# The Hart-Shelah example, in stronger logics. 

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#### Abstract

We generalize the Hart-Shelah example [1] to higher infinitary logics. We build, for each natural number $k \geq 2$ and for each infinite cardinal $\lambda$, a sentence $\psi_{k}^{\lambda}$ of the $\operatorname{logic} \mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$ that (modulo mild set theoretical hypotheses around $\lambda$ and assuming $2^{\lambda}<\lambda^{+m}$ ) is categorical in $\lambda^{+}, \ldots, \lambda^{+k-1}$ but not in $\beth_{k+1}(\lambda)^{+}$(or beyond); we study the dimensional encoding of combinatorics involved in the construction of this sentence and study various model-theoretic properties of the resulting abstract elementary class $\mathcal{K}^{*}(\lambda, \mathrm{k})=$ $\left(\operatorname{Mod}\left(\psi_{k}^{\lambda}\right), \prec_{\left(2^{\lambda}\right)^{+}, \omega}\right)$ in the finite interval of cardinals $\lambda, \lambda^{+}, \ldots, \lambda^{+k}$.


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The study of categoricity transfer has been central to model theory since Morley's theorem; the question of finding extensions of this theorem to infinitary contexts and to abstract elementary classes has been a major source of results. Many central concepts of stability theory, both in first order and in its generalizations, are essential byproducts of the theory built in order to generalize the original Morley theorem.

One of the most important landmarks along this path was the Categoricity Transfer result for $\mathrm{L}_{\omega_{1}, \omega}$ due to the first author: if a sentence $\psi$ is categorical in $\aleph_{n}$ for all $n<\omega$ and the weak GCH holds for the $\aleph_{n}$ 's $\left(2^{\aleph_{n}}<2^{\aleph_{n+1}}\right.$ for all $\left.n<\omega\right)$ then $\psi$ is categorical in

[^0]all cardinals (see [2] and [3]; although these are two references, they correspond to "Part A" and "Part B" of one big paper from 1983). Notice the unusually strong assumption!

An example from 1990 due to Bradd Hart and the first author of this paper [1] established the (surprising) necessity of that strong assumption: the existence of few models at all the $\aleph_{n}$ 's is needed to get the eventual categoricity transfer for $\mathrm{L}_{\omega_{1}, \omega}$ : they provide, for each positive $k \in \omega$, an example of a sentence $\psi_{k}$ in $L_{\omega_{1}, \omega}$ categorical in $\aleph_{0}, \aleph_{1}, \cdots, \aleph_{k}$ but not eventually categorical: there exists some cardinal greater than $\aleph_{k}$ where categoricity fails.

That important example has later been referred to as the Hart-Shelah example. In many ways, the existence of such sentences points to an interesting failure of the "categoricity transfer" (Morley's theorem for first order logic, for countable theories) at small cardinalities, in the absence of further set theoretical hypotheses. Our work extends those results.

The present paper shows that a similar example also exists for the stronger logic $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega^{2}}$. A corollary of our result is that any extension of the results from [2] and [3] to stronger logics will require to assume categoricity at all cardinalities $\lambda, \lambda^{+}, \ldots, \lambda^{+n}, \ldots$ for all $n<\omega$.

Later, the first author has attempted an extension of the main result from [2] and [3] to Abstract Elementary Classes. These are more general than classes axiomatized by the logic $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$.

Our construction provides, for each infinite cardinal $\lambda$ and each $k \in(2, \omega)$, a sentence $\psi_{k}$ of $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$ that is categorical in $\lambda, \lambda^{+}, \ldots, \lambda^{+\mathrm{k}}$ but is not categorical in any cardinality $\mu \geq \beth_{k+1}(\lambda)^{+}$.

The shift of focus from infinitary logic to abstract elementary classes entails in many cases using Galois (orbital) types instead of syntactic types; although this shift is natural, compactness and locality properties in general do not transfer to Galois types. In particular, tameness and type-shortness do not hold in general for Galois types. Tameness was isolated by Grossberg and VanDieren [4]; later, Baldwin and Shelah [5] constructed an example of failure of tameness, based on an almost free non-Whitehead group. More recently, Boney and Unger have provided serious set theoretic reasons for the failure of tameness in AECs [6].

In [7], Baldwin and Kolesnikov study again the Hart-Shelah example: they prove that for the sentence $\psi_{k}$ of $\mathrm{L}_{\omega_{1}, \omega}$ of the example, the corresponding AEC (for $k \geq 3$ )

$$
\mathcal{K}^{\mathrm{HS}}\left(\omega_{1}, k\right)=\left(\operatorname{Mod}\left(\psi_{\mathrm{k}}\right), \prec_{\omega_{1}, \omega}\right)
$$

- has disjoint amalgamation,
- is Galois stable exactly in $\aleph_{0}, \aleph_{1}, \ldots, \aleph_{k-1}$,
- is $\left(<\aleph_{0}, \leq \aleph_{k-1}\right)$-tame.

Moreover, the AEC axiomatized by their sentence $\psi_{k}$ fails $\left(\aleph_{k-1}, \aleph_{k}\right)$-tameness. This is an immediate consequence of the failure of categoricity transfer and the upward categoricity theorem for tame AECs due to Grossberg and VanDieren [8].

Baldwin and Kolesnikov really study a slight variant of the Hart-Shelah example, presented in the language of group actions and revealing the filiation to the early BaldwinLachlan example of an $\mathbb{N}_{1}$-categorical theory which is not almost strongly minimal.

More recently, Boney [9] has continued this study of the behavior of the Hart-Shelah example; he has proved that the class $\mathcal{K}^{\mathrm{HS}}\left(\omega_{1}, k\right)$ has a "good $\aleph_{m}$-frame" for all $m \leq k-1$ but cannot have a good frame above by the failure of stability. Then, Boney and Vasey [10] continue this study and show first that the frame at $\aleph_{k-1}$ cannot be "successful". They study good frames in connection with the Hart-Shelah example: for frames around the $\aleph_{n}$ 's $(n<\omega)$ the Hart-Shelah example is a natural place to look for "boundary properties": being "successful up to some point" but failing to be successful above.

Our generalization of the Hart-Shelah example addresses the question of how necessary an assumption similar to "few models in all the $\aleph_{n}$ 's" is for categoricity transfer in the case of stronger logics. Here of course the corresponding assumption would be of the form "few models in all the $\lambda^{+n}(n<\omega)$ ".

We build a sentence $\psi_{k}^{\lambda}$ in $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$, categorical in $\lambda, \lambda^{+}, \ldots, \lambda^{+\mathrm{k}}$ but not categorical in $\beth_{k+1}(\lambda)^{+}$. Here are two important differences between our approach and earlier ones:

- The sentences are constructed in all cases by first building a "standard model" and then extracting the sequence from it. In the Hart-Shelah example, one predicate Q "ties together" various copies of groups in a way that ends up linking the "dimension" of the predicate to the length of induction in the proof of categoricity. In our example, we need a large family of predicates $Q_{s}, s \in S=[\lambda]^{<X_{0}}$.
- The "failure of categoricity" argument at cardinals greater than or equal to $\beth_{k+1}(\lambda)^{+}$ here is done by using a regular filter $\mathfrak{D}$.

A natural question arises, on the "gap" between categoricity and failure of categoricity of $\psi_{\mathrm{k}}^{\lambda}$. Here, we can guarantee categoricity in the interval $\left[\lambda, \lambda^{+\mathrm{k}}\right]$ and failure of categoricity...at $\beth_{k+1}(\lambda)^{+}$. Admittedly, this is a very large gap, relatively much wider than what Baldwin and Kolesnikov have for their version of the Hart-Shelah sentence. The question remains open whether this gap may be reduced.

In our concluding remarks, we raise some questions connected with the tameness and frames, inspired by the paper [10]. In particular, we ask whether the methods from that paper (that worked for the Hart-Shelah sentence) may be generalized to our sentence $\psi_{k}^{\lambda}$.

A note on indexing: the previous papers dealing with constructing examples of sentences where categoricity "stops" are [1], [7] (which proved more model theoretic facts on a variant of the original example and studied the abstract elementary class determined by the example; in particular, Galois (=orbital) types, the amalgamation and tameness spectra
associated with the class), [9] and [10], in which the connection to frames is worked out (analyzing the Hart-Shelah example enables Boney and Vasey to study limitations to the existence of good frames). Now, for [1], the "critical" cardinality (the last cardinality of categoricity) is $\aleph_{k}$. In [7], because of the way they analyze the construction, it is more natural to work with $k \geq 3$ and with $k-2$ as the critical cardinality. The two other papers follow this.

Since our paper is directly a generalization of [1], it is more natural for us to revert to the choice of critical cardinality from there, of course adapted to our context. So, the last cardinality where we will have categoricity is $\lambda^{+\mathrm{k}-1}$.

Our notation is standard.
We thank John Baldwin, Will Boney, Rami Grossberg, Alexei Kolesnikov, Sebastien Vasey and Boban Velickovic for several remarks and valuable discussions concerning (directly or less directly) this work, as well as for pressing us to provide some clarification of the big construction. The second author is particularly indebted to John Baldwin for very interesting conversations of the connections between this example and the original group covers in the strongly minimal context, due to Baldwin and Lachlan [11]. We also thank Péter Komjath for helpful discussion on the negative partition relation in [12] useful in our theorem. We also thank the anonymous referee of an earlier version of this paper for extremely insightful and helpful comments. They (hopefully) led to our improving this paper. We also thank a second anonymous referee of the version prior to this for remarks that led to a substantial rewriting and what we believe is a much better presentation of the results.

## 1. Construction of the sentence $\psi_{k}^{\lambda}$, in $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$

Context 1.1. For the rest of the paper, we fix an infinite cardinal $\lambda$ and a natural number $k \geq 2$.

We build in this section a new sentence $\psi_{k}^{\lambda}$ in the logic $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega^{\prime}}$. Our construction of $\psi_{k}^{\lambda}$ requires first building a model we will call "canonical", $M_{\mathrm{I}}$, for an arbitrary index set $I$ and later taking a conjunction of the first order theory of $M_{I}$ along with several infinitary sentences describing the behavior of various components of $M_{I}$. The sentence $\psi_{k}^{\lambda}$ has some similarity to the Hart-Shelah sentence and may be seen as a generalization, but important differences are also present and will be apparent later (the regular group $G$ and the regular filter on $\lambda, \mathfrak{D})$. However, it is important to stress that prior knowledge of the Hart-Shelah is not necessary for an understanding of our construction, as we make it self-contained.

We will build the model $M_{I}$ around a "spine" I, essentially by coding interactions between $k$-element subsets of $(k+1)$-element subsets of $I$, in some combinatorial ways. Namely, we will define various groups and encode in the model their actions on those $k$
and $(k+1)$-element subsets of $I$, focusing especially on the way different $k$-subsets of a given ( $k+1$ )-subset of I interact. Finally, a collection of predicates (called $Q_{s}$ ) will "tie" those combinatorial interactions.

Definition 1.2. Notation and general construction tools. We fix the following basic objects to use in the construction later.

- $\mathrm{S}=\mathrm{S}_{\lambda}:=[\lambda]^{<\mathcal{N}_{0}}=\{\mathrm{u} \subset \lambda \mid \mathrm{u}$ is finite $\}$,
- $\mathfrak{D}=\mathfrak{D}_{\lambda}:=\left\{A \subset S \mid \exists \mathfrak{u}_{A} \in S \quad \forall v \in S\left(u_{A} \subset v \rightarrow v \in A\right)\right\}$, the regular filter on $S$ generated by sets of the form $\langle u\rangle=\{v \in S \mid u \subset v\}$,
- $\mathrm{G}^{+}=\mathrm{G}_{\lambda}^{+}:={ }^{\mathrm{S}}\left(\mathbb{Z}_{2}\right)$, as a group with the natural operation $(\mathrm{f}+\mathrm{g})(v)=\mathrm{f}(v)+\mathbb{Z}_{2} \mathrm{~g}(v)$,
- $\mathrm{G}=\mathrm{G}_{\lambda}:=\left\{\mathrm{f} \in{ }^{\mathrm{S}}\left(\mathbb{Z}_{2}\right) \mid \operatorname{ker}(\mathrm{f})=\{\mathbf{u} \in \mathrm{S} \mid \mathrm{f}(\mathrm{u})=0\} \in \mathfrak{D}\right\}$, as a subgroup of $\mathrm{G}^{+}(\mathrm{G} \leq$ $\mathrm{G}^{+}$since, if $\mathrm{f}, \mathrm{g} \in \mathrm{G}$, then $\operatorname{ker}(\mathrm{f}), \operatorname{ker}(\mathrm{g}) \in \mathfrak{D}$, so $\operatorname{ker}(\mathrm{f}+\mathrm{g}) \supset \operatorname{ker}(\mathrm{f}) \cap \operatorname{ker}(\mathrm{g}) \in \mathfrak{D}$, hence $\operatorname{ker}(\mathrm{f}+\mathrm{g}) \in \mathfrak{D}$ and $\mathrm{f}+\mathrm{g} \in \mathrm{G}$.

Note that $|G|=2^{\lambda}$.

### 1.1. The model $M_{I}$

Fix a set I for the rest of this section.
We define the group $H_{I}$ and then describe the model $M_{I}$.
Definition 1.3 (The group $\mathrm{H}_{\mathrm{I}}$ ). For our (fixed) I , we let $\mathrm{H}_{\mathrm{I}}:=\left[[\mathrm{I}]^{\mathrm{k}}\right]^{<\mathrm{N}_{0}}$. So, $\mathrm{H}_{\mathrm{I}}$ is the set of finite subsets of $[\mathrm{I}]^{\mathrm{k}}$, with the group operation $\mathrm{F}_{1}+\mathrm{F}_{2}:=\mathrm{F}_{1} \Delta \mathrm{~F}_{2}$ (symmetric difference). Equivalently, $\mathrm{H}_{\mathrm{I}}$ may be seen as the set of functions $[\mathrm{I}]^{\mathrm{k}} \rightarrow \mathbb{Z}_{2}$ with finite support. In this case, $\mathrm{F} \in \mathrm{H}_{\mathrm{I}}$ is coded by the function $\mathrm{h}_{\mathrm{F}}:[\mathrm{I}]^{\mathrm{k}} \rightarrow \mathbb{Z}_{2}$ with $\mathrm{h}_{\mathrm{F}}(\mathrm{u})=1$ iff $u \in \mathrm{~F}$ and the group operation is given by $\left(h_{1}+h_{2}\right)(u)=h_{1}(u)+\mathbb{Z}_{2} h_{2}(u)$.

