

Sets in Prikry and Magidor Generic Extensions

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Abstract

We continue [5] and study sets in generic extensions by the Magidor forcing and by the Prikry forcing with non-normal ultrafilters.

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Mathematical Subject Classification— 03E40, 03E55, 03E35, 03E99 .

Introduction

In the paper [5] by V. Kanovei, P. Koepke and the second author, subforcings of the Prikry forcing were characterized. Here we extend the analyzes to the Magidor forcing, introduced by M. Magidor in [8] (more recent account can be found in [3] or in a very recent nice and detailed paper by G. Fuchs [2]).

The following is shown:

Theorem 3.3 *Let \vec{U} be a coherent sequence in V , $\langle \kappa_1, \dots, \kappa_n \rangle$ be a sequence such that $o^{\vec{U}}(\kappa_i) < \min(\nu \mid 0 < o^{\vec{U}}(\nu))$, let G be $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ -generic¹ and let $A \in V[G]$ be a set of ordinals. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$, where C_G is the Magidor sequence of G .*

One of the main methods used in the proof was the construction of a forcing $\mathbb{M}_I[\vec{U}] \in V$, which is a projection of Magidor forcing $\mathbb{M}[\vec{U}]$. This forcing is a Magidor type forcing which uses only measures from \vec{U} with index $i \in I$. Moreover, $\mathbb{M}_I[\vec{U}]$ adds a prescribed subsequence $C_I := (C_G) \restriction I$

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¹ $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ is Magidor forcing with the coherent sequence \vec{U} above a condition which has $\langle \kappa_1, \dots, \kappa_n \rangle$ as its ordinal sequence.

as a generic object, where $I \subseteq \lambda_0$ is a set of indexes in $\lambda_0 = \text{otp}(C_G)$. Hence, we may examine the intermediate extensions $V \subseteq V[C_I] \subseteq V[C_G]$ as an iteration of two forcings, which resemble $\mathbb{M}[\vec{U}]$. A consequence of this theorem is the classification of all complete subforcings of $\mathbb{M}[\vec{U}]$, this result is stated in theorem 5.3.

Another direction addressed in this paper, is an attempt to extend the results of [5] to Prikry forcings with non-normal ultrafilters. The full generalization is not possible. Thus, for example, once κ is a κ -compact cardinal, then the Prikry forcing with a non-normal ultrafilter over κ can add a generic for any κ -distributive forcing of size κ , see [4] for more on this. Here we show that even from a single measurable, one can produce counter examples to generalizations of [5] to non-normal ultrafilters.

Namely the following is proved:

Theorem 7.1 *Suppose that V satisfies GCH and κ is a measurable cardinal. Then in a generic cofinality preserving extension there is a κ -complete ultrafilter U over κ such that the Prikry forcing with U adds a Cohen subset to κ over V . In particular, this forcing has a non-trivial subforcing which preserves regularity of κ .*

However, if one restricts to the Prikry forcing with P -point ultrafilters, then the following holds:

Theorem 6.7 *Let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega} \rangle$ consists of P -point ultrafilters over κ . Then for every new set of ordinals A in $V^{P(\mathbb{U})}$, κ has cofinality ω in $V[A]$, where $P(\mathbb{U})$ denotes the tree Prikry forcing with \mathbb{U} .*

The paper is organized as follows:

- Sections 1 – 5 present the results for Magidor forcing.
- Section 6 presents the generalization of [5] to the tree Prikry forcings with P -points.
- Section 7 presents the proof for Theorem 7.1.

Notations

- V denotes the ground model.
- For any set A , $V[A]$ denote the minimal model of ZFC containing V and $\{A\}$.
- $\prod_{j=1}^n A_j$ increasing sequences $\langle a_1, \dots, a_n \rangle$ where $a_i \in A_i$.
- $\prod_{i=1}^m \prod_{j=1}^n A_{i,j}$ left-lexicographically increasing sequences (which is denoted by \leq_{LEX}).
- $[\kappa]^\alpha$ increasing sequences of length α .
- $[\kappa]^{<\omega} = \bigcup_{n<\omega} [\kappa]^n$.
- ${}^\alpha[\kappa]$ not necessarily increasing sequences, i.e functions with domain α and range κ .
- ${}^{\omega>}[\kappa] = \bigcup_{n<\omega} {}^n[\kappa]$.
- $\langle \alpha, \beta \rangle$ an ordered pair of ordinals. (α, β) the interval between α and β .
- $\vec{\alpha} = \langle \alpha_1, \dots, \alpha_n \rangle$, $|\vec{\alpha}| = n$, $\vec{\alpha} \setminus \langle \alpha_i \rangle = \langle \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_n \rangle$.
- For every $\alpha < \beta$, The Cantor normal form (abbreviated C.N.F) equation is $\alpha + \omega^{\nu_1} + \dots + \omega^{\nu_m} = \beta$, $\nu_1 \geq \dots \geq \nu_m$ are unique. If $\alpha = 0$ this is the C.N.F of β , otherwise, this is the C.N.F difference of α, β .
- The limit order of α , denoted by $o_L(\alpha) = \gamma$, where $\alpha = \omega^{\gamma_1} + \dots + \omega^{\gamma_n} + \omega^\gamma$ (C.N.F).
- $\text{Lim}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) = \alpha\}$.
- $\text{Succ}(A) = \{\alpha \in A \mid \sup(A \cap \alpha) < \alpha\}$.
- $\biguplus_{i \in I} A_i$ is the union of $\{A_i \mid i \in I\}$ with the requirement that A_i 's are pairwise disjoint.
- If $f : A \rightarrow B$ is a function then for every $A' \subseteq A$, $B' \subseteq B$

$$f''A' = \{f(x) \mid x \in A'\}, f^{-1''}B' = \{x \in A \mid f(x) \in B'\}.$$

- Let $B \subseteq \langle \alpha_\xi \mid \xi < \delta \rangle = A$ be sequences of ordinals,

$$\text{Index}(B, A) = \{\xi < \delta \mid \exists b \in B \ \alpha_\xi = b\}.$$

- Let \mathbb{P} be a forcing notion, σ a formula in the forcing language and $p \in \mathbb{P}$. If \dot{A} is a \mathbb{P} -name, then

$$p \Vdash \dot{A} \text{ means "there is } a \in V \text{ such that } p \Vdash \dot{a} = \dot{A} \text{"}.$$

- Let $p, q \in \mathbb{P}$ then " p, q are compatible in \mathbb{P} " if there exists $r \in \mathbb{P}$ such that $p, q \leq_{\mathbb{P}} r$. Otherwise, if they are incompatible denote it by $p \perp q$.

- In any forcing notion, $p \leq q$ means " q extends p ".
- The notion of complete subforcing, complete embedding and projection is used as defined in [11].

1 Magidor forcing

Definition 1.1 A *coherent sequence* is a sequence

$\vec{U} = \langle U(\alpha, \beta) \mid \beta < o^{\vec{U}}(\alpha), \alpha \leq \kappa \rangle$ such that:

1. $U(\alpha, \beta)$ is a normal ultrafilter over α .
2. Let $j : V \rightarrow Ult(U(\alpha, \beta), V)$ be the corresponding elementary embedding, then

$$j(\vec{U}) \restriction \alpha = \vec{U} \restriction \langle \alpha, \beta \rangle$$

Where

$$\begin{aligned} \vec{U} \restriction \alpha &= \langle U(\gamma, \delta) \mid \delta < o^{\vec{U}}(\gamma), \gamma \leq \alpha \rangle \\ \vec{U} \restriction \langle \alpha, \beta \rangle &= \langle U(\gamma, \delta) \mid (\delta < o^{\vec{U}}(\gamma), \gamma < \alpha) \vee (\delta < \beta, \gamma = \alpha) \rangle \end{aligned}$$

■

Fix a coherent sequence of ultrafilters \vec{U} with maximal measurable κ . Assume that $o^{\vec{U}}(\kappa) < \min(\nu \mid o^{\vec{U}}(\nu) > 0) := \delta_0$. Let $\alpha \leq \kappa$ with $o^{\vec{U}}(\alpha) > 0$, define

$$\bigcap U(\alpha, i) = \bigcap_{i < o^{\vec{U}}(\alpha)} U(\alpha, i)$$

We will follow the description of Magidor forcing as presented in [3].

Definition 1.2 The *Magidor forcing*, denoted by $\mathbb{M}[\vec{U}]$, consist of conditions p of the form $p = \langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle$. For every $1 \leq i \leq n$, t_i is either an ordinal κ_i if $o^{\vec{U}}(\kappa_i) = 0$ or a pair $\langle \kappa_i, B_i \rangle$ if $o^{\vec{U}}(\kappa_i) > 0$ such that:

1. $B \in \bigcap_{\xi < o^{\vec{U}}(\kappa)} U(\kappa, \xi)$, $\min(B) > \kappa_n$.
2. for every $1 \leq i \leq n$:
 - (a) $\langle \kappa_1, \dots, \kappa_n \rangle \in [\kappa]^{<\omega}$.
 - (b) $B_i \in \bigcap_{\xi < o^{\vec{U}}(\kappa_i)} U(\kappa_i, \xi)$.
 - (c) $\min(B_i) > \kappa_{i-1}$ ($i > 1$).

■

We shall adopt the following notations:

1. $t_0 = 0, t_{n+1} = \langle \kappa, B \rangle$.
2. $o^{\vec{U}}(t_i) = o^{\vec{U}}(\kappa(t_i))$.
3. $o^{\vec{U}}(t_i) > 0$ then $t_i = \langle \kappa_i, B_i \rangle = \langle \kappa(t_i), B(t_i) \rangle$.
4. $o^{\vec{U}}(t_i) = 0$ then $t_i = \kappa_i = \kappa(t_i)$.
5. $\kappa(p) = \{\kappa(t_1), \dots, \kappa(t_n)\}$.
6. $B(p) = \biguplus_{i=1}^{n+1} B(t_i)$.

The ordinals κ_i are designated to form the eventual Magidor sequence and candidates for the sequence's missing elements in the interval $(\kappa(t_{i-1}), \kappa(t_i))$ (where $t_0 = 0, \kappa(t_{n+1}) = \kappa$) are provided by the sets $B(t_i)$.

Definition 1.3 Let $p = \langle t_1, t_2, \dots, t_n, \langle \kappa, B \rangle \rangle, q = \langle s_1, \dots, s_m, \langle \kappa, C \rangle \rangle \in \mathbb{M}[\vec{U}]$, define the *Magidor forcing order* by $p \leq q$ ("q extends p") iff:

1. $n \leq m$.
2. $B \supseteq C$.
3. $\exists 1 \leq i_1 < \dots < i_n \leq m$ such that for every $1 \leq j \leq m$:
 - (a) If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\kappa(t_r) = \kappa(s_{i_r})$ and $C(s_{i_r}) \subseteq B(t_r)$.
 - (b) Otherwise $\exists 1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then
 - i. $\kappa(s_j) \in B(t_r)$.
 - ii. $o^{\vec{U}}(s_j) < o^{\vec{U}}(t_r)$.
 - iii. $B(s_j) \subseteq B(t_r) \cap \kappa(s_j)$.

The *direct extension order* is defined by $p \leq^* q$ iff:

1. $p \leq q$.
2. $n = m$.

■

Remarks:

1. Let $p = \langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle$. Assume we would like to add an element s_j to p between t_{r-1} and t_r . It is possible only if $o^{\vec{U}}(t_r) > 0$. Moreover, let $\xi = o^{\vec{U}}(s_j)$, then

$$s_j \in \{\alpha \in B(t_r) \mid o^{\vec{U}}(\alpha) = \xi\}$$

If $s_j = \kappa(s_j)$ (i.e. $\xi = 0$), then any s_j satisfying this requirement can be added. If $s_j = \langle \kappa(s_j), B(s_j) \rangle$ (i.e. $\xi > 0$), Then according to definition 1.3 (3.b.iii) s_j can be added iff

$$B(t_r) \cap \kappa(s_j) \in \bigcap_{\xi' < \xi} U(\kappa(s_j), \xi')$$

2. If $p = \langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle \in \mathbb{M}[\vec{U}]$. Fix some $1 \leq j \leq n$ with $o^{\vec{U}}(t_j) > 0$. Then t_j yields a Magidor forcing in the interval $(\kappa(t_{j-1}), \kappa(t_j))$ with the coherent sequence $\vec{U} \restriction \kappa(t_j)$. t_j acts autonomously in the sense that the sequence produced by it is independent of how the sequence develops in other parts. This observation becomes handy when manipulating p , since we can make local changes at t_j with no impact on other t_i 's.

Let $Y = \{\alpha \leq \kappa \mid o^{\vec{U}}(\alpha) < \delta_0\}$. From the coherency of \vec{U} , it follows that $Y \in \bigcap U(\kappa, i)$. For every $\beta \in Y$ with $o^{\vec{U}}(\beta) > 0$, and $i < \delta_0$ define

$$Y(i) = \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) = i\} \text{ and } Y[\beta] = \biguplus_{i < o^{\vec{U}}(\beta)} Y(i)$$

It follows that for every $\beta \in Y$ and $i < o^{\vec{U}}(\beta)$, $Y(i) \cap \beta \in U(\beta, i)$. To see this take $\beta \leq \kappa$ in Y and $j_{\beta, i} : V \rightarrow Ult(U(\beta, i), V)$.

$$Y(i) \cap \beta \in U(\beta, i) \Leftrightarrow \beta \in j_{\beta, i}(Y(i) \cap \beta)$$

By coherency, $o^{j_{\beta, i}(\vec{U})}(\beta) = i$ and therefore

$$\beta \in j_{\beta, i}(Y(i) \cap \beta) = \{\alpha < j_{\beta, i}(\beta) \mid o^{j_{\beta, i}(\vec{U})}(\alpha) = j_{\beta, i}(i) = i\}.$$

Consequently, $Y[\beta] \cap \beta \in \bigcap_{i < o^{\vec{U}}(\beta)} U(\beta, i)$.

For $B \in \bigcap_{i < o^{\vec{U}}(\beta)} U(\beta, i)$ define recursively, $B^{(0)} = B$,

$$B^{(n+1)} = \{\alpha \in B^{(n)} \mid (o^{\vec{U}}(\alpha) = 0) \vee (B^{(n)} \cap \alpha \in \bigcap U(\alpha, i))\}$$

Let $B^\star = \bigcap_{n < \omega} B^{(n)}$ it follows by induction that for all $n < \omega$,

$$B^{(n)} \in \bigcap_{i < o^{\vec{U}}(\beta)} U(\beta, i)$$

By β -completeness $B^\star \in \bigcap_{i < o^{\vec{U}}(\beta)} U(\beta, i)$. B^\star has the feature that ,

$$\forall \alpha \in B^\star \quad \alpha \cap B^\star \in \bigcap_{i < o^{\vec{U}}(\alpha)} U(\alpha, i)$$

The previous paragraph indicates that by restricting to a dense subset of $\mathbb{M}[\vec{U}]$ we can assume that given $p = \langle t_1, t_2, \dots, t_n, \langle \kappa, B \rangle \rangle \in \mathbb{M}[\vec{U}]$, every choice of ordinal in $B(t_r)$ automatically satisfies the requirement that we discussed in remark (2). Formally, we work above $\langle \langle \rangle, \langle \kappa, Y \rangle \rangle$ and we directly-extend any $p = \langle t_1, t_2, \dots, t_n, \langle \kappa, B \rangle \rangle$ as follows:

For every $1 \leq r \leq n + 1$ and $i < o^{\vec{U}}(t_r)$ define

$$B(t_r, i) := Y(i) \cap B(t_r)^\star \in U(\kappa(t_r), i)$$

It follows that

$$B^\star(t_r) := \biguplus_{i < o^{\vec{U}}(t_r)} B(t_r, i) \in \bigcap_{i < o^{\vec{U}}(t_r)} U(\kappa(t_r), i).$$

Shrink $B(t_r)$ to $B^\star(t_r)$ to obtain

$$p \leq^* p^\star = \langle t'_1, \dots, t'_n, \langle \kappa, B^\star \rangle \rangle$$

$$t'_r = \begin{cases} t_r & o^{\vec{U}}(t_r) = 0 \\ \langle \kappa(t_r), B^\star(t_r) \rangle & \text{otherwise} \end{cases}$$

This dense subset also simplifies \leq to

$$p \leq q \text{ iff } \kappa(p) \subseteq \kappa(q) , B(p) \subseteq B(q)$$

When applying the revised approach regarding the large sets, it is apparent that $B(t_r, i)$ provide candidates, precisely, for the i -limit indices in the final sequence C_G i.e. of indices γ such that $o_L(\gamma) = i$ (for the definition of $o_L(\gamma)$ see Notations). This is stated formally in proposition 1.5.

Recall that:

- $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+ - c.c.$
- Let $p = \langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle \in \mathbb{M}[\vec{U}]$ and denote $\nu = \kappa(t_j)$ where j is the minimal such that $o^{\vec{U}}(t_j) > 0$. Then above p there is ν - \leq^* -closure.
- $\mathbb{M}[\vec{U}]$ satisfies the Prikry condition.

Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic, define

$$C_G = \bigcup \{ \kappa(p) \mid p \in G \}$$

We will abuse notation by considering C_G as a the canonical enumeration of the set C_G . C_G is closed and unbounded in κ . Therefore, the order type of C_G determines the cofinality of κ in $V[G]$. The next propositions can be found in [3].

Proposition 1.4 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic. Then G can be reconstructed from C_G as follows*

$$G = \{ p \in \mathbb{M}[\vec{U}] \mid (\kappa(p) \subseteq C_G) \wedge (C_G \setminus \kappa(p) \subseteq B(p)) \}$$

Therefore $V[G] = V[C_G]$.

■

Proposition 1.5 *Let G be $\mathbb{M}[\vec{U}]$ -generic and C_G the corresponding Magidor sequence. Let $\langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle \in G$, then*

$$\text{otp}((\kappa(t_i), \kappa(t_{i+1})) \cap C_G) = \omega^{o^{\vec{U}}(\kappa(t_{i+1}))}$$

Thus if $\kappa(t_{i+1}) = C_G(\gamma)$ then $o_L(\gamma) = o^{\vec{U}}(t_{i+1})$.

■

Corollary 1.6 *If $o^{\vec{U}}(\kappa) < \delta_0$, then $cf^V(o^{\vec{U}}(\kappa)) = cf^{V[G]}(o^{\vec{U}}(\kappa))$ and $cf^{V[G]}(\kappa) = cf(\omega^{o^{\vec{U}}(\kappa)})$.*

■

Let $p = \langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle \in G$. By proposition 1.5, for each $i \leq n$ one can determine the position of $\kappa(t_i)$ in C_G . Namely, $C_G(\gamma(t_i, p)) = \kappa(t_i)$ where

$$\gamma(t_i, p) := \sum_{j \leq i} \omega^{o^{\vec{U}}(t_j)} < \omega^{o^{\vec{U}}(\kappa)} \quad (*)$$

Addition and power are of ordinals. The equation $(*)$ induces a C.N.F equation

$$\gamma = \sum_{r=1}^m \omega^{o^{\vec{U}}(t_{j_r})} \quad (\text{C.N.F})$$

This indicates the close connection between Cantor normal form of the index γ in $\text{otp}(C_G)$ and the elements t_{j_1}, \dots, t_{j_m} in p which determines that $\gamma(t_i, p) = \gamma$. Let $q = \langle s_1, \dots, s_m, \langle \kappa, B' \rangle \rangle$ be another condition, by definition 1.3 (3.b.ii), if s_j is an element of q which was added to p in the interval $(\kappa(t_r), \kappa(t_{r+1}))$ then $o^{\vec{U}}(s_j) < o^{\vec{U}}(t_{r+1})$. Consequently, adding it to p , does not impact the Cantor normal form and $\gamma(t_{r+1}, p) = \gamma(t_{r+1}, q)$.

2 Combinatorial properties

The combinatorial nature of $\mathbb{M}[\vec{U}]$ is most clearly depicted through the language of step-extensions as presented below.

To perform a one step extension of $p = \langle t_1, t_2, \dots, t_n, \langle \kappa, B \rangle \rangle$:

1. Choose $1 \leq r \leq n + 1$ with $0 < o^{\vec{U}}(t_r)$.
2. Choose $i < o^{\vec{U}}(t_r)$.
3. Choose an ordinal $\alpha \in B(t_r, i)$.
4. Shrink the sets $B(t_s, j)$ to $C(t_s, j) \in U(t_s, j)$ for every $1 \leq s \leq n + 1$ and set

$$C(t_s) = \biguplus_{j < o^{\vec{U}}(t_i)} C_s(j)$$

5. For $j < o^{\vec{U}}(\alpha)$ pick $C(\alpha, j) \in U(\alpha, j)$, $C(\alpha, j) \subseteq B(t_r, j) \cap \alpha$ and set

$$C(\alpha) = \biguplus_{j < o^{\vec{U}}(\alpha)} C(\alpha, j)$$

6. Cut $C(t_r)$ above α .

Extend p to

$$p \frown \langle \alpha, (C(t_s))_{s=1}^{n+1}, C(\alpha) \rangle = \langle t'_1, \dots, t'_{i-1}, \langle \alpha, C(\alpha) \rangle, t'_i, \dots, t'_n, \langle \kappa, C(t_{n+1}) \rangle \rangle$$

$$t'_r = \begin{cases} t_r & o^{\vec{U}}(t_r) = 0 \\ \langle \kappa(t_r), C(t_r) \rangle & o.w. \end{cases}$$

It is clear that every extension of p with only one ordinal added is a one step extension. Next we introduce some notations which will describe a general step extension. The idea is simply to classify extensions according to the order of the measures the new elements of the sequence are chosen from.

Definition 2.1 Let $p = \langle t_1, t_2, \dots, t_n, \underbrace{\langle \kappa, B \rangle}_{t_{n+1}} \rangle \in \mathbb{M}[\vec{U}]$

1. For $1 \leq i \leq n + 1$ define the tree or order types $T_i(p) = {}^{\omega>}[o^{\vec{U}}(t_i)]$, with the order

$$\langle x_1, \dots, x_m \rangle \preceq \langle x'_1, \dots, x'_{m'} \rangle$$

iff there are $1 \leq i_1 < \dots < i_m \leq m'$ such that for every $1 \leq j \leq m'$:

- (a) If $\exists 1 \leq r \leq m$ such that $i_r = j$ then $x_r = x'_j$.
- (b) Otherwise $\exists 1 \leq r \leq n+1$ such that if $i_{r-1} < j < i_r$ then $x'_j < x_r$.

We think of x_r 's as placeholders of ordinals from $B(t_i, x_r)$ and the ordering is induced by definition 1.3 (3).

- 2. $T(p) = \prod_{i=1}^{n+1} T_i(p)$ with \preceq is the product order.
- 3. Let $X_i \in T_i(p)$ $1 \leq i \leq n+1$, $|X_i| = l_i$, $X = \langle X_1, \dots, X_{n+1} \rangle \in T(p)$.

(a) Define

$$\vec{\alpha}_i = \langle \alpha_1, \dots, \alpha_{l_i} \rangle \in B(p, X_i) := \prod_{j=1}^{l_i} B(t_i, X_i(j))$$

X_i is called an *extension-type* below t_i and $\langle \alpha_1, \dots, \alpha_{l_i} \rangle$ is of *type* X_i .

(b) Define

$$\vec{\alpha} = \langle \vec{\alpha}_1, \dots, \vec{\alpha}_{n+1} \rangle \in B(p, X) := \prod_{i=1}^{n+1} \prod_{j=1}^{l_i} B(t_i, X_i(j))$$

X is called an *extension-type* of p and $\vec{\alpha}$ is of *type* X . Notice that X is uniquely determined by $\vec{\alpha}$ and the $o^{\vec{U}}()$ function.

■

Notice that by our assumption $|T(p)| < \min(\nu | 0 < o^{\vec{U}}(\nu)) = \delta_0$. We also use:

- $|X_i| = l_i$.
- $l_x = \max(i \mid X_i \neq \emptyset)$.
- $x_{i,j} = X_i(j)$ $\alpha_{i,j} = \vec{\alpha}_i(j)$.
- $x_{i,l_i+1} = o^{\vec{U}}(t_i)$ and $\alpha_{i,n+1} = \kappa(t_i)$.
- $x_{mc} = x_{l_X, l_X}$ (i.e. the last element of X).
- $o^{\vec{U}}(\vec{\alpha}) = \langle o^{\vec{U}}(\alpha_{i,j}) \mid x_{i,j} \in X \rangle$ is the type of $\vec{\alpha}$.

A general extension of p of type X is of the form:

$$p \frown \langle \vec{\alpha}, (C(x_{i,j}))_{x_{i,j} \in X}, (C(t_r))_{r=1}^{n+1} \rangle = p \frown \langle \vec{\alpha}, (C(x_{i,j}))_{\substack{i \leq n+1 \\ j \leq l_i+1}} \rangle$$

where

$$p^\frown \langle \vec{\alpha}, (C(x_{i,j}))_{\substack{i \leq n+1 \\ j \leq l_i+1}} \rangle = \langle \vec{s}_1, t'_1, \dots, \vec{s}_n, t'_n, s_{n+1}, \langle \kappa, C \rangle \rangle$$

1. $\vec{\alpha} \in B(p, X)$.

2. $t'_s = \begin{cases} t_s & o^{\vec{U}}(t_s) = 0 \\ \langle \kappa(t_s), C(t_s) \rangle & o.w. \end{cases}$

For some pre chosen sets $C(t_s) \in \bigcap_{\xi < o^{\vec{U}}(t_s)} U(\kappa(t_s), \xi)$, $C(t_s) \subseteq B(t_s)$.

3. $\vec{s}_i(j) = \begin{cases} \alpha_{i,j} & x_{i,j} = 0 \\ \langle \alpha_{i,j}, C(x_{i,j}) \rangle & o.w. \end{cases}$

For some pre chosen sets $C(x_{i,j}) \in \bigcap_{\xi < x_{i,j}} U(\alpha_{i,j}, \xi)$, $C(x_{i,j}) \subseteq B(t_i) \cap \alpha_{i,j}$.

4. $C \in \bigcap_{\xi < o^{\vec{U}}(\kappa)} U(\kappa, \xi)$ and $\min(C) > \max(\vec{s}_{n+1})$.

