Big\langle \langle s, \sigma\rangle, \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle \Big\rangle$$

is seen to be a condition in $\mathcal{Q}$ (and thus in $\bar{\mathcal{Q}}$) as in the proof of Lemma 5.7. $\square$

From Lemma 5.9 we immediately obtain the following corollary, which will be used repeatedly.

Corollary 5.10. *$\mathcal{Q}$ is $<\kappa$-distributive in $\mathbf{V}$ and $\dot{\mathbb{P}}$ is $<\kappa$-distributive in $\mathbf{V}^{\vec{S}}$.*

Let us go back to $\mathbf{W}$ now. The following is another corollary of Lemma 5.9.

Corollary 5.11. *For every $p \in \mathbb{P}$, every collection $\mathcal{X}$ of size $< \kappa$ of $\mathbb{P}$-names for unbounded subsets of κ and every $\beta < \kappa$ there is $p' \in \mathbb{P}$ stronger than p such that for every $\dot{X} \in \mathcal{X}$ there is some $\gamma > \beta$ such that p' forces $\gamma \in \dot{X}$.* $\square$

Lemma 5.12. *Let $p = \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle \in \mathbb{P}$.*

(i) $\forall \bar{x} \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}\ \exists p' \leq p\ \ \bar{x} \in a^{p'}$.

(ii) $\forall \bar{x} \in a\, \forall \nu < \kappa\ \exists p' \leq p\ \Big[\nu < \max(c^{p'}_{\bar{x}})$ *and* $c^{p'}_{\bar{x}} \cap \nu = c_{\bar{x}} \cap \nu\Big]$.

Proof. For (i), let $\bar{x} \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus (a \cup \{\vec{0}\})$ be given. By Lemma 5.8, we may assume $\bar{x} \in W(\beta^{\prec + \succ})$. Let c^*_z be equal to c_z for $z \in a$ and let $c^*_{\bar{x}} = \emptyset$. If we set

$$p' = \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c^*_z \mid z \in a \cup \{\bar{x}\}\rangle\rangle,$$

then p' is easily seen to be a condition in $\mathbb{P}$ for $A^{p'} = A^p$, and p' is as desired.

For (ii), let $\bar{x} \in a$ be given. Using Lemma 5.8, we may assume that $\beta > \nu$. Pick a countable elementary substructure N of some $H(\theta)$ containing $\mathbb{P}$, p, $\bar{x}$, F and G_0 with θ sufficiently large, and such that $\nu' := \sup(N \cap \kappa) \notin S$. We build a decreasing $\langle N, \mathbb{P}\rangle$-generic sequence of conditions $\langle p_n \mid n \in \omega\rangle$ with $p_0 = p$. Note that $\langle \sup(c^{p_n}_{\bar{x}}) \mid n < \omega\rangle$ is either eventually constant or has supremum ν' by genericity, Lemma 5.8 and clause (ii) in the definition of the extension relation of $\mathbb{P}$. We build a condition q extending all p_n as in the proof of Lemma 5.7, except that we set $c^q_{\bar{x}} = \bigcup_{n\in\omega} c^{p_n}_{\bar{x}} \cup \{\nu'\}$, which is a closed subset of κ by the above. To argue that q is a condition, we only need to show that if $z \mathcal{W} \bar{x}$ is such that $z \in \mathscr{C}$ and z is $\mathcal{W}$-least such that $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ and $\nu'' < \nu'$, then there is $n < \omega$ such that $p_n \restriction z$ decides whether or not $\nu'' \in F(\bar{x})^{G_0}$. But $\nu'' < \nu^*$ for some $\nu^* \in N \cap \kappa$ and, by Claim 5.4 together with Corollary 5.10, there is a dense set $D \in N$ of conditions in P_z deciding $F(\bar{x})^{G_0} \cap \nu^*$. It then follows from the genericity of $\langle p_n \mid n < \omega\rangle$ that there is some n such that $p_n \restriction z$ decides for every $\xi \in \nu^*$ whether or not $\xi \in F(\bar{x})^{G_0}$. But of course ν'' is one such ξ. $\square$

Now let G be $\mathcal{Q}$-generic over $\mathbf{V}$, $G = G_0 * G_1$, where G_1 is $\mathbb{P}(S, \vec{s})$-generic over $\mathbf{W} := \mathbf{V}[G_0]$. Work in $\mathbf{V}[G]$. Let $t^G = \bigcup_{p \in G_1} t^p$ and $\vec{E}^G = \bigcup_{p \in G_1} \vec{e}^p$. For each $i < \kappa$, let $\vec{C}^{i,G} = \bigcup\{\vec{C}^{i,p} \mid p \in G_1,\, i < \beta^p\}$ and $\vec{D}^{i,G} = \bigcup\{\vec{D}^{i,p} \mid p \in G_1,\, i < \beta^p\}$. By the definition of $\mathcal{Q}$ and its extension relation and by Lemma 5.8, $\vec{E}^G$ is a ladder system defined on all of S and each $\vec{C}^{i,G}$ is a coherent club-sequence (which is witnessed by $\vec{D}^{i,G}$) with nonempty domain disjoint from S and such that every successor point of every member of range$(\vec{C}^{i,G})$ has countable cofinality.

Write $\vec{E}^G$ as $\vec{E}^G = \langle E_\delta \mid \delta \in S\rangle$ and let $c^G_{\bar{x}} = \bigcup\{c^p_{\bar{x}} \mid p \in G_1,\, \bar{x} \in a^p\}$ for all $\bar{x} \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$. By Lemma 5.12, each $c^G_{\bar{x}}$ is a club subset of κ in $\mathbf{V}[G]$. Also, by condition (ix) (a) in the definition of $\mathbb{P}$, for every $\delta \in c^G_{\bar{x}} \cap S$, if there is a $\mathcal{W}$-least $z\mathcal{W}\bar{x}$ with $z \in \mathscr{C}$ such that $F(\bar{x})^{G_0}$ is a P_z-name $\dot{C}$ for a club subset of κ and C is the G_1-interpretation of $\dot{C}$, then $E^G_\delta \setminus C$ is finite. Let $A^G = \bigcup_{p \in G_1} A^p$. t^G will have a canonical $\mathcal{Q}$-name in $\mathbf{V}$ which we will denote by $\dot{t}$. The partial evaluation of $\dot{t}$ by G_0 will be denoted by $\dot{t}^{G_0}$ and is a $\mathbb{P}$-name for t^G in $\mathbf{W}$. We will do the same for the other objects defined in $\mathbf{V}[G]$ above.

Lemma 5.13. *In $\mathbf{V}[G]$, S is stationary.*

Proof. We go back to working in $\mathbf{V}[G_0] = \mathbf{W}$. Let $p \in \mathbb{P}$ and let $\dot{C}$ be a $\mathbb{P}$-name for a club subset of κ. We want to find an extension p^* of p and some $\gamma \in S$ such that $p^* \Vdash_{\mathbb{P}} \gamma \in \dot{C}$. For this, let $\langle N_n \mid n < \omega\rangle$ be an $\in$-chain of countable elementary substructures of some large enough $H(\theta)$ containing $\mathbb{P}$, $\dot{C}$, p, F and G_0 such that $\gamma_n := \sup(N_n \cap \kappa) \notin S$ for all n and $\gamma := \sup_n \gamma_n \in S$. This can be done using Lemma 4.3. Let $\langle p_n \mid n < \omega\rangle$ be a decreasing sequence of conditions in $\mathbb{P}$ extending p such that for all n, $p_n \in N_{n+1}$ is a lower bound of a decreasing $\langle N_n, \mathbb{P}\rangle$-generic sequence of conditions in N_n extending p and extending p_{n-1} if $n > 0$. By Lemma 5.7, these lower bounds exist. For each n, p_{n+1} forces

(i) $\gamma_n \in \dot{C}$ and

(ii) $\gamma_n \in F(\bar{x})^{G_0}$ for all $\bar{x} \in a^{p_n}$ for which there is some $z\mathcal{W}\bar{x}$ such that $z \in \mathscr{C}$ and $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ.

This is true by Corollary 5.11 since p_{n+1} is a lower bound of an $\langle N_n, \mathbb{P}\rangle$-generic sequence. Let

- $\beta = \bigcup_{n<\omega} \beta^{p_n} = \gamma$,
- $t = (\bigcup_{n<\omega} t^{p_n}) \cup \{\langle \gamma, 0\rangle\}$,
- $\vec{e} = (\bigcup_{n<\omega} \vec{e}^{p_n}) \cup \{\langle \gamma, \{\gamma_n \mid n < \omega\}\rangle\}$,
- $\vec{C}^i = \bigcup\{\vec{C}^{p_n,i} \mid n < \omega,\, i < \beta^{p_n}\}$ and $\vec{D}^i = \bigcup\{\vec{D}^{p_n,i} \mid n < \omega,\, i < \beta^{p_n}\}$ for every $i < \beta$,
- $a = \bigcup_{n<\omega} a^{p_n}$, and
- $c_{\bar{x}} = \{\gamma\} \cup \bigcup\{c^{p_n}_{\bar{x}} \mid n < \omega,\, \bar{x} \in a^{p_n}\}$ for every $\bar{x} \in a$.

It is easy to check that $p^* = \langle t, \vec{e}, \langle\langle \vec{C}^i, \vec{D}^i\rangle \mid i < \beta\rangle, \langle c_{\bar{x}} \mid \bar{x} \in a\rangle\rangle$ is a condition extending p and forcing that $\gamma \in \dot{C}$ by condition (i) above. The proof of this is by induction on $\mathcal{W}$ as usual, i.e., one proves by induction on x that $p^* \restriction x$ is in P_x whenever $x \in \mathscr{C}$. The only nontrivial point is the verification of condition (ix) in the definition of P_x. But (ix) follows now thanks to (ii) above. $\square$

One could have verified the preceding lemma arguing in $\mathbf{V}$ instead of $\mathbf{W}$ and thus avoiding use of Lemma 4.3. We will perform this kind of argument in the proof of Claim 5.15 below. The following lemma is now easy.