We now start the (lengthy) description of $M_{I}$ : the universe, basic predicates, projections between them, other partial functions coded by relations and, crucially, the family of predicates $\mathrm{Q}_{\mathrm{s}}$ (for $s \in S$ ).

Definition 1.4 (Universe of $M_{I}$ ). The universe of $M_{I}$ is the union of seven different sorts:

$$
\left|M_{I}\right|=I \cup[I]^{k} \cup[I]^{k+1} \cup\left([I]^{k} \times S \times H_{I}\right) \cup\left([I]^{k} \times S \times \mathbb{Z}_{2}\right) \cup H_{I} \cup\left([I]^{k+1} \times G\right) .
$$

The following remarks on the universe of $M_{I}$ are important:

- The natural way to think about the universe of $M_{I}$ is as

$$
\left|M_{\mathrm{I}}\right|
$$

consisting of two parts: the "support of the model" (I, [I] $]^{k},[I]^{k+1}$ ) and many copies of (the domains of) the three groups $\mathrm{H}_{\mathrm{I}}, \mathbb{Z}_{2}$ and G , indexed by elements of S and of the support part:

$$
\overbrace{I \cup[I]^{k} \cup[I]^{k+1}}^{\text {ssupport part' }} \overbrace{\left([I]^{k} \times S \times H_{I}\right) \cup\left([I]^{k} \times S \times \mathbb{Z}_{2}\right) \cup H_{I} \cup\left([I]^{k+1} \times G\right)}^{H_{I} \text { and copies of the domains of } H_{I}, \mathbb{Z}_{2} \text { and } G} .
$$

- Notice that the intersection between all the pieces of the model is empty.
- The universe of $M_{I}$ depends directly on $I$ and on $G$, as is clear from the various pieces. In particular, when the cardinality of $I$ is $\geq \lambda$, the cardinality of $M_{I}$ will be equal to $|I|+2^{\lambda}$.
- The universe depends on $k$ as well. Of course in our standard model this dependence is immediate, as seen from the superindices $k$ and $k+1$. In general models later, we will need projection functions among the predicates in the model in order to axiomatize the connections between pieces corresponding to abstract versions of I, $[\mathrm{I}]^{\mathrm{k}}$, etc. This dependence on k will be crucial in the "dimension" analysis later.
- The universe also depends on $\lambda$, through the appearance of $S$ and $G$ among the pieces.

Definition 1.5 (Relations, functions of $M_{I}$ - the predicates $Q_{s}$ ). The structure of $M_{I}$ consists of the following items:

- $\lambda$-many predicates $\mathrm{P}_{0}^{\mathrm{M}}, \mathrm{P}_{1,1}^{\mathrm{M}}, \mathrm{P}_{1,2}^{\mathrm{M}}, \mathrm{P}_{2}^{\mathrm{M}},\left(\mathrm{P}_{2, \mathrm{~s}}^{\mathrm{M}}\right)_{s \in \mathrm{~S}}, \mathrm{P}_{3}^{\mathrm{M}},\left(\mathrm{P}_{3, \mathrm{~s}}^{\mathrm{M}}\right)_{s \in \mathrm{~S}}, \mathrm{P}_{4}^{\mathrm{M}}, \mathrm{P}_{5}^{\mathrm{M}}$,
- k-many projections $\pi_{\ell}^{0}: \mathrm{P}_{1,1}^{\mathrm{M}} \rightarrow \mathrm{P}_{0}^{\mathrm{M}}(\ell<\mathrm{k})$ and $\mathrm{k}+1$-many projections $\pi_{\ell}^{1}: \mathrm{P}_{1,2}^{\mathrm{M}} \rightarrow$ $\mathrm{P}_{0}^{\mathrm{M}}$,
- $2^{\lambda}$-many additional functions $F_{2}^{M}, F_{3}^{M}, F_{4}^{M}, F_{5}^{M},\left(F_{3, g^{*}}^{M}\right)_{g^{*} \in G}$,
- $a(3 \mathrm{k}+4)$-ary predicate $\mathrm{Q}_{\mathrm{s}}$, for each $\mathrm{s} \in \mathrm{S}$.

Each of these predicates and functions will be discussed in detail in the following paragraphs.
1.1.1. Descriptions of basic relations, functions, and the $\mathrm{Q}_{s}$-predicates

Basic Relations: these consist of a family of $\lambda$-many predicates

$$
P_{0}^{M}, P_{1,1}^{M}, P_{1,2}^{M}, P_{2}^{M},\left(P_{2, s}^{M}\right)_{s \in S}, P_{3}^{M},\left(P_{3, s}^{M}\right)_{s \in S}, P_{4}^{M}, P_{5}^{M}
$$

defined by

- $\mathrm{P}_{0}^{\mathrm{M}}=\mathrm{I}$,
- $\mathrm{P}_{1,1}^{\mathrm{M}}=[I]^{\mathrm{k}}$,
- $\mathrm{P}_{1,2}^{\mathrm{M}}=[\mathrm{I}]^{\mathrm{k}+1}$,
- $\mathrm{P}_{2}^{\mathrm{M}}=[\mathrm{I}]^{\mathrm{k}} \times \mathrm{S} \times \mathrm{H}_{\mathrm{I}}$,
- for $s \in S, P_{2, s}^{M}=\left\{(u, s, h) \in P_{2}^{M} \mid u \in[I]^{k}, h \in H_{I}\right\}=[I]^{k} \times\{s\} \times H_{I}$,
- $P_{3}^{M}=[I]^{k} \times S \times \mathbb{Z}_{2}$ (a copy of $\mathbb{Z}_{2}$ for each $b \in[I]^{k}, s \in S$ ),
- for $s \in S, P_{3, s}^{M}=\left\{(u, s, i) \in P_{3}^{M} \mid u \in[I]^{k}, i \in \mathbb{Z}_{2}\right\}=[I]^{k} \times\{s\} \times \mathbb{Z}_{2}$,
- $\mathrm{P}_{4}^{\mathrm{M}}=\mathrm{H}_{\mathrm{I}}$,
- $\mathrm{P}_{5}^{\mathrm{M}}=[\mathrm{I}]^{\mathrm{k}+1} \times \mathrm{G}$

Remark 1.6. The meaning of $P_{0}^{M}, P_{1,1}^{M}, P_{1,2}^{M}, P_{2}^{M}, P_{3}^{M}, P_{4}^{M}$ is clear. In the case of $\mathrm{P}_{2, s}^{M}$, the idea is that we stack "copies" of $\mathrm{H}_{\mathrm{I}}$ for each $\mathrm{b} \in[\mathrm{I}]^{\mathrm{k}}$ and each $s \in S$, and similarly for $\mathrm{P}_{3}^{\mathrm{M}}, \mathrm{P}_{3, \mathrm{~s}}^{\mathrm{M}}$. Another way of seeing this is thinking of the predicates as codifying families, as follows:

- $\mathrm{P}_{2}^{\mathrm{M}}$ corresponds to $\left(\mathrm{H}_{v, \mathrm{~s}}\right)_{v \in[I]^{\mathrm{k}}, \mathrm{s} \in \mathrm{S}}$,
- $P_{3}^{M}$ corresponds to $\left(\left(\mathbb{Z}_{2}\right)_{v, s}\right)_{v \in[I]^{k}, s \in S}$,
- $P_{5}^{M}$ corresponds to $\left(G_{\mathfrak{u}}\right)_{\mathfrak{u} \in[I]^{k+1}}$.

Projections: We also include, for $\ell<k$, all the projections $\pi_{\ell}^{0}: P_{1,1}^{M} \rightarrow P_{0}^{M}$ :

$$
\pi_{\ell}^{0}(\overline{\mathrm{a}})=\mathrm{a}_{\ell}
$$

and for $\ell<k+1$, the projections $\pi_{l}^{1}: P_{1,2}^{M} \rightarrow P_{0}^{M}$ :

$$
\pi_{\ell}^{1}(\bar{a})=a_{\ell}
$$

The role of these projections is to tie the predicates $P_{1,1}^{M}$ and $P_{1,2}^{M}$ to $P_{0}^{M}$ making them behave as the corresponding sets of ktuples or $\mathrm{k}+1$-tuples.

Other Partial Functions: We also include $2^{\lambda}$-many functions in $M_{I}$,

$$
F_{2}^{M}, F_{3}^{M}, F_{4}^{M}, F_{5}^{M},\left(F_{3, g^{*}}^{M}\right)_{g^{*} \in G}:
$$

- A unary function $F_{2}^{M}$ with domain $P_{2}^{M}$, given by

$$
F_{2}^{M}(u, s, h)=u
$$

- A unary function $F_{3}^{M}$ with domain $P_{3}^{M}$, given by

$$
F_{3}^{M}(u, s, i)=u
$$

- for $g^{*} \in G$, a unary function $F_{3, g^{*}}^{M}$ with domain $P_{5}^{M}$, given by

$$
F_{3, g^{*}}^{M}(u, g)=\left(u, g^{*}+g\right)
$$

- A binary function $F_{4}^{M}$ with domain $P_{2}^{M} \times P_{4}^{M}$, given by

$$
\mathrm{F}_{4}^{\mathrm{M}}\left((v, s, h), h_{1}\right)=\left(v, s, h+_{\mathrm{H}} \mathrm{~h}_{1}\right)
$$

- A unary function $F_{5}^{M}$ with domain $P_{5}^{M}$, given by

$$
F_{5}^{M}(u, g)=u
$$

$A(3 k+4)$-ary predicate $Q_{s}$, for each $s \in S$. This is the crux of the construction of the model $M_{\mathrm{I}}$. The predicate will encode interactions between the different parts of the model, in a way that will involve dimensional interactions between them. This predicate on the one hand enables later to move up in the proof of categoricity by induction $k-1$ times from $\lambda$ to $\lambda^{k}$ and on the other blocks the proof from moving up to $\lambda^{k+1}$. It is interpreted in $M_{I}$ as the set of tuples

$$
\left\langle a_{0}, \ldots, a_{k}, u_{0}, \ldots, u_{k}, x_{0}, \ldots, x_{k-1}, y_{k}, z\right\rangle
$$

satisfying (for fixed $s \in S$ !!) for all $h_{k} \in H_{I}, \mathfrak{i}_{\ell} \in \mathbb{Z}_{2}(\ell<k), g \in G$ :
( $\alpha) a_{\ell} \in I$ with no repetitions $(\ell \leq k)$,
$(\beta) u_{\ell}=\left\langle a_{m} \mid m \neq \ell\right\rangle \in P_{1,1}^{M}(\ell \leq k)$,
$(\gamma) y_{k}=\left(u_{k}, s, h_{k}\right) \in P_{2}^{M}$,
( $\delta) x_{\ell}$ has the form $\left(u_{\ell}, s, i_{\ell}\right) \in P_{3}^{M}(\ell<k)$ so $i_{\ell} \in \mathbb{Z}_{2}$,
( $\epsilon$ ) $z$ is of the form $(u, g) \in P_{5}^{M}$, where $u=\left(a_{0}, \ldots, a_{k}\right) \in[I]^{k+1}$ and
$(\zeta)$ (main point)

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(u_{0}\right)+g(s)
$$

Some general remarks on this definition of the model $M_{I}$ are in point, before giving a specific description for the case $k=2$.

Remark 1.7. - $(\zeta)$ is the crucial part of the definition of the predicates $Q_{s}$. It provides the connection between k copies of $\mathbb{Z}_{2}$, one copy of $\mathrm{H}_{\mathrm{I}}$, one copy of G and the $(\mathrm{k}+1)$ many k -element subsets of a set of size $\mathrm{k}+1$ in I .

- The role of $\mathrm{F}_{2}^{\mathrm{M}}$ is to project $\mathrm{P}_{2}^{\mathrm{M}}$ (essentially $\left(\mathrm{H}_{v, \mathrm{~s}}\right)_{v \in[I]^{\mathrm{k}}, s \in \mathrm{~S}}$ ) onto its first coordinate; to trace the k -element subset of I it corresponds to. Similarly for $\mathrm{F}_{3}^{\mathrm{M}}$ and $\mathrm{F}_{5}^{\mathrm{M}}$.
- The functions $\mathrm{F}_{3, g^{*}}^{\mathrm{M}}$ and $\mathrm{F}_{4}^{\mathrm{M}}$ encode the actions of the groups G and $\mathrm{H}_{\mathrm{I}}$ on the corresponding "fibers" over $u \in[I]^{k+1}$ or $(v, s) \in[I]^{k} \times S$. The model $M_{I}$ does not really include the group operations corresponding to G and $\mathrm{H}_{\mathrm{I}}$; it only has the effect of the group actions on the appropriate fibers.
- Notice that $+{ }^{\mathrm{H}}$ is definable - so in this case there is no need to add an analogue of $\mathrm{F}_{4}$ for copies of $\mathbb{Z}_{2}$ :

$$
F_{4}^{M}\left(F_{4}^{M}\left((u, s, h), h_{1}\right), h_{2}\right)=F_{4}^{M}\left((u, s, h), h_{3}\right) \Leftrightarrow H \models h_{1}+h_{2}=h_{3} .
$$

1.1.2. Illustration of the definition of $\mathrm{M}_{\mathrm{I}}$, when $\mathrm{k}=2$

As an example to visualize the situation, we momentarily fix $k=2$. We also fix $s \in S$ and choose some $u \in[I]^{k+1}=[I]^{2+1}, u=\left\langle a_{0}, a_{1}, a_{2}\right\rangle$. This determines automatically (using the projections) in the models we have described so far $a_{0}, a_{1}, a_{2}$ and $u_{0}=\left\langle a_{1}, a_{2}\right\rangle$, $u_{1}=\left\langle a_{0}, a_{2}\right\rangle, u_{2}=\left\langle a_{0}, a_{1}\right\rangle$.

We then have

- copies of $\mathbb{Z}_{2}$ over both $\mathfrak{u}_{0}$ and $\mathfrak{u}_{1}$,
- a copy of the domain of $\mathrm{H}_{\mathrm{I}}$ over $\mathrm{u}_{2}$, together with the action of $\mathrm{H}_{\mathrm{I}}$ on this copy,
- a copy of $G$ over $u$, again with the action of $G$ over this copy.