By the discussion succeeding definition 1.3, we conclude that

$$p^\frown \langle \vec{\alpha}, (C(x_{i,j}))_{\substack{i \leq n+1 \\ j \leq l_i+1}} \rangle \in \mathbb{M}[\vec{U}]$$

We will more frequently use $p^\frown \langle \vec{\alpha} \rangle$ with the same definition as above except we do not shrink any sets and simply take $\alpha_{i,j} \cap B(t_i) = C(x_{i,j})$. Define

$$p^\frown X = \{p^\frown \langle \vec{\alpha} \rangle \mid \vec{\alpha} \in B(p, X)\}$$

The set $p^\frown X$'s induce a partition of $\mathbb{M}[\vec{U}]$ above p as stated in the next proposition which is well known and follows directly from definition 1.3.

Proposition 2.2 *Let $p \in \mathbb{M}[\vec{U}]$ be any condition and $p \leq q \in \mathbb{M}[\vec{U}]$. Then there exists a unique $\vec{\alpha} \in B(p, X)$ such that $p^\frown \langle \vec{\alpha} \rangle \leq^* q$.*

■

Example: Let

$$p = \langle \underbrace{\langle \kappa(t_1), B(t_1) \rangle}_{t_1}, \underbrace{\kappa(t_2)}_{t_2}, \underbrace{\langle \kappa(t_3), B(t_3) \rangle}_{t_3}, \underbrace{\langle \kappa(t_4), B(t_4) \rangle}_{t_4}, \underbrace{\langle \kappa, B \rangle}_{t_5} \rangle$$

$$o^{\vec{U}}(t_1) = 1, o^{\vec{U}}(t_2) = 0, o^{\vec{U}}(t_3) = 2, o^{\vec{U}}(t_4) = 1, o^{\vec{U}}(\kappa) = 3$$

Extend p to q as follows:

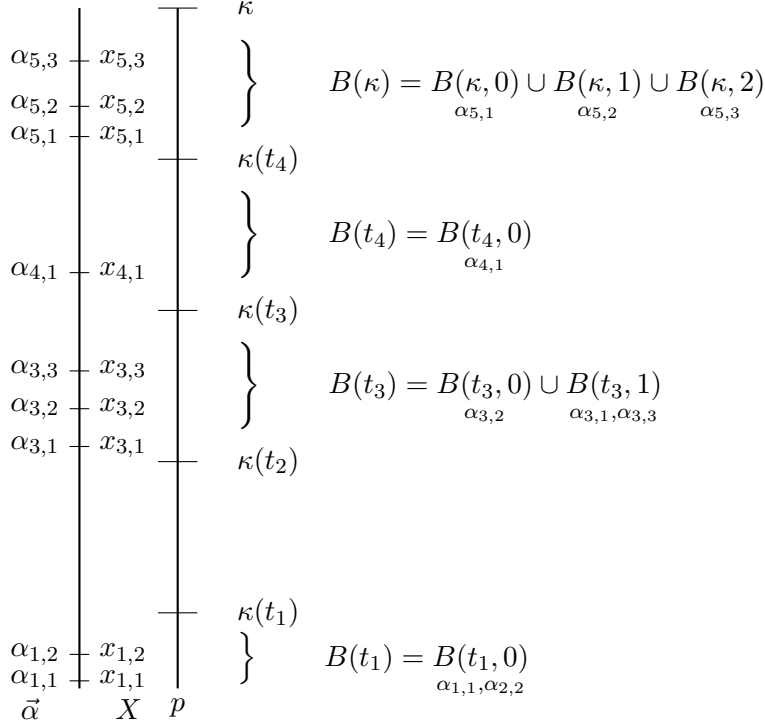
$$q = p^\frown \langle \underbrace{\langle \alpha_{1,1}, \alpha_{1,2} \rangle}_{\vec{\alpha}_1}, \underbrace{\langle \rangle}_{\vec{\alpha}_2}, \underbrace{\langle \alpha_{3,1}, \alpha_{3,2}, \alpha_{3,3} \rangle}_{\vec{\alpha}_3}, \underbrace{\langle \alpha_{4,1} \rangle}_{\vec{\alpha}_4}, \underbrace{\langle \alpha_{5,1}, \alpha_{5,2}, \alpha_{5,3} \rangle}_{\vec{\alpha}_5} \rangle$$

$$o^{\vec{U}}(\alpha_{i,j}) = \begin{cases} 0 & \langle i, j \rangle = \langle 1, 1 \rangle, \langle 1, 2 \rangle, \\ & \langle 3, 2 \rangle, \langle 4, 1 \rangle, \langle 5, 1 \rangle \\ 1 & \langle i, j \rangle = \langle 3, 1 \rangle, \langle 3, 3 \rangle, \\ & \langle 5, 2 \rangle \\ 2 & \langle i, j \rangle = \langle 5, 3 \rangle \end{cases}$$

Then the extension-type of $\langle \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3, \vec{\alpha}_4, \vec{\alpha}_5 \rangle$ is

$$X = \underbrace{\langle \langle 0, 0 \rangle \rangle}_{X_1}, \underbrace{\langle \rangle}_{X_2}, \underbrace{\langle \langle 1, 0, 1 \rangle \rangle}_{X_3}, \underbrace{\langle \langle 0 \rangle \rangle}_{X_4}, \underbrace{\langle \langle 0, 1, 2 \rangle \rangle}_{X_5}$$

This can be illustrated as follows:



As presented in proposition 2.2, a choice from the set $p \smallfrown X$ is essentially a choice from $B(p, X)$, which is of the form $\prod_{i=1}^n A_i$ where for every $1 \leq i \leq n$, $A_i \in U_i$ and U_i 's are normal measures on a non decreasing sequence of measurable $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$. For the rest of this section, we prove some combinatorial properties of such sets.

Lemma 2.3 *Let $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$ be any collection of measurable cardinals with normal*

measures U_1, \dots, U_n respectively. Assume $F : \prod_{i=1}^n A_i \longrightarrow \nu$ where $\nu < \kappa_1$ and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$ $H_i \in U_i$ such that $\prod_{i=1}^n H_i$ is homogeneous for F i.e. $F \upharpoonright \prod_{i=1}^n H_i$ is constant.

Proof. By induction on n . The case $n = 1$ is a well known property of normal measures. Assume that the lemma holds for $n - 1$, and fix $\vec{\eta} = \langle \eta_1, \dots, \eta_{n-1} \rangle \in \prod_{i=1}^{n-1} A_i$. Define

$$F_{\vec{\eta}} : A_n \setminus (\eta_{n-1} + 1) \longrightarrow \nu, \quad F_{\vec{\eta}}(\xi) = F(\eta_1, \dots, \eta_{n-1}, \xi)$$

By the case $n=1$ there exists an homogeneous $A_n \supseteq H(\vec{\eta}) \in U_n$ with color $C(\vec{\eta}) < \nu$. Define

$$\Delta_{\vec{\eta} \in \prod_{i=1}^{n-1} A_i} H(\vec{\eta}) =: H_n$$

By the induction hypotheses applied to $C : \prod_{i=1}^{n-1} A_i \rightarrow \nu$, there is an homogeneous set of the form $\prod_{i=1}^{n-1} H_i$ where $A_i \supseteq H_i \in U_i$. To see that $\prod_{i=1}^n H_i$ is homogeneous for F , let $\vec{\eta}', \vec{\eta} \in \prod_{i=1}^n H_i$, and denote by η_n, η'_n , the ordinals $\max(\vec{\eta}), \max(\vec{\eta}')$ respectively. It follows that:

$$F(\vec{\eta}) = F_{\vec{\eta} \setminus \langle \eta_n \rangle}(\eta_n) \underset{\eta_n \in H(\vec{\eta} \setminus \langle \eta_n \rangle)}{=} C(\vec{\eta} \setminus \langle \eta_n \rangle) \underset{\vec{\eta} \setminus \langle \eta_n \rangle, \vec{\eta}' \setminus \langle \eta'_n \rangle \in \prod_{i=1}^{n-1} H_i}{=} C(\vec{\eta}' \setminus \langle \eta'_n \rangle) = \dots = F(\vec{\eta}')$$

■

Lemma 2.4 *Let $\kappa_1 \leq \kappa_2 \leq \dots \leq \kappa_n$ be a non descending finite sequence of measurable cardinals with normal measures U_1, \dots, U_n respectively. Assume $F : \prod_{i=1}^n A_i \longrightarrow B$ where B is any set, and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$ $H_i \in U_i$ and set of coordinates $I \subseteq \{1, \dots, n\}$ such that for every $\vec{\eta}, \vec{\xi} \in \prod_{i=1}^n H_i$,*

$$F(\vec{\eta}) = F(\vec{\xi}) \leftrightarrow \vec{\eta} \upharpoonright I = \vec{\xi} \upharpoonright I$$

In other words, the function $F \upharpoonright \prod_{i=1}^n H_i$ is well defined modulo the equivalence relation:

$$\langle \alpha_1, \dots, \alpha_n \rangle \sim \langle \alpha'_1, \dots, \alpha'_n \rangle \quad \text{iff } \forall i \in I \quad \alpha_i = \alpha'_i$$

and the induced function, \bar{F} , is injective. We call the set I , a set of important coordinates.

Proof. By induction on n , if $n = 1$ then it is immediate since for any $f : A \rightarrow B$ such that $A \in U$ where U is a normal measure on a measurable cardinal κ , B is any set, then there exists $A \supseteq A' \in U$ for which $F \upharpoonright A'$ is either constant or injective. Assume that the lemma holds for $n - 1$, $n > 1$ and let $F : \prod_{i=1}^n A_i \rightarrow B$ be a function satisfying the conditions of the lemma. Define for every $x_1 \in A_1$, $F_{x_1} : \prod_{i=2}^n A_i \setminus (x_1 + 1) \rightarrow B$

$$F_{x_1}(x_2, \dots, x_n) = F(x_1, x_2, \dots, x_n)$$

By the induction hypothesis, for every $x_1 \in A_1$ there are $A_i \supseteq A_i(x_1) \in U_i$ and set of important coordinates $I(x_1) \subseteq \{2, \dots, n\}$. There is $A_1 \supseteq A'_1 \in U_1$ such that function $I : A_1 \rightarrow P(\{2, \dots, n\})$ is constant on A'_1 with value I' . For every $i = 2, \dots, n$ define $A'_i = \Delta_{x_1 \in A_1} A_i(x_1)$. So far, $\prod_{i=1}^n A'_i$ has the property:

$$(1) \quad \forall \langle x_1, x_2, \dots, x_n \rangle, \langle x_1, x'_2, \dots, x'_n \rangle \in \prod_{i=1}^n A'_i \text{ with the same first coordinate}$$

$$F(x_1, x_2, \dots, x_n) = F(x_1, x'_2, \dots, x'_n) \text{ iff } \forall i \in I'. x_i = x'_i$$

In particular, \bar{F} is a well defined function modulo $I' \cup \{1\}$. Next we determine if 1 is important. For every $\langle \alpha, \alpha' \rangle \in A'_1 \times A'_1$, define $t_{\langle \alpha, \alpha' \rangle} : \prod_{i=2}^n A'_i \setminus (\alpha' + 1) \rightarrow 2$

$$t_{\langle \alpha, \alpha' \rangle}(x_2, \dots, x_n) = 1 \iff F(\alpha, x_2, \dots, x_n) = F(\alpha', x_2, \dots, x_n)$$

By lemma 2.3, for $i = 2, \dots, n$ there are $A'_i \supseteq A_i(\alpha, \alpha') \in U_i$ such that $\prod_{i=2}^n A_i(\alpha, \alpha')$ is homogeneous for $t_{\langle \alpha, \alpha' \rangle}$ with color $C(\alpha, \alpha')$. Taking the diagonal intersection over $A'_1 \times A'_1$ of the sets $A_i(\alpha, \alpha')$, at each coordinate $i = 2, \dots, n$, we obtain $H_i \in U_i$ such that $\prod_{i=2}^n H_i$ is homogeneous for every $t_{\langle \alpha, \alpha' \rangle}$. Finally, the function $C : A'_1 \times A'_1 \rightarrow 2$ yield an homogeneous $A'_1 \supseteq H_1 \in U_1$ with color C' .

Case 1: $C' = 1$. Let us show that the important coordinates are I' .

Let $\langle x_1, \dots, x_n \rangle, \langle x'_1, \dots, x'_n \rangle \in \prod_{i=1}^n H_i$, then,

$$F(x_1, \dots, x_n) = F(x'_1, \dots, x'_n) \quad \begin{array}{c} \longleftrightarrow \\ \uparrow \\ F(x'_1, x'_2, \dots, x'_n) = F(x_1, x_2, \dots, x_n) \end{array} \quad F(x_1, x_2, \dots, x_n) = F(x_1, x'_2, \dots, x'_n) \quad \begin{array}{c} \longleftrightarrow \\ \uparrow \\ (1) \end{array} \quad \forall i \in I' \quad x_i = x'_i$$

Case 2: $C' = 0$. Then we have a second property:

$$(2) \quad \forall x_1, x'_1 \in H_1 \text{ and } \langle x_2, \dots, x_n \rangle \in \prod_{i=2}^n H_i. \quad x_1 = x'_1 \iff F(x_1, x_2, \dots, x_n) = F(x'_1, x_2, \dots, x_n)$$

We would like to claim that in this case the important coordinates are $I = I' \cup \{1\}$ but we still have to shrink H_i 's, to eliminate all remaining counter examples for \bar{F} not being injective i.e. $\langle x_1, \dots, x_n \rangle, \langle x'_1, \dots, x'_n \rangle \in \prod_{i=1}^n H_i$ such that

$$\langle x_1, \dots, x_n \rangle \neq \langle x'_1, \dots, x'_n \rangle \text{ mod } I \text{ and } F(x_1, \dots, x_n) = F(x'_1, \dots, x'_n)$$

Take Any counter example and set

$$\{x_1, \dots, x_n\} \cup \{x'_1, \dots, x'_n\} = \{y_1, \dots, y_k\} \text{ (increasing enumeration)}$$

To reconstruct $\{x_1, \dots, x_n\}, \{x'_1, \dots, x'_n\}$ from $\{y_1, \dots, y_k\}$ is suffices to know for example how $\{x_1, \dots, x_n\}$ are arranged between $\{x'_1, \dots, x'_n\}$. There are finitely many ways ² for Such an arrangement. Therefore, if we succeed with eliminating examples of a fixed arrangement, then by σ -completeness of the measures we will be able to eliminate all counter example.

Fix such an arrangement, the increasing sequence $\langle y_1, \dots, y_k \rangle$ is in the product of some k large sets $\prod_{i=1}^k H_{n_i}$. We have to be careful since the sequence of measurables induced by n_1, \dots, n_k is not necessarily non descending. To fix this we can cut the sets H_i such that in the sequence $\langle \kappa_i \mid i = 1, \dots, n \rangle$, wherever $\kappa_i < \kappa_{i+1}$ then $\min(H_{i+1}) > \kappa_i = \sup(H_i)$. Therefore, assume that $\langle \kappa_{n_i} \mid i = 1, \dots, k \rangle$ is non descending. Define $G : \prod_{i=1}^k H_{n_i} \rightarrow 2$

$$G(y_1, \dots, y_k) = 1 \Leftrightarrow F(x_1, \dots, x_n) = F(x'_1, \dots, x'_n)$$

By lemma 2.3 there must be $U_i \ni H'_i \subseteq H_i$ homogeneous for G with value D . If $D = 0$ we have eliminated from H_i 's all counter examples of that fixed ordering. Toward a contradiction, assume that $D = 1$, then every y_1, \dots, y_k yield a counter example $\langle x_1, \dots, x_n \rangle, \langle x'_1, \dots, x'_n \rangle$. By property (1), $x_1 = x'_1$, hence assume without loss of generality that $x_1 < x'_1$, fix $x < w < y_2 < \dots < y_n$, where $x, w \in H'_1$ and $y_i \in H'_{n_i}$ for $i = 2, \dots, k$. Since $D = 1$, it follows that $G(x, y_2, \dots, y_k) = G(w, y_2, \dots, y_k) = 1$, thus,

$$F(x, x_2, \dots, x_n) = F(x'_1, x'_2, \dots, x'_n) = F(w, x_2, \dots, x_n)$$

which is a contradiction to property (2). ■

²In general, the number of possibilities to arrange two counter examples into one increasing sequence depends on I . Nevertheless, there is an upper bound: Think of x_i 's as balls we would like to divide into $n + 1$ cells. The cells are represented by the intervals $(x'_{i-1}, x'_i]$ plus the cell for elements above x'_n . There are $\binom{2n}{n}$ such divisions. For any such division, we decide either the cell is $(x'_{i-1}, x'_i]$ or (x'_{i-1}, x'_i) . Hence, there are at most $\binom{2n}{n} \cdot 2^n$ such arrangements.

3 The Main Result Up to κ

As stated in corollary 1.6, Magidor forcing adds a closed unbounded sequence of length $\omega^{o\vec{U}(\kappa)}$ to κ . It is possible to obtain a family of forcings that adds a sequence of any limit length to some measurable cardinal, using a variation of Magidor forcing as we defined it³. Namely, let \vec{U} be a coherent sequence and $\lambda_0 < \min(\nu \mid o\vec{U}(\nu) > 0)$ a limit ordinal

$$(\text{not necessarily C.N.F}) \quad \lambda_0 = \omega^{\gamma_1} + \dots + \omega^{\gamma_n} \quad , \gamma_n > 0$$

Let $\langle \kappa_1, \dots, \kappa_n \rangle$ be an increasing sequence such that $o\vec{U}(\kappa_i) = \gamma_i$. Define the forcing $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ as follows:

The root condition will be

$$0_{\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]} = \langle \langle \kappa_1, B_1 \rangle, \dots, \langle \kappa_n, B_n \rangle \rangle$$

where B_1, \dots, B_n are as in the discussion following definition 1.3. The conditions of this forcing are any finite sequence that extends $0_{\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]}$ in the sense of definition 1.3. Since each $\langle \kappa_i, B_i \rangle$ acts autonomously, this forcing is essentially the same as $\mathbb{M}[\vec{U}]$. In fact, $\mathbb{M}[\vec{U}]$ is just $\mathbb{M}_{\langle \kappa \rangle}[\vec{U}]$. The notation we used for $\mathbb{M}[\vec{U}]$ can be extended to $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ since the conditions are also of the form $\langle t_1, \dots, t_r, \langle \kappa, B \rangle \rangle$. Let

$$\langle \langle \nu_1, C_1 \rangle, \dots, \langle \nu_m, C_m \rangle \rangle \in \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$$

then $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$ is an open subset of $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ (i.e. \leq -upwards closed). Moreover, if $G \subseteq \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ is any generic set with $\langle \langle \nu_1, C_1 \rangle, \dots, \langle \nu_m, C_m \rangle \rangle \in G$ then,

$$(G)_{\langle \nu_1, \dots, \nu_m \rangle} := G \cap \mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}] = \{p \in G \mid p \geq \langle \langle \nu_1, C_1 \rangle, \dots, \langle \nu_m, C_m \rangle \rangle\}$$

is generic for $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$. The filter $(G)_{\vec{\nu}}$ is essentially the same generic as G since it yield the same Magidor sequence and in particular $V[(G)_{\vec{\nu}}] = V[G]$.

From now on the set B in $\langle t_1, \dots, t_r, \langle \kappa, B \rangle \rangle$ will be suppressed and replaced by $t_{r+1} = \langle \kappa, B \rangle$. An alternative way to describe $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ is through the following product

$$\begin{aligned} \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}] &\simeq \mathbb{M}[\vec{U}]_{\langle \kappa_1 \rangle} \times (\mathbb{M}[\vec{U}]_{\langle \kappa_2 \rangle})_{>\kappa_1} \times \dots \times (\mathbb{M}[\vec{U}]_{\langle \kappa_n \rangle})_{>\kappa_{n-1}} \\ (\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}])_{>\alpha} &= \{\langle t_1, \dots, t_{r+1} \rangle \in \mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}] \mid \kappa(t_1) > \alpha\} \end{aligned}$$

³Magidor's original formulation of $\mathbb{M}[\vec{U}]$ in [8] gives such a family

This isomorphism is induced by the embeddings

$$i_r : ((\mathbb{M}[\vec{U}]_{\langle \kappa_r \rangle})_{>\kappa_{r-1}} \rightarrow \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}] \quad , \quad r = 1, \dots, n$$

$$i_r(\langle s_1, \dots, s_{k+1} \rangle) = \langle \langle \kappa_1, B_1 \rangle, \dots, \langle \kappa_{r-1}, B_{r-1} \rangle, s_1, \dots, s_k, \underbrace{\langle \kappa_r, B(s_{k+1}) \rangle}_{s_{k+1}}, \dots, \langle \kappa_n, B_n \rangle \rangle$$

From this embeddings, it is clear that the generic sequence produced by $(\mathbb{M}[\vec{U}]_{\langle \kappa_r \rangle})_{>\kappa_{r-1}}$ is just $C_G \cap (\kappa_{r-1}, \kappa_r)$.

The formula to compute coordinates still holds:

Let $p = \langle t_1, \dots, t_m, t_{m+1} \rangle \in \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$. For each $1 \leq i \leq m$, the coordinate of $\kappa(t_i)$ in any Magidor sequence extending p is $C_G(\gamma) = \kappa(t_i)$, where

$$\gamma = \sum_{j \leq i} \omega^{o_{\vec{U}}(t_j)} =: \gamma(t_i, p) < \lambda_0$$

Lemma 3.1 *Let G be generic for $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ and the sequence derived*

$$C_G = \bigcup \{ \{ \kappa(t_1), \dots, \kappa(t_l) \} \mid \langle t_1, \dots, t_l, t_{l+1} \rangle \in G \}$$

1. $\text{otp}(C_G) = \lambda_0$.
2. *If $\kappa_i < C_G(\gamma) < \kappa_{i+1}$ where γ is limit, then there exists $\vec{\nu} = \langle \nu_1, \dots, \nu_m \rangle$ such that $(G)_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}$ is generic for $\mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$, $C_G = C_{(G)_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}}$ and the sequences obtained by the split*

$$\mathbb{M}_{\vec{\nu}}[\vec{U}] \times (\mathbb{M}_{\langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}])_{>\nu_m} \simeq \mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$$

are $C_G \cap C_G(\gamma)$, $C_G \setminus C_G(\gamma)$. More accurately, if

$$\gamma = \underbrace{\omega^{\gamma_1} + \dots + \omega^{\gamma_i}}_{\xi} + \omega^{\gamma'_{i+1}} + \dots + \omega^{\gamma'_m} \quad (C.N.F)$$

then

$$\vec{\nu} = \langle \nu_1, \dots, \nu_m \rangle = \langle \kappa_1, \dots, \kappa_i, C_G(\xi + \omega^{\gamma'_{i+1}}), \dots, C_G(\gamma) \rangle$$

Proof. For (1), the same reasoning as in lemmas 1.5-1.6 should work. For (2), notice that by proposition 1.4, $0_{\mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}} \in G$. Thus $(G)_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}$ is generic for $\mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$. The embeddings

$$i_1 : \mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}] \rightarrow \mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$$

$$i_1(\langle t_1, \dots, t_{r+1} \rangle) = \langle t_1, \dots, t_{r+1}, \langle \kappa_{i+1}, B_{i+1} \rangle, \dots, \langle \kappa_n, B_n \rangle \rangle$$

and

$$i_2 : (\mathbb{M}_{\langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}])_{>\nu_m} \rightarrow \mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$$

$$i_2(\langle s_1, \dots, s_{k+1} \rangle) = \langle \langle \kappa_1, B_1 \rangle, \dots, \langle \kappa_i, B_i \rangle, s_1, \dots, s_{k+1} \rangle$$

induces the isomorphism of $\mathbb{M}_{\vec{\nu} \smallfrown \langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}]$ with the product. Therefore, $i_1^{-1}(G)$, $i_2^{-1}(G)$ are generic for $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$, $(\mathbb{M}_{\langle \kappa_{i+1}, \dots, \kappa_n \rangle}[\vec{U}])_{>\nu_m}$ respectively. By the definition of i_1, i_2 this generics obviously yield the sequences $C_G \cap C_G(\gamma)$ and $C_G \setminus C_G(\gamma)$. ■

In general we will identify G with $(G)_{\vec{\nu}}$ when using lemma 3.1.

Notice that, the information used in order to compute $\gamma(t_i, p)$ is just $o^{\vec{U}}(t_i)$ which is provided by the suitable extension type. Let X be an extension type of p , one can compute the coordinates of any extension $\vec{\alpha}$ of type X . In particular, for any $\alpha_{i,r}$ substituting $x_{i,r} \in X$ the coordinate of $\alpha_{i,r}$ is

$$\gamma = \gamma(t_{i-1}, p) + \omega^{x_{i,1}} + \dots + \omega^{x_{i,r}} =: \gamma(x_{i,r}, p \smallfrown X)$$

In this situation we say that X *unveils the γ -th coordinate*. If $x_{i,r} = x_{mc}$, we say that X *unveils γ as maximal coordinate*.