Lemma 5.14. *In $\mathbf{V}[G]$, $\vec{E}^G$ is a strongly guessing ladder system defined on the stationary set S.*

Proof. We have just seen that S is stationary. To see that $\vec{E}^G$ is strongly guessing, let $\dot{C} \in \mathbf{W}$ be a nice $\mathbb{P}$-name for a club subset of κ. By the κ^+-c.c. of $\mathbb{P}$ together with $\mathrm{cof}(\lambda) > \kappa$, we know that there is some $z\mathcal{W}\vec{0}$ in $\mathscr{C}$ such that $\dot{C}$ is in fact a P_z-name for a club subset of κ, and of course we also have $\dot{C} \in H(\kappa^+)$. Suppose also that z is $\mathcal{W}$-minimal with the above property. Since F is a book-keeping function, we can find $\bar{x} \neq z$ such that $z\mathcal{W}\bar{x}$ and $F(\bar{x})^{G_0} = \dot{C}$. Let $C = \dot{C}^{G_1}$. By the paragraph just before Lemma 5.13, for every δ in the intersection of the club $c^G_{\bar{x}}$ with S, $E^G_\delta \setminus C$ is finite, which finishes the proof. $\square$

Claim 5.15. *A^G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S, $\vec{s}$ and t^G as parameters.*

Proof. We claim that in $\mathbf{V}[G]$,

$$A^G = \{y \in {}^\kappa\kappa \setminus \{\vec{0}\} \mid \exists C \subseteq \kappa \text{ club} \, \forall \alpha \in C \cap S \, [\vec{s}(\alpha) \subseteq y \to t^G(\alpha) = 0]\}.$$

Now if $y \in A^G$, c^G_y is a club subset of κ and witnesses that y is an element of the set on the right hand side of the above equation. For the other direction, we need to combine the proofs of Theorem 3.5 and Lemma 5.13. Work in $\mathbf{V}$. Similar to the proof of Theorem 3.5, pick a $\mathcal{Q}$-name $\dot{y}$ for an element of ${}^\kappa\kappa \setminus \{\vec{0}\}$ and a $\mathcal{Q}$-name $\dot{C}$ for a club subset of κ and assume, towards a contradiction, that there is $p \in \mathcal{Q}$ with

$$p \Vdash \dot{y} \notin \dot{A} \wedge \forall \alpha \in \dot{C} \cap \dot{S} \ [\dot{s}(\alpha) \subseteq \dot{y} \to \dot{t}(\alpha) = 0]. \tag{3}$$

Let $\langle N_n \mid n < \omega \rangle$ be an $\in$-chain of countable elementary substructures of some large enough $H(\theta)$ containing $\mathcal{Q}$, $\dot{y}$, $\dot{C}$, F and p. Let $N = \bigcup_{n<\omega} N_n$, let $\gamma_n = \sup(N_n \cap \kappa)$ for every $n < \omega$ and let $\gamma = \sup(N \cap \kappa) = \bigcup_{n<\omega} \gamma_n$. From Lemma 5.9 and its proof, we can obtain a sequence $\langle p_n \mid n < \omega \rangle$ of conditions in $\mathcal{Q}$ below p such that for all n, $p_n \in N_{n+1}$ is a lower bound of a decreasing $(N_n, \mathcal{Q})$-generic sequence of conditions in N_n extending p_{n-1} if $n > 0$, such that $s^{p_n}(\gamma_n) = 0$. As in the proof of Theorem 3.5, there is $u \colon \gamma \to \kappa$ such that for every $n < \omega$, p_{n+1} forces

- $\dot{y} \restriction \gamma_n = u \restriction \gamma_n$ and such that
- $x \restriction \gamma_n \neq u \restriction \gamma_n$ for all $x \in a_{p_n}$.

Moreover, as in the proof of Lemma 5.13, p_{n+1} forces

- $\gamma_n \in \dot{C}$ and
- $\gamma_n \in F(\bar{x})$ for all $\bar{x} \in a^{p_n}$ for which there is some $z\mathcal{W}\bar{x}$ such that $z \in \mathscr{C}$ and $F(\bar{x})$ is a $\mathcal{Q} \cap (\vec{S} * \dot{P}_z)$-name for a club subset of κ.

Now we define

- $s = \{\langle \gamma, 1 \rangle\} \cup \bigcup_{n<\omega} s^n$.

- $\sigma \;=\; \{\langle\gamma,u\rangle\} \;\cup\; \bigcup_{n<\omega}\sigma^{p_n}$.
- $\beta \;=\; \bigcup_{n<\omega}\beta^{p_n} \;=\; \gamma$.
- $t \;=\; \{\langle\gamma,1\rangle\} \;\cup\; \bigcup_{n<\omega} t^{p_n}$.
- $\vec{e} \;=\; \{\langle\gamma,\{\gamma_n \mid n<\omega\}\rangle\} \;\cup\; \bigcup_{n<\omega}\vec{e}^{p_n}$.
- $\vec{C}^i = \bigcup_{n<\omega}\vec{C}^{p_n,i}$ and $\vec{D}^i = \bigcup_{n<\omega}\vec{D}^{p_n,i}$ for every $i<\beta$.
- $a \;=\; \bigcup_{n<\omega} a^{p_n}$.
- $c_x \;=\; \{\gamma\} \;\cup\; \bigcup_{n<\omega} c_x^{p_n}$ for all $x\in a$.

Then $q = \langle\langle s,\sigma\rangle, \langle t,\vec{e},\langle\langle\vec{C}^i,\vec{D}^i\rangle \mid i<\beta\rangle, \langle c_{\bar{x}} \mid \bar{x}\in a\rangle\rangle\rangle$ is a condition in $\mathcal{Q}$, because $u\not\subseteq x$ for all $x\in a$ and the other requirements on q can be verified as in the proof of Lemma 5.13. But $q\leq p$ and

$$q \Vdash \check{\gamma}\in\dot{C}\cap\dot{S} \;\wedge\; \dot{s}(\check{\gamma})\subseteq\dot{y} \;\wedge\; \dot{t}(\check{\gamma})=1,$$

contradicting (3). $\square$

Claim 5.16. *G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S, $\vec{s}$ and t^G as parameters.*

Proof. By Claim 5.15, A^G is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S, $\vec{s}$ and t^G. As a first step, we show that using A^G as a predicate, we can define t^G, $\vec{E}^G$, $\vec{C}^{i,G}$ and $\vec{D}^{i,G}$ for every $i<\kappa$ and $c^G_{\bar{x}}$ for every $\bar{x}\in({}^{\kappa}\kappa)^{\mathbf{V}}\setminus\{\vec{0}\}$ over $H(\kappa^+)^{\mathbf{V}[G]}$, for the following is ensured by forcing with $\mathcal{Q}$.

- $t^G = \{\gamma<\kappa \mid \langle 1+2\cdot\gamma\rangle^\frown\vec{0}\in A^G\}$.
- $\gamma_0\in E^G_{\gamma_1}$ iff $\langle 1+6\cdot\prec\gamma_0,\gamma_1\succ+1\rangle^\frown\vec{0}\in A^G$.
- $\gamma_0\in C^{G,i}_{\gamma_1}$ iff $\langle 1+6\cdot\prec\gamma_0,\gamma_1,i\succ+3\rangle^\frown\vec{0}\in A^G$.
- $\gamma_0\in D^{G,i}_{\gamma_1}$ iff $\langle 1+6\cdot\prec\gamma_0,\gamma_1,i\succ+5\rangle^\frown\vec{0}\in A^G$.
- $\alpha\in c_{\bar{x}}$ iff $\langle\alpha,1\rangle^\frown\bar{x}\in A^G$.

Furthermore we can define F^* and $\mathcal{W}^*$, and thus F and $\mathcal{W}$ over $H(\kappa^+)^{\mathbf{V}[G]}$, for

- $\bar{x}\in F^*$ iff $\langle 0,2\rangle^\frown\bar{x}\in A^G$ and
- $\bar{x}\in\mathcal{W}^*$ iff $\langle 0,3\rangle^\frown\bar{x}\in A^G$.