Furthermore, we have the predicate $\mathrm{Q}_{s}$ : it is in this case $3 \cdot 2+4=10$-ary. The 10 -uple associated with our $u$ is then of the form

$$
\left(a_{0}, a_{1}, a_{2}, u_{0}, u_{1}, u_{2}, x_{0}, x_{1}, y_{2}, z\right)
$$

with $x_{0}=\left(u_{0}, s, i_{0}\right), x_{1}=\left(u_{1}, s, i_{1}\right), y=\left(u_{2}, s, h_{2}\right)$ and $z=(u, g)$ for some $i_{0}, i_{1} \in \mathbb{Z}_{2}$, $h_{2} \in H_{I}$ and $g \in G$.

We want to describe when this tuple belongs to $\mathrm{Q}_{s}$. The following triangle summarizes the relevant information:


The tuple $\left(a_{0}, a_{1}, a_{2}, u_{0}, u_{1}, u_{2}, x_{0}, x_{1}, y_{2}, z\right)$ belongs to $Q_{s}$ if and only if

$$
\mathbb{Z}_{2} \models \mathfrak{i}_{0}+\mathfrak{i}_{1}=\mathrm{h}_{2}\left(\mathrm{u}_{0}\right)+\mathrm{g}(\mathrm{~s}) .
$$

Therefore, on top of the triangle $u$ we have (when $k=2$ ) four pieces of information playing: two elements ( $\mathfrak{i}_{0}, \mathfrak{i}_{1}$ ) of $\mathbb{Z}_{2}$ associated to two sides of the triangle, one element $h$ of $H_{I}$ associated to the third side of the triangle (and the value of $h$ at $u_{0}$ ) and finally one element $g$ of $G$ associated to the triangle $u$ itself - and the value of $g$ at...s.

### 1.2. The language, the sentence $\psi_{\mathrm{k}}^{\lambda}$ and the $\operatorname{AEC}^{*}(\lambda, \mathrm{k})$

We now build the sentence $\psi_{k}^{\lambda}$.
Definition 1.8. We deal with two vocabularies:

- Let $\tau^{-}$be the vocabulary of all the construction above, except the predicates $\left\{\mathrm{Q}_{s} \mid s \in \mathrm{~S}\right\}$ and
- let $\tau$ be the full vocabulary used in the construction of $\mathrm{M}_{\mathrm{I}}$.

Specifically,

$$
\begin{aligned}
& \text { let } \tau^{-}=\left\langle\mathrm{P}_{0}, \mathrm{P}_{1,1}, \mathrm{P}_{1,2}, \mathrm{P}_{2},\left(\mathrm{P}_{2, s}\right)_{s \in S}, \mathrm{P}_{3},\left(\mathrm{P}_{3, s}\right)_{s \in S}, \mathrm{P}_{4}, \mathrm{P}_{5},\right. \\
& \left.\qquad \pi_{0}^{0}, \ldots, \pi_{\mathrm{k}-1}^{0}, \pi_{0}^{1}, \ldots, \pi_{\mathrm{k}}^{1}, \mathrm{~F}_{2}, \mathrm{~F}_{3}, \mathrm{~F}_{4}, \mathrm{~F}_{5},\left(\mathrm{~F}_{3, \mathrm{~g}^{*}}\right)_{\mathrm{g}^{*} \in \mathrm{G}}\right\rangle \\
& \text { and let } \tau=\tau^{-} \cup\left\{\mathrm{Q}_{s} \mid \mathrm{s} \in \mathrm{~S}\right\} .
\end{aligned}
$$

Notice that $|\tau|=\left|G_{\lambda}\right|+|S|+\aleph_{0}=2^{\lambda}$, since $\left|G_{\lambda}\right|=2^{\lambda}$.
Definition 1.9 (The sentence $\psi_{\mathrm{k}}^{\lambda}$ ). The sentence $\psi_{\mathrm{k}}^{\lambda} \in \mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}(\tau)$ is the conjunction

$$
\psi_{\mathrm{k}}^{\lambda} \equiv \bigwedge \mathrm{T}_{0} \wedge \psi_{\mathrm{G}} \wedge \psi_{\mathbb{Z}_{2}} \wedge \psi_{\mathrm{H}}
$$

of the first order theory $\mathrm{T}_{0}$ of $\mathrm{M}_{\mathrm{I}}$ (for infinite I ) and the infinitary sentences

- $\psi_{G} \equiv \forall z_{1} z_{2}\left(\left[P_{5}\left(z_{1}\right) \wedge P_{5}\left(z_{2}\right) \wedge F_{5}\left(z_{1}\right)=F_{5}\left(z_{2}\right)\right] \rightarrow \bigvee_{g^{*} \in G} F_{3, g^{*}}\left(z_{1}\right)=z_{2}\right)$,
- $\psi_{\mathbb{Z}_{2}} \equiv \forall \mathrm{y}\left(\mathrm{P}_{2}(\mathrm{y}) \leftrightarrow \bigvee_{\mathrm{s} \in \mathrm{S}} \mathrm{P}_{2, \mathrm{~s}}(\mathrm{y})\right)$,
- $\psi_{\mathrm{H}} \equiv \forall \mathrm{y}\left(\mathrm{P}_{3}(\mathrm{y}) \leftrightarrow \bigvee_{\mathrm{s} \in \mathrm{S}} \mathrm{P}_{3, \mathrm{~s}}(\mathrm{y})\right)$.

We describe in more detail some parts of the previous definition.
$\psi_{G}$ says that $G$ acts transitively (through the functions $F_{3, g^{*}}$ ) on copies of $G$ (fibers of $P_{5}$ ).
$\psi_{H}$ says that there are no "non-standard fibers" in $P_{2}$ : every element of $P_{2}$ is in some $P_{2, s}$. $\psi_{\mathbb{Z}_{2}}$ says that there are no "non-standard fibers" in $P_{3}$ : every element of $P_{3}$ is in some $\mathrm{P}_{3, \mathrm{~s}}$

Note that, although there are $2^{2^{\lambda}}$ sentences in the logic, we are only using $2^{\lambda}$ of them, as witnessed by $|\mathrm{G}|=2^{\lambda}$.

We will also use the following variant on the standard model: for a set I and a function

$$
\mathrm{f}:[\mathrm{I}]^{\mathrm{k}+1} \times \mathrm{S} \rightarrow \mathbb{Z}_{2}
$$

we will now build models $M_{\mathrm{I}, \mathrm{f}}$ and $\mathrm{M}_{\mathrm{I}, \mathrm{f}}^{-}$.
Definition 1.10. [The models $M_{\mathrm{I}, \mathrm{f}}$ and $\mathrm{M}_{\mathrm{I}, \mathrm{f}}^{-}$Let $\mathrm{f}:[\mathrm{I}]^{\mathrm{k}+1} \times \mathrm{S} \rightarrow \mathbb{Z}_{2}$ and I a set. Then $\mathrm{M}_{\mathrm{I}, \mathrm{f}}$ is the $\tau$-model constructed just like $\mathrm{M}_{\mathrm{I}}$, with only one difference: the interpretation of $\mathrm{Q}_{\mathrm{s}}$, for $\mathrm{s} \in \mathrm{S}$, now is the set of tuples

$$
\left\langle a_{0}, \ldots, a_{k}, u_{0}, \ldots, u_{k}, x_{0}, \ldots, x_{k-1}, y_{k}, z\right\rangle
$$

(see page 8) with condition ( $\zeta$ ) replaced by

$$
(\zeta)_{f}^{*} \quad \mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(u_{0}\right)+g(s)+f(u, s) .
$$

The $\tau^{-}$-model $\mathrm{M}_{\mathrm{I}, \mathrm{f}}^{-}$is then defined as $\mathrm{M}_{\mathrm{I}, \mathrm{f}} \upharpoonright \tau^{-}$.
We will use the models $M_{I, f}$ later as canonical ways of describing variants in the choices of elements of the groups when studying models of the sentence $\psi_{k}^{\lambda}$.

We call a $\tau$-structure strongly standard if $M \upharpoonright \tau^{-}=M_{I} \upharpoonright \tau^{-}$for $I=P_{0}^{M}$.

## Definition 1.11. (Some abstract classes related to $\psi_{k}^{\lambda}$ )

1. Let $K_{1}:=\left\{M \mid M \approx M_{I, f}\right.$ for some infinite set $I$, for some $f$ as in 1.10$\}$. Then $K_{1}$ is a class of $\tau$-models.
2. Let $\mathrm{K}^{*}(\lambda, \mathrm{k}):=\operatorname{Mod}\left(\psi_{\mathrm{k}}^{\lambda}\right)$ with the strong substructure relation

$$
\prec_{\mathrm{K}^{*}(\lambda, k)}:=\prec_{\mathrm{L}_{\left(2^{\lambda}\right)+}+\omega} .
$$

3. $M$ from $K^{*}(\lambda, k)$ is standard if $P_{1,1}^{M}=\left[P_{0}^{M}\right]^{k}$ and $P_{1,2}^{M}=\left[P_{0}^{M}\right]^{k+1}$ and the $\pi_{\ell}^{\mathrm{t}}$,s correspond to the actual projections sending $u \in[I]^{k}$ to its $\ell^{\prime}$ 'th coordinate in I.

Claim 1.12. For any $M_{I} \models \psi_{k}^{\lambda}, M_{I} \approx M_{I, 0}$, for the function $\mathbf{0}$ : $[I]^{k+1} \times S \rightarrow \mathbb{Z}_{2}$ of constant value 0 .

The proof is immediate from the definition.
Claim 1.13. Every $N \models \psi_{k}^{\lambda}$ is isomorphic to a strongly standard $M$.
Proof $\quad$ Let $N \models \psi_{k}^{\lambda}$ and let $I:=P_{0}^{N}$. Then $N \upharpoonright \tau^{-} \approx M_{I} \upharpoonright \tau^{-}$(following the definition of the sorts of the vocabulary $\tau^{-}$). Then define the interpretations of the relevant predicates $\mathrm{Q}_{s}$ on N by mapping directly from their definition on the strongly standard model $\mathrm{M}_{\mathrm{I}}$.
Next, a straightforward observation.
Claim 1.14. $M_{\mathrm{I}, \mathrm{f}}$ is strongly standard.
Proposition 1.15. $\left(\mathcal{K}^{*}(\lambda, k), \prec_{K^{*}(\lambda, k)}\right)$ is an abstract elementary class with LöwenheimSkolem number $2^{\lambda}$.

We do not investigate properties of this AEC in this paper; however, we propose some conjectures at the end of the paper on their properties and on their connection with good frames and the work of Boney and Vasey [10].

## 2. Categoricity of $\psi_{k}^{\lambda}$ below $\lambda^{+k}$

In this section we study the categoricity spectrum of $\psi_{k}^{\lambda}$. The strategy consists of the following steps:

- Since the complexity of models of $\psi_{k}^{\lambda}$ hinges on the predicates $Q_{s}$, and these ultimately depend on choices of elements of the copies of the groups above the "supports" (in the standard case, $k$-element subsets of ( $k+1$ )-sets of the index set), we will develop a language of choice functions to deal with these.
- Furthermore, comparing different models will amount to dealing with correction functions associated to the choice functions. We also set up a language for these.
- Later (Lemma 2.5) we establish that for every model $N$ of $\psi_{k}^{\lambda}$ and global choice for N with correction function f there are an index set I and an isomorphism $\mathbf{h}$ between $N$ and $M_{I, f}$.
- Therefore, establishing categoricity in a cardinality $\mathrm{\kappa}$ amounts to showing that for every $N \models \psi_{k}^{\lambda}$ of cardinality $\kappa$ there is a global choice for $N$ with correction function 0 (for $\kappa<\lambda^{+k}$; see Theorem 2.14).
- The rest of the section is devoted to showing that if $M \models \psi_{k}^{\lambda}$ and $|M|<\lambda^{+k}$ then there is a global choice function for $M$ with correction function $0-$ with the cardinality restriction in place, we may conclude that $\psi_{\mathrm{k}}^{\lambda}$ is categorical in $\lambda^{+}, \lambda^{++}, \ldots, \lambda^{+\mathrm{k}-1}$. This part requires several lemmas on extending choice functions while keeping the correction function 0 ; these lemmas depend crucially on the cardinality being of the form $\lambda^{+m}$ for $m$ a natural number below $k$. This is why the proof in this section only provides categoricity up to $\lambda^{+k-1}$.


### 2.1. Solutions, choices and correction functions

We will now define choice functions and correction functions. These will be used to study models of $\psi_{k}^{\lambda}$ of cardinality $\lambda^{+}, \ldots, \lambda^{+k-1}$.

Expanding choices from partial to global ones is the crucial issue.
Definition 2.1 (Partial $M$ - $\left(J_{0}, J_{1}, J_{2}\right)$-choice). For $M \models \psi_{k}^{\lambda}$, we say $(\bar{x}, \bar{y}, \bar{z})$ is a partial $\mathrm{M}-\left(\mathrm{J}_{0}, \mathrm{~J}_{1}, \mathrm{~J}_{2}\right)$-choice if
(a) $\mathrm{J}_{0}, \mathrm{~J}_{1} \subset \mathrm{P}_{1,1}^{\mathrm{M}}, \mathrm{J}_{2} \subset \mathrm{P}_{1,2}^{M}$,
(so, in the case of standard models, $\mathrm{J}_{0}, \mathrm{~J}_{1} \subset[\mathrm{I}]^{\mathrm{k}}, \mathrm{J}_{2} \subset[\mathrm{I}]^{\mathrm{k}+1}$ )
(b) $\bar{x}=\left\langle x_{u, s} \mid s \in S, u \in J_{0}\right\rangle$, where

$$
x_{u, s} \in\left(P_{3, s}^{M}\right)^{-1}(u) \subset P_{3, s}^{M} .
$$

(c) $\bar{y}=\left\langle y_{u, s} \mid s \in S, u \in J_{1}\right\rangle$,

$$
y_{u, s} \in H_{u, s}^{M}:=\left(P_{2, s}^{M}\right)^{-1}(u) \subset P_{2, s}^{M} .
$$

(d) $\bar{z}=\left\langle z_{u} \mid u \in J_{2}\right\rangle$,

$$
z_{\mathfrak{u}} \in \mathrm{G}_{\mathfrak{u}}^{M}:=\left(\mathrm{F}_{5}^{\mathrm{M}}\right)^{-1}(\mathfrak{u}) \subset \mathrm{P}_{5}^{\mathrm{M}} .
$$

Therefore $\bar{x}$ essentially chooses an element $i$ in the corresponding copy of $\mathbb{Z}_{2}, \bar{y}$ chooses a $h$ in the corresponding copy of $\mathrm{H}_{\mathrm{I}}, \bar{z}$ chooses a g in the corresponding copy of G , for each relevant ( $u, s$ ).