Proposition 3.2 *Let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle \in \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ and γ such that for some $0 \leq i \leq n$, $\gamma(t_i, p) < \gamma < \gamma(t_{i+1}, p)$. Then there exists an extension-type X unveiling γ as maximal coordinate. Moreover, if*

$$\gamma(t_i, p) + \sum_{j \leq m} \omega^{\gamma_j} = \gamma \text{ (C.N.F)}$$

then the extension type is $X = \langle X_i \rangle$ where $X_i = \langle \gamma_1, \dots, \gamma_m \rangle$. ■

Example: Assume $\lambda_0 = \omega_1 + \omega^2 \cdot 2 + \omega$, let $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 = \kappa$ be such that

$$o^{\vec{U}}(\kappa_1) = \omega_1, o^{\vec{U}}(\kappa_2) = o^{\vec{U}}(\kappa_3) = 2 \text{ and } o^{\vec{U}}(\kappa) = 1$$

Let

$$p = \underbrace{\langle \nu_1, B(\nu_1) \rangle}_{t_1}, \underbrace{\nu_2}_{t_2}, \underbrace{\langle \kappa_1, B(\kappa_1) \rangle}_{t_3}, \underbrace{\langle \nu_4, B(\nu_3) \rangle}_{t_4}, \underbrace{\langle \kappa_2, B(\kappa_2) \rangle}_{t_5}, \underbrace{\langle \kappa_3, B(\kappa_3) \rangle}_{t_6}, \underbrace{\langle \kappa, B \rangle}_{t_7}$$

$$o^{\vec{U}}(t_1) = \omega, \quad o^{\vec{U}}(t_2) = 0, \quad o^{\vec{U}}(t_4) = 1$$

Let G be any generic with $p \in G$. Calculating $\gamma(t_i, p)$ for $i = 1, \dots, 7$:

1. $\gamma(t_1, p) = \omega^{o^{\vec{U}}(t_1)} = \omega^\omega \Rightarrow C_G(\omega^\omega) = \nu_1.$
2. $\gamma(t_2, p) = \omega^\omega + \omega^{o^{\vec{U}}(t_2)} = \omega^\omega + 1 \Rightarrow C_G(\omega^\omega + 1) = \nu_2.$
3. $\gamma(t_3, p) = \omega^\omega + 1 + \omega^{\omega_1} = \omega^{\omega_1} = \omega_1.$
4. $\gamma(t_4, p) = \omega_1 + \omega \Rightarrow C_G(\omega_1 + \omega) = \nu_3.$
5. $\gamma(t_5, p) = \omega_1 + \omega + \omega^2 = \omega_1 + \omega^2.$

To demonstrate proposition 3.2 let $\gamma = \omega^\omega + \omega^5 \cdot 3 + 5$, then,

$$\begin{aligned} \gamma(t_2, p) &= \omega^\omega + 1 < \gamma < \omega_1 = \gamma(t_3, p) \\ (\omega^\omega + 1) + \omega^5 \cdot 3 + 5 &= \gamma \end{aligned}$$

The extension type unveiling γ as maximal coordinate is then

$$X = \langle \langle \rangle, \langle \rangle, X_3 \rangle, \quad X_3 = \langle 5, 5, 5, 0, 0, 0, 0, 0 \rangle$$

i.e. every extension $\vec{\alpha} = \langle \alpha_{3,1}, \dots, \alpha_{3,8} \rangle \in B(p, X)$ will satisfy that

$$\gamma(\alpha_{mc}, p \frown \vec{\alpha}) = \gamma(\alpha_{3,8}, p \frown \alpha) = \gamma(x_{3,8}, p \frown X) = \gamma$$

Let us state the main theorem of this paper.

Theorem 3.3 *Let \vec{U} be a coherent sequence in V , $\langle \kappa_1, \dots, \kappa_n \rangle$ be a sequence such that $o^{\vec{U}}(\kappa_i) < \min(\nu \mid 0 < o^{\vec{U}}(\nu)) =: \delta_0$, let G be $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ -generic and let $A \in V[G]$ be a set of ordinals. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

We will prove Theorem 3.3 by induction on $\text{otp}(C_G)$. For $\text{otp}(C_G) = \omega$ it is just the Prikry forcing which is known by [5]. Let $\text{otp}(C_G) = \lambda_0$ be a limit ordinal,

$$\lambda_0 = \omega^{\gamma_n} + \dots + \omega^{\gamma_1} \quad (\text{C.N.F})$$

If $\sup(A) < \kappa$, then by lemma 5.3 in [8], $A \in V[C_G \cap \sup(A)]$. By lemma 3.1, $V[C_G \cap \sup(A)]$ is a generic extension of some $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$ with order type smaller than λ_0 , thus by induction we are done. In fact, if there exists $\alpha < \kappa$ such that $A \in V[C_G \cap \alpha]$ then the induction hypothesis works. Let us assume that $A \notin V[C_G \cap \alpha]$ for every $\alpha < \kappa$ this kind of set will be called *recent set*. Since $\kappa_1, \dots, \kappa_n$ will be fixed through the rest of this chapter we shall abuse notation and denote $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}] = \mathbb{M}[\vec{U}]$.

3.1 The Main Results for Sets of Cardinality Less Than κ

First let us show that for A with small enough cardinality the theorem holds regardless of the induction.

Lemma 3.4 *Let \mathfrak{x} be a $\mathbb{M}[\vec{U}]$ -name and $p \in \mathbb{M}[\vec{U}]$ such that $p \Vdash \mathfrak{x}$ is an ordinal. Then there exists $p \leq^* p^* \in \mathbb{M}[\vec{U}]$ and an extension-type $X \in T(p)$ such that*

$$(*) \quad \forall p^* \frown \langle \vec{\alpha} \rangle \in p^* \frown X \quad p^* \frown \langle \vec{\alpha} \rangle \Vdash \mathfrak{x}$$

Proof. Let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle \in \mathbb{M}[\vec{U}]$.

Claim 1 *There exists $p \leq^* p'$ such that for some extension type X*

$$\forall \vec{\alpha} \in B(p', X) \exists C(x_{i,j}) \text{ s.t. } p' \frown \langle \vec{\alpha}, (C(x_{i,j}))_{i,j} \rangle \Vdash \mathfrak{x}$$

Proof of Claim: Define sets $B_X(t_i, j)$, for any fixed $X \in T(p)$ as follows: Recall the notation l_X , x_{mc} and let $\vec{\alpha} \in B(p, X \setminus \langle x_{mc} \rangle)$. Define

$$B_X^{(0)}(\vec{\alpha}) = \{ \theta \in B(t_{l_X}, x_{mc}) \mid \exists (C(x_{i,j}))_{i,j} \quad p \frown \langle \vec{\alpha}, \theta, (C(x_{i,j}))_{i,j} \rangle \Vdash \mathfrak{x} \}$$

and $B_X^{(1)}(\vec{\alpha}) = B(t_{l_X}, x_{mc}) \setminus B_X^{(0)}(\vec{\alpha})$. One and only one of $B_X^{(0)}(\vec{\alpha})$, $B_X^{(1)}(\vec{\alpha})$ is in $U(\kappa(t_{l_X}), x_{mc})$. Set $B_X(\vec{\alpha})$ and $F_X(\vec{\alpha}) \in \{0, 1\}$ such that

$$B_X(\vec{\alpha}) = B_X^{(F_X(\vec{\alpha}))}(\vec{\alpha}) \in U(\kappa(t_{l_X}), x_{mc})$$

Define

$$B'_X(t_{l_X}, x_{mc}) = \bigtriangleup_{\vec{\alpha} \in B(p, X \setminus \langle x_{mc} \rangle)} B_X(\vec{\alpha})$$

Consider the function $F : B(p, X \setminus \langle x_{mc} \rangle) \rightarrow \{0, 1\}$. Applying lemma 2.3 to F , we get an homogeneous $\prod_{x_{i,j} \in X \setminus \langle x_{mc} \rangle} B'_X(t_i, x_{i,j})$ where

$$B'_X(t_i, x_{ij}) \subseteq B(t_i, x_{ij}), \quad B'_X(t_i, x_{ij}) \in U(t_i, x_{ij}), \quad x_{ij} \in X \setminus \langle x_{mc} \rangle$$

For $\xi \notin X_i$, Set

$$B'_X(t_i, \xi) = B(t_i, \xi)$$

Since $|T(p)| < \kappa(t_1)$, for each $1 \leq i \leq n+1$ and $\xi < o^{\vec{U}}(t_i)$

$$B'(t_i, \xi) := \bigcap_{X \in T(p)} B'_X(t_i, \xi) \in U(\kappa(t_i), \xi)$$

Finally, let $p' = \langle t'_1, \dots, t'_n, t'_{n+1} \rangle$ where

$$t'_i = \begin{cases} t_i & o^{\vec{U}}(t_i) = 0 \\ \langle \kappa(t_i), B'(t_i) \rangle & \text{otherwise} \end{cases}$$

It follows that $p \leq^* p' \in \mathbb{M}[\vec{U}]$.

Let H be $\mathbb{M}[\vec{U}]$ -generic, $p' \in H$. By the assumption on p , there exists $\delta < \kappa$ such that $V[H] \models (x)_H = \delta$. Hence, there is $p' \leq q \in M[\vec{U}]$ such that $q \Vdash \check{x} = \delta$. By proposition 2.2, there is a unique $p' \frown \langle \vec{\alpha}, \theta \rangle \in p' \frown X$ for some extension type X , such that $p' \frown \langle \vec{\alpha}, \theta \rangle \leq^* q$. X, p' are as wanted:

By the definition of p' it follows that $\vec{\alpha} \in B(p', X \setminus \langle x_{mc} \rangle)$ and $\theta \in B_X(\vec{\alpha})$. Since $q \Vdash \check{x} = \delta$, we have that $F_X(\vec{\alpha}) = 0$. Fix $\langle \vec{\alpha}', \theta' \rangle$ of type X . $\vec{\alpha}'$ and $\vec{\alpha}$ belong to the same homogeneous set, thus $F(\vec{\alpha}') = F(\vec{\alpha}) = 0$ and

$$\theta' \in B_X^{(0)}(\vec{\alpha}') \Rightarrow \exists (C(x_{i,j}))_{i,j} \text{ s.t. } p' \frown \langle \vec{\alpha}', \theta', (C(x_{i,j}))_{i,j} \rangle \Vdash \check{x}$$

■ *of claim*

For every $\vec{\alpha} \in B(p', X)$, fix some $(C_{i,j}(\vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}}$ such that

$$p' \frown \langle \vec{\alpha}, (C_{i,j}(\vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}} \rangle \Vdash \check{x}$$

It suffices to show that we can find $p' \leq^* p^*$ such that for every $\vec{\alpha} \in B(p^*, X)$

$$B(t_i^*) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}(\vec{\alpha}), \quad 1 \leq i \leq n+1, \quad 1 \leq j \leq l_i+1$$

Where α_s is the predecessor of $\alpha_{i,j}$ in $\vec{\alpha}$. In order to do that, define $p' \leq^* p_{i,j}$ $i \leq n+1, j \leq l_i+1$ then $p^* \geq^* p_{i,j}$ will be as wanted. Define $p_{i,j}$ as follows:

Fix $\vec{\beta} \in B(p', \langle x_{1,1}, \dots, x_{i,j} \rangle)$, by lemma 2.3, the function

$$C_{i,j}(\vec{\beta}, *) : B(p', X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle) \rightarrow P(\beta_{i,j})$$

has homogeneous sets $B^*(\vec{\beta}, x_{r,s}) \subseteq B(p', x_{r,s})$ for $x_{r,s} \in X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle$. Denote the constant value by $C_{i,j}^*(\vec{\beta})$. Define

$$B^*(t_r, x_{r,s}) = \bigtriangleup_{\vec{\beta} \in B(p', \langle x_{1,1}, \dots, x_{i,j} \rangle)} B^*(\vec{\beta}, x_{r,s}), \quad x_{r,s} \in X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle$$

Next, fix $\alpha \in B(t'_i, x_{i,j})$ and let

$$C_{i,j}^*(\alpha) = \bigtriangleup_{\vec{\alpha}' \in B(p', \langle x_{1,1}, \dots, x_{i,j-1} \rangle)} C_{i,j}^*(\vec{\alpha}', \alpha)$$

Thus $C_{i,j}^*(\alpha) \subseteq \alpha$. Moreover, $\kappa(t_i)$ is in particular an ineffable cardinal and therefore there are $B^*(t_i, x_{i,j}) \subseteq B(t'_i, x_{i,j})$ and $C_{i,j}^*$ such that

$$\forall \alpha \in B^*(t_i, x_{i,j}) \quad C_{i,j}^* \cap \alpha = C_{i,j}^*(\alpha)$$

By coherency, $C_{i,j}^* \in \bigcap U(t_i, \xi)$. Finally, define $p_{i,j} = \langle t_1^{(i,j)}, \dots, t_n^{(i,j)}, t_{n+1}^{(i,j)} \rangle$

$$B(t_i^{(i,j)}) = B^*(t_i) \cap (\bigcap_j C_{i,j}^*) \quad 1 \leq i \leq n+1$$

To see that p^* is as wanted, let $\vec{\alpha} \in B(p^*, X)$ and fix any i, j . Then $\vec{\alpha} \in B(p_{i,j}, X)$ and $\alpha_{i,j} \in B^*(t_i, x_{i,j})$. Thus

$$B(t_i^*) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}^* \cap \alpha_{i,j} \setminus \alpha_s = C_{i,j}^*(\alpha_{i,j}) \setminus \alpha_s \subseteq C_{i,j}^*(\alpha_{1,1}, \dots, \alpha_{i,j}) = C_{i,j}(\vec{\alpha})$$

■

Lemma 3.5 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be V -generic filter and $A \in V[G]$ be any set of ordinals, such that $|A| < \delta_0$. Then there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof. Let $A = \langle a_\xi \mid \xi < \delta \rangle \in V[G]$, where $\delta < \delta_0$ and $\underline{A} = \langle \underline{a}_\xi \mid \xi < \delta \rangle$ be a $\mathbb{M}[\vec{U}]$ -name for $\langle a_\xi \mid \xi < \delta \rangle$. Let $q \in G$ such that $q \Vdash \underline{A} \subseteq On$. We proceed by a density argument, fix $q \leq p \in \mathbb{M}[\vec{U}]$. By lemma 3.4, for each $\xi < \delta$ there exists $X(\xi)$ and $p \leq^* p_\xi^*$ satisfying (*). By $\delta^+ - \leq^*$ closure above p we have $p^* \in \mathbb{M}[\vec{U}]$ such that $\forall \xi < \delta \ p_\xi^* \leq p^*$. For each ξ , define $F_\xi : B(p^*, X(\xi)) \longrightarrow \kappa$

$$F_\xi(\vec{\alpha}) = \gamma \text{ for the unique } \gamma \text{ such that } p^* \frown \langle \vec{\alpha} \rangle \Vdash \dot{a}_\xi = \check{\gamma}.$$

Using lemma 2.4, we obtain for every $\xi < \delta$ a set of important coordinates

$$I_\xi \subseteq \{ \langle i, j \rangle \mid 1 \leq i \leq n+1, 1 \leq j \leq l_i \}$$

Example: Assume $o^{\vec{U}}(k) = 3$, $C_G = \langle C_G(\alpha) \mid \alpha < \omega^3 \rangle$.

$$a_0 = C_G(80), a_1 = C_G(\omega + 2) + C_G(3), a_2 = C_G(\omega^2 \cdot 2 + \omega + 1)$$

and

$$p = \langle \nu_0, \langle \nu_\omega, B(\nu_\omega, 0) \rangle, \underbrace{\langle \kappa, B(\kappa, 0) \cup B(\kappa, 1) \cup B(\kappa, 2) \rangle}_{B(\kappa)} \rangle$$

We use as index the coordinate in the final sequence to improve clarity. To determine a_0 , unveil the first 80 elements of the Magidor sequence i.e. any element of the form

$$p_0 = \langle \nu_0, \nu_1, \dots, \nu_{80}, \langle \nu_\omega, B(\nu_\omega, 0) \setminus \nu_{80} + 1 \rangle, \langle \kappa, B(\kappa) \rangle \rangle$$

will decide the value of a_0 . Thus the extension type $X(0)$ is

$$X(0) = \langle \langle \underbrace{0, \dots, 0}_{80 \text{ times}} \rangle, \langle \rangle \rangle$$

The important coordinates to decide the value of a_0 is only the 80th coordinate. It is easily seen to be one to one modulo the irrelevant coordinates $1, \dots, 79$. For a_1 , the form is

$$p_1 = \langle \nu_0, \nu_1, \nu_2, \nu_3, \langle \nu_\omega, B(\nu_\omega, 0) \setminus \nu_3 + 1 \rangle, \nu_{\omega+1}, \nu_{\omega+2}, \langle \kappa, B(\kappa) \setminus (\nu_{\omega+2} + 1) \rangle \rangle$$

The extension type is

$$X(1) = \langle \langle 0, 0, 0 \rangle, \langle 0, 0 \rangle \rangle$$

The important coordinates are the 3rd and the 5th. For a_2 we have

$$p_2 = \langle \nu_0, \langle \nu_\omega, B(\nu_\omega, 0) \rangle, \langle \nu_{\omega^2}, B(\nu_{\omega^2}) \rangle, \langle \nu_{\omega^2 \cdot 2}, B(\nu_{\omega^2 \cdot 2}) \rangle, \langle \nu_{\omega^2 \cdot 2 + \omega}, B(\nu_{\omega^2 \cdot 2 + \omega}) \rangle, \langle \kappa, B(\kappa) \setminus \nu_{\omega^2 \cdot 2 + \omega} \rangle \rangle$$

$$X(2) = \langle \langle \rangle, \langle 2, 2, 1 \rangle \rangle$$

Back to the proof, since G was generic, there is $\langle t_1, \dots, t_n, t_{n+1} \rangle = p^* \in G$ with such functions. Find $D_\xi \subseteq C_G$ such that

$$D_\xi \in B(p^*, X_\xi)$$

By proposition 1.4 and since $p^* \in G$, D_ξ exists. Since $V[G] \models (a_\xi)_G = a_\xi$ we conclude that,

$$p^* \frown \langle D_\xi \rangle \Vdash a_\xi = \dot{a}_\xi$$

hence, $F_\xi(D_\xi) = a_\xi$. Set $C_\xi = D_\xi \restriction I_\xi$ and $C' = \bigcup_{\xi < \delta} C_\xi$. Let us show that $V[\langle a_\xi \mid \xi < \delta \rangle] = V[C']$:

In $V[C']$, fix some enumeration of $C' = \{C'_i \mid i < \text{otp}(C')\}$. For each $\xi < \delta$, C_ξ can be extracted from C' and $\text{Index}(C_\xi, C') \in V$ (See the notation section for the definition of $\text{Index}(A, B)$). Since $\delta < \delta_0$, $\langle \text{Index}(C_\xi, C') \mid \xi < \delta \rangle \in V$, which implies that $\langle C_\xi \mid \xi < \delta \rangle \in V[C']$. Still in $V[C']$, for every $\xi < \delta$ find

$$D'_\xi \in B(p^*, X_\xi) \text{ such that } D'_\xi \restriction I_\xi = C_\xi$$

Such D'_ξ exists as D_ξ witnesses (the sequence $\langle D_\xi \mid \xi < \delta \rangle$ may not be in $V[C']$). Since $D'_\xi \sim_{I_\xi} D_\xi$, and by the property of I_ξ ,

$$F_\xi(D'_\xi) = F_\xi(D_\xi) = a_\xi$$

hence $\langle a_\xi \mid \xi < \delta \rangle = \langle F_\xi(D'_\xi) \mid \xi < \delta \rangle \in V[C']$.

In the other direction, Given $\langle a_\xi \mid \xi < \delta \rangle$, for each $\xi < \delta$ pick $D'_\xi \in F_\xi^{-1}(a_\xi)$ (Note that $F_\xi^{-1}(a_\xi) \neq \emptyset$ follows from the fact that $D_\xi \in \text{dom}(F_\xi)$ and $F_\xi(D_\xi) = a_\xi$). Since F_ξ is 1-1 modulo I_ξ and $F_\xi(D_\xi) = F_\xi(D'_\xi)$ we have

$$D_\xi \sim_{I_\xi} D'_\xi \text{ and } C_\xi = D_\xi \restriction I_\xi = D'_\xi \restriction I_\xi$$

Hence

$$\langle C_\xi \mid \xi < \delta \rangle = \langle D'_\xi \restriction I_\xi \mid \xi < \delta \rangle \in V[\langle a_\xi \mid \xi < \delta \rangle] \text{ and } C' \in V[\langle a_\xi \mid \xi < \delta \rangle].$$

■

3.2 The Main Result for Subsets of κ

We shall proceed by induction on $\sup(A)$ for a recent set A . As we have seen in the discussion following Theorem 3.3, if $A \subseteq \kappa$ is recent then $\sup(A) = \kappa$. For such A , the next lemma gives a sufficient conditions.

Lemma 3.6 *Let $A \in V[G]$, $\sup(A) = \kappa$. Assume that $\exists C^* \subseteq C_G$ such that*

1. $C^* \in V[A]$ and $\forall \alpha < \kappa \ A \cap \alpha \in V[C^*]$.
2. $cf^{V[A]}(\kappa) < \delta_0$.

Then $\exists C' \subseteq C_G$ such that $V[A] = V[C']$.

Proof. Let $cf^{V[A]}(\kappa) = \eta$ and $\langle \gamma_\xi \mid \xi < \eta \rangle \in V[A]$ be a cofinal sequence in κ . Work in $V[A]$, pick an enumerations of $P(\gamma_\xi) = \langle X_{\xi,i} \mid i < 2^{\gamma_\xi} \rangle \in V[C^*]$. Since $A \cap \gamma_\xi \in V[C^*]$, there exists $i_\xi < 2^{\gamma_\xi}$ such that $A \cap \gamma_\xi = X_{\xi,i_\xi}$. The sequences

$$C^*, \langle i_\xi \mid \xi < \eta \rangle, \langle \gamma_\xi \mid \xi < \eta \rangle$$

can be coded in $V[A]$ to a sequence $\langle x_\alpha \mid \alpha < \eta \rangle$. By lemma 3.5, $\exists C' \subseteq C_G$ such that $V[\langle x_\alpha \mid \alpha < \eta \rangle] = V[C']$. Let us argue that $V[A] = V[\langle x_\alpha \mid \alpha < \delta \rangle]$, clearly $V[A] \supseteq V[\langle x_\alpha \mid \alpha < \eta \rangle]$. For the other direction, note that $A = \bigcup_{\xi < \eta} X_{\xi,i_\xi} \in V[\langle x_\alpha \mid \alpha < \eta \rangle]$.

■

Let us consider two kind of subsets of κ :

1. $\exists \alpha^* < \kappa$ such that $\forall \beta < \kappa \ A \cap \beta \in V[A \cap \alpha^*]$ and we say that $A \cap \alpha$ stabilizes. An example of such A is a generic Prikry sequence $\{C_G(n) \mid n < \omega\}$, simply take $\alpha^* = 0$.
2. For all $\alpha < \kappa$ there exists $\beta < \kappa$ such that $V[A \cap \alpha] \subsetneq V[A \cap \beta]$ as example we can take Magidor forcing with $o^{\vec{U}}(\kappa) = 2$ and A the entire Magidor sequence C_G .

First we consider A 's such that $A \cap \alpha$ does not stabilize.

Lemma 3.7 *Assume that $A \cap \alpha$ does not stabilize, then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

Proof. Work in $V[A]$, define the sequence $\langle \alpha_\xi \mid \xi < \theta \rangle$:

$$\alpha_0 = \min(\alpha \mid V[A \cap \alpha] \not\supseteq V)$$

Assume that $\langle \alpha_\xi \mid \xi < \lambda \rangle$ has been defined and for every ξ , $\alpha_\xi < \kappa$. If $\lambda = \xi + 1$ then set

$$\alpha_\lambda = \min(\alpha \mid V[A \cap \alpha] \not\supseteq V[A \cap \alpha_\xi])$$

If $\alpha_\lambda = \kappa$, then α_λ satisfies that

$$\forall \alpha < \kappa \quad A \cap \alpha \in V[A \cap \alpha_{\lambda^*}]$$

Thus $A \cap \alpha$ stabilizes which contradicts our assumption.

If λ is limit, define

$$\alpha_\lambda = \sup(\alpha_\xi \mid \xi < \lambda)$$

if $\alpha_\lambda = \kappa$ define $\theta = \lambda$ and stop. The sequence $\langle \alpha_\xi \mid \xi < \theta \rangle \in V[A]$ is a continues, increasing unbounded sequence in κ . Therefore, $cf^{V[A]}(\kappa) = cf^{V[A]}(\theta)$. Let us argue that $\theta < \delta_0$. Work in $V[G]$, for every $\xi < \theta$ pick $C_\xi \subseteq C_G$ such that $V[A \cap \alpha_\xi] = V[C_\xi]$. This is a 1-1 function from θ to $P(C_G)$. The cardinal δ_0 is still a strong limit cardinal (since there are no new bounded subsets below this cardinal and it is measurable in V). Moreover, $\lambda_0 := \text{otp}(C_G) < \delta_0$, thus

$$\theta \leq |P(C_G)| = |P(\lambda_0)| < \delta_0$$

The only thing left to prove, is that we can find C^* as in Lemma 3.6. Work in $V[A]$, for every $\xi < \theta$, $C_\xi \in V[A]$ (The sequence $\langle C_\xi \mid \xi < \theta \rangle$ may not be in $V[A]$). C_ξ witnesses that

$$\exists d_\xi \subseteq \kappa \quad (|d_\xi| < 2^{\lambda_0} \text{ and } V[A \cap \alpha_\xi] = V[d_\xi])$$

So $d = \bigcup \{d_\xi \mid \xi < \theta\} \in V[A]$ and $|d| \leq 2^{\lambda_0}$. Finally, by lemma 3.5, there exists $C^* \subseteq C_G$ such that $V[C^*] = V[d] \subseteq V[A]$. Note that for every $\xi < \theta$, $\text{Index}(d_\xi, d) \in V$ and also since $\theta < \delta_0$, the sequence $\langle \text{Index}(d_\xi, x) \mid \xi < \theta \rangle \in V$. It follows that for every $\xi < \theta$, $d_\xi \in V[C^*]$, and in turn $A \cap \alpha_\xi \in V[C^*]$. Since α_ξ is unbounded in κ , for all $\alpha < \kappa$ $A \cap \alpha \in V[C^*]$. Apply 3.6, to conclude the lemma. ■

For the rest of this section, we assume that the sequence $A \cap \alpha$ stabilizes on α^* . Let C^* be such that $V[A \cap \alpha^*] = V[C^*]$ and $\kappa^* = \sup(C^*)$ is limit in C_G . Notice that, $\kappa^* < \kappa$, since if $\kappa^* = \kappa$, then κ is singular in $V[C^*]$, but on the other hand $A \cap \alpha^* \in V[C_G \cap \alpha^*]$ which implies κ is regular in $V[A \cap \alpha^*] = V[C^*]$.