This allows us to define $H(\kappa^+)^{\mathbf{V}} = \mathrm{dom}(\mathcal{W})\cup\mathrm{range}(\mathcal{W})$ over $H(\kappa^+)^{\mathbf{V}[G]}$. Thus by the definition of $\mathcal{Q}$, it is straightforward to see that $\mathcal{Q}$ is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using the above parameters. Now assume that

$$p = \Big\langle\langle s,\sigma\rangle, \langle t,\vec{e},\langle\langle\vec{C}^i,\vec{D}^i\rangle \mid i<\beta\rangle, \langle c_{\bar{x}} \mid \bar{x}\in a\rangle\rangle\Big\rangle$$

is a condition in $\mathcal{Q}$. $p\in G$ iff $s=S\restriction\gamma$ and $\sigma=\vec{s}\restriction\gamma$ for some $\gamma<\kappa$, $\beta<\kappa$, $t=t^G\restriction(\beta+1)$, $\vec{e}=\vec{E}^G\restriction(\beta+1)$, $\vec{C}^i=\vec{C}^{i,G}$ and $\vec{D}^i=\vec{D}^{i,G}$ for every $i<\beta$, $a\subseteq A^G\cap W(\beta^{\prec+\succ})$ is of size less than κ and $c_{\bar{x}}=c^G_{\bar{x}}\restriction(\beta+1)$ for every $\bar{x}\in a$. $\square$

Lemma 5.17. *In $\mathbf{V}[G]$ there is a well-order R of $H(\kappa^+)^{\mathbf{V}[G]}$ that is definable over $\langle H(\kappa^+), \in\rangle^{\mathbf{V}[G]}$ by a formula using S, $\vec{s}$ and t^G as parameters.*

Proof. By the proof of Claim 5.16, $\mathcal{W}$, $\mathcal{Q}$ and $H(\kappa^+)^{\mathbf{V}}$ are each definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using parameters S, $\vec{s}$ and t^G. But now we can obtain a well-order R of $H(\kappa^+)^{\mathbf{V}[G]}$ by setting xRy iff $\tilde{x}\mathcal{W}\tilde{y}$ where $\tilde{x}$ is the $\mathcal{W}$-least characteristic function of a subset of κ in $\mathbf{V}$ coding a collection $\dot{x}$ of pairs of the form $\langle a, \nu\rangle$ with $a \in H(\kappa^+)^{\mathbf{V}}$ and $\nu < \kappa$ and such that $\{\nu < \kappa \mid (\exists a \in G)(\langle a, \nu\rangle \in \dot{x})\}$ is a subset of κ coding x (and analogously for $\tilde{y}$ and y). By Claim 5.16, it follows that the relation R, which is clearly a well-order of $H(\kappa^+)^{\mathbf{V}[G]}$, is definable over $H(\kappa^+)^{\mathbf{V}[G]}$ using S, $\vec{s}$ and t^G as parameters. $\square$

Let $X^G \subseteq \kappa$ be defined by setting $\xi \in X^G$ iff one of the following holds.

(i) $\xi = 2 \cdot \bar{\xi}$ and $\bar{\xi} \in S$.

(ii) $\xi = 4 \cdot \bar{\xi} + 1$, $\bar{\xi} = \prec\xi_0, \xi_1\succ$ and $(\xi_0, \xi_1) \in s^*$.

(iii) $\xi = 4 \cdot \bar{\xi} + 3$, $s_{\bar{\xi}} \neq s_\zeta$ for all $\zeta < \bar{\xi}$ and $s_{\bar{\xi}} \subseteq t^G$.

S, s^* and t^G (and therefore also $\vec{s}$) are obviously definable in $H(\kappa^+)^{\mathbf{V}[G]}$ from X^G. Also, by the definition of $\mathcal{Q}$ together with Lemma 5.8 we have that for every $\xi < \kappa$, $\xi \in X^G$ if and only if there is some $i < \kappa$ such that $\mathrm{ht}(\vec{C}^{i,G}) = \eta_\xi$.

Let $[S]$ be the class of S in $\mathcal{P}(\kappa)/\mathsf{NS}_\kappa$, i.e., the collection of all $S' \subseteq \kappa$ such that the symmetric difference $S'\Delta S$ is non-stationary.

Lemma 5.18 is the final ingredient in the proof of Theorem 1.1. Its proof is essentially a copy of the proof of [3, Lemma 3.2].[14] We reproduce that proof here for the reader's benefit (with the appropriate notational changes).

Lemma 5.18. *$[S]$ and X^G are lightface definable in $H(\kappa^+)^{\mathbf{V}[G]}$. In fact,*

(i) $[S]$ can be defined as the unique class K in $\mathcal{P}(\kappa)/\mathsf{NS}_\kappa$ such that for every $S' \in K$ there is a strongly guessing ladder system defined on S',[15] and

(ii) X^G can be defined in $\mathbf{V}[G]$ as the set of all $\xi < \kappa$ such that there is a coherent strongly type-guessing club-sequence $\vec{C}$ of height η_ξ, with $\mathrm{dom}(\vec{C})$ a stationary subset of κ disjoint from S' for some $S' \in [S]$, and such that every successor point of every member of $\mathrm{range}(\vec{C})$ has countable cofinality.

Proof. We start by showing the following.

Claim 5.19. *In $\mathbf{V}[G]$, every $\vec{C}^{i,G}$ is a coherent strongly type-guessing club-sequence with $\mathrm{dom}(\vec{C}^{i,G})$ a stationary subset of κ disjoint from S, and such that every successor point of every member of $\mathrm{range}(\vec{C}^{i,G})$ has countable cofinality.*

Proof. We have already argued that $\vec{C}^{i,G}$ is a coherent club-sequence and it is immediate from the definition of $\mathbb{P}$ that its domain is disjoint from S and that every successor point of every member of its range has countable cofinality. To see that the domain of $\vec{C}^{i,G}$ is, in $\mathbf{V}[G]$, a stationary subset of κ, let us move

[14] [3, Lemma 3.2] is stated in the context of the construction in that paper. In particular, κ is ω_1 in that lemma. However, the same proof works for general κ with almost no changes.

[15] Note that 'x belongs to the unique class $K \in \mathcal{P}(\kappa)/\mathsf{NS}_\kappa$ such that ... ' is indeed first order expressible in $\langle H(\kappa^+), \in\rangle$ despite the fact that an equivalence class in $\mathcal{P}(\kappa)/\mathsf{NS}_\kappa$ is a proper class in $H(\kappa^+)$.

back to $\mathbf{V}$ once again, let $\dot{C} \in \mathbf{V}$ be a $\mathcal{Q}$-name for a club subset of κ and let $r = \langle\langle s, \sigma\rangle, p\rangle \in \mathcal{Q}$ with $i < \beta^p$. It will suffice to see that there is a condition $\bar{r} = \langle\langle \bar{s}, \bar{\sigma}\rangle, \bar{p}\rangle \in \mathcal{Q}$ extending r and forcing $\mathrm{dom}(\vec{C}^{i,\bar{p}}) \cap \dot{C} \neq \emptyset$.

Let $\eta = \mathrm{ht}(\vec{C}^{i,p})$ and let us fix a $\subseteq$-continuous chain $\langle N_\xi \mid \xi \leq \eta\rangle$ of HIA elementary substructures of $\langle H(\theta), \in, \Delta\rangle$ of size less than κ, for some large enough θ and some well-order Δ of $H(\theta)$, containing $\mathcal{Q}$, $\dot{C}$, F and p, and such that $N_\xi \cap \xi \in \kappa$ and $\langle N_{\xi'} \mid \xi' \leq \xi\rangle \in N_{\xi+1}$ for all $\xi < \eta$. Let $\delta_\xi = N_\xi \cap \kappa$ for all ξ. We make sure in addition that $\mathrm{cof}(\delta_{\xi+1}) = \omega$ for every ξ. We aim to build a decreasing sequence $\langle r_\xi \mid \xi \leq \eta\rangle$ of conditions extending r in such a way that for all ξ, letting $r_\xi = \langle\langle s_\xi, \sigma_\xi\rangle, p_\xi\rangle$,

(i) $r_\xi \in N_{\xi+1}$ is an $\langle N_\xi, \mathcal{Q}\rangle$-generic condition and $\beta^{p_\xi} = \delta_\xi$,

(ii) if $\xi < \eta$ is a limit ordinal, then $\delta_\xi \in dom(\vec{D}^{i,p_\xi})$ and $D^{i,p_\xi}_{\delta_\xi} = \{\delta_{\xi'} \mid \xi' < \xi\}$,

(iii) $\delta_\eta \in dom(\vec{C}^{i,p_\eta})$ and $C^{i,p_\eta}_{\delta_\eta} = D^{i,p_\eta}_{\delta_\eta} = \{\delta_\xi \mid \xi < \eta\}$.

This is enough: Since r_η is $\langle N_\eta, \mathcal{Q}\rangle$-generic, it forces $\delta_\eta \in \dot{C}$. Hence, r_η is a condition extending r and forces $\dot{C} \cap dom(\vec{C}^{i,p_\eta}) \neq \emptyset$. The construction of $\langle r_\xi \mid \xi \leq \eta\rangle$ is quite standard. Given ξ and assuming $r_{\xi'}$ has been built for all $\xi' < \xi$, we can find r_ξ in the following way.

Suppose ξ is 0 or a successor ordinal. Let $\langle r'_\xi \mid \xi < |N_\xi|\rangle$ be the Δ-least $\langle N_\xi, \mathcal{Q}\rangle$-generic sequence of length $|N_\xi|$ of conditions extending r (if $\xi = 0$) or r_{ξ_0} (if $\xi = \xi_0 + 1$). This generic sequence exists thanks to Lemma 5.9 together with Lemma 4.2. Let r_ξ be obtained from $\langle r_{\xi'} \mid \xi' < |N_\xi|\rangle$ by another application of Lemma 5.9. Certainly this r_ξ is in $N_{\xi+1}$ and is an $\langle N_\xi, \mathcal{Q}\rangle$-generic condition. Furthermore, by Lemma 5.8 we have that $\beta^{p_\xi} = \delta_\xi$.

If $\xi < \eta$ is a nonzero limit ordinal, we can let r_ξ be obtained from $\langle r_{\xi'} \mid \xi' < \xi\rangle$ as in the proof of Lemma 5.9 and put δ_ξ into $\mathrm{dom}(\vec{D}^{i,p_\xi})$, but not into $\mathrm{dom}(\vec{C}^{i,p_\xi})$, and let $D^{i,p_\xi}_{\delta_\xi} = \{\delta_{\xi'} \mid \xi' < \xi\}$. Again, $\beta^{p_\xi} = \delta_\xi$. In this case, the verification that r_ξ is a condition in $\mathcal{Q}$ and that it extends $r_{\xi'}$ for all $\xi' < \xi$ is exactly as in the proof of Lemma 5.9, using the fact that $\delta_\xi \notin \mathrm{dom}(\vec{C}^{i',p_\xi})$ for all $i' < \delta_\xi$. $r_\xi \in N_{\xi+1}$ and it is $\langle N_\xi, \mathcal{Q}\rangle$-generic because $N_\xi = \bigcup_{\xi'<\xi} N_{\xi'}$ and because each $r_{\xi'}$ is $\langle N_{\xi'}, \mathcal{Q}\rangle$-generic.