So, $\chi_{u, s}$ is some element in the 'fiber' of $u$ via $F_{3}^{M}$, and analogously for $\bar{y}$ and $\bar{z}$.
Definition 2.2. We call $(\bar{x}, \bar{y}, \bar{z})$ a partial M-J-choice if it is an $M-(J, J, J *)$-choice, where

$$
J_{*}^{M}:=\left\{\mathrm{a} \in \mathrm{P}_{1,2}^{\mathrm{M}} \mid \bigwedge_{\mathrm{m} \leq \mathrm{k}} \exists \mathrm{~b} \in \mathrm{~J}\left[\bigwedge_{\ell<\mathrm{m}}\left(\pi_{\ell}^{1}(\mathrm{a})=\pi_{\ell}^{0}(\mathrm{~b}) \wedge \bigwedge_{\ell \in[\mathrm{m}, \mathrm{k}[ } \pi_{\ell}^{0}(\mathrm{~b})=\pi_{\ell+1}^{1}(\mathrm{a})\right]\right\} .\right.
$$

Similarly, we say that $(\bar{x}, \bar{y}, \bar{z})$ is a global $M$-choice if it is a partial $M-P_{1,1}^{M}$-choice. We will sometimes just say "M-choice" (if clear from context).

The previous is a way of describing, in our language of projections, that (in the standard case) $J_{*}^{M}$ consists of the $k+1$-element sets such that all their ( $k+1$-many) $k$-element subsets are in J).

So, when $M$ is standard, we have that

$$
J_{*}^{\mathrm{M}}=\left\{\left\langle\mathfrak{a}_{\ell} \mid \ell \leq \mathrm{k}\right\rangle \mid \bigwedge_{\mathfrak{m} \leq \mathrm{k}}\left\langle\mathfrak{a}_{\ell} \mid \ell \neq \mathrm{m}\right\rangle \in \mathrm{J}\right\} .
$$

Definition 2.3. Fix a standard $M$ and a $M-\left(J_{0}, J_{1}, J_{2}\right)$-choice $(\bar{x}, \bar{y}, \bar{z})$. Then we let the correction function $f$ for $M$ and $(\bar{x}, \bar{y}, \bar{z})$ be the function such that

1. $\operatorname{Dom}(\mathrm{f})$ is the set of pairs $(\mathrm{u}, \mathrm{s})$ such that
$(\alpha) u=\left\langle a_{\ell} \mid \ell \leq k\right\rangle \in J_{2} \subset P_{1,2}^{M}$,
( $\beta$ ) if $u_{m}:=\left\langle a_{\ell} \mid \ell \leq k, \ell \neq m\right\rangle, u_{\ell} \in \mathrm{J}_{0}$ for $\ell<\mathrm{k}, \mathrm{u}_{\mathrm{k}} \in \mathrm{J}_{1} \subset \mathrm{P}_{1,1}^{M}$,
2. $\operatorname{rng}(f) \subset \mathbb{Z}_{2}$, and
3. (recall $\chi_{\mathfrak{u}_{\ell}, s}, y_{\mathfrak{u}_{k}, s}, z_{\mathfrak{u}_{k}}$ are from the choice)

$$
f(u, s)=0 \Leftrightarrow\left\langle a_{0}, \ldots, a_{k}, u_{0}, \ldots, u_{k}, x_{u_{0}, s}, \ldots, x_{u_{k-1}, s}, y_{u_{k}, s}, z_{u_{k}}\right\rangle \in Q_{s}^{M}
$$

The next claim is a general observation on correction functions and choices.
Claim 2.4. For every $M \in \operatorname{Mod}\left(\psi_{k}^{\lambda}\right)$, there is an $M$-choice $(\bar{x}, \bar{y}, \bar{z})$.
Proof Immediate: just construct the tuples. There the demands are on each choice separately. There are no demands connecting different choices.

The next lemma is a crucial step. It shows how to build possible isomorphisms from an arbitrary model $N$ of $\psi_{k}^{\lambda}$ to standard models $M_{\mathrm{I}, \mathrm{f}}$.

Lemma 2.5. Let $\mathrm{N} \in \operatorname{Mod}\left(\psi_{\mathrm{k}}^{\lambda}\right)$ and let $(\bar{x}, \bar{y}, \bar{z})$ be a global N -choice with correction function f . Then, there exist a set I and an isomorphism

$$
\mathbf{h}: N \rightarrow M_{\mathrm{I}, \mathrm{f}}
$$

Furthermore, the isomorphism behaves as follows on the global $N$-choice $(\bar{x}, \bar{y}, \bar{z})$ :

$$
\mathbf{h}\left(x_{u, s}\right)=\left(\mathbf{h}(u), s, 0_{\mathbb{Z}_{2}}\right), \quad \mathbf{h}\left(y_{u, s}\right)=\left(\mathbf{h}(u), s, 0_{H_{\mathrm{I}}}\right), \quad \mathbf{h}\left(z_{\mathfrak{u}}\right)=\left(\mathbf{h}(u), 0_{G}\right)
$$

Proof Let $N \models \psi_{k}^{\lambda}$, and fix a global $N$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function $f$. We build I and $\mathbf{h}$ as in the statement.

First, we extract the predicates for the model $M=M_{I, f}:$ let $I:=P_{0}^{N}$. Clearly, $P_{0}^{M}=$ $\mathrm{P}_{0}^{\mathrm{N}}$.

We now define $\mathbf{h}$, following the predicates of the domain of N (remember that the domain of N is the disjoint union

$$
P_{0}^{N} \cup P_{1,1}^{N} \cup P_{1,2}^{N} \cup P_{2}^{N} \cup P_{3}^{N} \cup P_{4}^{N} \cup P_{5}^{N}
$$

and the predicates $P_{2}^{N}$ and $P_{3}^{N}$ are each partitioned into classes $\left.P_{2, s}^{N}, P_{3, s}^{N}(s \in S)\right)$.

- $\mathbf{h}$ is the identity on $\mathrm{P}_{0}^{\mathrm{N}}=\mathrm{I}=\mathrm{P}_{0}^{\mathrm{M}}$.
- if $x \in P_{1,1}^{N}, \ell<k, \pi_{\ell}^{0}(x)=x_{\ell}\left(\in P_{0}^{N}\right)$, then $\mathbf{h}(x):=\left(\mathbf{h}\left(x_{0}\right), \ldots, \mathbf{h}\left(x_{k-1}\right)\right)$.
- similarly, if $x \in P_{1,2}^{N}, \ell<k+1, \pi_{\ell}^{1}(x)=x_{\ell}\left(\in P_{0}^{N}\right)$, then $\mathbf{h}(x):=\left(\mathbf{h}\left(x_{0}\right), \ldots, \mathbf{h}\left(x_{k}\right)\right)$.
- if $x \in P_{2, s}^{N}$ then $\mathbf{h}(x)=\left(\mathbf{h}\left(F_{2}^{N}(x)\right), s,-\right) \in[I]^{k} \times S \times H_{I}$. For now we only know the third coordinate must be an element of $\mathrm{H}_{\mathrm{I}}$. Also, as soon as we know the third coordinate of the image of one element $x_{0}$ of a fiber inside the predicate $P_{2, s}^{N}$, we also know the third coordinate for all other elements $x$ of that fiber: since the action given by $F_{4}^{N}$ is transitive (as encoded by $T_{0}$ ), there is some $h_{0} \in P_{4}^{N}$ such that $F_{4}^{N}\left(x_{0}, h_{0}\right)=x$. Then (if we also have a definition of $\mathbf{h}$ on elements of $\left.P_{4}^{N}\right)$, we have that $\mathbf{h}(\mathrm{x})=\mathbf{h}\left(\mathrm{F}_{4}^{\mathrm{N}}\left(\mathrm{x}_{0}, \mathrm{~h}_{0}\right)\right)=\mathrm{F}_{4}^{\mathrm{N}}\left(\mathbf{h}\left(\mathrm{x}_{0}\right), \mathbf{h}\left(\mathrm{h}_{0}\right)\right)$.
- Similarly, if $x \in P_{3, s}^{N}$, then $\mathbf{h}(x)=\left(\mathbf{h}\left(F_{3}^{N}(x)\right), s,-\right) \in[I]^{k} \times S \times \mathbb{Z}_{2}$ and just as before the value of $\mathbf{h}$ on one element of the fiber will determine the rest.
- And similarly, if $x \in P_{5}^{N}$, then $\mathbf{h}(x)=\left(F_{5}^{N}(x),-\right) \in[I]^{k+1} \times G$. Again, since $N \models \psi_{G}$, the action ("of $G$ ") encoded by the family of functions $F_{3, g^{*}}^{N}$ is transitive, and therefore knowing a second coordinate for one element of a fiber of $\mathrm{P}_{5}^{\mathrm{N}}$ implies knowing it for all elements of the corresponding fiber that predicate.

It therefore remains, in order to complete the definition, to make choices of images of elements of $\mathrm{P}_{4}^{\mathrm{N}}$ (images in $\mathrm{H}_{\mathrm{I}}$ - but this is easy, as $\mathrm{H}_{\mathrm{I}}$ is definable in our structure) and selecting, for each $s \in S$, one image in each one of the relevant fibers. We now use the correction function $f$ and the predicates $\mathrm{Q}_{s}$.

So fix $s \in S$. Checking the equivalence we are looking for, namely

$$
\begin{gathered}
Q_{s}^{N}\left(a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k} z\right) \\
\Uparrow \\
Q_{s}^{M_{I, f}}\left(\mathbf{h}\left(a_{0}\right) \ldots \mathbf{h}\left(a_{k}\right) \mathbf{h}\left(u_{0}\right) \ldots \mathbf{h}\left(u_{k}\right) \mathbf{h}\left(x_{0}\right) \ldots \mathbf{h}\left(x_{k-1}\right) \mathbf{h}\left(y_{k}\right) \mathbf{h}(z)\right)
\end{gathered}
$$

amounts to answering the question

$$
\begin{gathered}
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(u_{0}\right)+g(s)+f(u, s) \\
\hat{\mathbb{I}} ? \\
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathbf{h}\left(\mathfrak{i}_{\ell}\right)=h_{k}\left(\mathbf{h}\left(u_{0}\right)\right)+g(\mathbf{h}(s))+f(\mathbf{h}(u), \mathbf{h}(s))
\end{gathered}
$$

Now letting

$$
\left\{\begin{array}{l}
\mathbf{h}\left(x_{u, s}\right)=\left(\mathbf{h}(\mathfrak{u}), s, o_{\mathbb{Z}_{2}}\right), \\
\mathbf{h}\left(y_{u, s}\right)=\left(\mathbf{h}(u), s, o_{H}\right), \\
\mathbf{h}\left(z_{\mathfrak{u}}\right)=\left(\mathbf{h}(\mathfrak{u}), 0_{G}\right)
\end{array}\right.
$$

works for these equations: we are assigning 0 on the missing coordinates (third or second) - exactly to those elements of the fibers $\left(x_{u, s}, y_{u, s}, z(u)\right)$ that had already been picked by the choice function.
Why is this enough?
Well, our definition turns the equation (at the choices) into

$$
\mathbb{Z}_{2} \models 0=\sum_{\ell<k} 0=0(\star)+0(\star)+f(\star) .
$$

But, since f was a correction function for the choice function ( $\bar{x}, \bar{y}, \bar{z}$ ),

$$
f(u, s)=0 \Leftrightarrow\left\langle a_{0}, \ldots, a_{k}, u_{0}, \ldots, u_{k}, x_{u_{0}, s}, \ldots, x_{u_{k-1}, s}, y_{u_{k}, s}, z_{u_{k}}\right\rangle \in Q_{s}^{N}
$$

and therefore our definition of $\mathbf{h}$ works.
Definition 2.6 (Canonical choice). Fix $\mathrm{M}=\mathrm{M}_{\mathrm{I}, \mathrm{f}}$, and let $(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \bar{z})$ be the M -choice given by

$$
\begin{aligned}
x_{u, s} & =\left(u, s, 0_{\mathbb{Z}_{2}}\right), \\
y_{u, s} & =\left(u, s, 0_{H_{1}}\right), \\
z_{u} & =\left(u, 0_{G}\right) .
\end{aligned}
$$

This is by definition the canonical M-choice.
Claim 2.7. 1. If $(\bar{x}, \bar{y}, \bar{z})$ is a global $M$-choice, $M \models \psi_{k}^{\lambda}$, and $f$ is the $M$-correction function for ( $\bar{x}, \bar{y}, \bar{z}$ ), and f is identically zero, then $\mathrm{M} \approx \mathrm{M}_{\mathrm{I}}$ for some I .
2. If f above is zero on $\mathrm{P}_{1,1}^{\mathrm{M}}, \mathrm{P}_{1,2}^{M}$ and $\mathrm{f}=\mathrm{f}^{\prime} \upharpoonright \mathrm{J}_{2} \times \mathrm{S}$, then $\mathrm{M} \approx \mathrm{M}_{\mathrm{P}_{1}, \mathrm{f}^{\prime}}$.

Proof Part (1) is a consequence of 2.5 . Part (2) is clear.
Corollary 2.8. The correction function for $\mathrm{M}_{\mathrm{I}, \mathrm{f}}$ with the canonical M -choice $(\bar{x}, \bar{y}, \bar{z})$ is $f$.
Proof Similar to the previous: add zeroes to $f$ as in 2.7 .

### 2.2. Models of cardinality below $\lambda^{+\mathrm{k}}$

The rest of the section contains several extension lemmas for models of $\psi_{k}^{\lambda}$ of cardinalities $\lambda, \lambda^{+}$, etc.: the crucial issue is to build a choice function with null correction function. This may be started first at cardinality $\lambda$, and then pushed up. But each step up exacts an "amalgam of choices" possible only up to cardinality $\lambda^{+k-1}$.

The next lemma is the first step in the categoricity proof. It provides a specific kind of extension of choice: from an $M$-J-choice with correction function zero to a global Mchoice with correction function zero when J consists of k -subsets of the "support part" of $M$ that omit some fixed set $W$ of at most $k$-many elements. Also, it is worth stressing the lemma is about standard $M$.

For instance, when $m=2<k=3$, the lemma would mean we start with a choice function ( $\bar{x}, \bar{y}, \bar{z}$ ) for all "triangles" and "tetrahedra" omitting some fixed pair $\{a, b\} \ldots$ and then would extend the choice function (with correction function zero) to all triangles and tetrahedra.

