UNBOUNDED TOWERS AND PRODUCTS

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Abstract. We investigate products of sets of reals with combinatorial covering properties. A topological space satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ if for each sequence of point-cofinite open covers of the space, one can pick one element from each cover and obtain a point-cofinite cover of the space. We prove that, if there is an unbounded tower, then there is a nontrivial set of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers. In contrast to earlier results, our proof does not require any additional set-theoretic assumptions.

A topological space satisfies $\binom{\Omega}{\Gamma}$ (also known as Gerlits–Nagy's property γ) if every open cover of the space such that each finite subset of the space is contained in a member of the cover, contains a point-cofinite cover of the space. We investigate products of sets satisfying $\binom{\Omega}{\Gamma}$ and their relations to other classic combinatorial covering properties. We show that finite products of sets with a certain combinatorial structure satisfy $\binom{\Omega}{\Gamma}$ and give necessary and sufficient conditions when these sets are productively $\binom{\Omega}{\Gamma}$.

1. Introduction

Let $\mathbb{N}$ be the set of natural numbers and $[\mathbb{N}]^\infty$ be the set of infinite subsets of $\mathbb{N}$. In a natural way an element of $[\mathbb{N}]^\infty$ can be viewed as an increasing function in $\mathbb{N}^\mathbb{N}$. A subset of $[\mathbb{N}]^\infty$ is *unbounded* if for each element $a \in [\mathbb{N}]^\infty$, there is an element b in the set such that the set $\{\, n : a(n) > b(n) \,\}$ is infinite. A set a is an *almost subset* of a set b, denoted $a \subseteq^* b$, if the set $a \setminus b$ is finite.

Definition 1.1. Let κ be an uncountable ordinal number. A set $\{\, x_\alpha : \alpha < \kappa \,\} \subseteq [\mathbb{N}]^\infty$ is a *κ-unbounded tower* if it is unbounded and, for all ordinal numbers $\alpha, \beta < \kappa$ with $\alpha < \beta$, we have $x_\alpha \,{}^*\!\supseteq x_\beta$.

For an uncountable ordinal number κ, the existence of a κ-unbounded tower is independent of ZFC [15]. It turns out that such a set is a significant object used in constructions nontrivial sets with combinatorial covering properties.

By *space* we mean a topological space. A *cover* of a space is a family of proper subsets of the space whose union is the entire space, and an *open* cover is a cover whose members are open subsets of the space. A *γ-cover* of a space is an infinite cover such that each point of the space belongs to all but finitely many sets of the cover. For a space X, let $\Gamma(X)$ be the family of all open γ-covers of the space X. A space X satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma(X)$, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ such that $\{\, U_n : n \in \mathbb{N} \,\} \in \Gamma(X)$. This property was introduced by Scheepers [23] and it was studied in the context of local properties of functions spaces [24, 25].

A *set of reals* is a space homeomorphic to a subset of the real line. We restrict our consideration to the realm of sets of reals. We identify the family $\mathrm{P}(\mathbb{N})$ of all subsets of the

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set $\mathbb{N}$ with the Cantor space $\{0,1\}^{\mathbb{N}}$, via characteristic functions. Since the Cantor space is homeomorphic to Cantor's set, every subspace of the space $\mathrm{P}(\mathbb{N})$ is considered as a set of reals. Let Fin be the family of all finite subsets of $\mathbb{N}$. Let κ be an uncountable ordinal number. A set $X \cup \mathrm{Fin}$ is a *κ-unbounded tower set*, if the set X is a κ-unbounded tower in $[\mathbb{N}]^\infty$. Let $\mathfrak{b}$ be the minimal cardinality of an unbounded subset of $[\mathbb{N}]^\infty$ Each $\mathfrak{b}$-unbounded tower set satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ ([25, Theorem 6], [15, Proposition 2.5]). Miller and Tsaban proved that, assuming some additional set-theoretic assumption, each $\mathfrak{b}$-unbounded tower set satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers [15, Theorem 2.8]. We prove that the same statement remains true with no extra assumption used by Miller and Tsaban. The proof method is new.

An *ω-cover* of a space is an open cover such that, each finite subset of the space is contained in a set from the cover. A space satisfies $\binom{\Omega}{\Gamma}$ if every ω-cover of the space contains a γ-cover of the space. This property was introduced by Gerlits and Nagy [9]. A *pseudointersection* of a family of infinite sets is an infinite set a with $a \subseteq^* b$ for all sets b in the family. A family of infinite sets is *centered* if the finite intersections of its elements, are infinite. Let $\mathfrak{p}$ be the minimal cardinality of a subfamily of $[\mathbb{N}]^\infty$ that is centered and has no pseudointersection. By the result of Orenshtein and Tsaban ([17, Theorem 3.6], [18, Theorem 6]), each $\mathfrak{p}$-unbounded tower set satisfies $\binom{\Omega}{\Gamma}$. A set is *productively* $\binom{\Omega}{\Gamma}$ if its product space with any set satisfying $\binom{\Omega}{\Gamma}$, satisfies $\binom{\Omega}{\Gamma}$. Miller, Tsaban and Zdomskyy proved that, each ω_1-unbounded tower set is productively $\binom{\Omega}{\Gamma}$ [16, Theorem 2.8]. We show that, if each set of cardinality less than $\mathfrak{p}$ is productively $\binom{\Omega}{\Gamma}$, then each $\mathfrak{p}$-unbounded tower set is productively $\binom{\Omega}{\Gamma}$. Moreover, the product space of finitely many $\mathfrak{p}$-unbounded tower sets, satisfies $\binom{\Omega}{\Gamma}$. We also consider products of sets satisfying $\binom{\Omega}{\Gamma}$ and their relations to classic combinatorial covering properties.

2. Generalized towers

We generalize the notion of κ-unbounded tower for an uncountable ordinal number κ. Let n,m be natural numbers with $n<m$. Define $[n,m) := \{\, i \in \mathbb{N} : n \leq i < m \,\}$. A set $x \in [\mathbb{N}]^\infty$ *omits* the interval $[n,m)$, if $x \cap [n,m) = \emptyset$.

Lemma 2.1 (Folklore [29, Lemma 2.13]). *A set $X \subseteq [\mathbb{N}]^\infty$ is unbounded if and only if for each function $a \in [\mathbb{N}]^\infty$, there is a set $x \in X$ that omits infinitely many intervals $[a(n), a(n+1))$.* □

Let κ be an uncountable ordinal number, and $\{\, x_\alpha : \alpha < \kappa \,\}$ be a κ-unbounded tower in $[\mathbb{N}]^\infty$. Fix a function $a \in [\mathbb{N}]^\infty$. By Lemma 2.1, there is an ordinal number $\alpha < \kappa$ such that the set x_α omits infinitely many intervals $[a(n), a(n+1))$. Thus, the set

$$b := \{\, n \in \mathbb{N} : x_\alpha \cap [a(n), a(n+1)) = \emptyset \,\}$$

is an element of $[\mathbb{N}]^\infty$ and

$$x_\alpha \cap \bigcup_{n\in b} [a(n), a(n+1)) = \emptyset.$$

For each ordinal number $\beta < \kappa$ with $\alpha < \beta$, we have $x_\alpha \,{}^*\!\supseteq x_\beta$, and thus

$$x_\beta \cap \bigcup_{n\in b} [a(n), a(n+1)) \in \mathrm{Fin}.$$

This observation motivates the following definition.

Definition 2.2. Let κ be an uncountable ordinal number. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \kappa$ is a *κ-generalized tower* if for each function $a \in [\mathbb{N}]^\infty$, there are sets $b \in [\mathbb{N}]^\infty$ and $S \subseteq X$ with $|S| < \kappa$ such that

$$x \cap \bigcup_{n\in b} [a(n), a(n+1)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S$.

Let κ be an uncountable ordinal number. Every κ-unbounded tower in $[\mathbb{N}]^\infty$ is a κ-generalized tower. The forthcoming Lemma 2.3 shows that the notion of κ-unbounded tower may capture wider class of sets than κ-unbounded towers.

A set $B \subseteq [\mathbb{N}]^\infty$ is *groupwise dense* if:

(1) for all sets $y \in [\mathbb{N}]^\infty$ and $b \in B$, if $y \subseteq^* b$, then $y \in B$,
(2) for each function $b \in [\mathbb{N}]^\infty$, there is a set $c \in [\mathbb{N}]^\infty$ such that $\bigcup_{n\in c}[b(n), b(n+1)) \in B$.

The *groupwise density number* $\mathfrak{g}$ is the minimal cardinality of a family of groupwise dense sets in $[\mathbb{N}]^\infty$ with empty intersection. For function $f, g \in \mathbb{N}^\mathbb{N}$, let $f \circ g \in \mathbb{N}^\mathbb{N}$ be a function such that $(f \circ g)(n) := f(g(n))$ for all natural numbers n.