If $\xi = \eta$, we again build r_η from $\langle r_\xi \mid \xi < \eta\rangle$ as in the proof of Lemma 5.9, but this time putting δ_η into both $\mathrm{dom}(\vec{D}^{i,p_\eta})$ and $dom(\vec{C}^{i,p_\eta})$, and making $C^{i,p_\eta}_{\delta_\eta} = D^{i,p_\eta}_{\delta_\eta} = \{\delta_\xi \mid \xi < \eta\}$.

Let us momentarily work in an $\vec{S}$-generic extension of $\mathbf{V}$ for some $\vec{S}$-generic G_0 containing $\langle s_\eta, \sigma_\eta\rangle$. As usual we prove by induction along $\mathcal{W}$ that $p_\eta \restriction x$ is a P_x-condition extending all $p_\xi \restriction x$ for $\xi < \eta$. We proceed as in the proof of Lemma 5.7. The only problem could come up in the verification of property (ix) for p_η. For this, suppose $\bar{x} \in a^{p_\eta \restriction x}$ and suppose $z\mathcal{W}\bar{x}$ is $\mathcal{W}$-minimal with $z \in \mathscr{C}$ such that $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ. By the construction of $\langle r_\xi \mid \xi \leq \eta\rangle$ as a generic sequence of conditions it is clear that $p_\eta \restriction z$, which by induction hypothesis is a condition in P_z extending all $p_\xi \restriction z$, decides whether or not ν is in $F(\bar{x})^{G_0}$ for every $\nu < \sup(c^{p_\eta}_{\bar{x}})$. Hence we are left with checking that if $C_{\bar{x}}$ is the collection of all $\nu < \sup(c^{p_\eta}_{\bar{x}})$ such that $p_\eta \restriction z \Vdash_{P_z} \nu \in F(\bar{x})^{G_0}$, then $\mathrm{ot}(C^{i',p_\eta}_\delta \cap^* C_{\bar{x}}) = \mathrm{ht}(\vec{C}^{i',p_\eta})$ for every $i' < \beta^{p_\eta}$ and every $\delta \in c^{p_\eta}_{\bar{x}} \cap \mathrm{dom}(\vec{C}^{i',p_\eta})$.

The proof of this in the case when either there is some $\xi < \eta$ such that $\delta \in c_{\bar{x}}^{p_\xi}$ or $i' \neq i$ goes through easily from the way we have built $\langle r_\xi \mid \xi \leq \eta\rangle$.

The only nontrivial case is when $i' = i$ and $\delta = \delta_\eta$. We want to prove that $\mathrm{ot}(C_{\delta_\eta}^{i,p_\eta} \cap^* C_{\bar{x}}) = \eta$. In this case we argue that, since $F(\bar{x})^{G_0}$ is a P_z-name for a club subset of κ and each r_ξ (for $\xi < \eta$) is $(N_\xi, \mathcal{Q})$-generic, $p_\eta{\restriction}z$ forces $\delta_\xi \in F(\bar{x})^{G_0}$ for all $\xi < \eta$ such that $z, \bar{x} \in a^{p_\xi} \subseteq N_\xi$. Hence, we have in fact that a final segment of $C_{\delta_\eta}^{i,p_\eta}$ is contained in $C_{\bar{x}}$. Now that we have that r_η is a condition in $\mathcal{Q}$, checking that it extends all r_ξ for $\xi < \eta$, is straightforward.

We still need to check that $\vec{C}^{i,G}$ is strongly type-guessing in $\mathbf{V}[G]$. For this we argue very much as in the proof of Lemma 5.14: Let us go back to $\mathbf{W}$. Let $\dot{C}$ be a nice $\mathbb{P}$-name for a club subset of κ and let $p \in \mathbb{P}$. As in the proof of Lemma 5.14, we know that there is some $z\mathcal{W}\vec{0}$ such that $z \in \mathscr{C}$ and $\dot{C} \in H(\kappa^+)$ is a P_z-name for a club subset of κ. Now suppose z is $\mathcal{W}$-minimal with the above property and find $\bar{x}$, $z\mathcal{W}\bar{x}$, such that $F(\bar{x}) = \dot{C}$. Let $C = \dot{C}^{G_1}$.

By Lemma 5.12 (i) we may extend p to a condition p^* such that $\bar{x} \in a^{p^*}$. But by condition (ix) in the definition of $\mathbb{P}$ we know that every $p' \in \mathbb{P}$ extending p^* is such that $p'{\restriction}z$ decides, for every $\nu < \max(c_{\bar{x}}^{p'})$, whether or not ν is in $\dot{C}$. Furthermore, for every such p', letting $C_{\bar{x}}$ be the set of $\nu < \max(c_{\bar{x}}^{p'})$ such that $p'{\restriction}z \Vdash_{P_z} \nu \in \dot{C}$, we know that every $\delta \in c_{\bar{x}}^{p'} \cap \mathrm{dom}(\vec{C}^{i,p'})$ is such that $\mathrm{ot}(C_\delta^{i,p'} \cap^* C_{\bar{x}}) = \eta$. That is, p' forces $\mathrm{ot}(C_\delta^{i,\dot{G}} \cap^* \dot{C}) = \eta$ for every such δ. This shows that, in $\mathbf{V}[G]$, $\mathrm{ot}(C_\delta^{i,G} \cap^* C) = \eta$ for all $\delta \in c_{\bar{x}}^G \cap \mathrm{dom}(\vec{C}^{i,G})$. Hence, $c_{\bar{x}}^G$ is a witness for C to the fact that $\vec{C}^{i,G}$ is strongly type-guessing. $\square$

We have already seen that $\vec{E}^G$ is a strongly guessing ladder system. It remains to see that there is no strongly guessing ladder system whose domain is a stationary subset of κ disjoint from S (this is shown in Claim 5.20 below in its case $\eta = \omega$) and that if $\xi < \kappa$ is such that $\xi \notin X^G$, then in $\mathbf{V}[G]$ there is no coherent strongly type-guessing club-sequence of height η_ξ whose domain is a stationary subset of κ disjoint from S and such that every successor point of every member of its range has countable cofinality (this is shown in Claim 5.20 below in its case $\eta > \omega$). This will finish the proof of Lemma 5.18.

Claim 5.20. *In $\mathbf{V}[G]$, let $\vec{C} = \langle C_\delta \mid \delta \in \mathrm{dom}(\vec{C})\rangle$ be a coherent club-sequence with $\mathrm{dom}(\vec{C})$ a stationary subset of $\kappa \setminus S$. Let $\eta = \mathrm{ht}(\vec{C})$. Suppose either*

(i) $\eta = \omega$, or else

(ii) η is a perfect ordinal of countable cofinality such that $\xi \notin X^G$ if $\eta_\xi = \eta$ and every successor point of every member of $\mathrm{range}(\vec{C})$ has countable cofinality.

Then there is some $z \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ such that in $\mathbf{V}[G]$,

$$\{\delta \in \mathrm{dom}(\vec{C}) \mid \mathrm{ot}(C_\delta \cap^* c_x^G) < \eta\}$$

is a stationary subset of κ for all $x \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ such that $z\mathcal{W}x$.[16]

[16]The proof also shows, for $\vec{C}$ as in the hypothesis, that if the set

$$\mathrm{dom}(\vec{C})\setminus\bigcup\{\mathrm{dom}(\vec{C}^{i,G}) \mid i < \kappa,\ \mathrm{ht}(\vec{C}^{i,G}) < \eta\}$$

is in $\mathbf{V}[G]$ a stationary subset of κ, then in fact $\{\delta \in \mathrm{dom}(\vec{C}) \mid \sup(C_\delta \cap^* c_x^G) < \eta\}$ is stationary in $\mathbf{V}[G]$ for $\mathcal{W}$-cofinally many x in $({}^\kappa\kappa)^{\mathbf{V}}$ below $\vec{0}$.

Proof. Let us work in $\mathbf{W}$. Using the κ^+-chain condition of $\mathbb{P}$ we may fix some $z \in \mathscr{C} \setminus \{\vec{0}\}$ such that $\vec{C} = \tau^{G_1}$, where $\tau \in H(\kappa^+)$ is a P_z-name for a coherent club-sequence of height η, and some $\bar{p} \in P_z \cap G_1$ forcing (in $\mathbb{P}$) that $\xi \notin \dot{X}^{G_0}$ if η is a perfect ordinal of countable cofinality and $\eta = \eta_\xi$ - where $\dot{X}^{G_0}$ is a $\mathbb{P}$-name in $\mathbf{W}$ for X^G - and that $\mathrm{dom}(\tau)$ is a stationary subset of κ disjoint from S.

Let $x \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ be such that $z\mathcal{W}x$ and let $\dot{C}$ be a $\mathbb{P}$-name for a club subset of κ. Let p' be a condition extending $\bar{p}$ in $\mathbb{P}$. By Lemma 5.12 (i) we may assume that $x \in a^{p'}$. It will suffice to find a condition $q \leq p'$ and some $\delta \in c^q_x$ such that $q \Vdash_{\mathbb{P}} \delta \in \dot{C} \cap \mathrm{dom}(\tau)$ and such that $q \Vdash_{\mathbb{P}} \mathrm{ot}(\tau_\delta \cap^* c^q_x) < \eta$ (where τ_δ is a name for $\tau(\delta)$).