Lemma 2.9. [Extension property for W of size $\mathrm{m}<\mathrm{k},\left|\mathrm{P}_{0}^{\mathrm{M}}\right| \leq \lambda$ ]
Assume $\mathrm{m}<\mathrm{k}, \mathrm{M} \vDash \psi_{\mathrm{k}}^{\lambda}, M$ is strongly standard, $\left|\mathrm{P}_{0}^{M}\right| \leq \lambda, W \subset P_{0}^{M}, W=\left\{\mathrm{b}_{\ell} \mid \ell<\mathrm{m}\right\}$ with no repetition, $\mathrm{J}=\left\{\mathrm{u} \in \mathrm{P}_{1,1}^{\mathrm{M}} \mid \mathrm{W} \not \subset \mathfrak{u}\right\}$ (note that $u \in\left[\mathrm{P}_{0}^{\mathrm{M}}\right]^{\mathrm{k}}$, as M is standard), $(\bar{x}, \bar{y}, \bar{z})$ is an M-J-choice with correction function $\mathrm{f}_{0}$, identically zero. Then, we can extend ( $\bar{x}, \bar{y}, \bar{z}$ ) to an M-choice with correction function identically zero.

## Proof

Part A: Without loss of generality, by 1.13 , since $M$ is strongly standard, $I=P_{0}^{M}$. Let $\left\langle\overline{\mathrm{a}}^{\alpha} \mid \alpha<\beta^{*}\right\rangle$ list $\mathrm{P}_{1,1}^{\mathrm{M}}$ with $\left\langle\overline{\mathrm{a}}^{\alpha} \mid \alpha<\alpha^{*}\right\rangle$ listing $J$ (we have also used $u$ for naming these $\bar{a}^{\alpha}$ s). Let $\left\langle\bar{b}^{\gamma} \mid \gamma<\gamma^{*}\right\rangle$ list $\left\{\bar{a} \in[I]^{k+1} \mid \bar{a}\right.$ with no repetition and $\left.W \subset \operatorname{rng}(\bar{a})\right\}$ and $\gamma^{*}<\lambda^{+}$.
Our hypothesis is then that we have choice functions for all $u \in P_{1,1}^{M}$ such that $u \not \supset W$, with correction function zero.

We list these choice functions as follows: Let, for $\alpha<\alpha^{*}$,

$$
\begin{aligned}
& x_{\bar{a}^{\alpha}, s}=\left(\bar{a}^{\alpha}, s, i_{\alpha, s}\right) \in\left(\mathbb{Z}_{2}\right)_{\bar{a}^{\alpha}, s}, i_{\alpha, s} \in \mathbb{Z}_{2}, \\
& y_{\bar{a}^{\alpha}, s}=\left(\overline{\mathrm{a}}^{\alpha}, s, h_{\alpha, s}\right) \in H_{\bar{a}^{\alpha}, s}, h_{\alpha, s} \in H_{I}, \\
& z_{\bar{b}^{\gamma}}=\left(\bar{b}^{\gamma}, g^{\gamma}\right), g^{\gamma} \in G .
\end{aligned}
$$

We now have to extend these choice functions to those $u$ such that $u \supset W$.
We will now choose ${\overline{\bar{a}^{\alpha}}, s}=\left(\overline{\mathrm{a}}^{\alpha}, s, i_{\alpha, s}\right)$, $y_{\bar{a}^{\alpha}, s}=\left(\overline{\mathrm{a}}^{\alpha}, s, h_{\alpha, s}\right), z_{\bar{b} \gamma}=\left(\bar{b}^{\gamma}, g^{\gamma}\right)$ for $\alpha^{*} \leq \alpha<\beta^{*}$ and appropriate $\gamma$.
Without loss of generality, $\beta^{*} \leq \alpha^{*}+\lambda, \gamma^{*} \leq \lambda$. (Remember $S=[\lambda]^{<\chi_{0}}$.)
Part B: First, we choose $i_{\alpha, s}=0_{\mathbb{Z}_{2}}$ for $\alpha^{*} \leq \alpha<\beta^{*}, s \in S$. This provides the choices $x_{\overline{\mathrm{a}}}$, s ${ }^{\text {for }} \alpha^{*} \leq \alpha<\beta^{*}$.
Second, we choose the relevant $h$ functions. We try a value for $h_{\alpha, s}$ for $\alpha^{*} \leq \alpha<$ $\beta^{*}$ and $s \in S$ so that
(*) if $\gamma \in s \subset \lambda, \bar{b}^{\gamma}=\left\langle b_{\ell}^{\gamma} \mid \ell \leq k\right\rangle, u_{n}^{\gamma}=\left\langle b_{\ell}^{\gamma} \mid \ell \leq k, \ell \neq n\right\rangle$, let $\varepsilon(\gamma, n)<\beta^{*}$ be such that $\mathbf{u}_{n}^{\gamma}=\bar{a}^{\varepsilon(\gamma, n)}$ then

$$
\begin{gathered}
h_{\varepsilon(\gamma, k), s}\left(\bar{a}^{\varepsilon(\gamma, 0)}\right)=0 \\
\hat{\mathbb{I}} \\
\left\langle\mathbf{b}_{0}^{\gamma}, \ldots, \mathrm{b}_{\mathrm{k}}^{\gamma}, u_{0}^{\gamma}, \ldots, u_{k}^{\gamma}, \chi_{\varepsilon(\gamma, 0), s}, \ldots, \chi_{\varepsilon(\gamma, k-1), s},\left(\overline{\mathrm{a}}^{\varepsilon(\gamma, k)}, s, 0_{H}\right),\left(u^{\gamma}, 0_{G}\right)\right\rangle \in \mathrm{Q}_{s}^{M} .
\end{gathered}
$$

Note that all the elements in the bottom part of the previous have already been defined previously.
Let $t(\gamma, s)$ be 0 if the bottom statement is true, 1 otherwise. For our fixed $s \in S$, let $A_{s}$ be the (finite) set $\{\varepsilon(\gamma, k) \mid \gamma \in s\}$; we now define $h_{\alpha, s}$ for our fixed $s$ and at the relevant $u$. If $\alpha \notin A_{s}$, then let $h_{\alpha, s}(u)=0$ for all $u$. If $\alpha \in A_{s}$, we proceed as follows. First we consider the set $s_{\alpha}:=\{\gamma \in s \mid \varepsilon(\gamma, k)=\alpha\}$ and we then define

$$
h_{\alpha, s}(u)= \begin{cases}t(\gamma, s), & \text { if } u=\mathfrak{a}^{\varepsilon(\gamma, 0)} \text { for some } \gamma \in s_{\alpha} \\ 0, & \text { otherwise }\end{cases}
$$

Notice that these decisions are made for each $s$ separately, and that as we fix $s$ we really deal with one $\alpha \in\left[\alpha^{*}, \beta\right)$ : when we choose $h_{\alpha, s}$ we only have to consider $\gamma<\gamma^{*}$ such that $\varepsilon(\gamma, \ell) \in s$. There are only finitely many such $\gamma$ 's. Moreover, if $\gamma_{1} \neq \gamma_{2} \in s$ and $\varepsilon\left(\gamma_{1}, k\right)=\alpha=\varepsilon\left(\gamma_{2}, k\right)$ then necessarily $\varepsilon\left(\gamma_{1}, 0\right) \neq \varepsilon\left(\gamma_{2}, 0\right)$, as $\bar{b}^{\gamma}$ is reconstructible from $\alpha$ and $\varepsilon\left(\gamma_{1}, 0\right)$.
So, our definition of the functions $h_{\varepsilon(\gamma, \mathrm{k}), s}$ does not have contradictory demands; since the set $s_{\alpha}$ is finite, the function defined has finite support.

Part C: Having extended the choices $x$ and $y$, it only remains to extend the $z$ part. Let us now fix $\gamma$ and find a $g \in G$ that will provide a choice (with correction function zero) for the corresponding $\bar{b}^{\gamma}$. [Recall that if $\overline{\mathrm{b}} \in[\mathrm{I}]^{\mathrm{k}+1}$ is such that $\overline{\mathrm{b}} \supset W$, then $\overline{\mathrm{b}}=\overline{\mathrm{b}}^{\gamma}$ for some $\gamma<\gamma^{*}$.]
But then the set

$$
\begin{aligned}
& \mathrm{S}_{\gamma}^{*}=\left\{s \in S \mid M \models \mathrm{Q}_{s}\left(\mathrm{~b}_{0}^{\gamma}, \ldots, \mathrm{b}_{\mathrm{k}}^{\gamma}, \overline{\mathrm{a}}^{\epsilon(\gamma, 0)}, \ldots, \overline{\mathrm{a}}^{\epsilon(\gamma, k)},\right.\right. \\
&\left.\left.x_{u_{0}^{\gamma}, s}, \ldots, x_{u_{k-1}^{\gamma}, s}, \mathrm{y}_{u_{k}^{\gamma}, s},\left(u^{\gamma}, 0_{G}\right)\right)\right\}
\end{aligned}
$$

belongs to $\mathfrak{D}$. This last point holds by the regularity of $\mathfrak{D}$ : if $s_{0} \in S_{\gamma}^{*}$ then the tuple

$$
\left(\mathrm{b}_{0}^{\gamma}, \ldots, \mathrm{b}_{\mathrm{k}}^{\gamma}, \bar{a}^{\epsilon(\gamma, 0)}, \ldots, \overline{\mathrm{a}}^{\epsilon(\gamma, k)}, x_{u_{0}^{\gamma}, s_{0}}, \ldots, x_{u_{k-1}^{\gamma}, s_{0}}, y_{u_{k}^{\gamma}, s_{0}},\left(u^{\gamma}, o_{G}\right)\right)
$$

belongs to $\mathrm{Q}_{s_{0}}$; now, if $s \supset s_{0}$, the corresponding tuple

$$
\left(\mathrm{b}_{0}^{\gamma}, \ldots, \mathrm{b}_{\mathrm{k}}^{\gamma}, \overline{\mathrm{a}}^{\epsilon(\gamma, 0)}, \ldots, \overline{\mathrm{a}}^{\epsilon(\gamma, k)}, x_{u_{0}^{\gamma}, s}, \ldots, x_{u_{\mathrm{k}-1}^{\gamma}, s}, y_{\mathfrak{u}_{k}^{\gamma}, s},\left(\mathfrak{u}^{\gamma}, o_{G}\right)\right)
$$

will belong to $\mathrm{Q}_{\mathrm{s}}$.
Next choose $z_{\bar{b} \gamma}:=\left(\bar{b}^{\gamma}, \mathrm{g}\right)$ with g given by

$$
g(s)= \begin{cases}0 & \text { if } s \in S_{\gamma}^{*} \\ 1 & \text { if } s \notin S_{\gamma}^{*}\end{cases}
$$

Now then, with these $x, y$ and $z$, the equation holds.

We now deal with systems of choices, trying to obtain extensions with correction function zero at cardinalities above $\lambda$. In what follows, as usual, $\mathcal{P}^{-}\left(m_{2}\right)$ denotes $\mathcal{P}\left(m_{2}\right) \backslash$ $\left\{\mathrm{m}_{2}\right\}$.

Definition 2.10 (Compatible system of choices). Let $M \models \psi_{k}^{\lambda}$ be strongly standard, $A_{\emptyset} \subset$ $P_{0}^{M}, m_{1}+m_{2}<k$ and $a_{0}, \ldots, a_{m_{2}-1}$ different elements of $P_{0}^{M} \backslash A_{\emptyset}$. Then

$$
\left\langle A_{s},(\bar{x}, \bar{y}, \bar{z})_{s} \mid s \in \mathcal{P}^{-}\left(m_{2}\right)\right\rangle
$$

is a compatible $\lambda^{+m_{1}}-\mathcal{P}^{-}\left(m_{2}\right)$-system of choices iff

1. $\cup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}=A_{\emptyset} \cup\left\{a_{0}, \ldots, a_{m_{2}-1}\right\},\left|A_{\emptyset}\right| \leq \lambda^{+m_{1}}, A_{s}=A_{\emptyset} \cup\left\{a_{t} \mid t \in s\right\}$.
2. $(\bar{x}, \bar{y}, \bar{z})_{s}$ is a $M-\left[A_{s}\right]^{\mathrm{k}}$-choice, for each $\mathrm{s} \in \mathcal{P}^{-}\left(\mathrm{m}_{2}\right)$.
3. For every $s, t \in \mathcal{P}^{-}\left(m_{2}\right), s \subset t \Rightarrow(\bar{x}, \bar{y}, \bar{z})_{s} \subset(\bar{x}, \bar{y}, \bar{z})_{t}{ }^{4}$.

Lemma 2.11. If $\left\langle A_{s},(\bar{x}, \bar{y}, \bar{z})_{s} \mid s \in \mathcal{P}^{-}\left(m_{2}\right)\right\rangle$ is a compatible $\lambda$ - $\mathcal{P}^{-}\left(\mathrm{m}_{2}\right)$-system with $\mathrm{m}_{2}<$ $k$ (with correction function zero for each $s \in \mathcal{P}^{-}\left(m_{2}\right)$ ), then there is an $M-\bigcup_{s \in \mathcal{P}^{-}\left(m_{2}\right)} A_{s^{-}}$ choice $(\bar{x}, \bar{y}, \bar{z})$ extending all the $(\bar{x}, \bar{y}, \bar{z})_{s}$, for $s \in \mathcal{P}^{-}\left(m_{2}\right)$, with correction function zero.

Proof Let $m_{2}<k$ and let $\left\langle A_{s},(\bar{x}, \bar{y}, \bar{z})_{s} \mid s \in \mathcal{P}^{-}\left(m_{2}\right)\right\rangle$ be a compatible $\lambda-\mathcal{P}^{-}\left(m_{2}\right)-$ system, each choice in the system with correction function zero. Notice that

$$
u \in\left[\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}\right]^{k} \backslash \bigcup_{s \in \mathcal{P}^{-}\left(m_{2}\right)}\left[A_{s}\right]^{k},
$$

if and only if $\left\{a_{0} \ldots a_{m_{2}-1}\right\} \subset u$.
We first notice that by compatibility, the union of the choices $(\bar{x}, \bar{y}, \bar{z})_{s}$ along $\mathcal{P}^{-}\left(m_{2}\right)$ is an $M$-choice for $\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)}\left[A_{s}\right]^{k}$. It remains to extend that choice to an $M-\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}-$ choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero.
We may apply Lemma 2.9 (here, the set $W$ of cardinality $m_{2}<k$ is $\left\{a_{0}, \ldots, a_{\mathfrak{m}_{2}-1}\right\}$ and the lemma provides the extension from a $M-\bigcup_{s \in \mathcal{P}-\left(\mathfrak{m}_{2}\right)}\left[\mathcal{A}_{s}\right]^{\mathrm{k}}$-choice with correction function zero to a $M-\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero - we extend the choice from those $k$-sets omitting $W$ to all of them).