Lemma 2.3. *Let κ be an uncountable ordinal number and λ be an ordinal number with $\lambda < \mathfrak{g}$. Let $X := \bigcup_{\alpha<\lambda} X_\alpha$ be a union of κ-generalized towers.*

(1) *If $\lambda < \mathrm{cf}(\kappa)$, then X is a κ-generalized tower.*
(2) *If $\lambda \geq \mathrm{cf}(\kappa)$, then X is a $(\kappa \cdot \lambda)^+$-generalized tower.*

Proof. (1) Fix a function $a \in [\mathbb{N}]^\infty$ and an ordinal number $\alpha < \lambda$. Let B_α be the set of all sets $b \in [\mathbb{N}]^\infty$ such that

$$x \cap \bigcup_{n\in b} [a(n), a(n+1)) \in \mathrm{Fin}, \tag{2.3.1}$$

for all but less than κ sets $x \in X_\alpha$. The set B_α is groupwise dense: Fix a set $c \in [\mathbb{N}]^\infty$. There is a set $b \in [\mathbb{N}]^\infty$ such that

$$x \cap \bigcup_{n\in b} [(a \circ c)(n), (a \circ c)(n+1)) \in \mathrm{Fin},$$

for all but less than κ sets $x \in X$. Let

$$d := \bigcup_{n\in b} [c(n), c(n+1)).$$

Fix a natural number $i \in d$. There is a natural number $n \in b$ with $i \in [c(n), c(n+1))$. Then $[a(i), a(i+1)) \subseteq [a(c(n)), a(c(n+1)))$. Thus,

$$\bigcup_{i\in d} [a(i), a(i+1)) \subseteq \bigcup_{n\in b} [a(c(n)), a(c(n+1))) = \bigcup_{n\in b} [(a \circ c)(n), (a \circ c)(n+1)).$$

It follows that $d \in B_\alpha$.

Since $\lambda < \mathfrak{g}$, there is a set $b \in \bigcap_{\alpha<\lambda} B_\alpha$. For each ordinal number $\alpha < \lambda$, there is a set $S_\alpha \subseteq X_\alpha$ with $|\, S_\alpha \,| < \kappa$ such that condition (2.3.1) holds for all sets $x \in X_\alpha \setminus S_\alpha$. Then condition (2.3.1) holds for all sets $x \in X \setminus \bigcup_{\alpha<\lambda} S_\alpha$. Since $|\bigcup_{\alpha<\lambda} S_\alpha\,| < \kappa$, the set X is a κ-generalized tower.

(2) Proceed as in (1) with the exception that $|\bigcup_{\alpha<\lambda} S_\alpha\,| < (\kappa \cdot \lambda)^+$. $\square$

3. Products of sets satisfying $\mathsf{S}_1(\Gamma,\Gamma)$

For a class $\mathcal{A}$ of covers of spaces and a space X, let $\mathcal{A}(X)$ be the family of all covers of X from the class $\mathcal{A}$. Let $\mathcal{A}$, $\mathcal{B}$ be classes of covers of spaces. A space X satisfies $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ if for each sequence $\mathcal{U}_1,\mathcal{U}_2,\dots \in \mathcal{A}(X)$, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2,\dots$ such that $\{ U_n : n \in \mathbb{N} \} \in \mathcal{B}(X)$. A *Borel* cover of a space is a cover whose members are Borel subsets of the space. Let Γ_{Bor} be the class of all Borel γ-covers of spaces. For a property of spaces $\mathbf{P}$, let $\mathrm{non}(\mathbf{P})$ be the minimal cardinality of a set of reals with no property $\mathbf{P}$. We have [12, Theorem 4.7], [26, Theorem 27(2)]

$$\mathrm{non}(\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})) = \mathrm{non}(\mathsf{S}_1(\Gamma,\Gamma)) = \mathfrak{b}.$$

In the Laver model, all sets of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ have cardinality strictly smaller than $\mathfrak{b}$ [15, Theorem 3.6], and thus they are trivial; it implies that the properties $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ and $\mathsf{S}_1(\Gamma,\Gamma)$ are equivalent. In order to construct *nontrivial* examples of sets of reals with the above properties, additional set-theoretic assumptions are needed. A set of reals of cardinality at least $\mathfrak{b}$ is a $\mathfrak{b}$-*Sierpiński set*, if its intersection with every Lebesgue-null set has cardinality less than $\mathfrak{b}$. Every $\mathfrak{b}$-Sierpiński set satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ and assuming that $\mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{b}$, such a set exists (definitions of cardinal characteristics of the continuum and relations between them can be found in the Blass survey [5]). For functions $a,b \in [\mathbb{N}]^\infty$, we write $a \leq^* b$ if the set $\{ n : b(n) < a(n) \}$ is finite. A set $A \subseteq [\mathbb{N}]^\infty$ is *bounded* if there is a function $b \in [\mathbb{N}]^\infty$ such that $a \leq^* b$ for all functions $a \in A$, denoted $A \leq^* b$. A subset of $[\mathbb{N}]^\infty$ is *dominating* if for every function $a \in [\mathbb{N}]^\infty$, there is a function b from the set such that $a \leq^* b$. Let $\mathfrak{d}$ be the minimal cardinality of a dominating set in $[\mathbb{N}]^\infty$. Assuming $\mathfrak{p} = \mathfrak{b}$ or $\mathfrak{b} < \mathfrak{d}$, a $\mathfrak{b}$-unbounded tower exists [15, Lemma 2.2]. Also if for an infinite ordinal number κ, there is a κ-generalized tower, then there is a $\mathfrak{b}$-generalized tower [15, Proposition 2.4]. Each $\mathfrak{b}$-unbounded tower set satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ ([25, Theorem 6], [15, Proposition 2.5]), but not $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ [15, Corollary 2.10].

A subfamily of $[\mathbb{N}]^\infty$ is *open* if it is closed under almost subsets, and it is *dense* if each set from $[\mathbb{N}]^\infty$ has an almost subset in the family. The *density number* $\mathfrak{h}$ is the minimal cardinality of a collection of open dense families in $[\mathbb{N}]^\infty$ with empty intersection. Let $\mathrm{add}(\mathsf{S}_1(\Gamma,\Gamma))$ be the minimal cardinality of a family of sets of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ whose union does not satisfy $\mathsf{S}_1(\Gamma,\Gamma)$. By the result of Scheepers [24, Theorem 5] we have $\mathrm{add}(\mathsf{S}_1(\Gamma,\Gamma)) \geq \mathfrak{h}$. Miller and Tsaban proved that, assuming $\mathrm{add}(\mathsf{S}_1(\Gamma,\Gamma)) = \mathfrak{b}$, each $\mathfrak{b}$-unbounded tower set satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ in all finite powers. By the result of Miller, Tsaban and Zdomskyy, the product space of a $\mathfrak{b}$-unbounded tower set and a set satisfying $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$, satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ [16, Theorem 7.1]. In the forthcoming Theorem 3.1, we generalize the above results. In the proof we develop methods of Miller, Tsaban and Zdomskyy [16, Theorem 7.1] and combine them with tools invented by Tsaban and the first named author [27, Lemma 5.1].

Theorem 3.1. *The product space of finitely many $\mathfrak{b}$-generalized tower sets and a set satisfying $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$, satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

In order to prove Theorem 3.1, we need the following notions and auxiliary results. Let $\overline{\mathbb{N}}$ be the set $\mathbb{N} \cup \{\infty\}$ with the discrete topology and $n < \infty$ for all natural numbers n. In the space $\overline{\mathbb{N}}^{\mathbb{N}}$, define relation $\leq^*$, analogously as in $[\mathbb{N}]^\infty$. For a set $x \in \mathrm{P}(\mathbb{N})$, let $x^{\mathsf{c}} := \mathbb{N} \setminus x$.

Lemma 3.2. *Let m be a natural number, Y be a set satisfying $\mathsf{S}_1(\Gamma_{\mathrm{Bor}},\Gamma_{\mathrm{Bor}})$ and $\{ U_n : n \in \mathbb{N} \} \in \Gamma(\mathrm{Fin}^m \times Y)$ be a family of open sets in $\mathrm{P}(\mathbb{N})^m \times Y$.*

(1) *There are Borel functions* $f, g\colon Y \to \overline{\mathbb{N}}^{\mathbb{N}}$ *such that for all points* $x \in ([\mathbb{N}]^{\infty})^m$ *and* $y \in Y$, *and all natural numbers* n:

If $x_i \cap [f(y)(n), g(y)(n)) = \emptyset$ *for all natural numbers* $i \leq m$, *then* $(x, y) \in U_n$.

(2) *There is an increasing function* $c \in \mathbb{N}^{\mathbb{N}}$ *such that for each point* $y \in Y$, *we have*

$$c(n) \leq f(y)(c(n+1)) \leq g(y)(c(n+1)) < c(n+2),$$

for all but finitely many natural numbers n.

Proof. (1) Define a function $d\colon Y \to \mathbb{N}^{\mathbb{N}}$ as follows: Fix a point $y \in Y$. Let $d(y)(1)$ be the minimal natural number such that

$$\{\emptyset\}^m \times \{y\} \subseteq \bigcap_{k \geq d(y)(1)} U_k$$

and, for each natural number i, let $d(y)(i+1)$ be the minimal natural number greater than $d(y)(i)$ such that

$$\mathrm{P}([1, i+1))^m \times \{y\} \subseteq \bigcap_{k \geq d(y)(i+1)} U_k.$$

The function d is Borel: We have

$$\{\, y \in Y : d(y)(1) = 1 \,\} = \left\{ y \in Y : \{\emptyset\}^m \times \{y\} \subseteq \bigcap_{k \geq 1} U_k \right\},$$

and

$$\{\, y \in Y : d(y)(1) = j+1 \,\} = \left\{ y \in Y : \{\emptyset\}^m \times \{y\} \subseteq \bigcap_{k \geq j+1} U_k \setminus U_j \right\},$$

for all natural numbers j; the above sets are Borel. Fix natural numbers i, j, and assume that the set

$$\{\, y \in Y : d(y)(i) = j \,\}$$

is Borel. Fix a point $y \in Y$ such that $d(y)(i+1) = j+1$. We have

$$\mathrm{P}([1, i+1))^m \times \{y\} \subseteq \bigcap_{k \geq j+1} U_k.$$

If $\mathrm{P}([1, i+1))^m \times \{y\} \subseteq U_j$, then $d(y)(i) = j$, otherwise there is an element $t \in \mathrm{P}([1, i+1))^m$ with $(t, y) \notin U_j$. Thus,

$$\begin{aligned}\{\, y \in Y : d(y)(i+1) = j+1 \,\} = \bigcap_{t \in \mathrm{P}([1,i+1))^m} \{\, y \in Y : (t, y) \in \bigcap_{k \geq j+1} U_k \,\} \cap \\ \left(\{\, y \in Y : d(y)(i) = j \,\} \cup \bigcup_{t \in \mathrm{P}([1,i+1))^m} \{\, y \in Y : (t, y) \notin U_j \,\} \right).\end{aligned}$$

Then the set $\{\, y \in Y : d(y)(i+1) = j+1 \,\}$ is Borel.