In order to apply lemma 3.6, we only need to argue that for A which is recent, κ changes cofinality in $V[A]$. To do this, consider the initial segment $C_G \cap \kappa^*$ and assume that $\kappa_{j-1} \leq \kappa^* < \kappa_j$. Denote by

$$\mathbb{M}_{\leq \kappa^*} := \mathbb{M}_{\langle \nu_1, \dots, \nu_i, \kappa^* \rangle}[\vec{U}], \quad \mathbb{M}_{> \kappa^*}[\vec{U}] := (\mathbb{M}_{\langle \kappa_j, \dots, \kappa \rangle}[\vec{U}])_{> \kappa^*}$$

By lemma 3.1 we can split $\mathbb{M}[\vec{U}]$ to

$$\mathbb{M}_{\leq \kappa^*}[\vec{U}] \times (\mathbb{M}_{\langle \kappa_j, \dots, \kappa \rangle}[\vec{U}])_{> \kappa^*}$$

such that C_G is generic for $\mathbb{M}_{\leq \kappa^*}[\vec{U}] \times \mathbb{M}_{> \kappa^*}[\vec{U}]$ and $C_G \cap \kappa^*$ is generic for $\mathbb{M}_{\leq \kappa^*}[\vec{U}]$. By [7, Thm. 15.43], there is a forcing $\mathbb{P} \subseteq RO(\mathbb{M}[\vec{U}]_{\leq \kappa^*})$ ⁴, such that $V[C^*] = V[G^*]$ for some generic G^* of \mathbb{P} . Also there is a projection of $\pi : \mathbb{M}_{\leq \kappa^*}[\vec{U}] \rightarrow \mathbb{P}$. Recall that if $\pi : \mathbb{M}_{\leq \kappa^*}[\vec{U}] \rightarrow \mathbb{P}$ is the projection, then the quotient forcing is define:

$$\mathbb{M}_{\leq \kappa^*}[\vec{U}]/G^* = \pi^{-1''}[G^*]$$

In $V[G^*]$ define $\mathbb{Q} = \mathbb{M}_{\leq \kappa^*}[\vec{U}]/G^* \subseteq \mathbb{M}_{\leq \kappa^*}[\vec{U}]$. It is well known that $G \restriction \kappa^*$ is $V[C^*]$ -generic filter for \mathbb{Q} and clearly $V[C^*][C_G \cap \kappa^*] = V[C_G \cap \kappa^*]$. In section 4, we give a more concrete description of π and \mathbb{Q} , however, in this section we will only need the existence of such a forcing and the fact that the projection if on the part below κ^* which implies that \mathbb{Q} is of small cardinality.

Forcing $\mathbb{M}_{> \kappa^*}[\vec{U}]$ above $V[G \restriction \kappa^*]$ is essentially forcing a Magidor forcing adding a sequence to κ above κ^* . To see this, note that all the measures in \vec{U} above κ^* generates measures in $V[G \restriction \kappa^*]$. In conclusion, we have managed to find a forcing $\mathbb{Q} \times \mathbb{M}_{> \kappa^*}[\vec{U}] \in V[C^*]$ such that $V[G]$ is one of its generic extensions and $\forall \alpha < \kappa \ A \cap \alpha \in V[C^*]$.

Work in $V[C^*]$, let \dot{A} be a $\mathbb{Q} \times \mathbb{M}_{> \kappa^*}[\vec{U}]$ -name for A . Since A stabilizes, and by the definition of C^* , we can find $\langle q, p \rangle \in G$ such that

$$\langle q, p \rangle \Vdash \forall \alpha < \kappa \ \dot{A} \cap \alpha \text{ is old (where old means in } V[C^*])$$

Formally, the next argument is a density argument above $\langle q, p \rangle$. Nevertheless, in order to simplify notation, assume that $\langle q, p \rangle = 0_{\mathbb{Q} \times \mathbb{M}[\vec{U}]_{> \kappa^*}}$. Lemmas 3.8-3.9 prove that a certain property holds densely often in $\mathbb{M}[\vec{U}]_{> \kappa^*}$. In order to Make these lemmas more clear, we consider an ongoing example.

Example: Let $\lambda_0 = \text{otp}(C_G) = \omega^2$,

$$A = \{C_G(n) \mid n \leq \omega \text{ is even}\} \cup \{C_G(\omega \cdot n) + C_G(n) \mid 0 < n < \omega\}$$

⁴ $RO(\mathbb{Q})$ is the complete boolean algebra of regular open subsets of \mathbb{Q}

Therefore

$$C^* = \{C_G(2n) \mid n < \omega\}, \quad \kappa^* = C_G(\omega)$$

The forcing \mathbb{Q} can be thought of as adding the missing coordinates to $C_G \restriction \omega$ i.e. the odd coordinates. For the sake of the example, let

$$p = \langle \underbrace{\langle \nu_{\omega \cdot 2}, B_{\omega \cdot 2} \rangle}_{t_1}, \underbrace{\nu_{\omega \cdot 2 + 1}}_{t_2}, \underbrace{\langle \kappa, B(\kappa) \rangle}_{t_3} \rangle \in \mathbb{M}[\vec{U}]_{>\kappa^*}$$

Lemma 3.8 *For every $p \in \mathbb{M}[\vec{U}]_{>\kappa^*}$ there exists $p \leq^* p^*$ such that for every extension type X of p^* and $q \in \mathbb{Q}$ (Recall $\max(\vec{\alpha})\alpha_{mc}$, if there is $p^* \frown \vec{\alpha} \in p^* \frown X$ and $p^{**} \geq^* p^* \frown \vec{\alpha}$ such that $\langle q, p^{**} \rangle \parallel \tilde{A} \cap \alpha_{mc}$) \Rightarrow , then*

$$(*) \quad (\forall p^* \frown \vec{\alpha} \in p^* \frown X \quad \langle q, p^* \frown \vec{\alpha} \rangle \parallel \tilde{A} \cap \alpha_{mc} =: a(q, \vec{\alpha})) \quad (a \text{ property of } q, X)$$

Example: Let

$$q = \langle \nu_1, \nu_3, \langle \kappa^*, B(\kappa^*) \rangle \rangle, \quad X = \langle \underbrace{\langle 0, 0 \rangle}_{X_1}, \underbrace{\langle \rangle}_{X_2}, \underbrace{\langle 1, 0 \rangle}_{X_3} \rangle - \text{extension of } p$$

and let

$$\vec{\alpha} = \langle \langle \alpha_{\omega+1}, \alpha_{\omega+2} \rangle, \langle \rangle, \langle \alpha_{\omega \cdot 3}, \alpha_{\omega \cdot 3 + 1} \rangle \rangle \in B(p, X)$$

If H is any generic with $\langle q, p \frown \vec{\alpha} \rangle \in H$ then all the elements in q and $p \frown \vec{\alpha}$ have there coordinates in C_H as specified above, thus,

$$(\tilde{A})_H \cap \alpha_{mc} = (\tilde{A})_H \cap \alpha_{\omega \cdot 3 + 1} = \{C_H(n) \mid n \leq \omega \text{ is even}\} \cup \{C_H(\omega \cdot n) + C_H(n) \mid n < \omega\} \cap C_H(\omega \cdot 3 + 1)$$

If $\alpha_{\omega \cdot 3} + \nu_3 \geq \alpha_{\omega \cdot 3 + 1}$ then

$$a(q, \vec{\alpha}) = (\tilde{A})_H \cap \alpha_{mc} = C_H \restriction_{\text{even}} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2)\}$$

If $\alpha_{\omega \cdot 3} + \nu_3 < \alpha_{\omega \cdot 3 + 1}$ then

$$a(q, \vec{\alpha}) = (\tilde{A})_H \cap \alpha_{mc} = C_H \restriction_{\text{even}} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2), \alpha_{\omega \cdot 3} + \nu_3\}$$

Anyway, we have that $a(q, \vec{\alpha}) \in V[C^*]$ and therefore $\langle q, p \frown \vec{\alpha} \rangle \parallel \tilde{A} \cap \alpha_{mc}$ for every extension $\vec{\alpha}$ of type X. Namely, q, X satisfy (*).

Proof of 3.8: Let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle$. For every

$$X = \langle X_1, \dots, X_{n+1} \rangle\text{-extension of } p \quad , \quad q \in \mathbb{Q} \quad , \quad \vec{\alpha} \in B(p, X \setminus \langle x_{mc} \rangle)$$

Recall that $l_X = \min(i \mid X_i \neq \emptyset)$ and define $B_{(0)}^X(q, \vec{\alpha})$ to be the set

$$\{\theta \in B(t_{l_X}, x_{mc}) \mid \exists a \exists (C(x_{i,j}))_{i,j} \quad \langle q, p \frown \langle \vec{\alpha}, \theta, C(x_{i,j}) \rangle \Vdash \dot{A} \cap \theta = a \}$$

Also let $B_{(1)}^X(q, \vec{\alpha}) = B(t_{l_X}, x_{mc}) \setminus B_{(0)}^X(q, \vec{\alpha})$. One and only one of $B_{(1)}^X(q, \vec{\alpha}), B_{(0)}^X(q, \vec{\alpha})$ is in $U(t_{l_X}, x_{mc})$. Define $B^X(q, \vec{\alpha})$ and $F_q^X(\vec{\alpha}) \in \{0, 1\}$ such that

$$B^X(q, \vec{\alpha}) = B_{(F_q^X(\vec{\alpha}))}^X(q, \vec{\alpha}) \in U(t_{l_X}, x_{mc})$$

Since $|\mathbb{Q}| \leq 2^{\kappa^*} < \kappa(t_{l_X})$ we have $B^X(\vec{\alpha}) = \bigcap_q B^X(q, \vec{\alpha}) \in U(t_{l_X}, x_{mc})$. Define

$$B^X(t_{l_X}, x_{mc}) = \bigtriangleup_{\vec{\alpha}} B^X(\vec{\alpha}) \in U(t_{l_X}, x_{mc})$$

Use lemma 2.3 to find $B^X(t_i, x_{i,j}) \subseteq B(t_i, x_{i,j})$, $B^X(t_i, x_{i,j}) \in U(t_i, x_{i,j})$ homogeneous for every F_q^X . As before, if $\lambda \notin X_i$ set $B^X(t_i, \lambda) = B(t_i, \lambda)$. Let

$$p^* = p \frown \langle (B^*(t_i))_{i=1}^{n+1} \rangle, \quad B^*(t_i, \lambda) = \bigcap_X B^X(t_i, \lambda)$$

So far what we established the following property: if $q, \vec{\alpha}, (C(x_{i,j}))_{i,j}, a$ are such that

$$\langle q, p^* \frown \langle \vec{\alpha}, (C(x_{i,j}))_{i,j} \rangle \rangle \Vdash \dot{A} \cap \alpha_{mc} = a$$

since $\alpha_{mc} \in B^X(q, \vec{\alpha} \setminus \langle \alpha_{mc} \rangle)$ we conclude that $F_q^X(\vec{\alpha} \setminus \langle \alpha_{mc} \rangle) = 0$. Let $\vec{\alpha}'$ be another extension of type X, then $\vec{\alpha}' \setminus \langle \alpha'_{mc} \rangle$ and $\vec{\alpha} \setminus \langle \alpha_{mc} \rangle$ belong to the same homogeneous set, thus

$$F_q^X(\vec{\alpha}' \setminus \langle \alpha'_{mc} \rangle) = F_q^X(\vec{\alpha} \setminus \langle \alpha_{mc} \rangle) = 0$$

By the definition of $F_q^X(\vec{\alpha}' \setminus \langle \alpha'_{mc} \rangle)$ it follows that $\alpha'_{mc} \in B_{(0)}^X(q, \vec{\alpha}' \setminus \langle \alpha'_{mc} \rangle)$ as wanted. For every $\vec{\alpha} \in B(p', X)$ and $q \in \mathbb{Q}$ fix some $(C_{i,j}(q, \vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}}$ such that

$$\langle q, p^* \frown \langle \vec{\alpha}, (C_{i,j}(q, \vec{\alpha}))_{\substack{i \leq n+1 \\ j \leq l_i+1}} \rangle \rangle \Vdash \dot{A} \cap \alpha_{mc}$$

Let us argue that we can extend p^* to p^{**} such that for all $1 \leq i \leq n+1$, $1 \leq j \leq l_i+1$ and $\vec{\alpha} \in B(p^*, X)$,

$$B(t_i^{**}) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}(\vec{\alpha})$$

Where α_s is the predecessor of $\alpha_{i,j}$ in $\vec{\alpha}$. In order to do that, fix i, j and stabilize $C_{i,j}(\vec{\alpha})$ as follows:

Fix $\vec{\beta} \in B(p^*, \langle x_{1,1}, \dots, x_{i,j} \rangle)$ By lemma 2.3, the function

$$C_{i,j}(q, \vec{\beta}, *) : B(p^*, X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle) \rightarrow P(\beta_{i,j})$$

has homogeneous sets $B'(\vec{\beta}, x_{r,s}, q) \subseteq B(t_r^*, x_{r,s})$ for $x_{r,s} \in X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle$. Denote the constant value by $C_{i,j}^*(q, \vec{\beta})$. Define

$$B'(t_r^*, x_{r,s}) = \bigcup_{\substack{\vec{\beta} \in B(p^*, \langle x_{1,1}, \dots, x_{i,j} \rangle) \\ q \in \mathbb{Q}}} B'(\vec{\beta}, x_{r,s}, q), \quad \text{for } x_{r,s} \in X \setminus \langle x_{1,1}, \dots, x_{i,j} \rangle$$

Next, fix $\alpha \in B(t_i^*, x_{i,j})$ and let

$$C_{i,j}^*(\alpha) = \bigcup_{\substack{\vec{\alpha}' \in B(p^*, \langle x_{1,1}, \dots, x_{i,j-1} \rangle) \\ q \in \mathbb{Q}}} C_{i,j}^*(q, \vec{\alpha}', \alpha)$$

Thus $C_{i,j}^*(\alpha) \subseteq \alpha$. Since $\kappa(t_i)$ is ineffable, there is $B'(t_i^*, x_{i,j}) \subseteq B(t_i^*, x_{i,j})$ and $C_{i,j}^*$ such that for every $\alpha \in B'(t_i^*, x_{i,j})$, $C_{i,j}^* \cap \alpha = C_{i,j}^*(\alpha)$. By coherency, $C_{i,j}^* \in \bigcap U(t_i, \xi)$. Finally, define $p^{**} = \langle t_1^{**}, \dots, t_n^{**}, t_{n+1}^{**} \rangle$, where

$$B(t_i^{**}) = B'(t_i^*) \cap (\bigcap_j C_{i,j}^*) \quad 1 \leq i \leq n+1$$

To see that p^{**} is as wanted, let $\vec{\alpha} \in B(p^{**}, X)$ and fix any i, j . Then $\vec{\alpha} \in B(p^{**}, X)$ and $\alpha_{i,j} \in B(t_i^{**}, x_{i,j})$, hence for any i, j

$$B(t_i^{**}) \cap (\alpha_s, \alpha_{i,j}) \subseteq C_{i,j}^* \cap \alpha_{i,j} \setminus \alpha_s = C_{i,j}^*(\alpha_{i,j}) \setminus \alpha_s \subseteq C_{i,j}^*(\alpha_{1,1}, \dots, \alpha_{i,j}) = C_{i,j}^*(\alpha)$$

■

Lemma 3.9 *Let p^* be as in lemma 3.8 There exist $p^* \leq p^{**}$ such that for every extension type X of p^{**} and $q \in \mathbb{Q}$ that satisfies (*) there exists sets $A(q, \vec{\alpha}) \subseteq \kappa$, for $\vec{\alpha} \in B(p^{**}, X \setminus \langle x_{mc} \rangle)$, such that for all $\alpha \in B(p^{**}, x_{mc})$*

$$A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$$

Example: Recall that we have obtained the sets

$$a(q, \vec{\alpha}) = C_H \restriction_{\text{even}} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2)\} \cup b(q, \vec{\alpha})$$

$$b(q, \vec{\alpha}) = \begin{cases} \emptyset & \alpha_{\omega \cdot 3} + \nu_3 \geq \alpha_{mc} \\ \{\alpha_{\omega \cdot 3} + \nu_3\} & \alpha_{\omega \cdot 3} + \nu_3 < \alpha_{mc} \end{cases}$$

The element α_{mc} is chosen from the set $B(t_3, x_{mc}) = B(t_3, 0)$, by shrinking this set, we can directly extend p to p^* such that for every $\vec{\alpha} \in B(p^*, X)$, $\alpha_{\omega \cdot 3} + \nu_3 < \alpha_{mc}$. Therefore,

$$A(q, \vec{\alpha}) = C_H \restriction_{\text{even}} \cup \{C_H(\omega), C_H(\omega) + \nu_1, \nu_{\omega \cdot 2} + C_H(2), \alpha_{\omega \cdot 3} + \nu_3\}$$

Proof of 3.9: Fix q, X satisfying $(*)$ and $\vec{\alpha} \in B(p^*, X \setminus \langle x_{mc} \rangle)$, since $\kappa(t_i)$ is ineffable we can shrink the set $B(t_{i_X}^*, x_{mc})$ to $B'(q, \vec{\alpha})$ to find sets $A(q) \subseteq t_i$ such that

$$\forall \alpha \in B'(q, \vec{\alpha}) \quad A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha)$$

define $B_q(t_i^*, x_{mc}) = \bigtriangleup_{\vec{\alpha} \in B(p^*, X \setminus \langle x_{mc} \rangle)} B^{**}(q, \vec{\alpha})$. Intersect over all X, q and find $p^* \leq p^{**}$ as before. ■

Thus there exists $p_* \in G_{>\kappa^*}$ with the properties described in Lemma's 3.8-3.9. Next we would like to claim that for some sufficiently large family of $q \in \mathbb{Q}$ and extension-type X we have q, X satisfy $(*)$.

Lemma 3.10 *Let $p_* \in G_{>\kappa^*}$ be as above and let X be any extension-type of p_* . Then there exists a maximal antichain $Z_X \subseteq \mathbb{Q}$ and extension-types $X \preceq X_q$ for $q \in Z_X$, unveiling the same maximal coordinate as X such that for every $q \in Z_X$, q, X_q satisfy $(*)$.*

Example: The anti chain Z_X can be chosen as follows: For any possible ν_1, ν_3 choose a condition $\langle \nu_1, \nu_3, \langle \kappa^*, B^* \rangle \rangle \in \mathbb{Q}$. This set definitely form a maximal anti chain, and by the same method of the previous examples taking $X_q = X$ works. In general, if the maximal coordinate of X is some $\omega \cdot (2n + 1)$, Z_X will be the anti chain consisting of representative conditions for the $2n + 1$ first coordinates.

Proof. The existence of Z_X will follow from Zorn's Lemma and the method proving existence of X_q for some q . Fix any $\vec{\alpha} \in B(p_*, X)$, there exists a generic $H \subseteq \mathbb{Q} \times \mathbb{M}_{>\kappa^*}[\vec{U}]$ with $\langle 1_{\mathbb{Q}}, p_* \widehat{\vec{\alpha}} \rangle \in H = H_{\leq \kappa^*} \times H_{>\kappa^*}$. Consider the decomposition of $\mathbb{M}[\vec{U}]_{>\kappa^*}$ above $p_* \widehat{\vec{\alpha}}$ induced by α_{mc} and let $p_* \widehat{\vec{\alpha}} = \langle p_1, p_2 \rangle$, i.e. $\langle p_1, p_2 \rangle \in (\mathbb{M}[\vec{U}]_{>\kappa^*})_{\leq \alpha_{mc}} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{>\alpha_{mc}}$. H is generic for the forcing $\mathbb{Q} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{\leq \alpha_{mc}} \times (\mathbb{M}[\vec{U}]_{>\kappa^*})_{>\alpha_{mc}}$. Define $H_1 = H_{\leq \kappa^*} \times (H_{>\kappa^*})_{\leq \alpha_{mc}}$ and $H_2 = H_{>\alpha_{mc}}$. Denote by $(\dot{A})_{H_1} \in V[H_1]$ to be the name obtained by filtering only the part of H_1 . It is a name of A in the forcing $\mathbb{M}[\vec{U}]_{>\alpha_{mc}}$. Above p_2 we have sufficient closure to determine $(\dot{A})_{H_1} \cap \alpha_{mc}$

$$\exists p_2^* \geq^* p_2 \text{ s.t. } p_2^* \Vdash_{\mathbb{M}[\vec{U}]_{>\alpha_{mc}}} (\dot{A})_{H_1} \cap \alpha_{mc} = a$$

for some $a \in V[C^*]$. Hence there exists $\langle 1_{\mathbb{Q}_{\leq \kappa^*}}, p_1 \rangle \leq \langle q, p_1^* \rangle$ such that

$$\langle q, p_1^* \rangle \Vdash_{\mathbb{Q} \times \mathbb{M}_{\leq \alpha_{mc}}[\vec{U}]} \bigvee p_2^{**} \Vdash_{\mathbb{M}[\vec{U}]_{>\alpha_{mc}}} \dot{A} \cap \alpha_{mc} = a$$

It is clear that $\langle q, p_1^*, p_2^* \rangle \Vdash_{\mathbb{Q} \times \mathbb{M}_{>\kappa^*}[\vec{U}]} \dot{A} \cap \alpha_{mc}$. Finally, X_q is simply the extension type of p_1^* . Since $p_1^* \in \mathbb{M}_{\leq \alpha_{mc}}[\vec{U}]$, X_q unveils the same maximal coordinate as X . By lemma 3.8, X_q, q satisfies $(*)$.

■

Lemma 3.11 κ changes cofinality in $V[A]$.

Proof. Let $p_* = \langle t_1^*, \dots, t_n^*, t_{n+1}^* \rangle \in G_{>\kappa^*}$ be as before, $\lambda_0 = \text{otp}(C_G)$ and $\langle C_G(\xi) \mid \xi < \lambda_0 \rangle$ be the Magidor sequence corresponding to G . Work in $V[A]$, define a sequence $\langle \nu_i \mid \gamma(t_n^*, p_*) \leq i < \lambda_0 \rangle \subset \kappa$:

$$\nu_{\gamma(t_n^*, p_*)} = C_G(\gamma(t_n^*, p_*)) + 1 = \kappa(t_n^*) + 1$$

Assume that $\langle \nu_{\xi'} \mid \xi' < \xi < \lambda_0 \rangle$ is defined such that it is increasing and $\nu_{\xi'} < \kappa$. If ξ is limit define

$$\nu_\xi = \sup(\nu_{\xi'}) + 1.$$

If $\sup(\nu_{\xi'}) = \kappa$ we are done, since κ changes cofinality to $cf(\xi) < \lambda_0$ (which cannot hold for regular λ_0). Therefore, $\nu_\xi < \kappa$. If $\xi = \xi' + 1$, by proposition 3.2, there exist an extension type X_ξ of p_* unveiling ξ as maximal coordinate. By lemma 3.10 we can find Z_ξ and $X_\xi \preceq X_q$ unveiling ξ as maximal coordinate such that q, X_q satisfies (*). By lemma 3.9 there exists

$$A(q, \vec{\alpha})\text{'s for } q \in Z_\xi \text{ } \vec{\alpha} \in B(p^*, X_q \setminus \langle x_{mc} \rangle).$$

Since $A \notin V[C^*]$, $A \neq A(q, \vec{\alpha})$. Thus define $\eta(q, \vec{\alpha}) = \min(A(q, \vec{\alpha}) \Delta A) + 1$

$$\beta_\xi = \sup(\eta(q, \vec{\alpha}) \mid \vec{\alpha} \in [\nu_{\xi'}]^{<\omega} \cap B(p^*, X_q \setminus \langle x_{mc} \rangle), q \in Z_\xi)$$

It follows that $\beta_\xi \leq \kappa$. Assume $\beta_\xi = \kappa$, then κ changes cofinality but it might be to some other cardinal larger than δ_0 , this is not enough in order to apply 3.6⁵. Toward a contradiction, fix an unbounded and increasing sequence $\langle \eta(q_i, \vec{\alpha}_i) \mid i < \theta < \kappa \rangle$ for some $q_i \in Z_\xi$ and $\vec{\alpha}_i \in [\nu_{\xi'}]^{<\omega}$. Notice that since $\eta(q_i, \vec{\alpha}_i) < \eta(q_{i+1}, \vec{\alpha}_{i+1})$ it must be that $A(q_i, \vec{\alpha}_i) \neq A(q_{i+1}, \vec{\alpha}_{i+1})$ and

$$A(q_i, \vec{\alpha}_i) \cap \eta(q_i, \vec{\alpha}_i) = A \cap \eta(q_i, \vec{\alpha}_i) = A(q_{i+1}, \vec{\alpha}_{i+1}) \cap \eta(q_i, \vec{\alpha}_i)$$

Define $\eta_i = \min(A(q_i, \vec{\alpha}_i) \Delta A(q_{i+1}, \vec{\alpha}_{i+1})) \geq \eta(q_i, \vec{\alpha}_i)$. It follows that $\langle \eta_i \mid i < \theta \rangle$ is a short cofinal sequence in κ . This definition is independent of A and only involve $\langle \langle q_i, \vec{\alpha}_i \rangle \mid i < \theta < \kappa \rangle$, which can be coded as a bounded sequence of κ . By the induction hypothesis there is $C'' \subseteq C$, bounded in κ such that

$$V[C''] = V[\langle \langle q_i, \vec{\alpha}_i \rangle \mid i < \theta < \kappa \rangle]$$

Define $C' = C^* \cup C''$, the model $V[C']$ should keep κ measurable, since C' is bounded, but also include the sequence $\langle \eta_i \mid i < \theta \rangle$, contradiction.

⁵Actually, after proving Theorem 3.3, we can conclude that this phenomena cannot hold.