Lemma 2.12. Let $\mathfrak{m}_{1}+\mathfrak{m}_{2}<k$. If $\left\langle A_{s},(\bar{x}, \bar{y}, \bar{z})_{s} \mid s \in \mathcal{P}^{-}\left(\mathfrak{m}_{2}\right)\right\rangle$ is a compatible $\lambda^{+m_{1}}-$ $\mathcal{P}^{-}\left(m_{2}\right)$-system of choices with correction function zero, then there is $a \bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}$-choice $(\bar{x}, \bar{y}, \bar{z})$ with correction function zero such that $(\bar{x}, \bar{y}, \bar{z})_{s} \subset(\bar{x}, \bar{y}, \bar{z})$ for every $s \in \mathcal{P}^{-}\left(m_{2}\right)$.

[^1]Proof By induction on $m_{1}$. For $m_{1}=0$, this is lemma 2.11. For $m_{1}>0$, suppose $A_{s}=A_{\emptyset} \cup\left\{b_{j} \mid j \in s\right\}$. Enumerate $A_{\emptyset}$ as $\left\langle a_{\beta} \mid \beta<\lambda^{+m_{1}}\right\rangle$. Let $A_{\emptyset}^{\alpha}=\left\{a_{\beta} \mid \beta<\alpha\right\}$ and $A_{s}^{\alpha}=A_{\emptyset}^{\alpha} \cup\left\{b_{j} \mid j \in s\right\}$ for every $s \in \mathcal{P}^{-}\left(m_{2}\right)$. Finally, let $(\bar{x}, \bar{y}, \bar{z})_{s}^{\alpha}$ be the restriction of the choice we have $(\bar{x}, \bar{y}, \bar{z})_{s}$ from the compatible system (with correction function zero) to an $M-A_{s}^{\alpha}$-choice (also immediately with correction function zero).

The plan is to obtain an $M-\bigcup_{s \in \mathfrak{m}_{2}} A_{s}^{\alpha}$-choice $(\bar{x}, \bar{y}, \bar{z})_{\alpha}$ with correction function zero for each $\alpha<\lambda^{+m_{1}}$, such that $\alpha<\beta<\lambda^{+m_{1}}$ implies $(\bar{x}, \bar{y}, \bar{z})_{\alpha} \subset(\bar{x}, \bar{y}, \bar{z})_{\beta}$.

We build $(\bar{x}, \bar{y}, \bar{z})_{\alpha}$ by another induction, on $\alpha<\lambda^{+m_{1}}$. For $\alpha=0$, the empty choice function is an $M-\emptyset$-choice $\left(A_{s}^{0}=\emptyset\right.$ for each $\left.s\right)$. When $\alpha$ is a limit ordinal, the union of the chain of choices $\left((\bar{x}, \bar{y}, \bar{z})_{\beta}\right)_{\beta<\alpha}$ is a $M-\bigcup_{s \in \mathfrak{m}_{2}} A_{s}^{\alpha}$-choice with correction function zero. Finally, for $\alpha=\beta+1$, we proceed as follows: we already have, by induction hypothesis, an $M-\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}^{\beta}$-choice with correction function zero, $(\bar{x}, \bar{y}, \bar{z})_{\beta}$; consider also the choices $(\bar{x}, \bar{y}, \bar{z})_{s}^{\alpha}$ for $s \in \mathcal{P}^{-}\left(m_{2}\right)$. Since the cardinalities of all their domains are $<\lambda^{+m_{1}}$, we may without loss of generality regard the previous choices as forming a compatible $\lambda^{+\mathfrak{m}_{1}-1}-\mathcal{P}^{-}\left(\mathfrak{m}_{2}+1\right)$-system of choices with correction function zero: the set $\left\{b_{i} \mid i \in s\right\} \cup$ $\{\beta\}$ has cardinality $m_{2}+1$. Since $\left(m_{1}-1\right)+\left(m_{2}+1\right)=m_{1}+m_{2}<k$, we may apply the induction hypothesis; we obtain $(\bar{x}, \bar{y}, \bar{z})_{\alpha}$ an $M-\bigcup_{s \in \mathcal{P}-\left(m_{2}\right)} A_{s}^{\alpha}$-choice with correction function zero.

Having constructed this chain $(\bar{x}, \bar{y}, \bar{z})_{\alpha}$ for $\alpha<\lambda^{+m_{1}}$, we just let

$$
(\bar{x}, \bar{y}, \bar{z}):=\bigcup_{\alpha<\lambda^{+m_{1}}}(\bar{x}, \bar{y}, \bar{z})_{\alpha} .
$$

This is a $\bigcup_{s \in \mathcal{P}^{-}\left(\mathfrak{m}_{2}\right)}$-choice with correction function zero, extending all the choices in the system.

We may now obtain our general extension property.
Lemma 2.13. (Full extension)
Let $\mathrm{M} \models \psi$ be strongly canonical, $\mathrm{J}_{1} \subset \mathrm{~J}_{2} \subset \mathrm{P}_{0}^{\mathrm{M}}$, with $\left|\mathrm{J}_{2}\right|<\lambda^{+\mathrm{k}-1}$ and $(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \bar{z})$ an M- $\mathrm{J}_{1}$-choice with correction function identically zero. Then $(\bar{x}, \bar{y}, \bar{z})$ can be extended to an $\mathrm{M}-\mathrm{J}_{2}$-choice with correction function identically zero.
Proof Without loss of generality, $\mathrm{J}_{2}=\mathrm{J}_{1} \cup\{\mathrm{~b}\}$. If $\mathrm{J}_{1}$ has size $\leq \lambda$, this is lemma 2.9. Now suppose $\left|J_{1}\right|=\lambda^{+m_{1}}<\lambda^{+k-1}$ (therefore $m_{1}<k-1$ ) and enumerate $J_{1}$ as $\left\langle a_{\beta} \mid \beta<\lambda^{+m_{1}}\right\rangle$. Let $J_{1}^{\alpha}=\left\{a_{\beta} \mid \beta<\alpha\right\}$, and let $(\bar{x}, \bar{y}, \bar{z})_{\alpha}$ be the restriction of $(\bar{x}, \bar{y}, \bar{z})$ to an $M-J_{1}^{\alpha}$-choice. We define by induction $M-J_{1}^{\alpha}$ choices with correction function identically zero $(\bar{x}, \bar{y}, \bar{z})_{\alpha}^{\prime} \supset(\bar{x}, \bar{y}, \bar{z})_{\alpha}$.

We may use here lemma 2.12 for $m_{2}=2$ to extend $(\bar{x}, \bar{y}, \bar{z})_{\alpha}^{\prime} \cup(\bar{x}, \bar{y}, \bar{z})_{\alpha+1}$ to an M$J_{1}^{\alpha+1} \cup\{b\}$-choice with correction function identically zero: since $m_{1}<k-1$, along the induction the cardinality is $<\lambda^{+m_{1}}$, say $\lambda^{+m_{1}^{\prime}}$ for some $\mathfrak{m}_{1}^{\prime}<\mathfrak{m}_{1}$. Since we also have that $\mathrm{m}_{1}<\mathrm{k}-1$, then $\mathrm{m}_{1}^{\prime}+2<\mathrm{k}$ and we can use $\mathrm{m}_{2}=2$ when invoking lemma 2.12.

At limits take unions; finally,

$$
\left(\bigcup_{\alpha<\lambda^{+m_{1}}} \bar{x}_{\alpha}^{\prime}, \bigcup_{\alpha<\lambda^{+m_{1}}} \bar{y}_{\alpha}^{\prime}, \bigcup_{\alpha<\lambda^{+m_{1}}} \bar{z}_{\alpha}^{\prime}\right)
$$

is an $M$ - $J_{2}$-solution extending $(\bar{x}, \bar{y}, \bar{z})$.
Theorem 2.14. If $M \models \psi_{k}^{\lambda}$ is strongly standard and $|M|<\lambda^{+k}$ then there is an $M$-choice with correction function identically zero.

Proof We apply Lemma 2.13 (starting from the empty choice function, and taking unions at limits): the lemma gives an extension of a choice function with correction function zero from $\mathrm{J}_{1} \subset \mathrm{P}_{1,1}^{M}$ to $\mathrm{J}_{2}$ with $\mathrm{J}_{1} \subset \mathrm{~J}_{2} \subset \mathrm{P}_{1,1}^{M}$ provided $\left|\mathrm{J}_{2}\right|<\lambda^{+\mathrm{k}-1}$. Here $|M|$ may be equal to $\lambda^{+k-1}$ (at "worst"); if (in that case) we enumerate $P_{1,1}^{M}$ as $\left\{a_{\beta} \mid \beta<\lambda^{+k-1}\right\}$ then given $\alpha<\lambda^{+\mathrm{k}-1},\left|\left\{\mathrm{a}_{\beta} \mid \beta<\alpha\right\}\right|<\lambda^{+\mathrm{k}-1}$ and we can apply Lemma 2.13 to get an extension of the choice with correction function zero to $\left\{a_{\beta} \mid \beta<\alpha\right\}$.

Theorem 2.15. (Categoricity and amalgamation up to $\lambda^{+\mathrm{k}}$ )

1. For $\mathrm{m}<\mathrm{k}$, Mod $\left(\psi_{k}^{\lambda}\right)$ has a unique strongly standard model $\mathrm{M},\left|\mathrm{P}_{0}^{\mathrm{M}}\right|=\lambda^{+\mathrm{m}}$, modulo isomorphism.
2. For $\mathrm{m}<\mathrm{k}-1$, if $2^{\lambda} \leq \lambda^{+\mathrm{m}}$, then $\mathcal{K}^{*}(\lambda, k)$ has amalgamation in $\lambda^{+\mathrm{m}}$.
3. If $m<k, \lambda^{+m} \geq 2^{\lambda}$, then $\mathcal{K}^{*}(\lambda, k)$ is categorical in $\lambda^{+m}$.

## Proof

1. Let $N \models \psi_{k}^{\lambda}$ be a strongly standard model with $\lambda \leq\left|P_{0}^{N}\right|<\lambda^{+k}$. By Lemma 2.5, once we have ( $\bar{x}, \bar{y}, \bar{z}$ ) a global $N$-choice with correction function $f$, then $N \approx M_{I, f}$ for $I=P_{0}^{M}$. Now, since $N$ is standard and $\left|P_{0}^{N}\right| \in\left[\lambda, \lambda^{+k-1}\right]$, Theorem 2.14 gives a global N -choice ( $\overline{\mathrm{x}}, \bar{y}, \bar{z}$ ) with correction function identically zero (as N is strongly standard). So, $N \approx M_{I}$.
2. In the proofs of the previous lemmas, amalgamation of choices (along systems) with correction function zero is carried out in detail. These give rise to the corresponding embeddings and amalgams of models, if the size of these is controlled by the size of their $P_{0}^{M}$ parts. The only part of a standard model where this cardinality may increase is given by the coding of the action of $G$ (remember $|\mathrm{G}|=2^{\lambda}$ ). If $2^{\lambda} \leq \lambda^{+m}$, the model $M$ will have the same size as $P_{0}^{M}$.
3. Let $\mathfrak{m}<k, \lambda^{+m} \geq 2^{\lambda}$ and let $M$ be a model in $\mathscr{K}^{*}(\lambda, k)$ of size $\lambda^{+k}$. Then by Lemma $1.13, M$ is isomorphic to a strongly standard model $N$; also, since $2^{\lambda} \leq \lambda^{+m}$, $\left|P_{0}^{M}\right|=\lambda^{+m}$. Thus by part (1) $M \approx N \approx M_{I}$ for some I of size $\lambda^{+m}$.

## 3. Failure of categoricity of $\psi_{k}^{\lambda}$ at $\beth_{k+1}(\lambda)$

We have proved in 2.15 that $\psi$ is categorical in $\lambda^{+m}$ if $m<k$ and $2^{\lambda}<\lambda^{+m}$. We now prove that our sentence is not categorical in any cardinality $k \geq \mu=\beth_{\mathrm{k}+1}(\lambda)^{+}$. (It is also possible to show that $\psi_{\mathrm{k}}^{\lambda}$ has the maximal number of models possible in $\mu$ for each $\mu \geq \beth_{\mathrm{k}+1}(\lambda)^{+}$. We do not do that in this paper.)

As before, we use our terminology of "solutions and corrections functions" to count the number of models.

### 3.1. Combinatorial criteria for (failure of) isomorphism

In this section we prove a combinatorial criterion for non-isomorphism between two models of the form $M_{I, f}$.

Before giving the purely combinatorial criterion, we prove the following lemma (a criterion for isomorphism in terms of choices and correction functions).