For a point $y \in Y$, let $f(y) \in \overline{\mathbb{N}}^{\mathbb{N}}$ be a function such that:

$$f(y)(n) := \begin{cases} \infty, & \text{for } n \in [1, d(y)(1)), \\ i, & \text{for } n \in [d(y)(i), d(y)(i+1)), i \in \mathbb{N}. \end{cases}$$

For natural numbers i, n, we have

$$\{\, y \in Y : f(y)(n) = \infty \,\} = \{\, y \in Y : n < d(y)(1) \,\}$$

and

$$\{\, y \in Y : f(y)(n) = i \,\} = \{\, y \in Y : d(y)(i) \leq n < d(y)(i+1) \,\}.$$

Since the function d is Borel, the function $f\colon Y \to \overline{\mathbb{N}}^{\mathbb{N}}$, defined above, is Borel as well.

For each natural number n, let $U_1^{(n)}, U_2^{(n)}, \dots$ be an increasing sequence of clopen sets in $\mathrm{P}(\mathbb{N})^m \times Y$ such that $U_n = \bigcup_k U_k^{(n)}$. Fix a point $y \in Y$. Define a function $g(y) \in \overline{\mathbb{N}}^{\mathbb{N}}$ in the following way: Let n be a natural number. If $f(y)(n) = \infty$, then $g(y)(n) := \infty$. Assume that $f(y)(n) \in \mathbb{N}$. There is a natural number $j > f(y)(n)$ such that

$$\mathrm{P}([f(y)(n), j)^{\mathsf{c}})^m \times \{y\} \subseteq U_j^{(n)} :$$

Let k be the minimal natural number with

$$\mathrm{P}([1, f(y)(n)))^m \times \{y\} \subseteq U_k^{(n)}.$$

Let j be the minimal natural number with $j > f(y)(n)$ and

$$\mathrm{P}([f(y)(n), j)^{\mathsf{c}})^m \times \{y\} \subseteq U_k^{(n)}.$$

Since the sets $U_1^{(n)}, U_2^{(n)}, \dots$ are ascending, we have

$$\mathrm{P}([f(y)(n), j)^{\mathsf{c}})^m \times \{y\} \subseteq U_j^{(n)}.$$

Let $g(y)(n) > f(y)(n)$ be the minimal natural number with

$$\mathrm{P}([f(y)(n), g(y)(n))^{\mathsf{c}})^m \times \{y\} \subseteq U_{g(y)(n)}^{(n)}.$$

The set $\mathrm{P}([f(y)(n), g(y)(n))^{\mathsf{c}})^m$ is compact, and thus there is an open set V_n^y in Y such that

$$\mathrm{P}([f(y)(n), g(y)(n))^{\mathsf{c}})^m \times V_n^y \subseteq U_{g(y)(n)}^{(n)}.$$

The function g is Borel: Fix a natural number n. We have $\{\, y \in Y : g(y)(n) = \infty \,\} = \{\, y \in Y : f(y)(n) = \infty \,\}$. Since the function f is Borel, the latter set is Borel. For a natural number k, we have

$$\{\, y \in Y : g(y)(n) = k \,\} = \bigcup_{i<k} \{\, y \in Y : f(y)(n) = i \wedge g(y)(n) = k \,\}.$$

In order to show that such a set is Borel, it is enough to show that for natural numbers i, k with $i < k$, the set $\{\, y \in Y : f(y)(n) = i \wedge g(y)(n) = k \,\}$ is Borel: Fix natural numbers i, k with $i < k$. Let $y \in Y$ be a point such that $f(y)(n) = i$ and $g(y)(n) = k$. For each point $y' \in V_n^y$ such that $f(y')(n) = i$, we have

$$\mathrm{P}([f(y')(n), g(y)(n)^{\mathsf{c}})^m \times \{y'\} \subseteq \mathrm{P}([f(y)(n), g(y)(n))^{\mathsf{c}})^m \times V_n^y \subseteq U_{g(y)(n)}^{(n)}.$$

By the minimality of the number $g(y')(n)$, we have

$$g(y')(n) \leq g(y)(n). \tag{3.2.1}$$

If $k = i + 1$, then $g(y')(n) = g(y)(n)$, and thus the set $\{\, y \in Y : f(y)(n) = i \wedge g(y)(n) = k \,\}$ is Borel (in fact, it is even open). Now assume that for a natural number $k > i + 1$, the set $\{\, y \in Y : f(y)(n) = i \wedge g(y)(n) < k \,\}$ is Borel. By the above inequality (3.2.1), we have

$$\{\, y \in Y : f(y)(n) = i \wedge g(y)(n) = k \,\} =$$

$$\Big(\{\, y \in Y : f(y)(n) = i \,\} \cap \bigcup \{\, V_n^y : g(y)(n) = k \,\}\Big) \setminus \{\, y \in Y : f(y)(n) = i \wedge g(y)(n) < k \,\}.$$

Since the function f is Borel, the set $\{\, y \in Y : f(y)(n) = i \wedge g(y)(n) = k \,\}$ is Borel.

(2) Let d and g be the functions from the proof of (1). For each point $y \in Y$, we have $g(y)(n) = \infty$ for finitely many natural numbers n. Since the functions d and g are Borel and the set Y satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$, there is an increasing function $b \in \mathbb{N}^{\mathbb{N}}$ such that $\{\, d(y), g(y) : y \in Y \,\} \leq^* b$ [26, Theorem 1]. We may assume that $b(1) \neq 1$. Let $c(1) := b(1)$ and $c(n+1) := b(c(n))$ for all natural numbers n. For each point $y \in Y$, we have

$$c(n) \leq f(y)(c(n+1)) \leq g(y)(c(n+1)) < c(n+2)$$

for all but finitely many natural numbers n: Fix a point $y \in Y$. There is a natural number l such that

$$d(y)(n) \leq b(n) \text{ and } g(y)(n) \leq b(n),$$

for all natural numbers $n \geq l$. Let n be a natural number with $n \geq l$. Since

$$d(y)(c(n)) \leq b(c(n)) = c(n+1)$$

and the function $f(y)$ is nondecreasing for arguments greater than or equal to $d(y)(1)$, we have

$$c(n) = f(y)(d(y)(c(n))) \leq f(y)(c(n+1)) \leq g(y)(c(n+1)) \leq b(c(n+1)) = c(n+2). \quad \square$$

Lemma 3.3. *Let X be a space such that for every sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma(X)$, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ and a space $X' \subseteq X$ satisfying $\mathsf{S}_1(\Gamma, \Gamma)$ such that $\{\, U_n : n \in \mathbb{N} \,\} \in \Gamma(X \setminus X')$. Then the space X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.*

Proof. Let $\mathcal{U}_n = \{\, U^{(n)}_m : m \in \mathbb{N} \,\}$ for all natural numbers n. We may assume that for all natural numbers n, m, we have

$$U^{(n+1)}_m \subseteq U^{(n)}_m.$$

For each natural number n, we have $\{\, U^{(n)}_m : m \geq n \,\} \in \Gamma(X)$. Then there are a function $f \in \mathbb{N}^{\mathbb{N}}$ and a space $X' \subseteq X$ such that $\{\, U^{(n)}_{f(n)} : n \in \mathbb{N} \,\} \in \Gamma(X \setminus X')$. The range of the function f is infinite, and thus there is a function $d \in [\mathbb{N}]^\infty$ such that $(f \circ d) \in [\mathbb{N}]^\infty$. Let $f' := f \circ d$. Since $U^{(d(n))}_{f'(n)} \subseteq U^{(n)}_{f'(n)}$, we have $\{\, U^{(n)}_{f'(n)} : n \in \mathbb{N} \,\} \in \Gamma(X \setminus X')$.

Since the space X' satisfies $\mathsf{S}_1(\Gamma, \Gamma)$, the set

$$\left\{\, a \in [\mathbb{N}]^\infty : \{\, U^{(n)}_{a(n)} : n \in \mathbb{N} \,\} \in \Gamma(X') \,\right\}$$

is open and dense [24, Theorem 5]. Thus, there is a function $a \in [\mathbb{N}]^\infty$ with $a \subseteq f'$ such that $\{\, U^{(n)}_{a(n)} : n \in \mathbb{N} \,\} \in \Gamma(X')$. There is a function $c \in [\mathbb{N}]^\infty$ such that $a = f' \circ c$. Then $\{\, U^{(c(n))}_{a(n)} : n \in \mathbb{N} \,\} \in \Gamma(X \setminus X')$. Since $U^{(c(n))}_{a(n)} \subseteq U^{(n)}_{a(n)}$, we have $\{\, U^{(n)}_{a(n)} : n \in \mathbb{N} \,\} \in \Gamma(X)$. $\square$

Proof of Theorem 3.1. Let Y be a set satisfying $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$.