Let G_* be P_x-generic over $\mathbf{W}$ with $p' \restriction x \in G_*$. Note that, since P_x is a complete suborder of $\mathbb{P}$, every generic filter G' for $\mathbb{P}/G_*$ over $\mathbf{W}[G_*]$ - where $\mathbb{P}/G_*$ is the suborder of $\mathbb{P}$ consisting of those conditions q such that $q \restriction x \in G_*$ - is such that $G' \cap P_x = G_*$ and is $\mathbb{P}$-generic over $\mathbf{W}$ as a filter of $\mathbb{P}$, and that, conversely, every $\mathbb{P}$-generic filter G' over $\mathbf{W}$ with $G' \cap P_x = G_*$ is $\mathbb{P}/G_*$-generic over $\mathbf{W}[G_*]$.

We will temporarily work in $\mathbf{W}[G_*]$. Let $\vec{C}^* = \tau^{G_*}$ and let

$$\vec{C}^{*\,i} = \bigcup\{\vec{C}^{i,p} \mid p \in G_*,\, i < \beta^p\}$$

for all $i < \kappa$. Let θ be a sufficiently large regular cardinal and let Δ be a well-order of $H(\theta)^{\mathbf{W}[G_*]}$. Let $\langle N_\xi \mid \xi < \kappa\rangle$ be a $\subseteq$-continuous chain of elementary substructures of $\langle H(\theta)^{\mathbf{W}[G_*]}, \in, \Delta\rangle$ of size less than κ containing everything relevant such that $N_\xi \cap \kappa \in \kappa$ and $\langle N_{\xi'} \mid \xi' \leq \xi\rangle \in N_{\xi+1}$ for all $\xi < \kappa$.[17] Let $\delta_\xi = N_\xi \cap \kappa$ for all $\xi < \kappa$ and let $D_0 = \{\delta_\xi \mid \xi < \kappa\}$.

Subclaim 5.21. *There is a limit ordinal $\bar{\xi} < \kappa$ with $\delta_{\bar{\xi}} \in dom(\vec{C}^*)$, $\eta < \delta_{\bar{\xi}}$, with $(D_0 \cap \delta_{\bar{\xi}}) \setminus (C^*_{\delta_{\bar{\xi}}} \cup \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i}) \cup S)$ unbounded in $\delta_{\bar{\xi}}$, and such that $\mathrm{ot}(C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) = \mathrm{ot}(C^{*\,i}_{\delta_{\bar{\xi}}})$ in case $i < \kappa$ is such that $\delta_{\bar{\xi}} \in \mathrm{dom}(\vec{C}^{*\,i})$.*[18]

Proof. Note that, by Claim 5.19,

$$\mathcal{Y} := \{\delta < \kappa \mid (\forall i)\,(\delta \in dom(\vec{C}^{*\,i}) \to \mathrm{ot}(C^{*\,i}_\delta \cap^* D_0) = \mathrm{ot}(C^{*\,i}_\delta))\}$$

is forced by $\mathbb{P}/G_*$ to contain a club subset of κ. This is true because $\mathrm{dom}(\vec{C}^{*\,i}) \cap (i+1) = \emptyset$ for all i - for every i there is, in $\mathbf{W}^{\mathbb{P}/G_*}$, a club

$$C_i \subseteq \{\delta < \kappa \mid \delta \in \mathrm{dom}(\vec{C}^{*\,i}) \to \mathrm{ot}(C^{*\,i}_\delta \cap^* D_0) = \mathrm{ot}(C^{*\,i}_\delta)\}.$$

Now the required club can be taken to be the diagonal intersection $\Delta_{i<\kappa} C_i$. $Z := D_0 \setminus (S \cup \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i}))$ is unbounded in κ (for example by an argument as in the proof of Lemma 5.7), and therefore $D_1 = \{\delta \in D_0 \mid \mathrm{rank}_Z(\delta) > \eta\}$ is a club subset of κ. Since $\mathrm{dom}(\vec{C}^*)$ is forced by $\mathbb{P}/G_*$ to be a stationary subset of κ, it must have stationary intersection with $\mathcal{Y} \cap D_1$. Pick $\bar{\xi}$ s.t. $\delta_{\bar{\xi}} > \eta$ is in this intersection. This is enough since then $(D_0 \cap \delta_{\bar{\xi}}) \setminus (C^*_{\delta_{\bar{\xi}}} \cup \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i}) \cup S)$ must be unbounded in $\delta_{\bar{\xi}}$ as $\mathrm{rank}_Z(\delta_{\bar{\xi}}) > \eta$ and $\mathrm{ot}(C^*_{\delta_{\bar{\xi}}}) = \eta$.[19] □

[17] For this proof we do not need that the structures be HIA.

[18] There may or may not be such an i. If there is such an i, then of course it is unique.

[19] Note that $Z \setminus Y$ is unbounded in $\sup(Z)$ whenever Z and Y are sets of ordinals with $\mathrm{rank}_Z(\sup(Z)) > \mathrm{ot}(Y)$.

Let $\bar{\xi}$ be as given by Subclaim 5.21. We will find, in $\mathbf{W}$, a condition q extending p' and forcing both $\delta_{\bar{\xi}} \in \dot{C} \cap \mathrm{dom}(\tau)$ and $\mathrm{ot}(\tau_{\delta_{\bar{\xi}}} \cap^* c^q_x) < \eta$.

The proof of the following subclaim is standard.

Subclaim 5.22. *For every dense set $\mathcal{D} \subseteq \mathbb{P}/G_*$, $q \in \mathbb{P}/G_*$, and every $u \in a^q$ with either $u = x$ or $x\mathcal{W}u$, there is a club $C \subseteq \kappa$ with the property that for every $\delta \in C$ and every $\delta' < \delta$ there is a condition $q' \in \mathcal{D}$ extending q with $\beta^{q'} < \delta$ and such that $c^{q'}_u \backslash c^q_u \subseteq (\delta',\ \delta)$.*

Proof. We may take this club to be $\{M_j \cap \kappa \mid j < \kappa\}$ for an $\in$-chain $\langle M_j \mid j < \kappa\rangle$ of elementary substructures of $H(\chi)$ (for some large enough χ) of size less than κ, containing $\mathcal{D}$ and q, and such that $M_j \cap \kappa \in \kappa$ for all j. The subclaim then follows from an application of Clause (ii) of Lemma 5.12 within a relevant M_j. □

In order to find the desired q extending p' we distinguish three cases.

Case 1: There is (a unique) i such that $\delta_{\bar{\xi}} \in \mathrm{dom}(\vec{C}^{*\,i})$ and $\eta < \mathrm{ht}(\vec{C}^{*\,i})$.

Let $\mathcal{E}$ be the set of ordinals in $C^*_{\delta_{\bar{\xi}}}$ above $\min(C^*_{\delta_{\bar{\xi}}})$ which are not limit points of $C^*_{\delta_{\bar{\xi}}}$ and let $\langle t_k \mid k < \omega\rangle$ be an increasing sequence converging to the height of $\vec{C}^{*\,i}$. Since $i \in N_{\bar{\xi}}$ (because $i < \delta_{\bar{\xi}}$ using Clause (viii) in the definition of conditions), and therefore $N_{\bar{\xi}}$ can be assumed to contain $\langle t_k \mid k < \omega\rangle$, by disregarding an initial segment of $\langle N_\xi \mid \xi < \kappa\rangle$ if necessary we may, and will, assume that $\langle t_k \mid k < \omega\rangle$ is in N_0. Note that, since $\mathrm{ht}(\vec{C}^{*\,i})$ is a perfect ordinal above η, for every k there are unboundedly many ordinals δ in $\mathcal{E}$ such that $\mathrm{ot}((C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap J_\delta) \geq t_k$, where J_δ is the interval $(max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ \delta)$. Otherwise $\mathrm{ht}(\vec{C}^{*\,i}) = \mathrm{ot}(C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0)$ would be bounded by $t_k{\cdot}\eta$ for some k, which would contradict the fact that $\mathrm{ht}(\vec{C}^{*\,i})$ is perfect and that t_k and η are less than $\mathrm{ht}(\vec{C}^{*\,i})$. Since every ordinal in $C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0$ is of the form δ_ξ for some $\xi < \bar{\xi}$, it follows that we may find a strictly increasing sequence $(\xi_k)_{k<\omega}$ converging to $\bar{\xi}$ such that $\delta_{\xi_0} > i$ and such that

$$\mathrm{ot}((C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap (\max(C^*_{\delta_{\bar{\xi}}} \cap \bar{\delta}_k),\ \bar{\delta}_k)) > t_{k+1}$$

for $\bar{\delta}_k := \min(\mathcal{E} \backslash \delta_{\xi_k})$ (for all k). It follows that there is a function h defined on $\mathcal{E}$ such that $\max(C^*_{\delta_{\bar{\xi}}} \cap \delta) \leq h(\delta) < \delta$ for every $\delta \in \mathcal{E}$ and such that

$$\mathrm{ot}((C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap \bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta'),\ h(\delta'))) \geq t_k$$

whenever $k < \omega$, $\delta \in \mathcal{E}$ and $\delta_{\xi_k} \leq \delta$. We may assume that h is defined inductively as follows. If $k < \omega$ is minimal such that $\delta \in \mathcal{E} \cap \delta_{\xi_k}$, let $h(\delta)$ be the least ϵ in $(\max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ \delta)$ such that

$$\mathrm{ot}\left((C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap \left(\bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta'),\ h(\delta')) \cup (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ \epsilon)\right)\right) \geq \bar{t},$$

where $\bar{t}$ is the maximal member t of the set $\{0\} \cup \{t_{k'} \mid k' \leq k\}$ for which there is some ϵ, $\max(C^*_{\delta_{\bar{\xi}}} \cap \delta) < \epsilon < \delta$, such that

$$\mathrm{ot}\left((C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap \left(\bigcup_{\delta' \in \mathcal{E} \cap \delta} (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta'),\ h(\delta')) \cup (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ \epsilon) \right) \right) \geq t.$$