Lemma 3.1. If $M_{1}$ and $M_{2}$ are strongly standard, and $(\bar{x}, \bar{y}, \bar{z})_{\ell}$ is an $M_{\ell}$-choice for $M_{\ell}$ $(\ell=1,2), P_{0}^{M_{1}}=P_{0}^{M_{2}}$ with correction function $f_{\ell}$ for $\ell=1,2$ then the following are equivalent:
(a) there is an isomorphism from $M_{1}$ onto $M_{2}$ over the identity on $P_{0}^{M_{1}} \cup P_{1}^{M_{1}}$
(b) $)_{1}$ there is an $M_{2}$-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is $f_{1}$,
$(\mathbf{b})_{2}$ there is an $M_{1}$-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is $f_{2}$,
(c) there are functions $g_{1}, g_{2}, g_{3}$ ("to correct the choice of zeros"), with

1. $\mathrm{g}_{1}:[\mathrm{I}]^{\mathrm{k}} \times \mathrm{S} \rightarrow \mathbb{Z}_{2}$ (like the $\mathrm{x}_{\mathrm{u}, \mathrm{s}}$ 's above),
2. $\mathrm{g}_{2}:[\mathrm{I}]^{\mathrm{k}} \times \mathrm{S} \rightarrow \mathrm{H}_{\mathrm{I}}$ (like the $\mathrm{y}_{\mathrm{u}, \mathrm{s}}$ 's above),
3. $\mathrm{g}_{3}:[\mathrm{I}]^{\mathrm{k}+1} \rightarrow \mathrm{G}$ (like the $z_{\mathrm{u}}$ 's above),
4. if $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k}, z\right\rangle$ are like in Definition 1.1.1 for $M_{1}$, or $M_{2}$ then

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} i_{\ell}-h_{k}\left(u_{0}\right)-g(s)=\sum_{\ell<k} g_{1}\left(u_{\ell}, s\right)-g_{2}\left(u_{k}, s\right)\left(u_{0}\right)-g_{3}(u)(s)
$$

Proof
(a) $\rightarrow \mathbf{( b )})_{1}$ Recall that $M_{1} \upharpoonright \tau^{-}=M_{2} \upharpoonright \tau^{-}$, so $M_{1}$ and $M_{2}$ have the same universes. Fix $F: M_{1} \xrightarrow[P_{0}^{M_{1}} \cup P_{1}^{M_{1}}]{\approx} M_{2}$. We have, since $f_{1}$ is a correction function for $M$ for the choice $(\bar{x}, \bar{y}, \bar{z})_{1}$, that

$$
\mathrm{f}_{1}(\mathrm{u}, \mathrm{~s})=0 \Leftrightarrow\left\langle\mathrm{a}_{0} \ldots \mathrm{a}_{\mathrm{k}} \mathrm{u}_{0} \ldots \mathrm{u}_{\mathrm{k}} x_{\mathfrak{u}_{0} s}^{1} \ldots x_{\mathfrak{u}_{k-1} s}^{1} \mathrm{y}_{\mathfrak{u}_{\mathrm{k}} \mathrm{~s}}^{1} z_{\mathfrak{u}}^{1}\right\rangle \in \mathrm{Q}_{\mathrm{s}}^{M_{1}}
$$

But the right hand side holds iff

$$
\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} F\left(x_{u_{0} s}^{1}\right) \ldots F\left(x_{u_{k-1} s}^{1}\right) F\left(y_{u_{k} s}^{1}\right) F\left(z^{1} u\right)\right\rangle \in Q_{s}^{M_{2}}
$$

since $F$ is an isomorphism fixing $P_{0}^{M_{1}} \cup P_{1}^{M_{1}}$, and $a_{0}, \ldots, a_{k} \in P_{0}^{M_{1}}$. This gives us the $M_{2}$-choice for which $f_{1}$ is a correction function: given $\mathfrak{u}_{\ell} \subset \mathfrak{u}, \mathfrak{u}_{\ell} \in[I]^{k}$, $u \in[I]^{k+1}$, let $x_{\mathfrak{u}_{\ell}, s}^{\prime}=F\left(x_{\mathfrak{u}_{\ell}, s}^{1}\right), y_{u, s}^{\prime}=F\left(y_{\mathfrak{u}_{k}, s}^{1}\right), z_{u}^{\prime}=F\left(z_{\mathfrak{u}}^{1}\right)$.
(a) $\rightarrow$ (b) $)_{2}$ Same.
$(\mathbf{b})_{\ell} \rightarrow \mathbf{( c )}(\ell=1,2)$ The point of $(\mathrm{c})$ is that we may find concrete representations $\mathrm{g}_{1}, \mathrm{~g}_{2}, \mathrm{~g}_{3}$, that act independently from $M_{1}$ or $M_{2}$ as 'corrected choice functions' for the zeros for $f_{1}$ and $f_{2}$. So, suppose we have a $M_{2}$-choice ( $\bar{x}, \bar{y}, \bar{z}$ ) with correction function $f_{1}$. Then for any $u \in P_{0}^{M_{2}}$ and any $s \in S$, if $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k}, z\right\rangle$ are like in Definition 1.1.1

$$
\begin{gathered}
\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{u_{0} s} \ldots x_{\mathfrak{u}_{k-1} s} y_{u_{k} s} z_{\mathfrak{u}}\right\rangle \in Q_{s}^{M_{2}} \\
\mathbb{\imath} \\
f_{1}(u, s)=0 .
\end{gathered}
$$

But since $f_{1}$ is also a correction function for the $M_{1}$-choice $(\bar{x}, \bar{y}, \bar{z})_{1}$,

$$
\begin{gathered}
f_{1}(u, s)=0 \\
\hat{\mathbb{}} \\
\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{u_{0} s}^{1} \ldots x_{\mathfrak{u}_{k-1} s}^{1} y_{\mathfrak{u}_{k} s}^{1} z_{\mathfrak{u}}^{1}\right\rangle \in Q_{s}^{M_{1}} .
\end{gathered}
$$

So, we have both $\mathbb{Z} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(u_{0}\right)+g(s)$ and $\mathbb{Z} \models \sum_{\ell<k} i_{\ell}^{1}=h_{k}^{1}\left(u_{0}\right)+g^{1}(s)$, so setting

$$
g_{1}\left(u_{\ell}, s\right)=i_{\ell}^{1}, \quad g_{2}\left(u_{k}, s\right)=h_{k}^{1}, \quad g_{3}(u)=g^{1}
$$

yields

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}-h_{k}\left(u_{0}\right)-g(s)=\sum_{\ell<k} g_{1}\left(u_{\ell}, s\right)-g_{2}\left(u_{k}, s\right)\left(u_{0}\right)-g_{3}(u)(s) .
$$

Since $f_{1}$ does this for all possible $(k+1)$-tuples, we have all the compability we need.
(c) $\rightarrow$ (a) If the predicates are the same modulo $g_{1}, g_{2}$ and $g_{3}$ then obtaining (a) becomes a matter of building $F: M_{1} \xrightarrow{\approx} P_{0}^{M_{1}} \cup p_{1}^{M_{1}} M_{2}$. Clearly we can start by $F \upharpoonright P_{0}^{M_{1}}=i d$, and then extend its definition to all the other portions of the model. The only strong restriction to the extension of this to the whole model is given by the relations $\mathrm{Q}_{s}^{\mathrm{M}_{1}}$ and $Q_{s}^{M_{2}}$. But part (4) of (c) provides this: the functions $g_{1}, g_{2}, g_{3}$ provide the definition of the isomorphism. Precisely, let $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k}, z\right\rangle$ be a tuple from $M_{1}$; we use (4) to find simultaneously $F\left(x_{\ell}\right), F\left(y_{k}\right)$ and $F(z)$. Compute (in $\left.\mathbb{Z}_{2}\right)$ the value $\sum_{l<k} \mathfrak{i}_{\ell}-h_{k}\left(u_{0}\right)-g(s)$ corresponding to the tuple. For every $s \in S$, this value is 0 iff the tuple belongs to $Q_{s}$. By (4), this value is equal to $\sum_{\ell<k} g_{1}\left(u_{\ell}, s\right)-$ $g_{2}\left(u_{k}, s\right)\left(u_{0}\right)-g_{3}(u)(s)$. But also by (4), this value also corresponds to a corresponding tuple $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0}^{\prime} \ldots x_{k-1}^{\prime} y_{k}^{\prime}, z^{\prime}\right\rangle$ in $M_{2}$. This provides the values $F\left(x_{\ell}\right)=x_{\ell}^{\prime}, F\left(y_{k}\right)=y_{k}^{\prime}$ and $F(z)=z^{\prime}:\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k}, z\right\rangle \in Q_{s}^{M_{1}}$ if and only if $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0}^{\prime} \ldots x_{k-1}^{\prime} y_{k}^{\prime}, z^{\prime}\right\rangle \in Q_{s}^{M_{2}}$.

Remark 3.2. Counting the number of isomorphism types here is akin to the study of $\operatorname{Ext}(G, \mathbb{Z})$ in the work of the first author and Väisänen in [13].

Lemma 3.3. If $\mathrm{M}_{\mathrm{I}_{1}, \mathrm{f}_{1}}$ and $\mathrm{M}_{\mathrm{I}_{2}, \mathrm{f}_{2}}$ are models of $\psi$, and $\mathrm{h}: \mathrm{I}_{1} \rightarrow \mathrm{I}_{2}$ is one-to-one and onto, then the following are equivalent:
(a) there is an isomorphism from $\mathrm{M}_{\mathrm{I}_{1}, \mathrm{f}_{1}}$ onto $\mathrm{M}_{\mathrm{I}_{2}, \mathrm{f}_{2}}$ extending $h$.
(b) $)_{1}$ there is an $M_{\mathrm{I}_{2}, \mathrm{f}_{2}}$-choice $(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \bar{z})$ whose correction function is $\mathrm{f}_{1}$,
(b) $)_{2}$ there is an $\mathrm{M}_{\mathrm{I}_{1}, \mathrm{f}_{1}}$-choice $(\bar{x}, \bar{y}, \bar{z})$ whose correction function is $\mathrm{f}_{2}$,
(c) there are functions $\mathrm{g}_{1}, \mathrm{~g}_{2}, \mathrm{~g}_{3}$ ("to correct the choice of zeros"), with

1. $\mathrm{g}_{1}:[\mathrm{I}]^{\mathrm{k}} \times \mathrm{S} \rightarrow \mathbb{Z}_{2}$ (like the $\mathrm{x}_{\mathrm{u}, \mathrm{s}}$ 's above),
2. $\mathrm{g}_{2}:[\mathrm{I}]^{\mathrm{k}} \times \mathrm{S} \rightarrow \mathrm{H}_{\mathrm{I}}$ (like the $\mathrm{y}_{\mathrm{u}, \mathrm{s}}$ 's above),
3. $g_{3}:[\mathrm{I}]^{\mathrm{k}+1} \rightarrow \mathrm{G}$ (like the $z_{\mathfrak{u}}$ 's above),
4. if $\left\langle a_{0} \ldots a_{k} u_{0} \ldots u_{k} x_{0} \ldots x_{k-1} y_{k}, z\right\rangle$ are like in Definition 1.1.1 for $M_{I_{1}, f_{1}}$, or $\mathrm{M}_{\mathrm{I}_{2}, \mathrm{f}_{2}}$ then

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}-h_{k}\left(u_{0}\right)-g(s)=\sum_{\ell<k} g_{1}\left(u_{\ell}, s\right)-g_{2}\left(u_{k}, s\right)\left(u_{0}\right)-g_{3}(u)(s)
$$

[^2]Proof The proof is almost the same as that of the previous lemma (3.1). The main change is that now the identity on $I$ is replaced by a bijection from $I_{1}$ onto $I_{2}$; the rest of the proof amounts to a renaming via the bijection $\mathrm{F} \upharpoonright \mathrm{I}_{1}$.

An important special case of the previous lemma happens when $\mathrm{I}_{1}=\mathrm{I}=\mathrm{I}_{2}$ but the isomorphism is not the identity on I. In this case, the restriction of the isomorphism between $M_{\mathrm{I}, \mathrm{f}_{1}}$ and $M_{\mathrm{I}, \mathrm{f}_{2}}$ is a permutation of I . Our combinatorial criterion for non-isomorphism will focus on this case.

Recall $\mathfrak{D}$ is the regular filter on $S$ generated by sets of the form $\langle u\rangle=\{v \in S \mid u \subset v\}$, where $S=[\lambda]^{<N_{0}}$ :

$$
\mathfrak{D}=\mathfrak{D}_{\lambda}:=\left\{A \subset S \mid \exists u_{A} \in S \forall v \in S\left(u_{A} \subset v \rightarrow v \in A\right)\right\}
$$

(see definition 1.2).
The notion of an I-function, which we define next, is central to our combinatorial criterion.

Definition 3.4. $f:[I]^{k+1} \times S \rightarrow \mathbb{Z}_{2}$ is an I-function iff

$$
\left\{s \in S \mid f_{\mathfrak{u}}(s) \neq 0\right\} \in \mathfrak{D}, \text { for all } u \in[I]^{k+1}
$$

where $f_{u}: S \rightarrow \mathbb{Z}_{2}$ is given by $f_{u}(s)=f(u, s)$.

Lemma 3.5. Let $\mathrm{f}:[\mathrm{I}]^{\mathrm{k}+1} \times \mathrm{S} \rightarrow \mathbb{Z}_{2}$ be an I -function. Then, the following is a sufficient condition for

$$
M_{I, f} \not \approx M_{I}:
$$

( $\star$ ) for every $\mathrm{F}_{1}:[\mathrm{I}]^{\mathrm{k}} \rightarrow[\mathrm{I}]^{\leq \lambda}, \mathrm{F}_{2}:[\mathrm{I}]^{\mathrm{k}} \rightarrow{ }^{\mathrm{S}}\left(\mathbb{Z}_{2}\right)$ and $\pi$ a permutation of I , there exists $u=\left\{\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right\} \in[\mathrm{I}]^{\mathrm{k}+1}$ (i.e., with no repetitions) such that
( $\alpha$ ) $t_{k} \notin F_{1}\left(\left\{t_{0} \ldots t_{k-1}\right\}\right)$,
( $\beta$ ) $\mathrm{f}_{\pi\left\{\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right\}}-\sum_{\ell<\mathrm{k}} \mathrm{F}_{2}\left(\left\{\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right\} \backslash\left\{\mathrm{t}_{\ell}\right\}\right) \notin \mathrm{G}$.
Before proving Lemma 3.5, we note:

- First, $(\star)$ is a purely combinatorial statement; this will enable us to focus solely on combinatorial principles to prove the failure of categoricity.
- Also, by the definition of $G,(\beta)$ says that for $\mathfrak{D}$-few elements $s \in S$ do we have

$$
\mathrm{f}_{\pi\left(\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right)}(\mathrm{s})=\sum_{\ell<k} \mathrm{~F}_{2}\left(\left\{\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right\} \backslash\left\{\mathrm{t}_{\ell}\right\}\right)(\mathrm{s}) .
$$

Notice the role of the permutation $\pi$ of I in the combinatorics that follows.
Proof of 3.5. Assume that $M_{\mathrm{I}, \mathrm{f}} \approx M_{\mathrm{I}}$. Then, since $M_{\mathrm{I}} \approx M_{\mathrm{I}, \mathrm{o}}$ (the null correction function) we may apply Lemma 3.3 and (by (b) $)_{2}$ of that lemma) assume that ( $\bar{x}, \bar{y}, \bar{z}$ ) witnesses $M_{\mathrm{I}, \mathrm{f}} \approx M_{\mathrm{I}, 0}$, with correction function identically zero.

We construct $F_{1}, F_{2}$ such that ( $\star$ ) of 3.5 does not hold (for the permutation $\pi$ induced by the isomorphism between $M_{\mathrm{I}, \mathrm{f}}$ and $\mathrm{M}_{\mathrm{I}}$ ).