Therefore, $\beta_\xi < \kappa$, set $\nu_\xi = \beta_\xi + 1$. This concludes the construction of the sequence ν_ξ . To see that the sequence is unbounded in κ , let us show that $C_G(\xi) < \nu_\xi$:

Clearly $C_G(\gamma(t_n^*, p_*)) < \nu_{\gamma(t_n^*, p_*)}$. Inductively assume that $C_G(i) < \nu_i$, $\gamma(t_n^*, p_*) \leq i < \xi$. If ξ is limit, since Magidor generic sequences are closed,

$$C_G(\xi) = \sup(C_G(i) \mid i < \xi) \leq \sup(\nu_i \mid \gamma(t_n^*, p_*) \leq i < \xi) < \nu_\xi$$

If $\xi = \xi' + 1$ is successor, let $\{q_\xi\} = Z_\xi \cap G_{\leq \kappa^*}$

$$p_\xi = p_* \widehat{\langle C_G(i_1), \dots, C_G(i_n), C_G(\xi) \rangle} \in p_* \widehat{X_\xi} \cap G_{> \kappa^*}$$

By induction $C_G(i_r) < \nu_{\xi'}$, therefore, $\eta(q_\xi, \langle C_G(i_1), \dots, C_G(i_n) \rangle) < \nu_\xi$. Finally, $\langle q_\xi, p_\xi \rangle \in G$, $\langle q_\xi, p_\xi \rangle \Vdash \dot{A} \cap C_G(\xi) = A(q_\xi, \langle C_G(i_1), \dots, C_G(i_n) \rangle) \cap C_G(\xi)$, thus

$$A \cap C_G(\xi) = A(q_\xi, \langle C_G(i_1), \dots, C_G(i_n) \rangle) \cap C_G(\xi), \text{ hence } C_G(\xi) \leq \eta(q_\xi, \langle C_G(i_1), \dots, C_G(i_n) \rangle) < \nu_\xi.$$

■

4 The Main Result Above κ

In order to push the induction to sets above κ we will need a projection of $\mathbb{M}[\vec{U}]$ onto some forcing that adds a subsequence of C_G . The majority of this chapter is the definition of this projection and some of its properties. The inductive argument will continue at lemma 4.17.

Let G be generic and C_G the corresponding Magidor sequence. Let $C^* \subseteq C_G$ be a subsequence and $I = \text{Index}(C^*, C_G)$. Then I is a subset of $\text{otp}(C_G) := \lambda_0$, hence $I \in V$. Assume that $\kappa^* = \sup(C^*)$ is a limit point in C_G and that C^* is closed i.e. containing all of its limit points below κ^* ⁶. As we will see in the next lemma, one can find a forcing of the form $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$, such that $G \subseteq \mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$ is V -generic, which will be easier to project.

Proposition 4.1 *Let G be $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ -generic and $C^* \subseteq C_G$ such that C^* is closed and $\kappa^* = \sup(C^*)$ is a limit point of C_G . Then there exists $\langle \nu_1, \dots, \nu_m \rangle$ such that G is generic for $\mathbb{M}_{\langle \nu_1, \dots, \nu_m \rangle}[\vec{U}]$ and for all $1 \leq i \leq m$, $C^* \cap (\nu_{i-1}, \nu_i)$ is either empty or a club in ν_i . (as usual denote $\nu_0 = 0$)*

Example: Assume that $\lambda_0 = \omega_1 + \omega^2 \cdot 2 + \omega$, C^* is

$$C_G \restriction (\omega_1 + 1) \cup \{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\} \cup \{C_G(\omega_1 + \alpha) \mid \omega^2 \cdot 2 < \alpha < \lambda_0\}$$

Let $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4 = \kappa$ be such that $o^{\vec{U}}(\kappa_1) = \omega_1$, $o^{\vec{U}}(\kappa_2) = o^{\vec{U}}(\kappa_3) = 2$ and $o^{\vec{U}}(\kappa) = 1$. We have

1. $(0, \kappa_1) \cap C^* = C_G \restriction \omega_1$.
2. $(\kappa_1, \kappa_2) \cap C^* = \{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\}$.
3. $(\kappa_2, \kappa_3) \cap C^* = \emptyset$.
4. $(\kappa_3, \kappa_4) \cap C^* = \{C_G(\omega_1 + \alpha) \mid \omega^2 \cdot 2 < \alpha < \lambda_0\}$.

Then (1),(3),(4) are either empty or a club, but (2) is not. To fix this, we simply add $\{C_G(\omega_1 + \omega + 2), C_G(\omega_1 + \omega + 3)\}$ to $\kappa_1 < \kappa_2 < \kappa_3 < \kappa_4$.

Proof of 4.1: By induction on m , let us define a sequence

$$\vec{\nu}_m = \langle \nu_{1,m}, \dots, \nu_{n_m,m} \rangle$$

such that for every m , G is generic for $\mathbb{M}_{\vec{\nu}_m}[\vec{U}]$. Define $\vec{\nu}_0 = \langle \kappa_1, \dots, \kappa_n \rangle$. Assume that $\vec{\nu}_m$ is defined with G generic, if for every $1 \leq i \leq n_m + 1$ we have $C^* \cap (\nu_{i-1,m}, \nu_{i,m})$ is either empty or

⁶Note that for any $C \subseteq C_G$, we can take $Cl(C) = \{\alpha < \sup(C) \mid \alpha = \sup(C \cap \alpha)\} \cup C$, and $V[Cl(C)] = V[C]$, since $\text{Index}(C, Cl(C)) \in V$.

unbounded (and therefore a club), stabilize the sequence at m . Otherwise, let i be maximal such that $C^* \cap (\nu_{i-1,m}, \nu_{i,m})$ is nonempty and bounded. Thus,

$$\nu_{i-1,m} < \sup(C^* \cap (\nu_{i-1,m}, \nu_{i,m})) < \nu_{i,m}$$

Since C^* is closed, $C_G(\gamma) = \sup(C^* \cap (\nu_{i-1,m}, \nu_{i,m})) \in C^*$ for some γ . As in lemma 3.1 we can find

$$\vec{\nu}_{m+1} = \langle \nu_{1,m}, \dots, \nu_{i,m}, \xi_1, \dots, \xi_k, \nu_{i+1,m}, \dots, \nu_{n_m,m} \rangle \subseteq C_G$$

such that $C_G(\gamma) = \xi_k$ is unveiled and the forcing $\mathbb{M}_{\vec{\nu}_{m+1}}[\vec{U}] \subseteq \mathbb{M}_{\vec{\nu}_m}[\vec{U}]$ is a subforcing of $\mathbb{M}_{\vec{\nu}_m}[\vec{U}]$ with G one of its generic sets. Note that the maximal ordinal in the sequence $\vec{\nu}_{m+1}$ such that $C^* \cap (\nu_{j-1,m+1}, \nu_{j,m+1})$ is nonempty and bounded is strictly less than $\nu_{i,m}$. Therefore this iteration stabilizes at some $N < \omega$. Consider the forcing $\mathbb{M}_{\vec{\nu}_N}[\vec{U}]$, by the construction of the $\vec{\nu}_r$'s, for every $1 \leq i \leq n_N + 1$ $C^* \cap (\nu_{i-1,N}, \nu_{i,N})$ is either empty or unbounded (Since $\vec{\nu}_{N+1} = \vec{\nu}_N$).

■

Assume that $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$ and C^* satisfy the property of 4.1. Let us define a projection of

$$\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}] = \prod_{i=1}^n (\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}}$$

onto some forcing $\prod_{i=1}^n \mathbb{P}_i$. We can define such a projection, by projecting each factor

$$\pi_i : (\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}} \rightarrow \mathbb{P}_i \quad (1 \leq i \leq n)$$

and derive $\pi : \mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}] \rightarrow \prod_{i=1}^n \mathbb{P}_i$. First, if $C^* \cap (\kappa_{i-1}, \kappa_i)$ is empty, the projection is going to be to the trivial forcing. Otherwise, $C^* \cap (\kappa_{i-1}, \kappa_i)$ is a club at κ_i . In order to simplify notation, we will assume that $(\mathbb{M}_{\kappa_i})_{>\kappa_{i-1}} = \mathbb{M}[\vec{U}]_{\langle \kappa \rangle} = \mathbb{M}[\vec{U}]$ and $C^* = C^* \cap (\kappa_{i-1}, \kappa_i)$ is a club in κ . It seems natural that the projection will keep only the coordinates in $I = \text{Index}(C^*, C_G)$, namely:

Definition 4.2 Let $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}[\vec{U}]$, define the projection to the I coordinates by,

$$\pi_I(p) = \langle t'_i \mid \gamma(t_i, p) \in I \rangle \frown \langle t_{n+1} \rangle, \text{ where } t'_i = \begin{cases} \kappa(t_i) & \gamma(t_i, p) \in \text{Succ}(I) \\ t_i & \gamma(t_i, p) \in \text{Lim}(I) \end{cases}$$

Let us define a forcing notion $\mathbb{P}_i = \mathbb{M}_I[\vec{U}]$ (the range of the projection π_I) that will add the subsequence C^* , such that the forcing $\mathbb{M}[\vec{U}]$ (more precisely, a dense subset of $\mathbb{M}[\vec{U}]$) projects onto $\mathbb{M}_I[\vec{U}]$ via the projection π_I we have just defined in 4.2.

4.1 The forcing $\mathbb{M}_I[\vec{U}]$

Considering C^* as a function with domain I , we would like to have a function similar to $\gamma(t_i, p)$ that tells us which coordinate is currently unveiled. Given $p = \langle t_1, \dots, t_n, t_{n+1} \rangle$, define recursively:

1. $I(t_0, p) := 0$.
2. If $\{j \in I \setminus I(t_{i-1}, p) + 1 \mid o_L(j) = o^{\vec{U}}(t_i)\} = \emptyset$, then $I(t_i, p) = N/A$ undefined.
3. If $\{j \in I \setminus I(t_{i-1}, p) + 1 \mid o_L(j) = o^{\vec{U}}(t_i)\} \neq \emptyset$, define

$$I(t_i, p) := \min(j \in I \setminus I(t_{i-1}, p) + 1 \mid o_L(j) = o^{\vec{U}}(t_i))$$

If for every $0 \leq i \leq n$, $I(t_i, p) \neq N/A$, we say that I is defined on p .

Example: Consider Magidor forcing adding a sequence of length ω^2 i.e. $o^{\vec{U}}(\kappa) = 2$ and $C_G = \{C_G(\alpha) \mid \alpha < \omega^2\}$. Assume $C^* = \{C_G(0)\} \cup \{C_G(\alpha) \mid \omega \leq \alpha < \omega^2\}$, hence, $I = \{0\} \cup (\omega^2 \setminus \omega)$. The ω -th element of C_G is no longer limit in C^* . Let

$$p = \langle \underbrace{\langle \kappa(t_1), B(t_1) \rangle}_{t_1}, \underbrace{\langle \kappa, B(t_2) \rangle}_{t_2} \rangle$$

Where $o^{\vec{U}}(t_1) = 1$. Computing $I(t_1, p)$,

$$I(t_1, p) = \omega = \gamma(t_1, p)$$

Therefore $\pi_I(p) = \langle \kappa(t_1), t_2 \rangle$.

Definition 4.3 The *Magidor forcing adding a sequence prescribed by I* , denoted by $\mathbb{M}_I[\vec{U}]$, consist of conditions of the form $p = \langle t_1, \dots, t_{n+1} \rangle$ such that:

1. I is defined on p .
2. $\kappa(t_1) < \dots < \kappa(t_n) < \kappa(t_{n+1}) = \kappa$.
3. For $i = 1, \dots, n + 1$,
 - (a) If $I(t_i, p) \in \text{Succ}(I)$,
 - i. $t_i = \kappa(t_i)$.
 - ii. $I(t_{i-1}, p)$ is the predecessor of $I(t_i, p)$ in I .

iii. If $I(t_{i-1}, p) + \sum_{i=1}^m \omega^{\gamma_i} = I(t_i, p)$ (C.N.F) , then ⁷,

$$\left(Y(\gamma_1) \times \dots \times Y(\gamma_{m-1}) \right) \cap [(\kappa(t_{i-1}), \kappa(t_i))]^{<\omega} \neq \emptyset$$

(b) If $I(t_i, p) \in \text{Lim}(I)$, then,

i. $t_i = \langle \kappa(t_i), B(t_i) \rangle$. , $B(t_i) \in \bigcap_{\xi < o^{\vec{U}}(t_i)} U(t_i, \xi)$

ii. $I(t_{i-1}, p) + \omega^{o^{\vec{U}}(t_i)} = I(t_i, p)$.

iii. $\min(B(t_i)) > \kappa(t_{i-1})$.

Definition 4.4 Let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle, q = \langle s_1, \dots, s_m, s_{m+1} \rangle \in \mathbb{M}_I[\vec{U}]$. Define the *order* of $\mathbb{M}_I[\vec{U}]$, $\langle t_1, \dots, t_n, t_{n+1} \rangle \leq_I \langle s_1, \dots, s_m, s_{m+1} \rangle$ iff $\exists 1 \leq i_1 < \dots < i_n \leq m < i_{n+1} = m+1$ such that:

1. $\kappa(t_r) = \kappa(s_{i_r})$ and $B(s_{i_r}) \subseteq B(t_r)$.

2. If $i_k < j < i_{k+1}$, then

(a) $\kappa(s_j) \in B(t_{k+1})$.

(b) If $I(s_j, q) \in \text{Succ}(I)$, then,

$$\left(B(t_{k+1}, \gamma_1) \times \dots \times B(t_{k+1}, \gamma_{k-1}) \right) \cap [(\kappa(s_{j-1}), \kappa(s_j))]^{<\omega} \neq \emptyset$$

where $I(s_{i-1}, q) + \sum_{i=1}^k \omega^{\gamma_i} = I(s_i, q)$ (C.N.F).

(c) If $I(s_j, q) \in \text{Lim}(I)$ then $B(s_j) \subseteq B(t_{k+1}) \cap \kappa(s_j)$.

Definition 4.5 Let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle, q = \langle s_1, \dots, s_m, s_{m+1} \rangle \in \mathbb{M}_I[\vec{U}]$, q is a *direct extension* of p , denoted $p \leq_I^* q$, iff:

1. $p \leq_I q$.

2. $n = m$.

Remarks:

1. In definition 4.3 (b.i), although it seems superfluous to take all the measures corresponding to t_i as well as those which do not take an active part in the development of C^* , the necessity is apparent when examining definition 4.4 (2.b)- the γ_i 's may not be the measures taking active part in C^* . In lemma 4.10 this condition will be crucial when completing C^* to C_G .

⁷Recall that $Y(\gamma) = \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) = \gamma\}$

2. As we have seen in earlier chapters, the function $\gamma(t_i, p)$ returns the same value when extending p . $I(t_i, p)$ have the same property, let $p = \langle t_1, \dots, t_n, t_{n+1} \rangle$, $q = \langle s_1, \dots, s_m, s_{m+1} \rangle \in \mathbb{M}_I[\vec{U}]$, such that $p \leq_I q$, by 4.3 (2.b.ii), $I(t_r, p) = I(s_r, q)$.
3. In definition 4.5, since $n = m$ we only have to check (1) of definition 4.4.
4. Let $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$ be any condition. Assume we would like to unveil a new index $j \in I$ between $I(t_i, p)$ and $I(t_{i+1}, p)$. It is possible if for example j is the successor of $I(t_i, p)$ in I :
Assume $I(t_i, p) + \sum_{l=1}^m \omega^{\gamma_l} = j$ (C.N.F), then $\gamma_l < o^{\vec{U}}(t_{i+1})$. Extend p by choosing $\alpha \in B(t_{i+1}, \gamma_m)$ above some sequence

$$\langle \beta_1, \dots, \beta_k \rangle \in B(t_{i+1}, \gamma_1) \times \dots \times B(t_{i+1}, \gamma_{m-1})$$

Then

$$I(\alpha, p \smallfrown \langle \alpha \rangle) = \min(r \in I \setminus I(t_i, p) \mid o_L(r) = o_L(j)) = j$$

Another possible index is any $j \in \text{Lim}(I)$ such that $I(t_i, p) + \omega^{o_L(j)} = j$. For such j , extend p by picking $\alpha \in B(t_{i+1}, o_L(j))$ above some sequence $\langle \beta_1, \dots, \beta_k \rangle$, to obtain

$$p \leq_I \langle t_1, \dots, t_i, \langle \alpha, \bigcap_{\xi < o_L(j)} B(t_{i+1}, \xi) \cap \alpha \rangle, \langle \kappa(t_{i+1}), B(t_{i+1}) \setminus (\alpha + 1) \rangle, \dots, t_{n+1} \rangle$$

A routine verification of definition 4.3 asserts that in both cases the extension of p is in $\mathbb{M}_I[\vec{U}]$.

The forcing $\mathbb{M}_I[\vec{U}]$ has lots of the properties of $\mathbb{M}[\vec{U}]$, however, they are irrelevant for the proof. Therefore, we will state only few of them.

Lemma 4.6 $\mathbb{M}_I[\vec{U}]$ satisfy $\kappa^+ - c.c.$

Proof. Let $\{\langle t_{\alpha,1}, \dots, t_{\alpha,n_\alpha} \rangle = p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}_I[\vec{U}]$. Find $n < \omega$ and $E \subseteq \kappa^+$, $|E| = \kappa^+$ and $\langle \kappa_1, \dots, \kappa_n \rangle$ such that $\forall \alpha \in E$,

$$n_\alpha = n, \text{ and } \langle \kappa(t_{\alpha,1}), \dots, \kappa(t_{\alpha,n_\alpha}) \rangle = \langle \kappa_1, \dots, \kappa_n \rangle$$

Fix any $\alpha, \beta \in E$. Define $p^* = \langle t_1, \dots, t_n, t_{n+1} \rangle$ where

$$B^*(t_i) = B(t_{i,\alpha}) \cap B(t_{i,\beta}) \in \bigcap_{\xi < o^{\vec{U}}(\kappa_i)} U(\kappa_i, \xi)$$

$$t_i = \begin{cases} \langle \kappa_i, B^*(t_i) \rangle & I(t_i, p) \in \text{Lim}(I) \\ \kappa_i & \text{otherwise} \end{cases}$$

Since $p_\alpha, p_\beta \in \mathbb{M}_I[\vec{U}]$, it is clear that $p^* \in \mathbb{M}_I[\vec{U}]$ and also $p_\alpha, p_\beta \leq_I^* p^*$.

■

Lemma 4.7 *Let $G_I \subseteq \mathbb{M}_I[\vec{U}]$ be generic, define*

$$C_I = \bigcup \{ \{ \kappa(t_i) \mid i = 1, \dots, n \} \mid \langle t_1, \dots, t_n, t_{n+1} \rangle \in G_I \}$$

Then

1. $\text{otp}(C_I) = \text{otp}(I)$ (thus we may also think of C_I as a function with domain I).
2. G_I consist of all conditions $p = \langle t_1, \dots, t_n, t_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$ such that
 - (a) $C_I(I(t_i, p)) = \kappa(t_i)$.
 - (b) $C_I \cap (\kappa(t_{i-1}), \kappa(t_i)) \subseteq B(t_i) \quad 1 \leq i \leq n+1$.
 - (c) $\forall i \in \text{Succ}(I) \cap (I(t_r, p), I(t_{r+1}, p))$ with predecessor $j \in I$ such that $j + \sum_{l=1}^k \omega^{\gamma_l} = i$ (C.N.F) we have

$$[(C_I(j), C_I(i))]^{<\omega} \cap B(t_{r+1}, \gamma_1) \times \dots \times B(t_{r+1}, \gamma_{k-1}) \neq \emptyset$$

Proof. For (1), let us consider the system of ordered sets of ordinals $(\kappa(p), i_{p,q})_{p,q}$ where

$$\kappa(p) = \{ \kappa(t_1), \dots, \kappa(t_n) \} \text{ for } p = \langle t_1, \dots, t_{n+1} \rangle \in G_I$$

$i_{p,q} : \kappa(p) \rightarrow \kappa(q)$ are defined for $p = \langle t_1, \dots, t_{n+1} \rangle \leq_I \langle s_1, \dots, s_{m+1} \rangle = q$ as the inclusion:

$$i_{p,q}(\kappa(t_r)) = \kappa(t_r) = \kappa(s_{i_r}) \text{ (} i_r \text{ are as in the definition of } \leq_I \text{)}$$

Since G_I is a filter, $(\kappa(p), i_{p,q})_{p,q}$ form a directed system with a direct ordered limit

$$\varinjlim \kappa(p) = \bigcup_{p \in G_I} \kappa(p) = C_I \text{ and inclusions } i_p : \kappa(p) \rightarrow C_I$$

We already defined for $p, q \in G_I$ such that $p \leq_I q$, commuting functions

$$I(*, p) : \kappa(p) \rightarrow I, \quad (*, p) = I(*, q) \circ i_{p,q}$$

Thus $(I(*, p))_{p \in G}$ form a compatible system of functions, and by the universal property of directed limits, we obtain

$$I(*) : C_I \rightarrow I, \quad I(*) \circ i_p = I(*, p)$$

Let us show that I is an isomorphism of ordered set: Since $I(*, p)$ are injective $I(*)$ is also injective. Assume $\kappa_1 < \kappa_2 \in C_I$, find $p \in G_I$ such that $\kappa_1, \kappa_2 \in \kappa(p)$. Therefore, $I(\kappa_i, p) = I(\kappa_i)$ preserve the order of κ_1, κ_2 . Fix $i \in I$, it suffices to show that there exists some condition $p \in G_I$ such that $i \in \text{Im}(I(*, p))$. To do this, let us show that the set of all conditions $p \in \mathbb{M}_I[\vec{U}]$ with $i \in \text{Im}(I(*, p))$ is a dense subset of $\mathbb{M}_I[\vec{U}]$. Let $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$ be any condition, if $i \in \text{Im}(I(*, p))$ then we are done. Otherwise, there exists $0 \leq k \leq n$ such that,

$$I(t_k, p) < i < I(t_{k+1}, p)$$

therefore $I(t_{k+1}, p) \in \text{Lim}(I)$. By induction on i , let us argue that it is possible to extend p to a condition p' , such that $i \in \text{Im}(I(*, p'))$. If

$$\sum_{l=1}^k \omega^{\gamma_l} = i = \min(I) \quad (\text{C.N.F})$$

it follows that $i < I(t_1, p)$. By definition 4.3 (2.b.ii), $I(t_1, p) = \omega^{o_{\vec{U}}(t_1)}$. To extend p just pick any α above some sequence

$$\langle \beta_1, \dots, \beta_k \rangle \in B(t_1, \gamma_1) \times \dots \times B(t_1, \gamma_{k-1})$$

and

$$p \leq_I \langle \alpha, \langle \kappa(t_1), B(t_1) \setminus (\alpha + 1) \rangle, t_2, \dots, t_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$$

If $i \in \text{Succ}(I)$ with predecessor $j \in I$. By the induction hypothesis, we can assume that for some k , $j = I(t_k, p) \in \text{Im}(I(*, p))$. Thus by the remarks following definition 4.5 we can extend p by some α such that $i \in \text{Im}(I(*, p))$. Finally if $i \in \text{Lim}(I)$, then

$$i = \alpha + \omega^{o_L(i)}, \text{ where } \alpha := \sum_{i=1}^m \omega^{\gamma_i} \quad (\text{C.N.F})$$

therefore $\forall \beta \in (\alpha, i)$, $\beta + \omega^{o_L(i)} = i$. Take any $i' \in I \cap (\alpha, i)$. Just as before, it can be assumed that $i' = I(t_k, p)$, thus $I(t_k, p) + \omega^{o_L(i)} = i$. By the same remark, we can extend p to some $p' \in \mathbb{M}_I[\vec{U}]$ with $j \in \text{Im}(I(*, p'))$.

For (2), let $p = \langle t_1, \dots, t_{n+1} \rangle \in G_I$. (a) is satisfied by the argument in (1). Fix $\alpha \in C_I \cap (\kappa(t_i), \kappa(t_{i+1}))$, there exists $p \leq_I p' = \langle s_1, \dots, s_m \rangle \in G_I$ such that $\alpha \in \kappa(p')$ thus $\alpha \in B(t_{i+1})$ by definition. Moreover, if $I(\alpha, p') \in \text{Succ}(I)$ with predecessor $j \in I$, then by definition 4.3 (2.a.ii), there is s_k such that $j = I(s_k, p')$ and by definition 4.4 (2.b)

$$[(\kappa(s_{k-1}), \kappa(s_k))]^{<\omega} \cap B(t_{i+1}, \gamma_1) \times \dots \times B(t_{i+1}, \gamma_{k-1}) \neq \emptyset$$

From (a),

$$\kappa(s_k) = C_I(j) \text{ and } \kappa(s_{k+1}) = C_I(i)$$

In the other direction, if $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$ satisfies (a)-(c). By (a), there exists some $p'' \in G_I$ with $\kappa(p) \subseteq \kappa(p'')$. Set E to be

$$\{\langle w_1, \dots, w_{l+1} \rangle \in (\mathbb{M}_I[\vec{U}])_{\geq_I p''} \mid \kappa(w_j) \in B(t_i) \cup \{\kappa(t_i)\} \rightarrow B(w_j) \subseteq B(t_i)\}$$

E is dense in $\mathbb{M}_I[\vec{U}]$ above p'' . Find $p'' \leq_I p' = \langle s_1, \dots, s_{m+1} \rangle \in G_I \cap D$. Checking definition 4.4, Let us show that $p \leq_I p'$: For (1), since $\kappa(p) \subseteq \kappa(p')$ there is a natural injection $1 \leq i_1 < \dots < i_n \leq m$ which satisfy $\kappa(t_r) = \kappa(s_{i_r})$. Since $p' \in E$, $B(s_{i_r}) \subseteq B(t_r)$. (2a), follows from condition (b), (2b) follows from condition (c). Since $p' \in E$, if $i_r < j < i_{r+1}$ then $\kappa(s_j) \in B(t_{r+1})$, thus, (2c) holds. ■

Given a generic set G_I for $\mathbb{M}_I[\vec{U}]$, we have $V[C_I] = V[G_I]$. Once we will show that π_I is a projection, then for every $G \subseteq \mathbb{M}[\vec{U}]$ generic,

$$\pi_{I*}(G) := \{p \in \mathbb{M}_I[\vec{U}] \mid \exists q \in \pi_I'' G, p \leq_I q\}$$

will be generic for $\mathbb{M}_I[\vec{U}]$ and by the definition of π_I 4.2, we conclude that the corresponding sequence to $\pi_I(G)$ is C^* , this is stated formally on corollary 4.11. Let us turn to the proof that π_I is a projection.

Definition 4.8 Let D be the set of all

$$p = \langle t_1, \dots, t_n, t_{n+1} \rangle \in \mathbb{M}[\vec{U}] , \pi_I(p) = \langle t'_{i_1}, \dots, t'_{i_m}, t_{n+1} \rangle$$

such that:

1. If $\gamma(t_{i_j}, p) \in \text{Lim}(I)$ then $\gamma(t_{i_{j-1}}, p) = \gamma(t_{i_j-1}, p)$.
2. If $\gamma(t_{i_j}, p) \in \text{Succ}(I)$ then $\gamma(t_{i_{j-1}}, p)$ is the predecessor of $\gamma(t_{i_j}, p)$ in I .