Let X be a $\mathfrak{b}$-generalized tower and $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma((X \cup \mathrm{Fin}) \times Y)$ be a sequence of families of open sets in $\mathrm{P}(\mathbb{N}) \times Y$ such that $\mathcal{U}_k = \{\, U^{(k)}_n : n \in \mathbb{N} \,\}$ for all natural numbers k. For each natural number k, let f_k, g_k, c_k be functions from Lemma 3.2 applied to the family $\mathcal{U}_k$. Let $h \colon Y \to \mathbb{N}^{\mathbb{N}}$ be a function such that $h(y)(k)$ is the minimal natural number n such that for all natural numbers j with $j \geq n$, we have

$$c_k(j) \leq f_k(y)(c_k(j+1)) < g_k(y)(c_k(j+1)) < c_k(j+2).$$

Since the functions f_k and g_k are Borel, so is the function h, and hence there is an increasing function $z \in \mathbb{N}^{\mathbb{N}}$ such that $\{\, h(y) : y \in Y \,\} \leq^* z$. We may assume that

$$c_{k+1}(z(k+1)) > c_k(z(k) + 2)$$

for all natural numbers k. Since the set X is a $\mathfrak{b}$-generalized tower, there are a set $b \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|S| < \mathfrak{b}$ such that

$$x \cap \bigcup_{k \in b} [c_k(z(k)), c_k(z(k)+2)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S$.

We have $\{ U^{(k)}_{c_k(z(k+1))} : k \in \mathbb{N} \} \in \Gamma((X \setminus S) \times Y)$: Fix points $x \in X \setminus S$ and $y \in Y$. There is a natural number l such that for all natural numbers $k \geq l$, we have

$$c_k(z(k)) \leq f_k(y)(c_k(z(k)+1)) < g_k(y)(c_k(z(k)+1)) < c_k(z(k)+2),$$
$$x \cap [c_k(z(k)), c_k(z(k)+2)) = \emptyset,$$

and thus

$$x \cap [f_k(y)(z(k+1)), g_k(y)(z(k+1))) = \emptyset.$$

By Lemma 3.2(1),

$$(x, y) \in \bigcap_{k \geq l} U^{(k)}_{z(k+1)}.$$

Since $|S| < \mathfrak{b}$ and $\mathrm{add}(\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})) = \mathfrak{b}$ [28, Corollary 2.4], the product space $(S \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$. By Lemma 3.3, the product space $(X \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.

Let m be a natural number with $m > 1$ and assume that the statement is true for $m-1$ $\mathfrak{b}$-generalized tower sets. Let $X_1, \ldots, X_m$ be $\mathfrak{b}$-generalized towers in $[\mathbb{N}]^\infty$ and

$$\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma((X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y)$$

be a sequence of families of open sets in $\mathrm{P}(\mathbb{N})^m \times Y$ such that $\mathcal{U}_k = \{ U^{(k)}_n : n \in \mathbb{N} \}$ for all natural numbers k. We proceed analogously to the previous case. For each natural number k, let f_k, g_k, c_k be functions from Lemma 3.2 applied to the family $\mathcal{U}_k$. Let $h\colon Y \to \mathbb{N}^\mathbb{N}$ be a function such that $h(y)(k)$ is the minimal natural number n such that for all natural numbers j with $j \geq n$, we have

$$c_k(j) \leq f_k(y)(c_k(j+1)) < g_k(y)(c_k(j+1)) < c_k(j+2).$$

Since the functions f_k and g_k are Borel, so is the function h, and hence there is an increasing function $z \in \mathbb{N}^\mathbb{N}$ such that $\{ h(y) : y \in Y \} \leq^* z$. We may assume that

$$c_{k+1}(z(k+1)) > c_k(z(k)+2)$$

for all natural numbers k. By Lemma 2.3(1), the set $X := \bigcup_{i \leq m} X_i$ is a $\mathfrak{b}$-generalized tower. Then there are a set $b \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|S| < \mathfrak{b}$ such that

$$x \cap \bigcup_{k \in b} [c_k(z(k)), c_k(z(k)+2)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S$.

We have $\{ U^{(k)}_{c_k(z(k+1))} : k \in \mathbb{N} \} \in \Gamma((X \setminus S)^m \times Y)$: Fix points $x \in (X \setminus S)^m$ and $y \in Y$. There is a natural number l such that for all natural numbers $k \geq l$, we have

$$c_k(z(k)) \leq f_k(y)(c_k(z(k)+1)) < g_k(y)(c_k(z(k)+1)) < c_k(z(k)+2),$$
$$x_i \cap [c_k(z(k)), c_k(z(k)+2)) = \emptyset \text{ for all natural numbers } i \leq m,$$

and thus

$$x_i \cap [f_k(y)(z(k+1)), g_k(y)(z(k+1))) = \emptyset \text{ for all natural numbers } i \leq m.$$

By Lemma 3.2(1),

$$(x, y) \in \bigcap_{k \geq l} U^{(k)}_{z(k+1)}.$$

Since $|S| < \mathfrak{b}$ and $\operatorname{add}(\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})) = \mathfrak{b}$ [28, Corollary 2.4], the product space $((X_j \cap S) \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$ for all natural numbers $j \leq m+1$, and the set

$$\bigcup_{j \leq m} \prod_{\substack{i \leq m \\ i \neq j}} (X_i \cup \mathrm{Fin}) \times ((X_j \cap S) \cup \mathrm{Fin}) \times Y$$

satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. By Lemma 3.3, the product space $(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. $\square$

Corollary 3.4. *Each $\mathfrak{b}$-generalized tower set satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ in all finite powers.*

The properties $\mathsf{S}_1(\Gamma, \Gamma)$ and $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$ are closely related to local properties of functions spaces. Let X be a space and $\mathrm{C_p}(X)$ be the set of all continuous real-valued functions on X with the pointwise convergence topology. A sequence $f_1, f_2, \ldots \in \mathrm{C_p}(X)$ converges *quasinormally* to the constant zero function $\mathbf{0}$, if there is a sequence of positive real numbers $\epsilon_1, \epsilon_2, \ldots$ converging to zero such that for any point $x \in X$, we have $|f_n(x)| < \epsilon_n$ for all but finitely many natural numbers n. A space X is a *QN-space* (*wQN-space*) if every sequence $f_1, f_2, \ldots \in \mathrm{C_p}(X)$ converging pointwise to $\mathbf{0}$, converges (has a subsequence converging) quasinormally to $\mathbf{0}$. By a breakthrough result of Tsaban and Zdomskyy [31, Theorem 2], a space is a QN-space if and only if all Borel images of the space in $\mathbb{N}^{\mathbb{N}}$ are bounded [26, Theorem 1]. In consequence, a space satisfies $\mathsf{S}_1(\Gamma_{\mathrm{Bor}}, \Gamma_{\mathrm{Bor}})$ if and only if it is a QN-space. Every perfectly normal space satisfying $\mathsf{S}_1(\Gamma, \Gamma)$, is a wQN-space [7, Theorem 7]. QN-spaces, wQN-spaces and their variations were extensively studied by Bukovský, Haleš, Recław, Sakai and Scheepers [6, 7, 8, 11, 20, 21, 24]. We have the following corollary from Theorem 3.1.

Corollary 3.5. *The product space of finitely many $\mathfrak{b}$-generalized tower sets and a QN-space, is a wQN-space.*

These results can also be formulated as dealing with Arhangel'skiĭ's properties α_i for spaces of continuous real-valued functions [25, 1, 2].

4. Generalized towers and productivity of $\binom{\Omega}{\Gamma}$

A space is *Fréchet–Urysohn* if each point in the closure of a set is a limit of a sequence from the set. By the celebrating result of Gerlits and Nagy, for a set of reals X, the space $\mathrm{C_p}(X)$ is Fréchet–Urysohn if and only if the set X satisfies $\binom{\Omega}{\Gamma}$ [9, Theorem 2]. The property $\binom{\Omega}{\Gamma}$ is *productive* if each space satisfying $\binom{\Omega}{\Gamma}$, is productively $\binom{\Omega}{\Gamma}$. The property $\binom{\Omega}{\Gamma}$ is preserved by finite powers, but productivity of $\binom{\Omega}{\Gamma}$ is independent of ZFC. In the Laver model, all sets of reals satisfying $\binom{\Omega}{\Gamma}$ are countable ([9, Theorem 2], [23, Theorem 17], [13]), and thus $\binom{\Omega}{\Gamma}$ is productive; if the Continuum Hypothesis holds, then $\binom{\Omega}{\Gamma}$ is not productive [16, Theorem 3.2.]. Miller, Tsaban and Zdomskyy proved that, each ω_1-unbounded tower set is productively $\binom{\Omega}{\Gamma}$ [16, Theorem 2.8.]. The main result of this section is the following Theorem.

Theorem 4.1.

(1) *The product space of finitely many $\mathfrak{p}$-generalized tower sets, satisfies $\binom{\Omega}{\Gamma}$.*
(2) *Assume that there is a $\mathfrak{p}$-generalized tower in $[\mathbb{N}]^{\infty}$. The following assertions are equivalent:*

(a) *Each set of reals of cardinality smaller than* $\mathfrak{p}$ *is productively* $\binom{\Omega}{\Gamma}$.
(b) *Each* $\mathfrak{p}$*-generalized tower set is productively* $\binom{\Omega}{\Gamma}$.

Let Ω be the class of all ω-covers of spaces.