Note that, since $\vec{C}^{*\,i}$ and $\vec{C}^*$ are coherent sequences, $h \restriction (\mathcal{E} \cap \delta_\xi) \in N_{\xi+1}$ for every $\xi < \bar{\xi}$. The reason is that $N_{\xi+1}$ contains all initial segments of $\vec{C}^{*\,i}$ and of $\vec{C}^*$ of length less than $\delta_{\xi+1}$ and the sequences $\langle N_{\xi'} \mid \xi' \leq \xi \rangle$ and $\langle t_k \mid k < \omega \rangle$. Let

$$\Sigma = \{\xi < \bar{\xi} \mid \delta_\xi \in (C^{*\,i}_{\delta_{\bar{\xi}}} \cap^* D_0) \cap \bigcup_{\delta \in \mathcal{E}} (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ h(\delta))\}$$

and let $\bar{\Sigma}$ be the closure of Σ. Note that $\mathrm{ot}(\Sigma) = \mathrm{ht}(\vec{C}^{*\,i})$ and that Σ does not contain any of its accumulation points. In fact, if $\xi \in \Sigma$, then δ_ξ is a member of $C^{*\,i}_{\delta_{\bar{\xi}}}$ which is not a limit point of $C^{*\,i}_{\delta_{\bar{\xi}}}$ (by the definition of $\cap^*$). Note also that $\max(\bar{\Sigma}) = \bar{\xi}$, that $\delta_\xi \notin S$ for any $\xi \in \bar{\Sigma}$ and that $\delta_\xi \notin \mathrm{dom}(\vec{C}^{*\,j})$ for any $\xi \in \bar{\Sigma} \cap \delta_{\bar{\xi}}$ and $j < \kappa$, using Clause (viii) in the definition of conditions.

Now we can inductively build a decreasing sequence $\langle p_\xi \mid \xi \in \bar{\Sigma} \rangle$ of conditions in $\mathbb{P}/G_*$ extending p' such that the following hold for each $\xi \in \bar{\Sigma}$.

(i) $p_\xi \in N_{\xi+1}$.

(ii) If $\xi \in \bar{\Sigma} \backslash \Sigma$, then p_ξ is a lower bound of $\langle p_{\xi'} \mid \xi' \in \Sigma \cap \xi \rangle$.

(iii) If $\xi \in \Sigma$, then p_ξ is a lower bound of a certain decreasing ω-sequence $\langle q^\xi_k \mid k < \omega \rangle$ of conditions in N_ξ (see below) and forces $\delta_\xi \in \dot{C}$.

(iv) Given any two $\xi_0 < \xi_1$ in $\bar{\Sigma}$ and any $\bar{x} \in a^{p_{\xi_0}}$, if there is a minimal $\bar{z} \mathcal{W} \bar{x}$ in $\mathscr{C}$ such that $F(\bar{x})^{G_0}$ is a $P_{\bar{z}}$-name for a club subset of κ, then $p_{\xi_1} \restriction \bar{z}$ forces $\delta_{\xi_1} \in F(\bar{x})^{G_0}$.

(v) If $\xi \in \Sigma$ and $\delta \in \mathcal{E}$ is such that $\delta_\xi \in (\max(C^*_{\delta_{\bar{\xi}}} \cap \delta),\ h(\delta))$, then

$$\max(C^*_{\delta_{\bar{\xi}}} \cap \delta) < \min(c^{p_\xi}_x \backslash (c^{p'}_x \cup (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\} + 1))).$$

We want to show first, given any $\xi \in \Sigma$ and assuming $p_{\xi'}$ has been built for all $\xi' \in \bar{\Sigma} \cap \xi$, how to find p_ξ in $N_{\xi+1}$ so that (iii) and (v) hold about p_ξ, and if $\xi' = \max(\bar{\Sigma} \cap \xi)$, so that (iv) holds about the pair (ξ', ξ). Moreover we want to show how to perform the construction in a uniformly definable way.

p_ξ can be built as a lower bound in $N_{\xi+1} \cap \mathbb{P}/G_*$ of a decreasing sequence $\langle q^\xi_k \mid k < \omega \rangle$ of $\mathbb{P}/G_*$-conditions in N_ξ extending $p_{\max(\bar{\Sigma} \cap \xi)}$ (if $\bar{\Sigma} \cap \xi \neq \emptyset$) or extending p' (if ξ is the first member of Σ) such that, for a suitable sequence $\langle \mathcal{D}_k \mid k < \omega \rangle$ of dense subsets of $\mathbb{P}/G_*$, all of them belonging to N_ξ,

(a) $q^\xi_k \in \mathcal{D}_k$ for all k,

(b) $\sup_{k' \geq k} \max(c^{q^\xi_{k'}}_u) = \delta_\xi$ for every k and every $u \in a^{q^\xi_k}$, and

(c) if $\delta \in \mathcal{E}$ is such that $\delta_\xi \in (\max(C^*_{\delta_{\bar\xi}} \cap \delta),\ h(\delta))$, then q^ξ_0 puts some ordinal above $\max(C^*_{\delta_{\bar\xi}} \cap \delta)$, but no new ordinal below $\max(C^*_{\delta_{\bar\xi}} \cap \delta) + 1$, inside $c^{q^\xi_0}_x$.[20]

Since $\delta_\xi \notin S$ and $\delta_\xi \notin \vec{C}^{*\,j}$ for any $j < \kappa$, the sequence $\langle q^\xi_k \mid k < \omega\rangle$ has a definable lower bound r such that $r\restriction x \in G_*$, by the arguments in the proof of Lemma 5.7. Conditions (a)-(c) can be met simultaneously once $\mathcal{D}_k$ has been fixed since, by correctness, N_ξ contains a club as given by Subclaim 5.22 for $\mathcal{D} = \mathcal{D}_0$ and for q being either $p_{\max(\bar\Sigma\cap\xi)}$ or p' and since δ_ξ, being a non-accumulation point of $C^{*\,i}_{\delta_{\bar\xi}}$, has countable cofinality.

Let $\langle \epsilon_k \mid k < \omega\rangle \in N_{\xi+1}$ be the Δ-first ω-sequence of ordinals in N_ξ converging to δ_ξ. We take each $\mathcal{D}_k$ to be the set of conditions q' forcing some ordinal above ϵ_k to be in $F(\bar x)^{G_0} \cap \dot C \cap c^{q'}_{\bar x}$ whenever $\bar x \in a^q$ (for the right choice of q) and there exists $\bar u \in \mathscr{C}$ with $\bar u \mathcal{W} \bar x$ such that $F(\bar x)^{G_0}$ is a $P_{\bar u}$-name for a club subset of κ. All choices can be made in a uniform way using Δ to pick all relevant objects to be Δ-least possible with the desired properties.

The sequence of conditions $\langle p_\xi \mid \xi \in \bar\Sigma\rangle$ can now be built in a uniform way, again by the usual argument involving the well-order Δ. At limit stages $\xi < \bar\xi$ of the construction, we extend all conditions built up to that point by considering a lower bound p_ξ of the sequence with $p_\xi\restriction x \in G_*$. This lower bound exists by the proof of Lemma 5.7 because $\delta_\xi \notin S$ and $\delta_\xi \notin \mathrm{dom}(\vec{C}^{*\,j})$ for any $j < \kappa$. The fact that $h\restriction(\mathcal{E} \cap \delta_\xi) \in N_{\xi+1}$ for every ξ ensures that the choice of p_ξ takes place inside $N_{\xi+1}$.

Now if $\delta \in \mathcal{E} \cap c^{p_{\bar\xi}}_x$, consider the least $\xi \in \bar\Sigma \setminus \{\bar\xi\}$ such that $\delta \in \mathcal{E} \cap c^{p_\xi}_x$. Then if δ^* is the successor of δ in $\mathcal{E}$, ξ is also least in $\bar\Sigma \setminus \{\bar\xi\}$ such that $\delta_\xi \in (\delta, \delta^*)$. But then by condition (v) in the construction of the p_ξ, it follows that $\delta \in \mathcal{E} \cap c^{p'}_x$. Considering that $\mathcal{E}$ is the set of successor points of $C^*_{\delta_{\bar\xi}}$, it follows that there is $\bar\delta < \delta_{\bar\xi}$ such that

($\star$) $C^*_{\delta_{\bar\xi}} \cap^* c^{p_\xi}_x \subseteq \bar\delta$ for each ξ.

Also, given any $x' \in ({}^\kappa\kappa)^{\mathbf{V}} \setminus \{\vec{0}\}$ such that $x \mathcal{W} x'$ and any $\xi_0 \in \Sigma$, if $x' \in a^{p_{\xi_0}}$, then $x' \in N_{\xi_0+1}$. Hence, if $\bar z \mathcal{W} x'$ is minimal such that $\bar z \in \mathscr{C}$ and $F(x')^{G_0}$ is a $P_{\bar z}$-name for a club subset of κ, then each δ_ξ (for $\xi \in \Sigma$, $\xi > \xi_0$) is a member of $C^{*\,i}_{\delta_{\bar\xi}}$ which is not a limit point of $C^{*\,i}_{\delta_{\bar\xi}}$ and which is forced by p_ξ to be in $F(x')^{G_0}$. This allows us to obtain $p_{\bar\xi}$ [21] and extend it to a condition q such that $q\restriction x \in G_*$ and such that q forces $\tau_{\delta_{\bar\xi}} = C^*_{\delta_{\bar\xi}}$. This finishes the proof in this case, since $q \Vdash_{\mathbb{P}} \delta_{\bar\xi} \in \dot C$ and since $q \Vdash_{\mathbb{P}} \tau_{\delta_{\bar\xi}} \cap^* c^q_z \subseteq \bar\delta$ (by ($\star$)).