Condition (1) is to be compared with definition 4.3 (2.b.ii) and condition (2) with (2.a.ii). The following example justifies the necessity of D.

Example: Assume that

$$\lambda_0 = \omega^2 \text{ and } I = \{2n \mid n \leq \omega\} \cup \{\omega + 2, \omega + 3\} \cup \{\omega \cdot n \mid n < \omega\}$$

let p be the condition

$$\langle \underbrace{\langle \nu_\omega, B_\omega \rangle}_{t_1}, \underbrace{\nu_{\omega+1}}_{t_2}, \underbrace{\langle \nu_{\omega \cdot 2}, B_{\omega \cdot 2} \rangle}_{t_3}, \underbrace{\langle \kappa, B \rangle}_{t_4} \rangle$$

$$\pi_I(p) = \langle \underbrace{\langle \nu_\omega, B_\omega \rangle}_{t_1 \mapsto t'_{i_1}}, \underbrace{\nu_{\omega \cdot 2}}_{t_3 \mapsto t'_{i_2}}, \underbrace{\langle \kappa, B \rangle}_{t_4} \rangle$$

The $\omega + 2, \omega + 3$ -th coordinates cannot be added. On one hand, they should be chosen below $\nu_{\omega \cdot 2}$, on the other hand, there is no large set associated to $\nu_{\omega \cdot 2}$. In D , this situation is impossible due to condition (2) of definition 4.8, which p fails to satisfy:

$$\omega \cdot 2 \in \text{Succ}(I) \text{ but } \omega + 3 \in I \text{ is the predecessor and } \gamma(t_{i_2}) = \omega \cdot 2$$

Notice that we can extend p to

$$\langle \langle \nu_\omega, B_\omega \rangle, \nu_{\omega+1}, \nu_{\omega+2}, \nu_{\omega+3}, \langle \nu_{\omega \cdot 2}, B_{\omega \cdot 2} \rangle, \langle \kappa, B \rangle \rangle$$

to fix this problem.

Next consider

$$I = \{2n \mid n \leq \omega\} \cup \{\omega + 2, \omega + 3\} \cup \{\omega \cdot n \mid n < \omega, n \neq 2\}$$

and let p be the condition

$$\begin{aligned} & \langle \underbrace{\langle \nu_\omega, B_\omega \rangle}_{t_1}, \underbrace{\langle \nu_{\omega \cdot 2}, B_{\omega \cdot 2} \rangle}_{t_2}, \underbrace{\langle \nu_{\omega \cdot 3}, B_{\omega \cdot 3} \rangle}_{t_3}, \underbrace{\langle \kappa, B \rangle}_{t_4} \rangle \\ & \pi_I(p) = \langle \underbrace{\langle \nu_\omega, B_\omega \rangle}_{t_1 \mapsto t'_{i_1}}, \underbrace{\langle \nu_{\omega \cdot 3}, B_{\omega \cdot 3} \rangle}_{t_3 \mapsto t'_{i_2}}, \underbrace{\langle \kappa, B \rangle}_{t_4} \rangle \end{aligned}$$

Once again the coordinates $\omega + 2, \omega + 3$ cannot be added since $\min(B_{\omega \cdot 3}) > \nu_{\omega \cdot 2}$. This problem points out condition (1) of definition 4.8, which p fails to satisfy:

$$\gamma(t_{i_1}, p) = \omega < \omega \cdot 2 = \gamma(t_{i_2-1}, p)$$

As before, we can extend p to avoid this problem.

Proposition 4.9 *D is dense in $\mathbb{M}[\vec{U}]$.*

Proof. Fix $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}[\vec{U}]$, define recursively $\langle p_k \mid k < \omega \rangle$ as follows:

First, $p_0 = p$. Assume that $p_k = \langle t_1^{(k)}, \dots, t_{n_k}^{(k)}, t_{n_k+1}^{(k)} \rangle$ is defined. If $p_k \in D$, define $p_{k+1} = p_k$. Otherwise, there exists a maximal $1 \leq i_j =: i_j(k) \leq n' + 1$ such that $\gamma(t_{i_j}^{(k)}, p_k) \in I$ which fails to satisfy (1) or fails to satisfy (2) of definition 4.8. Let us split into two cases accordingly:

1. Assume $\neg(1)$, Thus

$$\gamma(t_{i_j}^{(k)}, p_k) \in \text{Lim}(I) \text{ and } \gamma(t_{i_{j-1}}^{(k)}, p_k) < \gamma(t_{i_j}^{(k)}, p_k)$$

Since $\gamma(t_{i_j}^{(k)}, p_k) \in \text{Lim}(I)$ there exists $\gamma \in I \cap (\gamma(t_{i_{j-1}}^{(k)}, p_k), \gamma(t_{i_j}^{(k)}, p_k))$. Use proposition 3.2 to find $p_{k+1} \geq p_k$ with γ added and the only other coordinates added are below γ , thus if $t_{i_j}^{(k)} = t_r^{(k+1)}$ then $\gamma = \gamma(t_{r-1}^{(k+1)}, p_{k+1})$. Thus, every $l \geq r$ satisfies (1) and 2). If $p_{k+1} \notin D$ then the problem must accrue below $\gamma(t_{i_j}^{(k)}, p_k)$.

2. Assume $\neg(2)$, thus

$$\gamma(t_{i_j}^{(k)}, p) \in \text{Succ}(I) \text{ and } \gamma(t_{i_{j-1}}^{(k)}, p) \text{ is not the predecessor of } (\gamma(t_{i_j}^{(k)}, p))$$

Let γ be the predecessor in I of $\gamma(t_{i_j}^{(k)}, p)$. By proposition 3.2, there exist $p_{k+1} \geq p_k$ with γ added and the only other coordinates added are below γ . As before, if $t_{i_j}^{(k)} = t_r^{(k+1)}$ then $\gamma = \gamma(t_{r-1}^{(k+1)}, p_{k+1})$ and for every $l \geq r$, $\gamma(t_l^{(k+1)}, p_{k+1})$ satisfies (1) and (2).

The sequence $\langle p_k \mid k < \omega \rangle$ is defined. It necessarily stabilizes, otherwise the sequence $\gamma(t_{i_{j(k)}}^{(k)}, p_k)$ form a strictly decreasing infinite sequence of ordinals. Let p_{n^*} be the stabilized condition, it is an extension of p in D .

■

Lemma 4.10 $\pi_I \upharpoonright D : D \rightarrow \mathbb{M}_I[\vec{U}]$ is a projection, i.e:

1. π_I is onto.
2. $p_1 \leq p_2 \Rightarrow \pi_I(p_1) \leq_I \pi_I(p_2)$ (also \leq^* is preserved).
3. $\forall p \in \mathbb{M}[\vec{U}] \forall q \in \mathbb{M}_I[\vec{U}] (\pi_I(p) \leq_I q \rightarrow \exists p' \geq p (q = \pi_I(p'))$.

Proof. Let $p \in D$, such that $\pi_I(p) = \langle t'_{i_1}, \dots, t'_{i_{n'}}, t_{n+1} \rangle$

Claim: $\pi_I(p)$ computes I correctly i.e. for every $0 \leq j \leq n'$, we have the equality $\gamma(t_{i_j}, p) = I(t'_{i_j}, \pi_I(p))$.

Proof of claim: By induction on j , for $j = 0$, $\gamma(0, p) = 0 = I(0, \pi_I(p))$. For $j > 0$, assume $\gamma(t_{i_{j-1}}, p) = I(t'_{i_{j-1}}, \pi_I(p))$ and $\gamma(t_{i_j}, p) \in \text{Succ}(I)$. Since $p \in D$, $\gamma(t_{i_{j-1}}, p)$ is the predecessor of $\gamma(t_{i_j}, p)$ in I . Use the induction hypothesis to see that

$$I(t'_{i_j}, \pi_I(p)) = \min(\beta \in I \setminus \gamma(t_{i_{j-1}}, p) + 1 \mid o_L(\beta) = o_{\vec{U}}(t_{i_j})) = \gamma(t_{i_j}, p)$$

For $\gamma(t_{i_j}, p) \in \text{Lim}(I)$, use condition (1) of definition 4.8. to conclude that $\gamma(t_{i_{j-1}}, p) + \omega^{o^{\vec{U}}(t_{i_j})} = \gamma(t_{i_j}, p)$. Thus

$$\forall r \in I \cap (\gamma(t_{i_{j-1}}, p), \gamma(t_{i_j}, p)) \quad (o_L(r) < o^{\vec{U}}(t_{i_j}))$$

In Particular,

$$I(t'_{i_j}, \pi_I(p)) = \min(\beta \in I \setminus \gamma(t_{i_{j-1}}, p) + 1 \mid o_L(\beta) = o^{\vec{U}}(t_{i_j})) = \gamma(t_{i_j}, p)$$

■_{of claim}

Checking definition 4.3, show that $\pi_I(p) \in \mathbb{M}_I[\vec{U}]$: (1), (2.a.i), (2.b.i), (2.b.iii) are immediate from the definition of π_I . Use the claim to verify that (2.a.ii), (2.b.ii) follows from (1),(2) in D respectively. For (2.a.iii), let $1 \leq j \leq n'$, write

$$\gamma(t_{i_{j-1}}, p) + \sum_{i_{j-1} < l \leq i_j} \omega^{o^{\vec{U}}(t_l)} = \gamma(t_{i_j}, p)$$

This equation induces a C.N.F equation

$$I(t_{i_{j-1}}, \pi_I(p)) + \sum_{k=1}^{n_0} \omega^{o^{\vec{U}}(t_{l_k})} = I(t_{i_j}, \pi_I(p)) \quad (\text{C.N.F})$$

Thus

$$\langle \kappa(t_{l_1}), \dots, \kappa(t_{l_{n_0-1}}) \rangle \in Y(o^{\vec{U}}(t_{l_1})) \times \dots \times Y(o^{\vec{U}}(t_{l_{n_0-1}})) \cap [(\kappa(t_{i_{j-1}}), \kappa(t_{i_j}))]^{<\omega}$$

For (1), let $q = \langle t'_1, \dots, t'_{n+1} \rangle \in \mathbb{M}_I[\vec{U}]$. For every t'_j such that $I(t'_j, q) \in \text{Succ}(I)$, use definition 4.3 (2.a.iii) to find $\vec{s}_j = \langle s_{j,1}, \dots, s_{j,m_j} \rangle$ such that

$$\langle \kappa(s_{j,1}), \dots, \kappa(s_{j,m_j}) \rangle \in Y(\gamma_1) \times \dots \times Y(\gamma_{m-1}) \cap [(\kappa(t'_{i_r-1}), \kappa(t'_{i_r}))]^{<\omega}$$

where $I(t'_{i_r-1}, q) + \sum_{i=1}^m \omega^{\gamma_i} = I(t'_{i_r}, q)$ (C.N.F).

For each $i = 1, \dots, n$ such that $o^{\vec{U}}(t'_i) > 0$ and $\kappa(t'_i) \in \text{Succ}(I)$ pick some $B(t'_i) \in \bigcap_{\xi < o^{\vec{U}}(t'_i)} U(t_i, \xi)$.

Define $p = \langle t_1, \dots, t_{n+1} \rangle \frown \langle \vec{s}_r \mid I(t_r, q) \in \text{Succ}(I) \rangle$

$$t_i = \begin{cases} \langle \kappa(t'_i), B(t'_i) \setminus \kappa(s_{i,m_i}) + 1 \rangle & o^{\vec{U}}(t'_i) > 0 \\ \kappa(t'_i) & otherwise \end{cases}$$

Once we prove that $\gamma(s_{r,j}, p) \notin I$ and that p computes I correctly i.e. $\gamma(t_i, p) = I(t'_i, q)$, it will follow that $\pi_I(p) = \langle t'_i \mid \gamma(t_i, p) \in I \rangle = q$. By induction on i , for $i = 0$ it is trivial. Let $0 < i$ and assume the statement holds for i . If $I(t'_{i+1}, q) \in \text{Lim}(I)$, then by 4.3 (b.ii)

$$I(t'_{i+1}, q) = I(t'_i, q) + \omega^{o^{\vec{U}}(t'_{i+1})} = \gamma(t_i, p) + \omega^{o^{\vec{U}}(t_{i+1})} = \gamma(t_{i+1}, p)$$

If $I(t'_{i+1}, q) \in \text{Succ}(I)$, then from 4.3 (a.ii) it follows that $I(t'_i, q)$ is the predecessor of $I(t'_{i+1}, q)$. By the choice of \vec{s}_{i+1} ,

$$\begin{aligned} \gamma(t_{i+1}, p) &= \gamma(t_i, p) + \sum_{i=1}^{m-1} \omega^{\gamma_1} n_i + \omega^{\gamma_m} (n_m - 1) + \omega^{o^{\vec{U}}(t_{i+1})} = \\ &= I(t'_i, q) + \sum_{i=1}^{m-1} \omega^{\gamma_1} n_i + \omega^{m_1} (n_{m_1} - 1) + \omega^{o^{\vec{U}}(t'_{i+1})} = I(t'_{i+1}, q) \end{aligned}$$

Also, for all $1 \leq r \leq m_{i+1}$, $\gamma(s_{i+1,r}, p)$ is between two successor ordinals in I , hence $\gamma(s_{i+1,r}, p) \notin I$. Finally, $p \in D$ follows from 4.4 (a.ii) and 4.8 condition (1). If $\gamma(t_i, p) \in \text{Lim}(I)$ we did not add \vec{s}_i . Thus $i_{j-1} = i_j - 1$.

For (2), assume that $p, q \in D$, $p \leq q$. Using the claim, the verification of definition 4.4 it similar to (1).

As for (3), let us prove it for a simpler case to ease the notation. Nevertheless, the general statement is very similar and only require suitable notation. Let $p = \langle t_1, \dots, t_{n+1} \rangle \in \mathbb{M}[\vec{U}]$. Assume that

$$\pi_I(p) = \langle t'_{i_1}, \dots, t'_{i_n} \rangle \leq_I \langle t'_{i_1}, \dots, t'_{i_{j-1}}, s_1, \dots, s_m, t'_{i_j}, \dots, t'_{i_n} \rangle = q' \in \mathbb{M}_I[\vec{U}]$$

For every $l = 1, \dots, m$ such that $I(s_l, \pi_I(p)) \in \text{Succ}(I)$ use definition 4.4 (2b) to find $\vec{s}_l = \langle s_{l,1}, \dots, s_{l,m_l} \rangle$ such that

$$\langle \kappa(s_{l,1}), \dots, \kappa(s_{l,m_l}) \rangle \in B(t_{i_j}, \gamma_1) \times \dots \times B(t_{i_j}, \gamma_{m-1}) \cap [(\kappa(s_{l-1}), \kappa(s_l))]^{<\omega}$$

where $I(s_{l-1}, \pi_I(p)) + \sum_{i=1}^m \omega^{\gamma_i} = I(s_l, \pi_I(p))$ (C.N.F). Define $p \leq p'$ to be the extension $p' = p \frown \langle s'_1, \dots, s'_m \rangle \frown \langle \vec{s}_l \mid I(s_l, \pi_I(p)) \in \text{Succ}(I) \rangle$ where

$$s'_i = \begin{cases} \langle \kappa(s_i), B_i \setminus \kappa(s_{i,m_i}) + 1 \rangle & o^{\vec{U}}(s_i) > 0 \\ s_i & otherwise \end{cases}$$

As in (1), $\pi_I(p') = \langle t'_{i_1}, \dots, t'_{i_{j-1}}, (s'_1)', \dots, (s'_m)', \dots, t_{i_n'} \rangle$. Notice that since we only change s_l such that $I(s_l, \pi_I(p)) \in \text{Succ}(I)$, $(s'_l)' = s_l$. Thus $\pi_I(p') = q$ and $p' \in D$ follows. ■

From the discussion previous to 4.8, we have the following corollary:

Corollary 4.11 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V -generic filter, and let $C' \subseteq C_G$ be a closed subset, then there is $G_I \subseteq \mathbb{M}_I[\vec{U}]$ such that $V[C'] = V[G_I]$ and $C_I = C'$, where $I = \text{Index}(C', C_G)$.* ■

4.2 The Quotient forcing $\mathbb{M}[\vec{U}]/G_I$

Definition 4.12 Let G_I be $\mathbb{M}_I[\vec{U}]$ generic, the *quotient forcing* is

$$\mathbb{M}[\vec{U}]/G_I = \pi_I^{-1''} G_I = \{p \in \mathbb{M}[\vec{U}] \mid \pi_I(p) \in G_I\}$$

The forcing $\mathbb{M}[\vec{U}]/G_I$ completes $V[G_I]$ to $V[G]$ in the sense that if $G \subseteq \mathbb{M}[\vec{U}]$ is V -generic, and $\pi_I^*(G) = G_I$, then $G \subseteq \mathbb{M}[\vec{U}]/G_I$ is $V[G_I]$ -generic. Moreover, if $G \subseteq \mathbb{M}[\vec{U}]/G_I$ is $V[G_I]$ -generic, then $G \subseteq \mathbb{M}[\vec{U}]$ is V -generic, and $\pi_I^*(G) = G_I$.

Corollary 4.13 *Let $G_I \subseteq \mathbb{M}_I[\vec{U}]$ be V -generic, then there is $G \subseteq \mathbb{M}[\vec{U}]$ such that $C_G \restriction I = C_I$.*

The following proposition is straightforward:

Proposition 4.14 *Let $x, p \in \mathbb{M}[\vec{U}]$ and $q \in \mathbb{M}_I[\vec{U}]$, then*

1. $\pi_I(p) \leq_I q \Rightarrow q \Vdash_{\mathbb{M}_I[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/G_I$.
 2. $q \Vdash_{\mathbb{M}_I[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/G_I \Rightarrow \pi_I(p), q$ are compatible.
 3. $x \Vdash_{\mathbb{M}[\vec{U}]} \check{p} \in \mathbb{M}[\vec{U}]/G_I \Rightarrow \pi_I(p), \pi_I(x)$ are compatible.
-

Lemma 4.15 *Let $G_I \subseteq \mathbb{M}_I[\vec{U}]$ be V -generic. Then the forcing $\mathbb{M}[\vec{U}]/G_I$ satisfies κ^+ -c.c. in $V[G_I]$.*

Proof. Fix $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}[\vec{U}]/G_I$ and let

$$r \in G_I, \quad r \Vdash_{\mathbb{M}_I[\vec{U}]} \forall \alpha < \kappa^+ \quad p_\alpha \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$$

Let us argue that

$$E = \{q \in \mathbb{M}_I[\vec{U}] \mid (q \perp r) \vee (q \Vdash_{\mathbb{M}_I[\vec{U}]} \exists \alpha, \beta < \kappa^+ (p_\alpha, p_\beta \text{ are compatible}))\}$$

is a dense subset of $\mathbb{M}_I[\vec{U}]$. Assume $r \leq_I r'$, for every $\alpha < \kappa^+$ pick some $r' \leq_I q_\alpha^* \in \mathbb{M}_I[\vec{U}]$, $p_\alpha^* \in \mathbb{M}[\vec{U}]$ such that:

1. $\pi_I(p_\alpha^*) = q_\alpha^*$.
2. $q_\alpha^* \Vdash p_\alpha \leq p_\alpha^* \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$.

There exists such q_α^*, p_α^* : Find $r' \leq_I q'_\alpha$ and p'_α such that $q'_\alpha \Vdash p'_\alpha = p_\alpha$ then by the proposition 4.14 (2), there is $q_\alpha^* \geq_I \pi_I(p'_\alpha), q'_\alpha$. By lemma 4.10 (3) there is $p_\alpha^* \geq p'_\alpha$ such that $q_\alpha^* := \pi_I(p_\alpha^*)$. It follows from proposition 4.14 (1) that

$$q_\alpha^* \Vdash p_\alpha \leq p_\alpha^* \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$$

Denote $p_\alpha^* = \langle t_{1,\alpha}, \dots, t_{n_\alpha,\alpha}, t_{n_\alpha+1,\alpha} \rangle$, $q_\alpha^* = \langle t_{i_1,\alpha}, \dots, t_{i_{m_\alpha},\alpha}, t_{n_\alpha+1,\alpha} \rangle$. Find $S \subseteq \kappa^+$, $n < \omega$ and $\langle \kappa_1, \dots, \kappa_n \rangle$ such that $|S| = \kappa^+$ and for any $\alpha \in S$, $n_\alpha = n$ and

$$\langle \kappa(t_{1,\alpha}), \dots, \kappa(t_{n_\alpha,\alpha}) \rangle = \langle \kappa_1, \dots, \kappa_n \rangle.$$

Since $\pi_I(p_\alpha^*) = q_\alpha^*$ it follows that

$$\langle \kappa(t_{i_1,\alpha}), \dots, \kappa(t_{i_{m_\alpha},\alpha}) \rangle = \langle \kappa_{i_1}, \dots, \kappa_{i_m} \rangle$$

for some $m < \omega$ and $1 \leq i_1 < \dots < i_m \leq n$.

Fix any $\alpha, \beta \in S$ and let $p^* = \langle t_1, \dots, t_n, t_{n+1} \rangle$ where

$$t_i = \begin{cases} \langle \kappa_i, B(t_{i,\alpha}) \cap B(t_{i,\beta}) \rangle & o_{\vec{U}}(t_{i,\alpha}) > 0 \\ \kappa_i & \text{otherwise} \end{cases}$$

Denote $p_\alpha^* \cap p_\beta^* = p^*$. Set

$$q^* = \pi_I(p^*) = \langle t'_{i_1}, \dots, t'_{i_m} \rangle$$

Then $r' \leq_I q^* \cap q^* = \pi_I(p^*_\alpha) \cap \pi_I(p^*_\beta) = \pi_I(p^*_\alpha \cap p^*_\beta) = \pi_I(p^*) = q^*$. It follows that $q^* \in E$ since by proposition 4.14 (1) $q^* \Vdash_{\mathbb{M}_I[\vec{U}]} p^* \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$ and

$$q^* \Vdash_{\mathbb{M}_I[\vec{U}]} p_\alpha \leq p^*_\alpha \leq^* p^* \wedge p_\beta \leq p^*_\beta \leq^* p^*$$

The rest is routine. ■

Lemma 4.16 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be V -generic. Then the forcing $\mathbb{M}[\vec{U}]/G_I$ satisfies $\kappa^+ - c.c.$ in $V[G]$.*

Proof. Fix $\{p_\alpha \mid \alpha < \kappa^+\} \subseteq \mathbb{M}[\vec{U}]/G_I$ in $V[G]$ and let

$$r \in G, \ r \Vdash_{\mathbb{M}[\vec{U}]} \forall \alpha < \kappa^+ \ p_\alpha \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$$

Similar to lemma 4.15 let us argue that

$$E = \{x \in \mathbb{M}[\vec{U}] \mid (q \perp r) \vee (q \Vdash_{\mathbb{M}[\vec{U}]} \exists \alpha, \beta < \kappa^+ (p_\alpha, p_\beta \text{ are compatible}))\}$$

is a dense subset of $\mathbb{M}[\vec{U}]$. Assume $r \leq r'$, for every $\alpha < \kappa^+$ pick some $r' \leq x'_\alpha \in \mathbb{M}[\vec{U}]$, $p'_\alpha \in \mathbb{M}[\vec{U}]$ such that $x'_\alpha \Vdash_{\mathbb{M}[\vec{U}]} p_\alpha = p'_\alpha$. By proposition 4.14 (3), we can find $\pi_I(x'_\alpha), \pi_I(p'_\alpha) \leq_I y_\alpha$. By lemma 4.10 (3), There is $x'_\alpha \leq x^*_\alpha, p'_\alpha \leq p^*_\alpha$ such that

$$\pi_I(x'_\alpha), \pi_I(p''_\alpha) \leq_I y_\alpha = \pi_I(p^*_\alpha) = \pi_I(x^*_\alpha)$$

Denote by

$$x^*_\alpha = \langle s_{1,\alpha}, \dots, s_{k_\alpha,\alpha}, s_{k_\alpha+1,\alpha} \rangle, \ p^*_\alpha = \langle t_{1,\alpha}, \dots, t_{n_\alpha,\alpha}, t_{n_\alpha+1,\alpha} \rangle$$

and

$$\pi_I(x^*_\alpha) = \langle t'_{i_1,\alpha}, \dots, t'_{i_{k'_\alpha},\alpha}, t'_{k'_\alpha+1,\alpha} \rangle = \pi_I(p_\alpha)$$

Find $S \subseteq \kappa^+ \mid |S| = \kappa^+$ and $\langle \kappa_1, \dots, \kappa_n \rangle, \langle \nu_1, \dots, \nu_k \rangle$ such that for any $\alpha \in S$,

$$\langle \kappa(t_{1,\alpha}), \dots, \kappa(t_{n_\alpha,\alpha}) \rangle = \langle \kappa_1, \dots, \kappa_n \rangle, \ \langle \kappa(s_{1,\alpha}), \dots, \kappa(s_{k_\alpha,\alpha}) \rangle = \langle \nu_1, \dots, \nu_k \rangle$$

Fix any $\alpha, \beta \in S$ and let $p^* = p^*_\alpha \cap p^*_\beta$, $x^* = x^*_\alpha \cap x^*_\beta$. Then $p'_\alpha, p'_\beta \leq^* p^*$ and $x_\alpha, x_\beta \leq^* x^*$. Finally claim that $x^* \in E$:

$$\pi_I(p^*) = \pi_I(p^*_\alpha) \cap \pi_I(p^*_\beta) = \pi_I(x^*_\alpha) \cap \pi_I(x^*_\beta) = \pi_I(x^*)$$

thus $x^* \Vdash_{\mathbb{M}[\vec{U}]} p^* \in \mathbb{M}[\vec{U}]/\mathcal{G}_I$. Moreover, $x_\alpha \leq^* x^*$ which implies that

$$x^* \Vdash_{\mathbb{M}[\vec{U}]} p^* \geq p_\alpha, p_\beta.$$

■

4.3 The Argument for General Sets

Let us conclude first the main result for subsets of κ^+ .