Lemma 4.2 (Galvin–Miller[10, Lemma 1.2])**.** *Let* $\mathcal{U}$ *be a family of open sets in* $\mathrm{P}(\mathbb{N})$ *such that* $\mathcal{U} \in \Omega(\mathrm{Fin})$*. There are a function* $a \in [\mathbb{N}]^\infty$ *and sets* $U_1, U_2, \ldots \in \mathcal{U}$ *such that for each set* $x \in [\mathbb{N}]^\infty$ *and all natural numbers* n*:*

$$\text{If } x \cap [a(n), a(n+1)) = \emptyset, \text{ then } x \in U_n.$$

For spaces X and Y, let $X \sqcup Y$ be the *disjoint union* of these spaces. The product space $X \times Y$ satisfies $\binom{\Omega}{\Gamma}$ if and only if $X \sqcup Y$ satisfies $\binom{\Omega}{\Gamma}$ [14, Proposition 2.3]. For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq b$ if $a(n) \leq b(n)$ for all natural numbers n.

Lemma 4.3. *Let* $X \subseteq [\mathbb{N}]^\infty$ *be a* $\mathfrak{p}$*-generalized tower and* Y *be a set such that for every subset* $S \subseteq X$ *with* $|\,S\,| < \mathfrak{p}$*, the product space* $S \times Y$ *satisfies* $\binom{\Omega}{\Gamma}$*. Then the product space* $(X \cup \mathrm{Fin}) \times Y$ *satisfies* $\binom{\Omega}{\Gamma}$*.*

Proof. Let $\mathcal{U} \in \Omega((X \cup \mathrm{Fin}) \sqcup Y)$ be a family of open sets in $\mathrm{P}(\mathbb{N}) \sqcup \mathrm{P}(\mathbb{N})$. Let $S_1 := \mathrm{Fin}$. Fix a natural number $k > 1$, and assume that the set $S_{k-1} \subseteq X$ with $\mathrm{Fin} \subseteq S_{k-1}$ and $|\,S_{k-1}\,| < \mathfrak{p}$ has been already defined. Since $|\,S_{k-1}\,| < \mathfrak{p}$, there is a subfamily $\mathcal{U}'$ of $\mathcal{U}$ with $\mathcal{U}' \in \Gamma(S_{k-1} \sqcup Y)$. Apply Lemma 4.2 to the family $\mathcal{U}'$. Then there are a function $a_k \in [\mathbb{N}]^\infty$ and sets $U_1^{(k)}, U_2^{(k)}, \ldots \in \mathcal{V}$ such that for each set $x \in [\mathbb{N}]^\infty$ and all natural numbers n:

$$\text{If } x \cap [a_k(n), a_k(n+1)) = \emptyset, \text{ then } x \in U_n^{(k)}. \tag{4.3.1}$$

Since the set X is a $\mathfrak{p}$-generalized tower, there are a set $b_k \in [\mathbb{N}]^\infty$ and a set $S_k \subseteq X$ with $S_{k-1} \subseteq S_k$ and $|\,S_k\,| < \mathfrak{p}$ such that

$$x \cap \bigcup_{n \in b_k} [a_k(n), a_k(n+1)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S_k$. Since $\mathcal{U}' \in \Gamma(Y)$, we have

$$\{\, U^{(k)}_{b_k(j)} : j \in \mathbb{N} \,\} \in \Gamma(((X \setminus S_k) \cup S_{k-1}) \sqcup Y).$$

There is a function $a \in [\mathbb{N}]^\infty$ such that for each natural number k, we have

$$|(a_k \circ b_k) \cap [a(n), a(n+1))| \geq 2,$$

for all but finitely many natural numbers n. Since the set X is a $\mathfrak{p}$-generalized tower, there are a set $b \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|\,S\,| < \mathfrak{p}$ such that

$$x \cap \bigcup_{n \in b} [a(n), a(n+1)) \in \mathrm{Fin}. \tag{4.3.2}$$

for all sets $x \in X \setminus S$. We may assume that $\bigcup_k S_k \subseteq S$. The sets

$$b'_k := \left\{\, i \in b_k : [a_k(i), a_k(i+1)) \subseteq \bigcup_{n \in b} [a(n), a(n+1)) \,\right\} \tag{4.3.3}$$

are infinite for all natural numbers k. Thus,

$$\left\{\, U^{(k)}_{b'_k(j)} : j \in \mathbb{N} \,\right\} \in \Gamma(((X \setminus S_k) \cup S_{k-1}) \sqcup Y).$$

Since the sequence of the sets S_k is increasing, we have $X = \bigcup_k (X \setminus S_k) \cup S_{k-1}$ and each point of X belongs to all but finitely many sets $(X \setminus S_k) \cup S_{k-1}$. For each point $x \in S$, define

$$g_x(k) := \begin{cases} 0, & \text{if } x \notin (X \setminus S_k) \cup S_{k-1}, \\ \min\{\, j : x \in \bigcap_{i\geq j} U^k_{b'_k(i)} \,\}, & \text{if } x \in (X \setminus S_k) \cup S_{k-1}. \end{cases}$$

Since $|\,S\,| < \mathfrak{p}$, there is a function $g \in \mathbb{N}^{\mathbb{N}}$ with $\{\, g_x : x \in S \,\} \leq^* g$ and

$$a_k(b'_k(g(k) + 1)) < a_{k+1}(b'_{k+1}(g(k + 1))), \tag{4.3.4}$$

for all natural numbers k. Let

$$\mathcal{W}_k := \left\{ U^{(k)}_{b'_k(j)} : j \geq g(k) \right\}$$

for all natural numbers k. Then $\mathcal{W}_1, \mathcal{W}_2, \ldots \in \Gamma(Y)$. We may assume that families $\mathcal{W}_k$ are pairwise disjoint. Since the properties $\binom{\Omega}{\Gamma}$ and $\mathsf{S}_1(\Omega, \Gamma)$ are equivalent [9, Theorem 2], the set Y satisfies $\mathsf{S}_1(\Omega, \Gamma)$. Then there is a function $h \in \mathbb{N}^{\mathbb{N}}$ such that $g \leq h$ and

$$\left\{ U^{(k)}_{b'_k(h(k))} : k \in \mathbb{N} \right\} \in \Gamma(S \sqcup Y).$$

Fix a set $x \in X \setminus S$. By (4.3.3), for each natural number k, we have

$$\bigcup_{n\in b'_k} [a_k(n), a_k(n + 1)) \subseteq \bigcup_{n\in b} [a(n), a(n + 1)).$$

By (4.3.2), (4.3.4) and the fact that $g \leq h$, the set x omits all but finitely many intervals

$$\big[a_k(b'_k(h(k))), a_k(b'_k(h(k)) + 1)\big).$$

By (4.3.1), we have

$$\left\{ U^{(k)}_{b'_k(h(k))} : k \in \mathbb{N} \right\} \in \Gamma(X \setminus S).$$

Thus,

$$\left\{ U^{(k)}_{b'_k(h(k))} : k \in \mathbb{N} \right\} \in \Gamma((X \cup \mathrm{Fin}) \sqcup Y). \qquad \square$$

Proof of Theorem 4.1. (1) We prove a formally stronger assertion that the product space of finitely many $\mathfrak{p}$-generalized tower sets and a set of cardinality less than $\mathfrak{p}$, satisfies $\binom{\Omega}{\Gamma}$. By (a) and Lemma 4.3, the product space of a $\mathfrak{p}$-generalized tower set and a set of cardinality smaller than $\mathfrak{p}$, satisfies $\binom{\Omega}{\Gamma}$. Fix a natural number $m > 1$. Let $X_1, \ldots, X_m$ be $\mathfrak{p}$-generalized towers in $[\mathbb{N}]^\infty$ and Y be a set with $|Y| < \mathfrak{p}$. Assume that the product space

$$Z := (X_1 \cup \mathrm{Fin}) \times \cdots \times (X_{m-1} \cup \mathrm{Fin}) \times Y$$

satisfies $\binom{\Omega}{\Gamma}$. Fix a set $S \subseteq X_m$ with $|\,S\,| < \mathfrak{p}$. Since $|\,S \times Y\,| < \mathfrak{p}$, by the inductive assumption, the product space

$$S \times Z = (X_1 \cup \mathrm{Fin}) \times \cdots \times (X_{m-1} \cup \mathrm{Fin}) \times (S \times Y)$$

satisfies $\binom{\Omega}{\Gamma}$. By Lemma 4.3, the product space $(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y$ satisfies $\binom{\Omega}{\Gamma}$.

(2) ($\Rightarrow$) Apply Lemma 4.3.