Case 2: There is (a unique) i such that $\delta_{\bar\xi} \in \mathrm{dom}(\vec{C}^{*\,i})$ and $\eta > \mathrm{ht}(\vec{C}^{*\,i})$.

Let $\Sigma = \{\xi < \bar\xi \mid \delta_\xi \in C^{*\,i}_{\delta_{\bar\xi}}, \eta < \delta_\xi\}$. This time we build a decreasing sequence $\langle p_\xi \mid \xi \in \Sigma\rangle$ of conditions in $\mathbb{P}/G_*$ extending p' and satisfying the following.

[20] This means that $c^{q^\xi_0}_x \setminus (c^r_x \cup (\max(C^*_{\delta_{\bar\xi}} \cap \delta) + 1)) \neq \emptyset$ and $c^{q^\xi_0}_x \cap (\max(C^*_{\delta_{\bar\xi}} \cap \delta) + 1) = c^r_x \cap (\max(C^*_{\delta_{\bar\xi}} \cap \delta) + 1)$, where $r = p_{\max(\bar\Sigma\cap\xi)}$ if $\bar\Sigma \cap \xi \neq \emptyset$ and $r = p'$ if $\xi = \min(\Sigma)$.

[21] The crucial point here is that by the above, (b) in Condition (ix) in the definition of $\mathbb{P}$ holds for $p_{\bar\xi}$. (a) in Condition (ix) and (vi) in the definition of $\mathbb{P}$ hold trivially, for $\delta_{\bar\xi} \notin S$.

(i) $p_\xi \in N_{\xi+1}$ for every ξ.

(ii) For every limit point ξ of Σ, p_ξ is a lower bound of $\langle p_{\xi'} \mid \xi' \in \Sigma \cap \xi \rangle$.

(iii) Given any successor point ξ of Σ, p_ξ is a lower bound of a certain decreasing ω-sequence $\langle q^\xi_k \mid k < \omega \rangle$ of conditions in N_ξ and forces $\delta_\xi \in \dot{C}$.

(iv) Given any $\xi_0 < \xi_1$ in Σ and any $x' \in a^{p_{\xi_0}}$, if $\bar{z}\mathcal{W}x'$ is $\mathcal{W}$-minimal in $\mathscr{C}$ such that $F(x')^{G_0}$ is a $P_{\bar{z}}$-name for a club subset of κ, then p_{ξ_1} forces $\delta_{\xi_1} \in F(x')^{G_0}$.

(v) Given any successor point ξ of Σ,

$$c_x^{p_\xi} \cap (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\},\, \delta_\xi) \cap C^*_{\delta_{\bar{\xi}}} = \emptyset.$$

For any successor point ξ of Σ, assuming $p_{\xi'}$ for all $\xi' \in \Sigma \cap \xi$ has been defined, the choice of $p_\xi \in N_{\xi+1}$ can be made as in the previous case: p_ξ can be taken to be a lower bound of a decreasing sequence $\langle q^\xi_k \mid k < \omega \rangle$ of conditions in N_ξ meeting the members of a suitably chosen sequence $\langle \mathcal{D}_k \mid k < \omega \rangle$ of dense subsets of $\mathbb{P}/G_*$ in N_ξ. This lower bound will exist exactly by the same reasons as in the previous case. This time we pick the conditions q^ξ_k in such a way that, for all k,

(a) $q^\xi_k \in \mathcal{D}_k$,

(b) $\sup_{k' \geq k} max(c_u^{q^\xi_{k'}}) = \delta_\xi$ for every $u \in a^{q^\xi_k}$, and

(c) q^ξ_k does not put any ordinal in $C^*_{\delta_{\bar{\xi}}} \backslash (\sup\{\delta_{\xi'} \mid \xi' \in \Sigma \cap \xi\} + 1)$ inside $c_x^{q^\xi_k}$.

We again use that $\mathrm{cof}(\delta_\xi) = \omega$. Conditions (a)-(c) can be met, once $\mathcal{D}_k$ has been fixed, since $\mathrm{ht}(\vec{C}^*) < \delta_\xi = N_\xi \cap \kappa$ and since N_ξ contains a club as given by Subclaim 5.22 for $\mathcal{D} = \mathcal{D}_k$ and for q being p_j or q^ξ_{k-1} if $k > 0$. The intersection of this club with δ_ξ has order type $\delta_\xi > \mathrm{ht}(\vec{C}^*)$, so there are unboundedly many points δ in it such that $[\nu,\, \delta) \cap C^*_{\delta_{\bar{\xi}}} = \emptyset$ for some $\nu < \delta$. The choice of $\langle \mathcal{D}_k \mid k < \omega \rangle$ is as in Case 1.

This is enough since then, by the same reasons as before, $\langle p_\xi \mid \xi \in \Sigma \rangle$ has a lower bound $p_{\bar{\xi}}$ in $\mathbb{P}/G_*$ (using (iv) as in Case 1) such that $(c_x^{p_{\bar{\xi}}} \cap C^*_{\delta_{\bar{\xi}}}) \backslash (\delta_{\xi_0} + 1) \subseteq \{\delta_{\xi_j} \mid 0 < j < \mathrm{ht}(\vec{C}^{*\,i})\}$ (by (v)), and forcing $\delta_{\bar{\xi}} \in \dot{C}$ (by (iii)). As in the previous case, we can extend $p_{\bar{\xi}}$ to a condition q, with $q \restriction x \in G_*$, forcing $\tau_{\delta_{\bar{\xi}}} = C^*_{\delta_{\bar{\xi}}}$. This is enough, since then q forces $\mathrm{ot}(\tau_{\delta_{\bar{\xi}}} \cap^* c_x^q) \leq \mathrm{ht}(\vec{C}^{*\,i}) < \eta$.[22]

Case 3: $\delta_{\bar{\xi}} \notin \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i})$.

The proof is now easier than in the previous two cases. Let $\langle \xi_k \mid k < \omega \rangle$ be a strictly increasing sequence converging to $\bar{\xi}$ and with $\{\delta_{\xi_k} \mid k < \omega\}$ disjoint from $C^*_{\delta_{\bar{\xi}}} \cup \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i})$.

We can now build by recursion a decreasing sequence $\langle p_k \mid k < \omega \rangle$ of conditions in $\mathbb{P}/G_*$ extending p' such that, for each k,

[22] In fact, q forces $\mathrm{ot}((\tau_{\delta_{\bar{\xi}}} \backslash (\delta_{\xi_0} + 1)) \cap c_z^q) \leq \mathrm{ht}(\vec{C}^{*\,i}) < \eta$.

(i) $p_k \in N_{\xi_k+1}$,

(ii) p_k forces $\rho \in \dot{C}$ for some $\rho > \delta_{\xi_{k-1}}$, and

(iii) if $k > 0$, then $\min(c_x^{p_k} \setminus (\delta_{\xi_{k-1}} + 1)) > \max(C^*_{\delta_{\bar{\xi}}} \cap \delta_{\xi_k})$.

Finally, since $\delta_{\bar{\xi}} \notin S \cup \bigcup_{i<\kappa} \mathrm{dom}(\vec{C}^{*\,i})$, $\langle p_k \mid k < \omega \rangle$ has a lower bound $\tilde{q}$ with $\tilde{q} \restriction x \in G_*$ and forcing that $\delta_{\bar{\xi}}$ is in $\dot{C}$. Again, we can extend $\tilde{q}$ to a condition q such that $q \restriction x \in G_*$ and forcing $\tau_{\delta_{\bar{\xi}}} = C^*_{\delta_{\bar{\xi}}}$. It follows that q forces that $\tau_{\delta_{\bar{\xi}}} \cap^* c_x^q$ (and in fact $\tau_{\delta_{\bar{\xi}}} \cap c_x^q$) is bounded in $\delta_{\bar{\xi}}$.

The construction in this last case finishes the proof of Claim 5.20. $\square$

This completes the proof of Lemma 5.18. $\square$

It follows now, from Lemmas 5.17 and 5.18, that in $\mathbf{V}[G]$ there is a lightface definable well-order of $H(\kappa^+)^{\mathbf{V}[G]}$. This concludes the proof of Theorem 1.1. $\square$

6 The global iteration

In Section 5, given $\kappa \geq \omega_1$ with $\kappa^{<\kappa} = \kappa$, we obtained a partial order $\mathcal{Q}$ which has a $<\kappa$-directed closed dense subset, the κ^+-cc, is a subset of $H(\kappa^+)$ and forces the existence of a lightface definable well-order of $H(\kappa^+)$. Note that the definition of $\mathcal{Q}$ actually depended on the choice of an arbitrary well-order $\bar{\mathcal{W}}$ of ${}^\kappa\kappa$ and an arbitrary bookkeeping function $\bar{F}$ for $H(\kappa^+)$. Let us denote by $\mathcal{Q}_\kappa(\bar{\mathcal{W}}, \bar{F})$ the forcing $\mathcal{Q}$ at κ relative to a particular choice of well-order $\bar{\mathcal{W}}$ and bookkeeping function $\bar{F}$. We may also assume that $\mathcal{Q}_\kappa(\bar{\mathcal{W}}, \bar{F})$ is $<\kappa$-directed closed (by passing to its $<\kappa$-directed closed dense subset). Now let $\mathcal{Q}_\kappa$ be the two-step iteration which in the first step performs a lottery of all well-orders of ${}^\kappa\kappa$ and all bookkeeping functions for $H(\kappa^+)$ and thus chooses some particular $\bar{\mathcal{W}}_\kappa$ and $\bar{F}_\kappa$, and in the second step forces with $\mathcal{Q}_\kappa(\bar{\mathcal{W}}_\kappa, \bar{F}_\kappa)$.