Lemma 4.17 *If $A \in V[G]$, $A \subseteq \kappa^+$ then there exists $C^* \subseteq C_G$ such that $V[A] = V[C^*]$.*

Proof. Work in $V[G]$, for every $\alpha < \kappa^+$ find subsequences $C_\alpha \subseteq C_G$ such that $V[C_\alpha] = V[A \cap \alpha]$ using the induction hypothesis. The function $\alpha \mapsto C_\alpha$ has range $P(C_G)$ and domain κ^+ which is regular in $V[G]$. Therefore there exist $E \subseteq \kappa^+$ unbounded in κ^+ and $\alpha^* < \kappa^+$ such that for every $\alpha \in E$, $C_\alpha = C_{\alpha^*}$. Set $C^* = C_{\alpha^*}$, then

1. $C^* \subseteq C_G$.
2. $C^* \in V[A \cap \alpha^*] \subseteq V[A]$.
3. $\forall \alpha < \kappa^+. A \cap \alpha \in V[C^*]$.

Assume that C^* is a club⁸. Unlike A 's that were subsets of κ , for which we added another piece of C_G to C^* to obtain C' such that $V[A] = V[C']$, here we argue that $V[A] = V[C^*]$.

By (2), $C^* \in V[A]$. For the other direction, denote by I the indexes of C^* in C and consider the forcings $\mathbb{M}_I[\vec{U}]$, $\mathbb{M}[\vec{U}]/G_I$. Toward a contradiction, assume that $A \notin V[C^*]$, and let $\check{A} \in V[C^*]$ be a $\mathbb{M}[\vec{U}]/G_I$ -name for A , where $\pi''_I G = G_I$. Work in $V[G_I]$, by lemma 4.7 (2), $V[G_I] = V[C^*]$. For every $\alpha < \kappa^+$ define

$$X_\alpha = \{B \subseteq \alpha \mid \|\check{A} \cap \alpha = B\| \neq 0\}$$

where the truth value is taken in $RO(\mathbb{M}[\vec{U}]/G_I)$ ⁹. By lemma 4.15,

$$\forall \alpha < \kappa^+ |X_\alpha| \leq \kappa$$

For every $B \in X_\alpha$ define $b(B) = \|\check{A} \cap \alpha\|$. Assume that $B' \in X_\beta$ and $\alpha \leq \beta$ then $B = B' \cap \alpha \in X_\alpha$. Switching to boolean algebra notation ($p \leq_B q$ means p extends q) $b(B') \leq_B b(B)$. Note that for such B, B' if $b(B') <_B b(B)$, then there is

$$0 < p \leq_B (b(B) \setminus b(B')) \leq_B b(B)$$

Therefore,

$$p \cap b(B') \leq_B (b(B) \setminus b(B')) \cap b(B') = 0$$

⁸ $Cl(C^*)$ clearly satisfy (1) – (3)

⁹ $RO(\mathbb{Q})$ denotes the complete boolean algebra of regular open subsets of \mathbb{Q}

Hence $p \perp b(B')$. Work in $V[G]$, denote $A_\alpha = A \cap \alpha$. Recall that

$$\forall \alpha < \kappa^+ \ A_\alpha \in V[C^*]$$

thus $A_\alpha \in X_\alpha$. Consider the \leq_B -non-increasing sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+ \rangle$. If there exists some $\gamma^* < \kappa^+$ on which the sequence stabilizes, define

$$A' = \bigcup \{B \subseteq \kappa^+ \mid \exists \alpha \ b(A_{\gamma^*}) \Vdash \dot{A} \cap \alpha = B\} \in V[C^*]$$

To see that $A' = A$, notice that if B, B', α, α' are such that $\alpha \leq \alpha'$, and

$$b(A_{\gamma^*}) \Vdash \dot{A} \cap \alpha = B, \ b(A_{\gamma^*}) \Vdash \dot{A} \cap \alpha' = B'$$

then $B' \cap \alpha = B$ otherwise, the non zero condition $b(A_{\gamma^*})$ would force contradictory information. Consequently, for every $\xi < \kappa^+$ there exists $\xi < \gamma < \kappa^+$ such that $b(A_{\gamma^*}) \Vdash \dot{A} \cap \gamma = A \cap \gamma$, hence $A' \cap \gamma = A \cap \gamma$. This is a contradiction to $A \notin V[C^*]$.

Therefore, the sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+ \rangle$ does not stabilize. By regularity of κ^+ , there exists a subsequence $\langle b(A_{i_\alpha}) \mid \alpha < \kappa^+ \rangle$ which is strictly decreasing. By the observation we made in the last paragraph, find $p_\alpha \leq_B b(A_{i_\alpha})$ such that $p_\alpha \perp b(A_{i_{\alpha+1}})$. Since $b(A_{i_\alpha})$ are decreasing, for any $\beta > \alpha$ $p_\alpha \perp b(A_{i_\beta})$ and in turn $p_\alpha \perp p_\beta$. This shows that $\langle p_\alpha \mid \alpha < \kappa^+ \rangle \in V[G]$ is an anti chain of size κ^+ which contradicts Lemma 4.16. Thus $V[A] = V[C^*]$. ■

End of the proof of Theorem 3.3: By induction on $\sup(A) = \lambda > \kappa^+$. It suffices to assume that λ is a cardinal.

case1: Assume $cf^{V[G]}(\lambda) > \kappa$, then the arguments of lemma 4.17 works.

case2: Assume $cf^{V[G]}(\lambda) \leq \kappa$, since $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+ - c.c.$ we must have that $\nu := cf^V(\lambda) \leq \kappa$. Fix $\langle \gamma_i \mid i < \nu \rangle \in V$ cofinal in λ . Work in $V[A]$, for every $i < \nu$ find $d_i \subseteq \kappa$ such that $V[d_i] = V[A \cap \gamma_i]$. By induction, there exists $C^* \subseteq C_G$ such that $V[\langle d_i \mid i < \nu \rangle] = V[C^*]$, therefore

1. $\forall i < \nu \ A \cap \gamma_i \in V[C^*]$.
2. $C^* \in V[A]$.

Work in $V[C^*]$, for $i < \nu$ define $X_i = \{B \subseteq \alpha \mid \|\dot{A} \cap \gamma_i = B\| \neq 0\}$. By lemma 4.15, $|X_i| \leq \kappa$. For every $i < \nu$ fix an enumeration

$$X_i = \langle X(i, \xi) \mid \xi < \kappa \rangle \in V[C^*]$$

There exists $\xi_i < \kappa$ such that $A \cap \gamma_i = X(i, \xi_i)$. Moreover, since $\nu \leq \kappa$ the sequence $\langle A \cap \gamma_i \mid i < \nu \rangle = \langle X(i, \xi_i) \mid i < \nu \rangle$ can be coded in $V[C^*]$ as a sequence of ordinals below κ . By induction there exists $C'' \subseteq C_G$ such that $V[C''] = V[\langle \xi_i \mid i < \nu \rangle]$. It follows that,

$$V[C'', C^*] = (V[C^*])[\langle \xi_i \mid i < \nu \rangle] = V[A]$$

Finally, we can take for example, $C' = C'' \cup C^* \subseteq C_G$ to obtain $V[A] = V[C']$

■ *theorem 3.3*

5 Classification of subforcing of Magidor

Now that we have Classified models of the form $V[A]$, we can conclude the following:

Corollary 5.1 *Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V -generic filter, and let M be a transitive model of ZFC such that $V \subseteq M \subseteq V[G]$. Then there is $C_M \subseteq C_G$ such that $V[C_M] = M$.*

Proof. By [7, Thm. 15.43], there is $D \in V[G]$ such that $M = V[D]$. By theorem 3.3, there is $C_M \subseteq C_G$ such that $V[C_M] = V[D] = M$.

As we have seen in the previous section, the models $V[C_M]$ are generic extensions for the forcings $\mathbb{M}_I[\vec{U}]$ which in turn are projection of $\mathbb{M}[\vec{U}]$, this yield the classification of subforcings. Although the classification can naturally be extended to a the class of forcings $\mathbb{M}_{\langle \kappa_1, \dots, \kappa_n \rangle}[\vec{U}]$, we present here only the classification of subforcings of $\mathbb{M}[\vec{U}]$.

Definition 5.2 Recall definition 4.3 of $\mathbb{M}_I[\vec{U}]$. The forcings

$$\{\mathbb{M}_I[\vec{U}] \mid I \in P(\omega^{o_{\vec{U}}(\kappa)}), I \text{ is closed}\}$$

is the family of *Magidor-type forcings* with the coherent sequence \vec{U} .

In practice, Magidor-type forcings are just Magidor forcing with a subsequence of \vec{U} ; If I is any closed subset of indices, we can read the measures of \vec{U} from which the elements of the final sequence are chosen from, using the sequence $\langle o_L(i) \mid i \in I \rangle$ (recall that $o_L(i) = \gamma_n$ where $i = \omega^{\gamma_1} + \dots + \omega^{\gamma_n}$ C.N.F).

Example: Assume that $o_{\vec{U}}(\kappa) = 2$ and let a

$$I = \{1, \omega, \omega + 1\} \cup (\omega \cdot 3 \setminus \omega \cdot 2) \cup \{\omega \cdot 3, \omega \cdot 4, \dots\} \in P(\omega^2)$$

Then $\langle o_L(i) \mid i \in I \rangle = \langle 0, 1, \underbrace{0, 0, 0, \dots}_{\omega}, \underbrace{1, 1, 1, \dots}_{\omega} \rangle$. Therefore $\mathbb{M}_I[\vec{U}]$ is just Prikry forcing with $U(\kappa_1, 0)$

for some measurable $\kappa_1 < \kappa$ followed by Prikry forcing with $U(\kappa, 1)$.

Although in this example the noise at the beginning can be neglected, there are I 's for which we do not get "pure" Magidor forcing which uses one measure at a time and combine several measure. For example we can obtain the Tree-Prikry forcing- let $I = \langle \omega^n \mid n < \omega \rangle$ then $\langle o_L(i) \mid i \in I \rangle = \langle n \mid n < \omega$, conditions in the forcing are of the form $\langle t_1, \dots, t_n, \langle \kappa, B \rangle \rangle$ the extensions is from the measures $U(\kappa, m)$, $m > n$ which is essentially $P_T(\langle U(\kappa, n) \mid n < \omega \rangle)$ the tree Prikry forcing such that at level n the tree splits on a large set in $U(\kappa, n)$.

For the definition of complete subforcing see [11].

Theorem 5.3 *Let $\mathbb{P} \subseteq \mathbb{M}[\vec{U}]$ be a complete subforcing of $\mathbb{M}[\vec{U}]$ then there exists a maximal antichain $Z \subseteq \mathbb{P}$ and $\langle I_p \mid p \in Z \rangle$ such that $\mathbb{P}_{\geq p}$ (the forcing \mathbb{P} above p) is forcing equivalent to the Magidor-type forcing $\mathbb{M}_{I_p}[\vec{U}]_{\geq q_p}$ for some condition $q_p \in RO(\mathbb{M}_{I_p}[\vec{U}])$.*

Proof. Let $H \subseteq \mathbb{P}$ be generic, then there exists $G \subseteq \mathbb{M}[\vec{U}]$ generic such that $H = G \cap \mathbb{P}$, in particular $V \subseteq V[H] \subseteq V[G]$. By Theorem 3.3, there is a closed $C' \subseteq C_G$ such that $V[C'] = V[H]$, and let $I = \text{Index}(C', C_G)$. The assumption $o^{\vec{U}}(\kappa)$ is crucial to claim that $I \in V$. By corollary 4.11, there is $G_I \subseteq \mathbb{M}_I[\vec{U}]$ such that $C_I = C'$. Let $\tilde{C}'_0, \tilde{H}_0$ be a \mathbb{P} -name of C', H . Let $p \in \mathbb{P}$ such that

$$p \Vdash \tilde{C}' \text{ is generic sequence for } \mathbb{M}_I[\vec{U}] \text{ and } V[\tilde{H}] = V[\tilde{C}']^{10}$$

Denote by $I_p := I$. For the other direction, let $\tilde{C}'_1, \tilde{H}_1$ be $\mathbb{M}_{I_p}[\vec{U}]$ -names for C', H and let $q_p \in RO(\mathbb{M}_{I_p}[\vec{U}])$ be the truth value:

$$\tilde{H}_1 \subseteq \mathbb{P} \text{ is } V\text{-generic}, p \in \tilde{H}_1 \text{ and } V[\tilde{H}_1] = V[\tilde{C}'_1]$$

Clearly, $\mathbb{M}_{I_p}[\vec{U}]_{\geq q_p}$ and $\mathbb{P}_{\geq p}$ have the same generic extensions and therefore forcing equivalent. ■

¹⁰This is indeed a formula in the forcing language since for any set A , $V[A] = \bigcup_{z \subseteq \text{ord}, z \in V} L[z, A]$ where $L[z, A]$ is the class of all constructible sets relative to z, A .

6 Prikry forcings with non-normal ultrafilters.

Let κ be a measurable cardinal and let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega} \rangle$ be a tree consisting of κ -complete non-trivial ultrafilter over κ .

Recall the definition due to Prikry of the tree Prikry forcing with \mathbb{U} .

Definition 6.1 The *Tree Prikry forcing* $P(\mathbb{U})$, consist of all pairs $\langle p, T \rangle$ such that:

1. p is a finite increasing sequence of ordinals below κ .
2. $T \subseteq [\kappa]^{<\omega}$ is a tree with trunk p such that for every $q \in T$ with $q \geq_T p$, the set of the immediate successors of q in T , i.e. $Suc_T(q)$ is in U_q .

The orders \leq, \leq^* are defined in the usual fashion.

For every $a \in [\kappa]^{<\omega}$, let π_a be a projection of U_a to a normal ultrafilter. Namely, let $\pi_a : \kappa \rightarrow \kappa$ be a function which represents κ in the ultrapower by U_a , i.e. $[\pi_a]_{U_a} = \kappa$. Once U_a is a normal ultrafilter, then let π_a be the identity.

By passing to a dense subset of $P(\mathbb{U})$, we can assume that for each $\langle p, T \rangle \in P(\mathbb{U})$, for every $\langle \nu_1, \dots, \nu_n \rangle \in T$ we have

$$\nu_1 < \pi_{\langle \nu_1 \rangle}(\nu_2) \leq \nu_2 < \dots \leq \nu_{n-1} < \pi_{\langle \nu_1, \dots, \nu_{n-1} \rangle}(\nu_n)$$

and for every $\nu \in Suc_T(\langle \nu_1, \dots, \nu_n \rangle)$, $\pi_{\langle \nu_1, \dots, \nu_n \rangle}(\nu) > \nu_n$.

Note that once the measures over a certain level (or certain levels) are the same - say for some $n < \omega$ and U , for every $a \in [\kappa]^n$, $U_a = U$, then a modified diagonal intersection

$$\Delta_{\alpha < \kappa}^* A_\alpha := \{ \nu < \kappa \mid \forall \alpha < \pi_k(\nu) (\nu \in A_\alpha) \} \in U,$$

once $\{A_\alpha \mid \alpha < \kappa\} \subseteq U$, can be used to avoid or to simplify the tree structure.

For example, if $\langle \mathcal{V}_n \mid n < \omega \rangle$ is a sequence of κ -complete ultrafilters over κ , then the Prikry forcing with it $P(\langle \mathcal{V}_n \mid n < \omega \rangle)$ is defined as follows:

Definition 6.2 The *tree Prikry forcing with an ω -sequence of ultrafilters*, $P(\langle \mathcal{V}_n \mid n < \omega \rangle)$, is the set of all pairs $\langle p, \langle A_n \mid |p| < n < \omega \rangle \rangle$ such that:

1. $p = \langle \nu_1, \dots, \nu_k \rangle$ is a finite sequence of ordinals below κ , such that $\nu_j < \pi_i(\nu_i)$, whenever $1 \leq j < i \leq k$.
2. $A_n \in \mathcal{V}_n$, for every $n, |p| < n < \omega$.

3. $\pi_{k+1}(\min(A_{k+1})) > \max(p)$, where $\pi_n : \kappa \rightarrow \kappa$ is a projection of \mathcal{V}_n to a normal ultrafilter, i.e. π_n is a function which represents κ in the ultrapower by \mathcal{V}_n , $[\pi]_{\mathcal{V}_n} = \kappa$.

A simpler case is once all \mathcal{V}_n are the same, say all of them are U . Then we will have the Prikry forcing with U :

Definition 6.3 The *Prikry forcing with general ultrafilter* $P(U)$ is the set of all pairs $\langle p, A \rangle$ such that

1. $p = \langle \nu_1, \dots, \nu_k \rangle$ is a finite sequence of ordinals below κ , such that $\nu_j < \pi(\nu_i)$, whenever $1 \leq j < i \leq k$.
2. $A \in U$.
3. $\pi(\min(A)) > \max(p)$, where π is a projection of U to a normal ultrafilter.

Let G be a generic for $\langle P(\mathbb{U}), \leq \rangle$. Set

$$C = \bigcup \{p \mid \exists T \quad \langle p, T \rangle \in G\}.$$

It is called a Prikry sequence for \mathbb{U} .

For every natural $n \geq 1$ we would like to define a κ -complete ultrafilter U_n over $[\kappa]^n$ which correspond to the first n -levels of trees in $P(\mathbb{U})$.

If $n = 1$, set $U_1 = U_{\langle \rangle}$.

Deal with the next step $n = 2$. Here for each $\nu < \kappa$ we have U_ν .

Consider the ultrapower by U_1 :

$$i_1 := i_{\langle \rangle} : V \rightarrow M_{\langle \rangle}.$$

Then the sequence $i_{\langle \rangle}(\langle U_{\langle \nu \rangle} \mid \nu < \kappa \rangle)$ will have the length $i_{\langle \rangle}(\kappa)$.

Let $U_{\langle [id]_{U_{\langle \rangle}} \rangle}$ be its $[id]_{U_{\langle \rangle}}$ ultrafilter in $M_{\langle \rangle}$ over $i_{\langle \rangle}(\kappa)$. Consider its ultrapower

$$i_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} : M_{\langle \rangle} \rightarrow M_{\langle [id]_{U_{\langle \rangle}} \rangle}$$

Set

$$i_2 = i_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} \circ i_{\langle \rangle}.$$

Then

$$i_2 : V \rightarrow M_{\langle [id]_{U_{\langle \rangle}} \rangle}.$$

Note that if all of $U_{\langle \nu \rangle}$'s are the same or just for a set of ν 's in $U_{\langle \rangle}$ they are the same, then this is just an ultrapower by the product of $U_{\langle \rangle}$ with this ultrafilter. In general it is an ultrapower by

$$U_{\langle \rangle} - \text{Lim} \langle U_{\langle \nu \rangle} \mid \nu < \kappa \rangle,$$

where

$$X \in U_{\langle \rangle} - \text{Lim} \langle U_{\langle \nu \rangle} \mid \nu < \kappa \rangle \text{ iff } [id]_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} \in i_2(X).$$

Note that once most of $U_{\langle \nu \rangle}$'s are normal, then $U_{\langle [id]_{U_{\langle \rangle}} \rangle}$ is normal as well, and so, $[id]_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} = i_{\langle \rangle}(\kappa)$.

Define an ultrafilter U_2 on $[\kappa]^2$ as follows:

$$X \in U_2 \text{ iff } \langle [id]_{U_{\langle \rangle}}, [id]_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} \rangle \in i_2(X).$$

Define also for $k = 1, 2$, ultrafilters U_2^k over κ as follows:

$$X \in U_2^1 \text{ iff } [id]_{U_{\langle \rangle}} \in i_2(X),$$

$$X \in U_2^2 \text{ iff } [id]_{U_{\langle [id]_{U_{\langle \rangle}} \rangle}} \in i_2(X).$$

Clearly, then $U_2^1 = U_1$ and $U_2^2 = U_1 - \text{Lim} \langle U_{\langle \nu \rangle} \mid \nu < \kappa \rangle$. Also U_2^1 is the projection of U_2 to the first coordinate and U_2^2 to the second.

Let $\langle \langle \rangle, T \rangle \in P(\mathbb{U})$. It is not hard to see that $T \restriction 2 \in U_2$.

Continue and define in the similar fashion the ultrafilter U_n over $[\kappa]^n$ and its projections to the coordinates U_n^k for every $n > 2, 1 \leq k \leq n$. We will have that for any $\langle \langle \rangle, T \rangle \in P(\mathbb{U})$, $T \restriction n \in U_n$. Also, if $1 \leq n \leq m < \omega$, then the natural projection of U_m to $[\kappa]^n$ will be U_n .

It is easy to see that C is a Prikry sequence for $\langle U_n^n \mid 1 \leq n < \omega \rangle$, in a sense that for every sequence $\langle A_n \mid n < \omega \rangle \in V$, with $A_n \in U_n^n$, there is $n_0 < \omega$ such that for every $n > n_0$, $C(n) \in U_n^n$. However, it does not mean that C is generic for the forcing $P(\langle U_n^n \mid 1 \leq n < \omega \rangle)$ defined above (Definition 6.2). The problem is with projection to normal. All U_n^n 's have the same normal U_1 .

Suppose now that we have an ultrafilter W over $[\kappa]^\ell$ which is Rudin-Keisler below some \mathfrak{V} over $[\kappa]^k$ ($W \leq_{RK} \mathfrak{V}$), for some $k, \ell, 1 \leq \ell, k < \omega$. This means that there is a function $F : [\kappa]^k \rightarrow [\kappa]^\ell$ such that

$$X \in W \text{ iff } F^{-1} X \in \mathfrak{V}.$$

So F projects \mathfrak{V} to W . Let us denote this by $W = F_* \mathfrak{V}$.

The next statement characterizes ω -sequences in $V[C]$.

Theorem 6.4 *Let $\langle \alpha_k \mid k < \omega \rangle \in V[C]$ be an increasing cofinal in κ sequence. Then $\langle \alpha_k \mid k < \omega \rangle$ is a Prikry sequence for a sequence in V of κ -complete ultrafilters which are Rudin-Keisler below*

$\langle U_n \mid n < \omega \rangle$.¹¹

Moreover, there exist a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega \rangle$ and a sequence of functions $\langle F_k \mid k < \omega \rangle$ in V , $F_k : [\kappa]^{n_k} \rightarrow \kappa$, $(k < \omega)$, such that

1. $\alpha_k = F_k(C \restriction n_k)$, for every $k < \omega$.
2. Let $\langle n_{k_i} \mid i < \omega \rangle$ be the increasing subsequence of $\langle n_k \mid k < \omega \rangle$ such that:
 - (a) $\{n_{k_i} \mid i < \omega\} = \{n_k \mid k < \omega\}$.
 - (b) $k_i = \min\{k \mid n_k = n_{k_i}\}$.

Set $\ell_i = |\{k \mid n_k = n_{k_i}\}|$. Then $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i} \rangle$ will be a Prikry sequence for $\langle W_i \mid i < \omega \rangle$, i.e. for every sequence $\langle A_i \mid i < \omega \rangle \in V$, with $A_i \in W_i$, there is $i_0 < \omega$ such that for every $i > i_0$, $\langle F_k(C \restriction n_{k_i}) \mid i < \omega, n_k = n_{k_i} \rangle \in A_i$, where each W_i is an ultrafilter over $[\kappa]^{\ell_i}$ which is the projection of $U_{n_{k_i}}$ by $\langle F_{k_i}, \dots, F_{k_i+\ell_i-1} \rangle$.

Proof. Work in V . Given a condition $\langle q, S \rangle$, we will construct by induction, using the Prikry property of the forcing $P(\mathbb{U})$, a stronger condition $\langle p, T \rangle$ which decides α_k once going up to a certain level n_k of T . Let us assume for simplicity that q is the empty sequence.

Build by induction $\langle \langle \rangle, T \rangle \geq^* \langle \langle \rangle, S \rangle$ and a non-decreasing sequence of natural numbers $\langle n_k \mid k < \omega \rangle$ such that for every $k < \omega$

1. for every $\langle \eta_1, \dots, \eta_{n_k} \rangle \in T$ there is $\rho_{\langle \eta_1, \dots, \eta_{n_k} \rangle} < \kappa$ such that:
 - (a) $\langle \langle \eta_1, \dots, \eta_{n_k} \rangle, T_{\langle \eta_1, \dots, \eta_{n_k} \rangle} \rangle \Vdash \alpha_k = \rho_{\langle \eta_1, \dots, \eta_{n_k} \rangle}$.
 - (b) $\rho_{\langle \eta_1, \dots, \eta_{n_k} \rangle} \geq \pi_{\langle \eta_1, \dots, \eta_{n_{k-1}} \rangle}(\eta_{n_k})$.
2. there is no $n, n_k \leq n < n_{k+1}$ such that for some $\langle \eta_1, \dots, \eta_n \rangle \in T$ and E , the condition $\langle \langle \eta_1, \dots, \eta_n \rangle, E \rangle$ decides the value of α_{k+1} ,

Now, using the density argument and making finitely many changes, if necessary, we can assume that such $\langle \langle \rangle, T \rangle$ in the generic set.

For every $k < \omega$, define a function $F_k : Lev_{n_k}(T) \rightarrow \kappa$ by setting

$$F_k(\eta_1, \dots, \eta_{n_k}) = \nu \text{ if } \langle \langle \eta_1, \dots, \eta_{n_k} \rangle, T_{\langle \eta_1, \dots, \eta_{n_k} \rangle} \rangle \Vdash \alpha_k = \nu.$$

■

¹¹Let $\langle \mathcal{V}_k \mid k < \omega \rangle$ be such sequence of ultrafilters over κ . We do not claim that $\langle \alpha_k \mid k < \omega \rangle$ is Prikry generic for the forcing $P(\langle \mathcal{V}_k \mid k < \omega \rangle)$, but rather that for every sequence $\langle A_k \mid k < \omega \rangle \in V$, with $A_k \in \mathcal{V}_k$, there is $k_0 < \omega$ such that for every $k > k_0$, $\alpha_k \in \mathcal{V}_k$.

We restrict now our attention to ultrafilters U which are P-points. This will allow us to deal with arbitrary sets of ordinals in $V[C]$. Recall the definition.