($\Leftarrow$) Let $A \subseteq [\mathbb{N}]^\infty$ be a set with $|A| < \mathfrak{p}$ and Y be a set satisfying $\binom{\Omega}{\Gamma}$. Since $|A| < \mathfrak{p}$, there is an element $b \in [\mathbb{N}]^\infty$ such that $A \leq^* b$. Let $X \subseteq [\mathbb{N}]^\infty$ be a $\mathfrak{p}$-generalized tower such that $x \subseteq b$ for all sets $x \in X$. Then the set $Z := X \cup A$ is a $\mathfrak{p}$-generalized tower. We have

$$A = Z \cap \{\, x \in [\mathbb{N}]^\infty : b \leq^* x \,\},$$

and thus the set A is an F_σ subset of Z. The property $\binom{\Omega}{\Gamma}$ is preserved by taking F_σ subsets [10, Theorem 3]. Since the space $(Z\cup\mathrm{Fin})\sqcup Y$ satisfies $\binom{\Omega}{\Gamma}$, the space $A\sqcup Y$ satisfies $\binom{\Omega}{\Gamma}$, too. $\square$

5. Products of sets satisfying $\binom{\Omega}{\Gamma}$ and the properties $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{S}_1(\Omega,\Omega)$

We already mentioned that the properties $\mathsf{S}_1(\Gamma,\Gamma)$ and $\binom{\Omega}{\Gamma}$ were considered in the context of local properties of functions spaces. This is also the case for property $\mathsf{S}_1(\Omega,\Omega)$. A space Y has *countable strong fan tightness* [19] if for each point $y\in Y$ and each sequence $A_1,A_2,\dots$ of subsets of the space Y with $y\in\bigcap_n\overline{A_n}$, there are points $a_1\in A_1,a_2\in A_2,\dots$ such that $y\in\overline{\{\,a_n : n\in\mathbb{N}\,\}}$. For a set of reals X, the space $\mathrm{C_p}(X)$ has countable strong fan tightness if and only if the set X satisfies $\mathsf{S}_1(\Omega,\Omega)$ [19].

Let O be the class of all open covers of spaces. A space X satisfies Menger's property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ if for each sequence $\mathcal{U}_1,\mathcal{U}_2,\dots\in\mathrm{O}(X)$, there are finite sets $\mathcal{F}_1\subseteq\mathcal{U}_1,\mathcal{F}_2\subseteq\mathcal{U}_2,\dots$ such that $\bigcup_n\mathcal{F}_n\in\mathrm{O}(X)$. In this section, we consider products of sets satisfying $\binom{\Omega}{\Gamma}$ and their relations to the properties $\mathsf{S}_1(\Gamma,\Gamma)$, $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. We have the following implications between considered properties.

$$\begin{tikzcd} \mathsf{S}_1(\Gamma,\Gamma) \arrow[r] & \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}) \\ \binom{\Omega}{\Gamma} \arrow[u] \arrow[r] & \mathsf{S}_1(\Omega,\Omega) \arrow[u] \end{tikzcd}$$

The properties $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{S}_1(\Omega,\Omega)$ are much stronger than $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. Indeed, all sets of reals satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ or $\mathsf{S}_1(\Omega,\Omega)$ are *totally imperfect* [12, Theorem 2.3], i.e., they do not contain uncountable compact subsets; each compact set satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. The existence of a nontrivial set of reals satisfying $\mathsf{S}_1(\Omega,\Omega)$ is independent of *ZFC*: In the Laver model all sets of reals satisfying $\mathsf{S}_1(\Omega,\Omega)$ are countable([23, Theorem 17], [13]), and assuming that the Continuum Hypothesis holds, there is a nontrivial set satisfying $\mathsf{S}_1(\Omega,\Omega)$ [12, Theorem 2.13]. In ZFC, there is a nontrivial totally imperfect set satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [4, Theorem 16].

Miller, Tsaban and Zdomskyy proved that there are two sets of reals satisfying $\binom{\Omega}{\Gamma}$ whose product space does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [16, Theorem 3.2.]. In Theorem 4.1, we gave a necessarily and sufficient condition when a $\mathfrak{p}$-generalized tower set is productively $\binom{\Omega}{\Gamma}$. Now, we show that the product space of a $\mathfrak{p}$-generalized tower set and a set satisfying $\binom{\Omega}{\Gamma}$, satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ and $\mathsf{S}_1(\Omega,\Omega)$ in all finite powers (in fact, the property $\mathsf{S}_1(\Omega,\Omega)$ is preserved by finite powers [19]).

Let $\mathbf{P}$ be a property of spaces. A space is *productively* $\mathbf{P}$, if its product space with any space satisfying $\mathbf{P}$, satisfies $\mathbf{P}$.

Theorem 5.1. *Let κ be an uncountable ordinal number such that each set of reals of cardinality less than κ is productively $\mathsf{S}_1(\Gamma,\Gamma)$. The product space of finitely many κ-generalized tower sets and a set satisfying $\binom{\Omega}{\Gamma}$, satisfies $\mathsf{S}_1(\Gamma,\Gamma)$.*

Let X, Y be spaces and m be a natural number. Identifying the space X with $X\sqcup\emptyset$ and the space Y with $\emptyset\sqcup Y$, the product space $X^m\times Y$ is a closed subset of $(X\sqcup Y)^{m+1}$.

Lemma 5.2. *Let $X,Y\subseteq\mathrm{P}(\mathbb{N})$ and m be a natural number. Let $\mathcal{U}\in\Omega(\mathrm{Fin}^m\times Y)$ be a family of open sets in $\mathrm{P}(\mathbb{N})^{m+1}$. There is a family $\mathcal{V}\in\Omega(\mathrm{Fin}\sqcup Y)$ of open sets in $\mathrm{P}(\mathbb{N})\sqcup\mathrm{P}(\mathbb{N})$ such that the family $\{\,V^{m+1}\cap(\mathrm{P}(\mathbb{N})^m\times Y) : V\in\mathcal{V}\,\}$ refines the family $\mathcal{U}$.*

Proof. There are families $\mathcal{W}$ and $\mathcal{W}'$ of open sets in $\mathrm{P}(\mathbb{N})$ such that the family

$$\{ W^m \times W' : W \in \mathcal{W}, W' \in \mathcal{W}' \} \in \Omega(\mathrm{Fin}^m \times Y)$$

refines the family $\mathcal{U}$. We have

$$\{ W \sqcup W' : W \in \mathcal{W}, W' \in \mathcal{W}' \} \in \Omega(\mathrm{Fin} \sqcup Y),$$

and

$$W^m \times W' = (W \sqcup W')^{m+1} \cap (\mathrm{P}(\mathbb{N})^m \times Y)$$

for all sets $W \in \mathcal{W}$, $W' \in \mathcal{W}'$. $\square$

Lemma 5.3. *Let κ be an uncountable ordinal number. Let $X \subseteq [\mathbb{N}]^\infty$ be a κ-generalized tower and Y be a set satisfying $\binom{\Omega}{\Gamma}$. For each family $\mathcal{U} \in \Omega(\mathrm{Fin} \sqcup Y)$ of open sets in $\mathrm{P}(\mathbb{N}) \sqcup \mathrm{P}(\mathbb{N})$, there are a set $S \subseteq X$ with $|S| < \kappa$ and a subfamily $\mathcal{V}$ of the family $\mathcal{U}$ with $\mathcal{V} \in \Gamma((X \setminus S) \sqcup Y)$.*

Proof. Let $\mathcal{U} \in \Omega(\mathrm{Fin} \sqcup Y)$ be a family of open sets in $\mathrm{P}(\mathbb{N}) \sqcup \mathrm{P}(\mathbb{N})$. The space $\mathrm{Fin} \sqcup Y$ satisfies $\binom{\Omega}{\Gamma}$, and thus there is a subfamily $\mathcal{U}'$ of the family $\mathcal{U}$ with $\mathcal{U}' \in \Gamma(\mathrm{Fin} \sqcup Y)$. Apply Lemma 4.2 to the family $\mathcal{U}'$. Then there are a function $a \in [\mathbb{N}]^\infty$ and sets $U_1, U_2, \ldots \in \mathcal{U}'$ such that for each set $x \in [\mathbb{N}]^\infty$ and all natural numbers n:

$$\text{If } x \cap [a(n), a(n+1)) = \emptyset, \text{ then } x \in U_n.$$

Since X is a κ-generalized tower, there are a set $b \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|S| < \kappa$ such that

$$x \cap \bigcup_{n \in b} [a(n), a(n+1)) \in \mathrm{Fin}$$

for all sets $x \in X \setminus S$. We have $\{ U_n : n \in b \} \in \Gamma(X \setminus S)$. Thus,

$$\{ U_n : n \in b \} \in \Gamma((X \setminus S) \sqcup Y). \qquad \square$$

Lemma 5.4. *Let κ be an uncountable ordinal number. Let $X \subseteq [\mathbb{N}]^\infty$ be a κ-generalized tower, Y be a set satisfying $\binom{\Omega}{\Gamma}$, and m be a natural number. For each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Omega(\mathrm{Fin}^m \times Y)$ of families of open sets in $\mathrm{P}(\mathbb{N})^{m+1}$, there are a set $S \subseteq X$ with $|S| < \kappa$ and sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ such that*

$$\{ U_n : n \in \mathbb{N} \} \in \Gamma\big(((X \setminus S) \cup \mathrm{Fin})^m \times Y\big).$$

Proof. Fix a natural number m. We may assume that the family $\mathcal{U}_{n+1}$ refines the family $\mathcal{U}_n$ for all natural numbers n. By Lemma 5.2, there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots \in \Omega(\mathrm{Fin} \sqcup Y)$ such that the family

$$\{ V^{m+1} \cap (X^m \times Y) : V \in \mathcal{V}_n \}$$

refines the family $\mathcal{U}_n$ for all natural numbers n. Since the set $\mathrm{Fin} \sqcup Y$ satisfies $\binom{\Omega}{\Gamma}$ and the properties $\binom{\Omega}{\Gamma}$ and $\mathsf{S}_1(\Omega, \Gamma)$ are equivalent [9, Theorem 2], there are sets $V_1 \in \mathcal{V}_1, V_2 \in \mathcal{V}_2, \ldots$ such that $\{ V_n : n \in \mathbb{N} \} \in \Gamma(\mathrm{Fin} \sqcup Y)$. By Lemma 5.3, there are a set $a \in [\mathbb{N}]^\infty$ and a set $S \subseteq X$ with $|S| < \kappa$ such that $\{ V_n : n \in a \} \in \Gamma(((X \setminus S) \cup \mathrm{Fin}) \sqcup Y)$. For each natural number $n \in a$, there is a set $U_n \in \mathcal{U}_n$ such that $V^{m+1} \cap (X^m \times Y) \subseteq U_n$. For each natural number $n \in a^{\mathrm{c}}$, there is a set $U_n \in \mathcal{U}_n$ such that $U_n \supseteq U_k$ for some natural number $k \in a$ with $n < k$. We have $\{ U_n : n \in \mathbb{N} \} \in \Gamma(((X \setminus S) \cup \mathrm{Fin})^m \times Y)$. $\square$

Proof of Theorem 5.1. Let Y be a set satisfying $\binom{\Omega}{\Gamma}$.