We are now ready to give the definition of the global iteration $\mathbb{P}$ that will serve as a witness for Theorem 1.2 and Theorem 1.3. For either theorem, the forcing $\mathbb{P}$ is defined as follows.

Definition 6.1. Let $\mathbb{P}$ be the reverse Easton iteration which is trivial at stage κ unless κ is an uncountable cardinal satisfying $\kappa^{<\kappa} = \kappa$ that is not a counterexample to the SCH, in which case we force with $\mathcal{Q}_\kappa$ at stage κ.

Proof of Theorem 1.2 and Theorem 1.3. That $\mathbb{P}$ preserves ZFC, cofinalities and the continuum function under the assumption of SCH are standard arguments. That $\mathbb{P}$ introduces the relevant lightface definable well-orders follows by Theorem 1.1 and the fact that tails of $\mathbb{P}$ are sufficiently closed. The slightly more complicated statements when one doesn't assume the SCH follow by the same arguments (except that they may now fail at *counterexamples to the* SCH). $\square$

7 Large Cardinal Preservation

Various large cardinals can be preserved using (sometimes slight variations of) arguments to be found in the literature. We will state some of the resulting lemmas and mostly refer to the relevant articles for the proofs. Note that it is immediate that $\mathbb{P}$ preserves the strong inaccessibility of all strongly inaccessible cardinals.

Lemma 7.1. *Assume κ is λ-supercompact, $\kappa \leq \lambda$, $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$ and λ is not a counterexample to the* SCH. *Then forcing with $\mathbb{P}$ preserves the λ-supercompactness of κ.*

Proof-Sketch: The proof of this lemma is essentially as for [2, Theorem 4.1]. The only additional argument needed is that below any condition $p \in \mathbb{P}$, there is $q \leq p$ such that q chooses $\bar{\mathcal{W}}_\nu$ and $\bar{F}_\nu$ for every ν with $\theta \leq \nu \leq \lambda$. This implies that $\mathbb{P}(q)_\lambda$ has a dense subset of size λ and also $\mathcal{Q}_\lambda(q(\lambda))$ is sufficiently small (in a sense specified in (1) in the statement of [2, Theorem 4.1]) for the proof of [2, Theorem 4.1] to go through. Moreover one has to verify (in a straightforward way) that a suitable adaption of (2) from [2, Theorem 4.1] holds for $\mathcal{Q}_\lambda(\bar{\mathcal{W}}_\lambda, \bar{F}_\lambda)$.[23] $\square$

Under sufficient GCH hypothesis, this shows that $\mathbb{P}$ preserves all supercompact cardinals as in [2]. Note however that one may well be in a situation where no λ as above exists. Thus we do not know whether it can be shown that $\mathbb{P}$ preserves supercompact cardinals in general. Our below results however will show (under stronger large cardinal assumptions) that it is consistent for supercompact cardinals to exist after forcing with $\mathbb{P}$. The next two lemmas are based on large cardinal preservation results presented in [4].

Lemma 7.2. *Given a hyperstrong cardinal κ, there is a condition in $\mathbb{P}$ forcing that the hyperstrength of κ is preserved. The same is true with hyperstrength replaced (in both the assumption and conclusion of the above statement) by n-superstrength for any n with $2 \leq n < \omega$.*

Proof. Exactly as in the proof of [4, Theorem 9] (note that no GCH assumption is either made or needed there). $\square$

Lemma 7.3. *Given an ω-superstrong cardinal κ, there is a condition in $\mathbb{P}$ forcing that the ω-superstrength of κ is preserved.*

Proof. Essentially as in the proof of [4, Theorem 2]. Let $j\colon \mathbf{V} \to \mathbf{M}$ be the embedding witnessing that κ is ω-superstrong. Note that $\mathbb{P}$ is trivial at the singular cardinal $j^\omega(\kappa)$ and therefore the tail of the iteration starting from $j^\omega(\kappa)$ is $(j^\omega(\kappa))^+$-closed (and therefore one can use the argument of [4, Lemma 3] to generate the tail of the generic starting from $j^\omega(\kappa)$). However for the proof of [4, Lemma 4] to go through, one needs to choose (slightly different to the proof in [4], letting $G_{j^\omega(\kappa)}$ denote the $\mathbb{P}_{j^\omega(\kappa)}$-generic filter as in [4]) $p \in G_{j^\omega(\kappa)}$ such that p reduces $f(\bar{a})$ below $j^{n+1}(\kappa)$ whenever $\bar{a}$ belongs to $V_{j^n(\kappa)}$ and $f(\bar{a})$ is open dense on $\mathbb{P}_{j^\omega(\kappa)}$.[24] That such p exists is a standard reduction argument using that $V_{j^n(\kappa)}$ has size $j^n(\kappa)$, that $\mathbb{P}_{j^{n+1}(\kappa)}$ is $j^{n+1}(\kappa)$-cc and that the iteration $\mathbb{P}$ starting from $j^{n+1}(\kappa)$ is $j^{n+1}(\kappa)$-closed.[25] $\square$

[23] Since we consider the present lemma to be rather weak (it requires instances of the GCH to hold while our interest here is to work in a non-GCH context), we do not want to go into any further details of its proof but rather concentrate on other notions of large cardinals in the following.

[24] This means that for every relevant $\bar{a}$, there is an open dense subset $f^*(\bar{a})$ of $\mathbb{P}_{j^{n+1}(\kappa)}$ such that whenever $q \leq p$ is such that $q \restriction j^{n+1}(\kappa) \in f^*(\bar{a})$, then $q \in f(\bar{a})$.

[25] For example such a reduction argument is performed, in a somewhat more complicated context, in [5, Claim 23].

Note that starting with a proper class of large cardinals of any of the above kinds (i.e. hyperstrong or n-superstrong for $2 \leq n \leq \omega$) the arguments in [4] in fact show that the relevant large cardinal property is preserved for class-many of the given large cardinals by an easy density argument.[26]

The above leaves out one type of large cardinal treated in [4] for which the corresponding arguments seem not to work in our present context, namely superstrong cardinals. As is the case with supercompacts, we do not know whether superstrong cardinals can be preserved by the forcing in general.

8 Other global Iterations

In Section 6, we provided a class sized iteration that introduces a lightface definable well-order of $H(\kappa^+)$ whenever this is possible by the methods developed in Section 5. We could however define *sparser* iterations that only introduce lightface definable well-orders of $H(\kappa^+)$ for certain κ. This can of course allow for better results concerning preservation of large cardinals. For example, one could use this to give an alternative proof of the main result of [5] (using the large cardinal preservation techniques of [5]) by using the reverse Easton iteration $\mathbb{P}$ that only forces with $\mathcal{Q}_\kappa$ at stage κ if κ is inaccessible. We will restate this result here.

Theorem 8.1 (Friedman-Holy-Lücke, [5]). *Assume* SCH *holds at singular fixed points of the $\beth$-function. There is a class sized notion of forcing $\mathbb{P}$ such that the following hold.*

(i) Forcing with $\mathbb{P}$ introduces a lightface definable well-order of $H(\kappa^+)$ for every inaccessible κ.

(ii) $\mathbb{P}$ is cofinality-preserving and preserves the continuum function.

(iii) $\mathbb{P}$ preserves the supercompactness of all supercompact cardinals.

(iv) If κ is ω-superstrong then there is a condition in $\mathbb{P}$ that forces κ to remain ω-superstrong.

References

[1] Uri Abraham and Saharon Shelah, *Coding with ladders a well ordering of the reals*, The Journal of Symbolic Logic, vol. 67 (2002), 579-597.

[2] David Asperó and Sy D. Friedman, *Large cardinals and locally defined well-orders of the universe.* Annals of Pure and Applied Logic, vol. 157, 1 (2009), 1-15.

[3] David Asperó and Sy D. Friedman, *Definable well-orders of $H(\omega_2)$ and* GCH, The Journal of Symbolic Logic, vol. 77, 4 (2012), 1101-1121.

[26] That is, if ϕ denotes the large cardinal property in question, for every $p \in \mathbb{P}$ and every ordinal δ, there is $\kappa > \delta$ with $\phi(\kappa)$ and $q \leq p$ such that $q \Vdash \phi(\kappa)$.

[4] Sy D. Friedman, *Large Cardinals and L-like Universes*, Set theory: recent trends and applications, Quaderni di Matematica, vol. 17 (2007), pp. 93-110.

[5] Sy D. Friedman, Peter Holy and Philipp Lücke, *Large Cardinals and Lightface Definable Well-Orders, without the GCH*, submitted. Preprint available online at http://www.math.uni-bonn.de/people/pluecke/pub/wo_code.pdf.

[6] Sy D. Friedman and Philipp Lücke, *Large Cardinals and Definable Well-Orders, without the GCH*, accepted for Annals of Pure and Applied Logic. Preprint available online at http://www.math.uni-bonn.de/people/pluecke/pub/nogch.pdf.

[7] Peter Holy and Philipp Lücke, *Locally Σ_1-definable well-orders of $H(\kappa^+)$*, submitted. Preprint available online at http://www.math.uni-bonn.de/people/pluecke/pub/Sigma1WO.pdf.

[8] Peter Holy and Philipp Lücke, *Simplest possible locally definable Wellorders*, in preparation.

[9] Thomas Jech, *Set Theory: The Third Millenium Edition, Revised and Expanded*, Springer, Berlin (2002).

[10] Ronald B. Jensen and Robert M. Solovay, *Some applications of almost disjoint sets.* In *Mathematical Logic and Foundations of Set Theory (Proc. Internat. Colloq., Jerusalem, 1968)*, pp. 84-104. North-Holland, Amsterdam, 1970.

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