Definition 6.5 U is called a *P-point* iff every non-constant (mod U) function $f : \kappa \rightarrow \kappa$ is almost one to one (mod U), i.e. there is $A \in U$ such that for every $\delta < \kappa$,

$$|\{\nu \in A \mid f(\nu) = \delta\}| < \kappa.$$

Note that, in particular, the projection to the normal ultrafilter π is almost one to one. Namely,

$$|\{\nu < \kappa \mid \pi(\nu) = \alpha\}| < \kappa,$$

for any $\alpha < \kappa$.

Denote by U^{nor} the projection of U to the normal ultrafilter.

Lemma 6.6 Assume that $\mathbb{U} = \langle U_a \mid 1 \leq a \in [\kappa]^{<\omega} \rangle$ consists of *P-point* ultrafilters. Suppose that $A \in V[C] \setminus V$ is an unbounded subset of κ . Then κ has cofinality ω in $V[A]$.

Proof. Work in V . Let \mathcal{A} be a name of A and $\langle s, S \rangle \in P(\mathbb{U})$. Suppose for simplicity that s is the empty sequence. Define by induction a subtree T of S . For each $\nu \in Lev_1(S)$ pick some subtree S'_ν of $S_{\langle \nu \rangle}$ and $a_\nu \subseteq \pi_{\langle \rangle}(\nu)$ such that

$$\langle \langle \nu \rangle, S'_\nu \rangle \Vdash_{\mathcal{A}} \mathcal{A} \cap \pi_{\langle \rangle}(\nu) = a_\nu.$$

Let $S(0)'$ be a subtree of S obtained by replacing $S_{\langle \nu \rangle}$ by S'_ν , for every $\nu \in Lev_1(S)$.

Consider the function $\nu \rightarrow a_\nu$, where $\nu \in Suc_S(\langle \rangle)$. By normality of $\pi_{\langle \rangle * U_{\langle \rangle}}$ it is easy to find $A(0) \subseteq \kappa$ and $T(0) \subseteq Lev_1(S(0)'), T(0) \in U_{\langle \rangle}$ such that $A(0) \cap \pi_{\langle \rangle}(\nu) = a_\nu$, for every $\nu \in T(0)$. Set the first level of T to be $T(0)$. Set $S(0)$ to be a subtree of $S(0)'$ obtained by shrinking the first level to $T(0)$.

Let now $\langle \nu_1, \nu_2 \rangle \in Lev_2(S(0))$. So, $\pi_{\langle \nu_1 \rangle}(\nu_2) > \nu_1$. Find a subtree S'_{ν_1, ν_2} of $S(1)_{\langle \nu_1, \nu_2 \rangle}$, and $a_{\nu_1, \nu_2} \subseteq \pi_{\langle \nu_1 \rangle}(\nu_2)$ such that

$$\langle \langle \nu_1, \nu_2 \rangle, S'_{\nu_1, \nu_2} \rangle \Vdash_{\mathcal{A}} \mathcal{A} \cap \pi_{\langle \nu_1 \rangle}(\nu_2) = a_{\nu_1, \nu_2}.$$

Let $S(1)'$ be a subtree of $S(0)$ obtained by replacing $S_{\langle \nu_1, \nu_2 \rangle}$ by S'_{ν_1, ν_2} , for every $\langle \nu_1, \nu_2 \rangle \in Lev_2(S(0))$.

Again, we consider the function $\nu \rightarrow a_\nu$, where $\nu \in Suc_{S(1)}(\langle \nu_1 \rangle)$. By normality of $\pi_{\langle \nu_1 \rangle * U_{\langle \nu_1 \rangle}}$, it is easy to find $A(\nu_1) \subseteq \kappa$ and $T(\nu_1) \subseteq Suc_{S(1)}(\langle \nu_1 \rangle)$, $T(\nu_1) \in U_{\langle \nu_1 \rangle}$ such that $A(\nu_1) \cap \pi_{\langle \nu_1 \rangle}(\nu) = a_{\nu_1, \nu}$, for every $\nu \in T(\nu_1)$.

Define the set of the immediate successors of ν_1 to be $T(\nu_1)$, i.e. $Suc_T(\nu_1) = T(\nu_1)$. Let $S(1)$ be a subtree of $S(1)'$ obtained this way.

This defines the second level of T . Continue similar to define further levels of T .

We will have the following property:

(*) for every $\langle \eta_1, \dots, \eta_n \rangle \in T$,

$$\langle \langle \eta_1, \dots, \eta_n \rangle, T_{\langle \eta_1, \dots, \eta_n \rangle} \rangle \Vdash A \cap \pi_{\langle \eta_1, \dots, \eta_{n-1} \rangle}(\eta_n) = A(\eta_1, \dots, \eta_{n-1}) \cap \pi_{\langle \eta_1, \dots, \eta_{n-1} \rangle}(\eta_n).$$

A simple density argument implies that there is a condition which satisfies (*) in the generic set. Assume for simplicity that already $\langle \langle \rangle, T \rangle$ is such a condition. Then, C is a branch through T^* . Let $\langle \kappa_n \mid n < \omega \rangle = C$. So, for every $n < \omega$,

$$A \cap \pi_{\langle \kappa_0, \dots, \kappa_{n-1} \rangle}(\kappa_n) = A(\kappa_0, \dots, \kappa_{n-1}) \cap \pi_{\langle \kappa_0, \dots, \kappa_{n-1} \rangle}(\kappa_n).$$

Let us work now in $V[A]$ and define by induction a sequence $\langle \eta_n \mid n < \omega \rangle$ as follows. Consider $A(0)$. It is a set in V , hence $A(0) \neq A$. So there is η such that for every $\nu \in Lev_1(T)$ with $\pi_\emptyset(\nu) \geq \eta$ we have $A \cap \pi_\emptyset(\nu) \neq A(0) \cap \pi_\emptyset(\nu)$. Set η_0 to be the least such η .

Turn to η_1 . Let $\xi \in Lev_1(T)$ be such that $\pi_\emptyset(\xi) < \eta_0$. Consider $A(\xi)$. It is a set in V , hence $A(\xi) \neq A$. So there is η such that for every $\nu \in Lev_2(T_{\langle \xi \rangle})$ with $\pi_{\langle \xi \rangle}(\nu) \geq \eta$ we have $A \cap \pi_{\langle \xi \rangle}(\nu) \neq A(\xi) \cap \pi_{\langle \xi \rangle}(\nu)$. Set $\eta(\xi)$ to be the least such η . Now define η_1 to be $\sup(\{\eta(\xi) \mid \pi_1(\xi) < \eta_0\})$. The crucial point now is that the number of ξ 's with $\pi_\emptyset(\xi) < \eta_0$ is less than κ , since U_\emptyset is a P-point.

If $\eta_1 = \kappa$, then the cofinality of κ (in $V[A]$) is at most η_0 . So it must be ω since the Prikry forcing used does not add new bounded subsets to κ , and we are done.

Let us argue however that this cannot happen and always $\eta_1 < \kappa$.

Claim 2 $\eta_1 < \kappa$.

Proof. Suppose otherwise. Then

$$\sup(\{\eta(\xi) \mid \pi_\emptyset(\xi) < \eta_0\}) = \kappa.$$

Hence for every $\alpha < \kappa$ there will be ξ with $\pi_\emptyset(\xi) < \eta_0$ such that

$$A \cap \alpha = A(\xi) \cap \alpha.$$

Then, for every $\alpha < \kappa$ there will be ξ, ξ' with $\pi_\emptyset(\xi), \pi_\emptyset(\xi') < \eta_0$ such that

$$A(\xi) \cap \alpha = A(\xi') \cap \alpha.$$

Now, in V , set $\rho_{\xi, \xi'}$ to be the least $\rho < \kappa$ such that

$$A(\xi) \cap \rho \neq A(\xi') \cap \rho,$$

if it exists and 0 otherwise, i.e. if $A(\xi) = A(\xi')$. Let

$$Z = \{\rho_{\xi, \xi'} \mid \pi_\emptyset(\xi), \pi_\emptyset(\xi') < \eta_0\}.$$

Then $|Z|^V < \kappa$, since the number of possible ξ, ξ' is less than κ . But Z should be unbounded in κ due to the fact that for every $\alpha < \kappa$ there will be ξ with $\pi_\emptyset(\xi) < \eta_0$ such that $A \cap \alpha = A(\xi) \cap \alpha$ and $A \neq A(\xi)$. Contradiction.

■ of the claim

Suppose that $\eta_0, \dots, \eta_n < \kappa$ are defined. Define η_{n+1} . Let $\langle \xi_0, \dots, \xi_n \rangle$ be in T . Consider $A(\xi_0, \dots, \xi_n)$. It is a set in V , hence $A(\xi_0, \dots, \xi_n) \neq A$. So there is η such that for every $\nu \in \text{Lev}_{n+2}(T_{\langle \xi_0, \dots, \xi_n \rangle})$ with $\pi_{\langle \xi_0, \dots, \xi_n \rangle}(\nu) \geq \eta$ we have $A \cap \pi_{\langle \xi_0, \dots, \xi_n \rangle}(\nu) \neq A(\xi_0, \dots, \xi_n) \cap \pi_{\langle \xi_0, \dots, \xi_n \rangle}(\nu)$. Set $\eta(\xi_0, \dots, \xi_n)$ to be the least such η . Now define η_{n+1} to be $\sup(\{\eta(\xi_0, \dots, \xi_n) \mid \pi_{\langle \rangle}(\xi_0) < \eta_0, \dots, \pi_{\langle \xi_0, \dots, \xi_{n-1} \rangle}(\xi_n) < \eta_n\})$. Each relevant ultrafilter is a P-point, and so, the number of relevant ξ_0, \dots, ξ_n is bounded in κ . So, $\eta_{n+1} < \kappa$, as in the claim above.

This completes the definition of the sequence $\langle \eta_n \mid n < \omega \rangle$.
Let us argue that it is cofinal in κ .
Suppose otherwise.

Note that the sequence $\langle \pi_{\langle \kappa_0, \dots, \kappa_{n-1} \rangle}(\kappa_n) \mid n < \omega \rangle$ is unbounded in κ .
Let k be the least such that $\pi_{\langle \kappa_0, \dots, \kappa_{k-1} \rangle}(\kappa_k) > \sup(\{\eta_n \mid n < \omega\})$. Then

$$A \cap \pi_{\langle \kappa_0, \dots, \kappa_{k-1} \rangle}(\kappa_k) = A(\kappa_0, \dots, \kappa_{k-1}) \cap \pi_{\langle \kappa_0, \dots, \kappa_{k-1} \rangle}(\kappa_k).$$

This is impossible, since $\eta_k < \pi_{\langle \kappa_0, \dots, \kappa_{k-1} \rangle}(\kappa_k)$. ■

Theorem 6.7 *Let $\mathbb{U} = \langle U_a \mid a \in [\kappa]^{<\omega} \rangle$ consists of P-point ultrafilters over κ . Then for every new set of ordinals A in $V^{P(\mathbb{U})}$, κ has cofinality ω in $V[A]$.*

Proof. Let A be a new set of ordinals in $V[G]$, where $G \subseteq P(\mathbb{U})$ is generic. By Lemma 6.6, it is enough to find a new subset of A of size κ .

Suppose that every subset of A of size κ is in V . Let us argue that then A is in V as well. Let $\lambda = \sup(A)$.

The argument is similar to [5](Lemma 0.7).

Note that $(\mathcal{P}_{\kappa^+}(\lambda))^V$ remains stationary in $V[G]$, since $P(\mathbb{U})$ satisfies κ^+ -c.c. For each $x \in (\mathcal{P}_{\kappa^+}(\lambda))^V$ pick $\langle s_x, S_x \rangle \in G$ such that

$$\langle s_x, S_x \rangle \Vdash_{\mathbb{A}} A \cap x = A \cap x.$$

There are a stationary $E \subseteq (\mathcal{P}_{\kappa^+}(\lambda))^V$ and $s \in [\kappa]^{<\omega}$ such that for each $x \in E$ we have $s = s_x$. Now, in V , we consider

$$H = \{ \langle s, T \rangle \in P(U) \mid \exists x \in \mathcal{P}_{\kappa^+}(\lambda) \exists a \subseteq x \quad \langle s, T \rangle \Vdash_{\mathbb{A}} A \cap x = a \}.$$

Note that if $\langle s, T \rangle, \langle s, T' \rangle \in P(U)$ and for some $x \subseteq y$ in $\mathcal{P}_{\kappa^+}(\lambda)$, $a \subseteq x, b \subseteq y$ we have

$$\langle s, T \rangle \Vdash_{\mathbb{A}} A \cap x = a \text{ and } \langle s, T' \rangle \Vdash_{\mathbb{A}} A \cap y = b,$$

then $b \cap x = a$. Just conditions of this form are compatible, and so they cannot force contradictory information.

Apply this observation to H . Let

$$X = \{ a \subseteq \lambda \mid \exists \langle s, S \rangle \in H \quad \exists x \in \mathcal{P}_{\kappa^+}(\lambda) \langle s, T \rangle \Vdash_{\mathbb{A}} A \cap x = a \}.$$

Then necessarily, $\bigcup X = A$.

■

We do not know whether $V[A]$ for $A \in V[C] \setminus V$ is equivalent to a single ω -sequence even for $A \subseteq \kappa^+$. The problematic case is once U_n 's have κ^+ -many different ultrafilters below in the Rudin-Keisler order.

Theorem 6.8 *Assume that there is no inner model with $o(\alpha) = \alpha^{++}$. Let U be κ -complete ultrafilter over κ and $V = L[\vec{E}]$, for a coherent sequence of measures \vec{E} . Force with the Prikry forcing with U . Suppose that A is a new set of ordinals in a generic extension. Then the cofinality of κ is ω in $V[A]$.*

Proof. Consider

$$i_U : V \rightarrow M \simeq V^\kappa/U.$$

By Mitchell [9], i_U is an iterated ultrapower using measures from \vec{E} and images of \vec{E} . In addition we have that ${}^\kappa M \subseteq M$. Hence it should be a finite iteration using measures from \vec{E} . Since κ is the critical point, no measures below κ are involved and the first one applied is a measure on κ in \vec{E} . Denote it by E_0 and let

$$i_0 : V \rightarrow M_1$$

be the corresponding embedding. Let $\kappa_1 = i_0(\kappa)$. Rearranging, if necessary, we can assume that the next step was to use a measure E_1 over κ_1 from $i_0(\vec{E})$. So, it is either the image of one of the measures of \vec{E} or $E_0 - \text{Lim}\langle E^\xi \mid \xi < \kappa \rangle$, where $\langle E^\xi \mid \xi < \kappa \rangle$ is a sequence of measures over κ from \vec{E} which represents in M_1 the measure used over κ_1 .

Let

$$i_1 : M_1 \rightarrow M_2$$

be the corresponding embedding and $\kappa_2 = i_1(\kappa_1)$.

κ_2 can be moved further in our iteration, but only finitely many times. Suppose for simplicity that it does not move.

If nothing else is moved then U is equivalent to $E_0 - \text{Lim}\langle E^\xi \mid \xi < \kappa \rangle$ and theorem 6.7 easily provides the desired conclusion.

Suppose $i_1 \circ i_0$ is not i_U . Then some measures from $i_1 \circ i_0(\vec{E})$ with critical points in the intervals $(\kappa, \kappa_1), (\kappa_1, \kappa_2)$ are applied. Again, only finitely many can be used.

Thus suppose for simplicity that only one is used in each interval. The treatment of a general case is more complicated only due to notation.

So suppose that a measure E_2 with a critical point $\delta \in (\kappa, \kappa_1)$ is used on the third step of the iteration.

Let

$$i_2 : M_2 \rightarrow M_3$$

be the corresponding embedding. Note that the ultrafilter \mathcal{V} defined by

$$X \in \mathcal{V} \text{ iff } i_2(\delta) \in i_2 \circ i_1 \circ i_0(X)$$

is P -point. Thus, a function $f : \kappa \rightarrow \kappa$ which represents δ in M_1 , i.e. $\delta = i_0(f)(\kappa)$, will witness this.

Similar an ultrafilter used in the interval (κ_1, κ_2) will be P -point in M_1 , and so, in V , it will be equivalent to a limit of P -points.
So such situation is covered by 6.7.

■

7 Prikry forcing may add a Cohen subset.

Our aim here will be to show the following:

Theorem 7.1 *Suppose that V satisfies GCH and κ is a measurable cardinal. Then in a generic cofinality preserving extension there is a κ -complete ultrafilter U over κ such that the Prikry forcing with U adds a Cohen subset to κ over V . In particular, this forcing has a non-trivial subforcing which preserves regularity of κ .*

By [5] such F cannot be normal and by 6.6 F cannot be a P-point ultrafilter, since in any Cohen extension, κ stays regular.

Note that the above situation is impossible in $L[\mu]$. Just every κ -complete ultrafilter over the measurable κ is Rudin-Kiesler equivalent to μ^n , for some $n, 1 \leq n < \omega$ (see [7, Lemma 19.21]). But the Prikry forcing with μ^n is the same as the Prikry forcing with μ which is a normal measure.

We start with a GCH model with a measurable. Let κ be a measurable and U a normal measure on κ .

Denote by $j_U : V \rightarrow N \simeq \text{Ult}(V, U)$ the corresponding elementary embedding.

Define an iteration $\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ with Easton support as follows. Set $P_0 = 0$. Assume that P_α is defined. Set Q_α to be the trivial forcing unless α is an inaccessible cardinal. If α is an inaccessible cardinal, then let $Q_\alpha = Q_{\alpha 0} * \tilde{Q}_{\alpha 1}$, where $Q_{\alpha 0}$ is an atomic forcing consisting of three elements $0_{Q_{\alpha 0}}, x_\alpha, y_\alpha$, such that x_α, y_α are two incompatible elements which are stronger than $0_{Q_{\alpha 0}}$. Let $\tilde{Q}_{\alpha 1}$ be trivial once y_α is picked and let it be the Cohen forcing at α , i.e.

$$\text{Cohen}(\alpha, 2) = \{f : \alpha \rightarrow 2 \mid |f| < \alpha\}$$

once x_α was chosen.

Let $G_\kappa \subseteq P_\kappa$ be a generic. We extend now the embedding

$$j_U : V \rightarrow N,$$

in $V[G_\kappa]$, to

$$j_U^* : V[G_\kappa] \rightarrow N[G_\kappa * G_{[\kappa, j_U(\kappa))}],$$

for some $G_{[\kappa, j_U(\kappa))} \subseteq P_{[\kappa, j_U(\kappa))}$ which is $N[G_\kappa]$ -generic for $P_{j_U(\kappa)}/G_\kappa$. This can be done easily, once over κ itself in $Q_{\kappa 0}$, we pick y_κ , which makes the forcing Q_κ a trivial one.

This shows, in particular, that κ is still a measurable in $V[G_\kappa]$, as witnessed by an extension of U .

Consider now the second ultrapower $N_2 \simeq \text{Ult}(N, j_U(U))$. Denote j_U by j_1 , N by N_1 . Let

$$j_{12} : N_1 \rightarrow N_2$$

denotes the ultrapower embedding of N_1 by $j_1(U)$. Let $j_2 = j_{12} \circ j_1$. Then

$$j_2 : V \rightarrow N_2.$$

Let us extend, in $V[G_\kappa]$, the embedding

$$j_{12} : N_1 \rightarrow N_2$$

to

$$j_{12}^* : N_1[G_\kappa * G_{[\kappa, j_1(\kappa)]}] \rightarrow N_2[G_\kappa * G_{[\kappa, j_1(\kappa)]} * G_{[j_1(\kappa), j_2(\kappa)]}]$$

in a standard fashion, only this time we pick $x_{j_1(\kappa)}$ at stage $j_1(\kappa)$ of the iteration. Then a Cohen function should be constructed over $j_1(\kappa)$, which is not at all problematic to find in $V[G_\kappa]$.

Now we will have

$$j_2 \subseteq j_2^* : V[G_\kappa] \rightarrow N_2[G_\kappa * G_{[\kappa, j_1(\kappa)]} * G_{[j_1(\kappa), j_2(\kappa)]}]$$

which is the composition of j_1^* with j_{12}^* .

Define a κ -complete ultrafilter W over κ as follows:

$$X \in W \text{ iff } X \subseteq \kappa \text{ and } j_1(\kappa) \in j_2^*(X).$$

Proposition 7.2 *W has the following basic properties:*

1. $W \cap V = U$.
2. $\{\alpha < \kappa \mid x_\alpha \text{ was picked at the stage } \alpha \text{ of the iteration}\} \in W$.
3. if $C \subseteq \kappa$ is a club, then $C \in W$. Moreover

$$\{\nu \in C \mid \nu \text{ is an inaccessible}\} \in W.$$

Proof. (1) and (2) are standard. Let us show only (3). Let $C \subseteq \kappa$ be a club. Then, in N_2 , $j_2(C)$ is a club at $j_2(\kappa)$. In addition, $j_2(C) \cap \kappa_1 = j_1(C)$. Now, $j_1(C)$ is a club in $j_1(\kappa)$. It follows that $j_1(\kappa) \in j_2(C)$.

In order to show that

$$\{\nu \in C \mid \nu \text{ is an inaccessible}\} \in W,$$

just note that $j_1(\kappa)$ is an inaccessible in N_2 , and so W concentrates on inaccessibles.

■

Force with $\text{Prikry}(W)$ over $V[G_\kappa]$.

Let

$$C = \langle \eta_n \mid n < \omega \rangle$$

be a generic Prikry sequence.

By (2) in the previous proposition, there is $n^* < \omega$ such that for every $m \geq n^*$, at the stage η_m of the forcing P_κ , x_{η_m} was picked, and, hence, a Cohen function $f_{\eta_m} : \eta_m \rightarrow 2$ was added.

Define now $H : \kappa \rightarrow 2$ in $V[G_\kappa, C]$ as follows:

$$H = f_{\eta_{n^*}} \cup \bigcup_{n^* \leq m < \omega} f_{\eta_{m+1}} \upharpoonright [\eta_m, \eta_{m+1}).$$

Proposition 7.3 *H is a Cohen generic function for κ over $V[G_\kappa]$.*

Proof. Work in $V[G_\kappa]$. Let $D \in V[G_\kappa]$ be a dense open subset of $\text{Cohen}(\kappa)$. Consider a set

$$C = \{\alpha < \kappa \mid \text{if } \alpha \text{ is an inaccessible, then } D \cap V_\alpha[G_\alpha] \text{ is a dense open subset of } \text{Cohen}(\alpha) \text{ in } V[G_\alpha]\}.$$

claim 1 C is a club.

Proof. Suppose otherwise. Then $S = \kappa \setminus C$ is stationary. It consists of inaccessible cardinals by the definition of C .

Pick a cardinal χ large enough and consider an elementary submodel X of $\langle H_\chi, \in \rangle$ such that

1. $X \cap (V_\kappa)^{V[G_\kappa]} = (V_\delta)^{V[G_\kappa]}$, for some $\delta \in S$.
2. $\kappa, P_\kappa, D \in X$.

Note that it is possible to find such X due to stationarity of S . Note also that $(V_\kappa)^{V[G_\kappa]} = V_\kappa[G_\kappa]$ and $(V_\delta)^{V[G_\kappa]} = V_\delta[G_\delta]$, since the iteration P_κ splits nicely at inaccessibles.

Let us argue that $D \cap V_\delta[G_\delta]$ is a dense open subset of $\text{Cohen}(\delta)$ in $V[G_\delta]$. Just note that

$$D \cap X = D \cap X \cap (V_\kappa)^{V[G_\kappa]} = D \cap (V_\delta)^{V[G_\kappa]} = D \cap V_\delta[G_\delta].$$

So let $q \in (\text{Cohen}(\delta))^{V_\delta[G_\delta]}$. Then $q \in X$. Remember $X \preceq H_\chi$. So,

$$X \models D \text{ is dense open ,}$$

hence there is $p \geq q, p \in D \cap X$. But then, $p \in D \cap V_\delta[G_\delta]$, and we are done. Contradiction.

■ of the claim.

It follows now that $C \in W$. Hence there is $n^{**} \geq n^*$ such that for every $m, n^{**} \leq m < \omega$,

$$\eta_m \in C.$$

So, for every $m, n^{**} \leq m < \omega$,

$$f_{\eta_m} \in D,$$

since D is open.

It is almost what we need, however $H \restriction \eta_m$ need not be f_{η_m} , since an initial segment may have been changed.

In order to overcome this, let us note the following basic property of the Cohen forcing:

Claim 2 Let E be a dense open subset of $Cohen(\kappa, 2)$, then there is a dense subset E^* of E such that for every $p \in E^*$ and every inaccessible cardinal $\tau \in \text{dom}(p)$ for every $q : \delta \rightarrow 2, p \restriction [\delta, \kappa) \cup q \in E^*$.

The proof is an easy use of κ -completeness of the forcing.

Now we can finish just replacing D by its dense subset which satisfies the conclusion of the claim. Then, $H \restriction \eta_m$ will belong to it as a bounded change of f_{η_m} .

So we are done.

■

8 Further directions.

One of possible further directions is to extend our results from the Magidor forcing to the Radin forcing. Note that we cannot claim that every subforcing of the Radin forcing is equivalent to Radin forcing. Thus, the negation of $o^{\vec{U}}(\kappa_i) < \min(\nu \mid 0 < o^{\vec{U}}(\nu))$ provides a counterexample. However, it is still reasonable that every set in a Radin extension is equivalent to a subsequence of the Radin sequence. We conjecture that this is the case.

An other direction is to proceed further with the Prikry forcing with P -points ultrafilters and to prove that every subforcing of it is equivalent to a Prikry forcing. The complications starts once a P -point has more than κ many generators. In such situations it is easy to construct a subset of κ^+ which is not equivalent to any of its initial segments. The opposite was crucial for the arguments of [5] with a normal ultrafilter. We conjecture that it is possible to overcome this problem and that every subforcing of the Prikry forcing with P -points ultrafilters is indeed equivalent to Prikry forcing.

Let us conclude with few questions.

Question 1. *Is every set in a Radin extension equivalent to a subsequence of the Radin sequence?*

Question 2. *Is every subforcing of the Prikry forcing with P -points ultrafilters equivalent to a Prikry forcing?*

Question 3. *Characterize all κ -complete ultrafilters U over κ such that κ changes its cofinality in $V[A]$, for any new set A in the Prikry extension with U .*

In section 6 a rather large class of such ultrafilters was presented. It includes P -points, their products and limits. But are there other ultrafilters like this?

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