Let $X \subseteq [\mathbb{N}]^\infty$ be a κ-generalized and $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma((X \cup \mathrm{Fin}) \times Y)$ be families of open sets in $\mathrm{P}(\mathbb{N})^2$. By Lemma 5.4, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ and a set $S \subseteq X$ with $|\, S \,| < \kappa$ such that

$$\{\, U_n : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin}) \times Y).$$

Since $|\, S \,| < \kappa$ and, by the assumption, each set of cardinality smaller than κ is productively $\mathsf{S}_1(\Gamma, \Gamma)$, the product space $(S \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. By Lemma 3.3, the product space $(X \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$.

Fix a natural number $m > 1$ and assume that the statement is true for $m-1$ κ-generalized tower sets. Let $X_1, \ldots, X_m \subseteq [\mathbb{N}]^\infty$ be κ-generalized tower sets and

$$\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Gamma((X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y)$$

be families of open sets in $\mathrm{P}(\mathbb{N})^{m+1}$. By Lemma 2.3(1), the set $X := \bigcup_{i \leq m} X_i$ is a κ-generalized tower. By Lemma 5.4, there are sets $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \ldots$ and a set $S \subseteq X$ with $|\, S \,| < \kappa$ such that

$$\{\, U_n : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin})^m \times Y).$$

The set

$$Z := ((X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y) \setminus (((X \setminus S) \cup \mathrm{Fin})^m \times Y)$$

satisfies $\mathsf{S}_1(\Gamma, \Gamma)$: Fix a natural number $i \leq m$. By the inductive assumption, the product space

$$\prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times Y$$

satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. By the assumption, each set of cardinality smaller than κ is productively $\mathsf{S}_1(\Gamma, \Gamma)$. Since $|\, X_i \cap S \,| < \kappa$, the product space

$$\prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times (X_i \cap S) \times Y$$

satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. A finite union of spaces satisfying $\mathsf{S}_1(\Gamma, \Gamma)$, satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ [24, Theorem 5], and thus the set

$$Z = \bigcup_{i \leq m} \prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times (X_i \cap S) \times Y,$$

satisfies $\mathsf{S}_1(\Gamma, \Gamma)$, too.

By Lemma 3.3, the product space $(X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y$ satisfies $\mathsf{S}_1(\Gamma, \Gamma)$. □

The property $\binom{\Omega}{\Gamma}$ is preserved by finite powers and each set of cardinality less than $\mathfrak{p}$ is productively $\mathsf{S}_1(\Gamma, \Gamma)$ [5, Proposition 6.8], [24, Theorem 5]. Thus, we have the following result.

Corollary 5.5. *The product space of a* $\mathfrak{p}$*-generalized tower set and a set satisfying* $\binom{\Omega}{\Gamma}$*, satisfies* $\mathsf{S}_1(\Gamma, \Gamma)$ *in all finite powers.*

A set $X \subseteq \mathbb{N}^\mathbb{N}$ is *guessable* if there is a function $a \in \mathbb{N}^\mathbb{N}$ such that the sets $\{\, n : a(n) = x(n) \,\}$ are infinite, for all functions $x \in X$. Let $\mathrm{cov}(\mathcal{M})$ be the minimal cardinality of a family of meager subsets of the Baire space $\mathbb{N}^\mathbb{N}$, that covers $\mathbb{N}^\mathbb{N}$. The minimal cardinality of a subset of $\mathbb{N}^\mathbb{N}$ that is no guessable, is equal to $\mathrm{cov}(\mathcal{M})$ [5, Theorem 5.9].

Theorem 5.6. *Let κ be an uncountable ordinal number with $\kappa \leq \mathrm{cov}(\mathcal{M})$ such that κ is regular or $\kappa < \mathrm{cov}(\mathcal{M})$. The product space of finitely many κ-generalized tower sets and a set satisfying $\binom{\Omega}{\Gamma}$, satisfies $\mathsf{S}_1(\Omega, \Omega)$.*

We need the following Lemma.

Lemma 5.7. *A union of less than $\mathrm{cov}(\mathcal{M})$ sets satisfying $\binom{\Omega}{\Gamma}$, satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$.*

Proof. Fix an ordinal number $\lambda < \mathrm{cov}(\mathcal{M})$. Let $X := \bigcup_{\alpha<\lambda} X_\alpha$ be a union of sets satisfying $\binom{\Omega}{\Gamma}$ and $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Omega(X)$. Assume that $\mathcal{U}_n = \{\, U^{(n)}_m : m \in \mathbb{N} \,\}$ for all natural numbers n. For each ordinal number $\alpha < \lambda$, there is a function $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\{\, U^n_{f_\alpha(n)} : n \in \mathbb{N} \,\} \in \Gamma(X_\alpha)$. Since $\lambda < \mathrm{cov}(\mathcal{M})$, the set $\{\, f_\alpha : \alpha < \kappa \,\}$ is guessable. Then there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that the sets $\{\, n : f_\alpha(n) = g(n) \,\}$ are infinite, for all ordinal numbers $\alpha < \kappa$. Thus, $\{\, U^{(n)}_{g(n)} : n \in \mathbb{N} \,\} \in \mathrm{O}(X)$. $\square$

Proof of Theorem 5.6. A set of reals satisfies $\mathsf{S}_1(\Omega, \Omega)$ if and only if it satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ in all finite powers [19]. The properties $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ and $\mathsf{S}_1(\Omega, \mathrm{O})$ are equivalent [23, Theorem 17] and property $\binom{\Omega}{\Gamma}$ is preserved by finite powers. Thus, it is enough to show that the statement is true, when consider the property $\mathsf{S}_1(\Omega, \mathrm{O})$ instead of $\mathsf{S}_1(\Omega, \Omega)$. We prove a formally stronger assertion that the product space of finitely many κ-generalized tower sets, a set satisfying $\binom{\Omega}{\Gamma}$ and a set of cardinality less than $\mathrm{cov}(\mathcal{M})$, satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$.

Let Y be a set satisfying $\binom{\Omega}{\Gamma}$ and Z be a set with $|Z| < \mathrm{cov}(\mathcal{M})$.

Let $X \subseteq [\mathbb{N}]^\infty$ be a κ-generalized tower and $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Omega((X \cup \mathrm{Fin}) \times Y \times Z)$ be families of open sets in $\mathrm{P}(\mathbb{N})^3$, where $\mathcal{U}_n = \{\, U^{(n)}_m : m \in \mathbb{N} \,\}$ for all natural numbers n. Fix a point $z \in Z$. By Lemma 5.4, there are a function $g_z \in \mathbb{N}^{\mathbb{N}}$ and a set $S_z \subseteq X$ with $|S_z| < \kappa$ such that

$$\{\, U^{(n)}_{g_z(n)} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S_z) \cup \mathrm{Fin}) \times Y \times \{z\})$$

Since $|Z| < \mathrm{cov}(\mathcal{M})$, there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that the sets $\{\, n : g(n) = g_z(n) \,\}$ are infinite for all points $z \in Z$. Let $S := \bigcup\{\, S_z : z \in Z \,\}$. We have

$$\{\, U^{(n)}_{g(n)} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin}) \times Y \times Z)$$

Then there are sets $U_1 \in \mathcal{U}_1, U_3 \in \mathcal{U}_3, \ldots$ such that

$$\{\, U_{2n-1} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin}) \times Y \times Z).$$

By the assumption about the ordinal number κ, we have $|S| < \mathrm{cov}(\mathcal{M})$. Thus, $|S \times Z| < \mathrm{cov}(\mathcal{M})$. By Lemma 5.7, the product space $S \times Y \times Z$ satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$. There are sets $U_2 \in \mathcal{U}_2, U_4 \in \mathcal{U}_4, \ldots$ such that

$$\{\, U_{2n} : n \in \mathbb{N} \,\} \in \mathrm{O}(S \times Y \times Z).$$

Finally, we have

$$\{\, U_n : n \in \mathbb{N} \,\} \in \mathrm{O}((X \cup \mathrm{Fin}) \times Y \times Z).$$

Fix a natural number $m > 1$ and assume that the statement is true for $m-1$ κ-generalized tower sets. Let $X_1, \ldots, X_m \subseteq [\mathbb{N}]^\infty$ be κ-generalized towers. Let

$$\mathcal{U}_1, \mathcal{U}_2, \ldots \in \Omega((X_1 \cup \mathrm{Fin}) \times (X_m \cup \mathrm{Fin}) \times Y \times Z)$$

be a sequence of families of open sets in $\mathrm{P}(\mathbb{N})^{m+2}$, where $\mathcal{U}_n = \{\, U^{(n)}_m : m \in \mathbb{N} \,\}$ for all natural numbers n. By Lemma 2.3(1), the set $X := \bigcup_{i \leq m} X_i$ is a κ-generalized tower. Fix a point

$z \in Z$. By Lemma 5.4, there are a function $g_z \in \mathbb{N}^{\mathbb{N}}$ and a set $S_z \subseteq X$ with $|S_z| < \kappa$ such that

$$\{\, U^{(n)}_{g_z(n)} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S_z) \cup \mathrm{Fin})^m \times Y \times \{z\}).$$

Since $|Z| < \mathrm{cov}(\mathcal{M})$, there is a function $g \in \mathbb{N}^{\mathbb{N}}$ such that the sets $\{\, n : g(n) = g_z(n) \,\}$ are infinite for all points $z \in Z$. Let $S := \bigcup \{\, S_z : z \in Z \,\}$. We have

$$\{\, U^{(n)}_{g(n)} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin})^m \times Y \times Z).$$

Then there are sets $U_1 \in \mathcal{U}_1, U_3 \in \mathcal{U}_3, \dots$ such that

$$\{\, U_{2n-1} : n \in \mathbb{N} \,\} \in \Gamma(((X \setminus S) \cup \mathrm{Fin})^m \times Y \times Z).$$

The set

$$T := ((X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y \times Z) \setminus (((X \setminus S) \cup \mathrm{Fin})^m \times Y \times Z)$$

satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$: Fix a natural number $i \leq m$. By the inductive assumption, the product space

$$\prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times Y \times Z$$

satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$. By the assumption about the ordinal number κ, we have $|S| < \mathrm{cov}(\mathcal{M})$. Since $|\, X_i \cap S \,| < \mathrm{cov}(\mathcal{M})$, we have $|\, (X_i \cap S) \times Z \,| < \mathrm{cov}(\mathcal{M})$. By the inductive assumption, the product space

$$\prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times Y \times (X_i \cap S) \times Z$$

satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$. A finite union of sets satisfying $\mathsf{S}_1(\Omega, \mathrm{O})$, satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$ [3, Theorem 2.3.9], and thus the set

$$T = \bigcup_{i \leq m} \prod_{\substack{j \leq m \\ j \neq i}} (X_j \cup \mathrm{Fin}) \times Y \times (X_i \cap S) \times Z,$$

satisfies $\mathsf{S}_1(\Omega, \mathrm{O})$, too. There are sets $U_2 \in \mathcal{U}_2, U_4 \in \mathcal{U}_4, \dots$ such that

$$\{\, U_{2n} : n \in \mathbb{N} \,\} \in \mathrm{O}(T).$$

Finally, we have

$$\{\, U_n : n \in \mathbb{N} \,\} \in \mathrm{O}((X_1 \cup \mathrm{Fin}) \times \cdots \times (X_m \cup \mathrm{Fin}) \times Y \times Z). \qquad \square$$

Corollary 5.8. *Let κ be an uncountable ordinal number with $\kappa \leq \mathrm{cov}(\mathcal{M})$ such that κ is regular or $\kappa < \mathrm{cov}(\mathcal{M})$. Each κ-generalized tower set satisfies $\mathsf{S}_1(\Omega, \Omega)$.*

Since the ordinal number $\mathfrak{p}$ is regular and $\mathfrak{p} \leq \mathrm{cov}(\mathcal{M})$, we have the following result.

Corollary 5.9. *The product space of a $\mathfrak{p}$-generalized tower set and a set satisfying $\binom{\Omega}{\Gamma}$, satisfies $\mathsf{S}_1(\Omega, \Omega)$.*

6. Remarks and open problems

6.1. **Around Scheepers's Conjecture.** A *clopen* cover of a space is a cover whose members are clopen subsets of the space. Let Γ_{clp} be the class of all clopen γ-covers of spaces. A set of reals is a wQN-space if and only if it satisfies $\mathsf{S}_1(\Gamma_{\mathrm{clp}}, \Gamma_{\mathrm{clp}})$ [6, Theorem 9]. The following conjecture was formulated by Scheepers.

Conjecture 6.1 ([22])**.** *The properties* $\mathsf{S}_1(\Gamma, \Gamma)$ *and* $\mathsf{S}_1(\Gamma_{\mathrm{clp}}, \Gamma_{\mathrm{clp}})$ *are equivalent.*

The property $\mathsf{S}_1(\Gamma, \Gamma)$ describes a local property of functions spaces: Let $\mathbb{R}$ be the real line with the usual topology. Let X be a set of reals. A function $f\colon X \to \mathbb{R}$ is *upper semicontinuous* if the sets $\{\, x \in X : f(x) < a \,\}$ are open for all real numbers a. By the result of Bukovský [7, Theorem 13], the set X satisfies $\mathsf{S}_1(\Gamma, \Gamma)$ if and only if it is an *SPP** *space*, that is, for each sequence $\langle f_{1,m} \rangle_{m\in\mathbb{N}}, \langle f_{2,m} \rangle_{m\in\mathbb{N}}, \dots$ of sequences of upper continuous functions on X, each of them converging pointwise to the constant zero function $\mathbf{0}$, there is a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ of natural numbers such that the sequence $\langle f_{n,m_n} \rangle_{n\in\mathbb{N}}$ converges to $\mathbf{0}$. Thus, in the language of functions spaces, the above Scheepers conjecture asks whether the properties wQN and SSP* are equivalent.

6.2. **Generalized towers and productivity of $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$.** A space X satisfies Hurewicz's property $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$ if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots \in \mathrm{O}(X)$ there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\{\, \bigcup \mathcal{F}_n : n \in \mathbb{N} \,\} \in \Gamma(X)$. We have the following implications between considered properties

$$\mathsf{S}_1(\Gamma, \Gamma) \longrightarrow \mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma) \longrightarrow \mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O}),$$

and the property $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$ is strictly in between properties $\mathsf{S}_1(\Gamma, \Gamma)$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ ([12, Theorems 2.2, 2.4], [30, Theorem 3.9]). Let cF be the set of all cofinite subsets of $\mathbb{N}$. For functions $z, t \in [\mathbb{N}]^\infty$, we write $z \leq^\infty t$ if $t \not\leq^* z$.

Definition 6.2 ([27, Definition 4.1])**.** A set $X \subseteq [\mathbb{N}]^\infty$ is a cF-*scale* if for each element $z \in [\mathbb{N}]^\infty$, there is an element $t \in [\mathbb{N}]^\infty$ such that

$$z \leq^\infty t \leqslant^* x$$

for all but less than $\mathfrak{b}$ functions $x \in X$.

Lemma 6.3. *Each* $\mathfrak{b}$*-generalized tower is a* cF*-scale.*

Proof. Let $X \subseteq [\mathbb{N}]^\infty$ be a $\mathfrak{b}$-generalized tower and $z \in [\mathbb{N}]^\infty$ be an element such that $z(1) \neq 1$. Define an element $\tilde{z}$ such that $\tilde{z}(1) := z(1)$, and $\tilde{z}(n+1) := z(\tilde{z}(n))$ for all natural numbers n. There is a set $b \in [\mathbb{N}]^\infty$ such that

$$x \cap \bigcup_{n\in b} [\tilde{z}(n), \tilde{z}(n+1)) \in \mathrm{Fin}$$

for all but less than $\mathfrak{b}$ many elements $x \in X$. We have $b^{\mathsf{c}} \in [\mathbb{N}]^\infty$. Then the set $t := \bigcup_{n\in b^{\mathsf{c}}} [\tilde{z}(n), \tilde{z}(n+1))$ omits infinitely many intervals $[\tilde{z}(n), \tilde{z}(n+1))$ and $x \subseteq^* t$ for all but less than $\mathfrak{b}$ elements $x \in X$. We have

$$z(\tilde{z}(n)) \leq \tilde{z}(n+1) \leq t(\tilde{z}(n)),$$

for all natural numbers $n \in b^{\mathsf{c}}$. Thus, $z \leqslant^\infty t$. For an element $x \in [\mathbb{N}]^\infty$ such $x \subseteq^* t$ and $t \setminus x \in [\mathbb{N}]^\infty$, we have $t \leq^* x$. There are only countably many elements $x \in [\mathbb{N}]^\infty$ with $t \setminus x \in \mathrm{Fin}$. Thus,

$$z \leq^\infty t \leq^* x$$

for all but less than $\mathfrak{b}$ elements $x \in X$. □

By the result of Tsaban and the first named author [27, Theorem 5.4], we have the following corollary.

Corollary 6.4. *Let κ be an uncountable ordinal number with $\kappa \leq \mathfrak{b}$. Each κ-generalized tower set is productively $\mathsf{U}_{\mathrm{fin}}(\mathcal{O}, \Gamma)$.*

6.3. **Questions.**

Problem 6.5. *Is a $\mathfrak{b}$-unbounded tower, provably, productively $\mathsf{S}_1(\Gamma, \Gamma)$? Is this the case assuming the Continuum Hypothesis?*

Problem 6.6. *Assume Martin Axiom and the negation of the Continuum Hypothesis. Is each set of cardinality less than $\mathfrak{c}$ productively $\binom{\Omega}{\Gamma}$?*

Problem 6.7. *Is it consistent that $\mathfrak{p} > \omega_1$, each set of cardinality less than $\mathfrak{p}$ is productively $\binom{\Omega}{\Gamma}$ and there is a $\mathfrak{p}$-generalized tower?*

Problem 6.8. *Let κ be an uncountable ordinal number. Does the existence of a κ-generalized tower imply the existence of a κ-unbounded tower?*

Problem 6.9. *Let κ be an uncountable ordinal number. Is a union of less than $\mathfrak{b}$ many κ-generalized towers, a κ-generalized tower?*

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