We first let $\mathrm{F}_{1}:[\mathrm{I}]^{\mathrm{k}} \rightarrow[\mathrm{I}]^{\leq \lambda}$ be

$$
F_{1}(v)=\bigcup\left\{w \in[I]^{\mathrm{k}} \mid \text { for some } s_{1} \in S, y_{v, s_{1}}(w) \neq 0\right\} .
$$

This is well defined, as $F_{1}(u)$ is a union of $|S|$-many finite sets. Also, let

$$
F_{2}(v)=\left\langle x_{v, s} \mid s \in S\right\rangle .
$$

We will show that no $u \in[I]^{k+1}$ satisfies both $(\alpha)$ and $(\beta)$ of condition $(\star)$.
Suppose otherwise; let then $u=\left\{t_{0}, \ldots, t_{k}\right\} \in[I]^{k+1}$ satisfy $(\alpha)+(\beta)$. Let as usual $u_{\ell}=u \backslash\left\{t_{\ell}\right\}$. By $(\alpha)$, for each $s$,

$$
y_{u_{k}, s}\left(u_{0}\right)=0 .
$$

[Why? Just notice that by $(\alpha)$,

$$
t_{k} \notin F_{1}\left(u_{k}\right)=\bigcup\left\{v \in[I]^{k} \mid \text { for some } s_{1} \in S, y_{u, s_{1}}(v) \neq 0\right\},
$$

so for all $v \in[I]^{k}$, if $t_{k} \in v$, then for all $s_{1} \in S$ we have $y_{u_{k}, s_{1}}(v)=0$. In particular, as $\left.t_{k} \in u_{0}, y_{u_{k}, s_{1}}\left(u_{0}\right)=0.\right]$

Now, since ( $\bar{x}, \bar{y}, \bar{z}$ ) is an $M_{I, f}$-choice with correction function identically zero, for each $s \in S$ we have that

$$
\left\langle a_{0}, \ldots, a_{k}, u_{0}, \ldots, u_{k}, x_{u_{0}, s}, \ldots, x_{u_{k-1}, s}, y_{u_{k}, s}, z_{u}\right\rangle
$$

belongs to $Q_{s}^{M_{1, f}}$ if and only if (by the definition of this predicate in the model $M_{I, f}$ )

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} x_{\mathfrak{u}_{\ell}, s}=y_{\mathfrak{u}_{k}, s}\left(u_{0}\right)+z_{\mathfrak{u}}(s)+f_{\pi(u)}(s) .
$$

But
But we also have that $z_{\mathfrak{u}}(s)=0$ for the $\mathfrak{D}$-majority of $s \in S$ (by the definition of $G$ ); since we also have that $\mathfrak{y}_{\mathfrak{u}_{k}, s}\left(u_{0}\right)=0$ for our particular $\mathfrak{u}$,
(*) For the $\mathfrak{D}$-majority of $s \in S$

$$
\sum_{\ell<k} x_{\mathfrak{u}_{\ell}, s}=f_{\pi(u)}(s) .
$$

But this contradicts ( $\beta$ ).
Remark 3.6. We can then regard $\mathrm{F}_{2}$ as

$$
\mathrm{F}_{2}:[\mathrm{I}]^{\mathrm{k}} \rightarrow^{\mathrm{S}}\left(\mathbb{Z}_{2}\right) / \mathrm{G}
$$

Corollary 3.7. If $f_{1}, f_{2}$ are $I$-functions, and $f=f_{1}-f_{2}$ (coordinatewise) satisfies $(*)$, then $M_{\mathrm{I}, \mathrm{f}_{1}} \not \approx \mathrm{M}_{\mathrm{I}, \mathrm{f}_{2}}$.

Proof $\quad$ We already know that since $f$ satisfies $(\star), M_{I, f} \not \approx M_{\mathrm{I}}$. Now suppose we have an isomorphism $F: M_{I, f_{1}} \xrightarrow{\approx} M_{I, f_{2}}$. As before, $\pi=F \upharpoonright I$ is a permutation of $I$, and the automorphism lifts in a natural way to all components of $M_{\mathrm{I}, \mathrm{f}}$ in the vocabulary $\tau^{-}$. Now, the remaining part of $\tau$ : if $s \in S$, then a tuple
$\left\langle a_{0}, \ldots, a_{k}, \mathfrak{u}_{0}, \ldots, \mathfrak{u}_{k}, x_{\mathfrak{u}_{0}, s}, \ldots, x_{\mathfrak{u}_{k-1}, s}, y_{\mathfrak{u}_{k}, s}, z_{\mathfrak{u}}\right\rangle$ belongs to $Q_{s}$ in $M_{I, f}$ if and only if (for the corresponding indices)

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(\mathfrak{u}_{0}\right)+g(s)+f(u, s)
$$

but this holds if and only if

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}=h_{k}\left(u_{0}\right)+g(s)+f_{1}(u, s)-f_{2}(u, s) .
$$

Now, since $F$ is an isomorphism, this is true if and only the tuple
$\left\langle F\left(a_{0}\right), \ldots, F\left(a_{k}\right), F\left(u_{0}\right), \ldots, F\left(u_{k}\right), F\left(x_{u_{0}, s}\right), \ldots, F\left(x_{u_{k-1}, s}\right), F\left(y_{u_{k}, s}\right), F\left(z_{u}\right)\right\rangle$ belongs to $Q_{s}$ in $M_{I}$, this is if and only if

$$
\mathbb{Z}_{2} \models \sum_{\ell<k} \mathfrak{i}_{\ell}^{\prime}=\mathrm{h}_{\mathrm{k}}^{\prime}\left(\mathrm{u}_{0}^{\prime}\right)+\mathrm{g}^{\prime}(\mathrm{s})
$$

where the primes denote the values corresponding to the F -images of components of the long tuple. But this witnesses that $F$ is also an isomorphism between $M_{I, f_{1}}$ and $M_{I, f_{2}}$, which contradicts the hypothesis.

We now have what we need for a proof of failure of categoricity at some $\mu$ above the categoricity cardinals. Notice we do not give an optimal (minimal) such $\mu$; this is left for (possible) later work.

Theorem 3.8. For some $\mu>\lambda^{+\mathrm{k}}$, the sentence $\psi_{\mathrm{k}}^{\lambda}$ is not categorical in $\mu$.
Proof Let $\mu$ be a cardinal with the following properties:
$\otimes_{1} \mu \rightarrow(\omega)_{2^{\lambda}}^{k}$,
$\otimes_{2} \mu \nrightarrow(\omega)_{2^{\lambda}}^{k+1}$,
$\otimes_{3} \mu$ regular.
The existence of such a $\mu$ uses the Erdös-Rado theorem (the partition

$$
\beth_{k}(\lambda)^{+} \rightarrow\left(\left(2^{\lambda}\right)^{+}\right)_{2^{\lambda}}^{k}
$$

is an instance) for $\otimes_{1}$ and the negative partition relation $\beth_{k+1}(\lambda) \nrightarrow(k+2)_{2^{\lambda}}^{k+1}$ (a consequence of [12, Lemma 24.1(e)]) for $\otimes_{2}$; we may therefore take $\mu$ as $\beth_{k}(\lambda)^{+}$.

Let then I have cardinality $\mu$ and let $\mathrm{f}:[\mu]^{\mathrm{k}+1} \rightarrow \mathrm{G}^{+} / \mathrm{G}$ be an I-function witnessing $\otimes_{2}$ (recall that $\mathrm{G}^{+}$denotes the group ${ }^{\mathrm{S}} \mathbb{Z}_{2}$ ). We use our criterion 3.5 to show that $M_{\mathrm{I}, \mathrm{f}}$ and $M_{I}$ can not be isomorphic from which we conclude that the sentence $\psi_{\mathrm{k}}^{\lambda}$ is not categorical in $\mu$.

Let $\mathrm{F}_{1}:[\mathrm{I}]^{\mathrm{k}} \rightarrow[\mathrm{I}]^{\leq \lambda}, \mathrm{F}_{2}:[\mathrm{I}]^{\mathrm{k}} \rightarrow \mathrm{G}^{+}$and $\pi$ a permutation of I .
Now find $\mathrm{E} \subset \mu$ club such that

$$
\alpha_{0}<\cdots<\alpha_{k} \in E \Longrightarrow\left\{\begin{array}{l}
F_{1}\left(\alpha_{0}, \ldots, \alpha_{k-1}\right) \subset \alpha_{k} \\
\pi\left(\alpha_{0}\right), \ldots, \pi\left(\alpha_{k-1}\right)<\alpha_{k} .
\end{array}\right.
$$

This is possible by the regularity of $\mu$.
Now apply $\otimes_{1}$ to $F_{2} \upharpoonright E$ : since $\mu \rightarrow(\omega)_{2^{\lambda}}^{k}$, there must be an infinite $\omega$-sequence $\mathrm{T}=\left\{\alpha_{0}<\alpha_{1}<\ldots \alpha_{n}<\ldots\right\}$ such that $\mathrm{F}_{2} \upharpoonright[\mathrm{~T}]^{\mathrm{k}}$ is constant. Therefore we have, for $u=\left\{\alpha_{0}, \ldots, \alpha_{k}\right\}$ and $u_{\ell}=u \backslash\left\{\alpha_{\ell}\right\}:$

- $\alpha_{k} \notin F_{1}\left(\left\{\alpha_{0}, \ldots, \alpha_{k-1}\right\}\right)$ (since these are elements from the club $E$ ) and
- the equation $f_{\pi\left\{\alpha_{0}, \ldots, \alpha_{k}\right\}}(s)=\sum_{\ell<k} F_{2}\left(u_{\ell}\right)(s)$ holds for $\mathfrak{D}$-few elements $s:$ as $F_{2}$ is constant on $\mathfrak{u}_{\ell}$ from the monochromatic sequence, the sum on the right hand side will be 0 when $k$ is even (and 1 when $k$ is odd) whereas the value on the left hand side will not be constant, by $\otimes_{2}$ applied to $f$.

The previous two properties correspond to $(\alpha)$ and ( $\beta$ ) of the criterion from Lemma 3.5. Therefore, $M_{\mathrm{I}, \mathrm{f}} \not \approx \mathrm{M}_{\mathrm{I}}$ and the sentence $\psi_{\mathrm{k}}^{\lambda}$ is not categorical in $\mu$.

The result also holds for all $\kappa \geq \beth_{k+1}(\lambda)^{+}$(we will show monotonicity of the crucial criterion).

Conclusion 3.9. The sentence $\psi_{k}^{\lambda}$ is not categorical in any $k \geq \beth_{k+1}(\lambda)^{+}$.
Proof Let $\kappa \geq \mu=\beth_{k+1}(\lambda)^{+}$. If $\kappa=\mu$, Theorem 3.8 shows how to get two nonisomorphic models. If $\kappa>\mu$ then let $J$ be a set of cardinality $\kappa$.

We show that as $k>\mu$ we may pick a J-function $f:[J]^{k+1} \times S \rightarrow \mathbb{Z}_{2}$ satisfying the criterion ( $\star$ ) of Lemma 3.5 (which will enable us to conclude that $M_{\mathrm{J}, \mathrm{f}} \not \approx M_{\mathrm{J}}$, and thus conclude failure of categoricity at $\kappa$ ).

Let first $F_{1}:[J]^{k} \rightarrow[J] \leq \lambda, F_{2}:[J]^{k} \rightarrow{ }^{s} \mathbb{Z}_{2}$ and $\pi$ a permutation of $J$. Let $I \subset J$ with $|\mathrm{I}|=\mu$, I closed under $\pi$ and such that $\mathrm{F}_{1} \upharpoonright[\mathrm{I}]^{k}:[\mathrm{I}]^{\mathrm{k}} \rightarrow[\mathrm{I}]^{\leq \lambda}$. [Such an I exists by closing first under iterating taking the unions of $\mathrm{F}_{1}$-images of k -tuples from I and taking the union of $\mu$ many sets of cardinality $\leq \lambda<\mu$ - after an $\omega$-iteration the result is closed under $\mathrm{F}_{1}$-images. Similarly, we close under images and preimages under the permutation $\pi$ and alternate these closure operations $\omega$ many times.]

Let now $f:[J]^{k+1} \times S \rightarrow \mathbb{Z}_{2}$ be a J-function that witnesses $\otimes_{2}$ on the set $I$; as $|I|=\mu$, this is possible.

Furthermore, for the set I , the functions $\mathrm{F}_{1} \upharpoonright \mathrm{I}, \mathrm{F}_{2}$ and $\pi \upharpoonright \mathrm{I}$ are in the situation of Lemma 3.5. The proof of Theorem 3.8 applies then, as $|\mathrm{I}|=\mu$. We then obtain $u=$ $\left\{\mathrm{t}_{0}, \ldots, \mathrm{t}_{\mathrm{k}}\right\} \in[\mathrm{I}]^{\mathrm{k}+1}$ such that $(\alpha)$ and $(\beta)$ of the criterion hold. But these properties are also true of the original $F_{1}, F_{2}, \pi$. Therefore, $M_{J, f} \not \approx M_{\mathrm{J}}$.
Remark 3.10. Here are some important differences between the structure of this proof and that of [1]:

1. The use of the filter $\mathfrak{D}$ is central here - it is not needed there.
2. The way the group itself is used is slightly different at the end of the proof.

We conjecture that the class has the maximal number of models at all $\mu>\beth_{k+1}(\lambda)^{+}$.

## 4. Further directions

After our generalization of the original Hart-Shelah example to the stronger logic $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$, we have the following situation:

- Any generalization of the early results from 1983 for the logic $\mathrm{L}_{\left(2^{\lambda}\right)^{+}, \omega}$ must necessarily use as hypothesis few models in all cardinalities $\lambda, \lambda^{+}, \ldots, \lambda^{+k}, \ldots$ for all $k<\omega$. The first author has written several papers in this direction (see [14]), in the (wider) context of AECs.
- On the other hand, the necessity of an interval of $\aleph_{0}$-many cardinals with few models to start the machinery for categoricity transfer seems interesting per se; and even more so the fact that this would happen along all strengthenings of $\mathrm{L}_{\omega_{1}, \omega}$ (inside $\left.\mathrm{L}_{\infty, \omega}\right)$.
- Finally, we conjecture that our sentence may be analyzed in terms of building frames, in the spirit of the work of Boney and Vasey [10]. Specifically, we conjecture that our abstract elementary class

$$
\mathcal{K}^{*}(\lambda, k)=\left(\operatorname{Mod}\left(\psi_{\mathrm{k}}^{\lambda}\right), \prec \prec_{\left(2^{\lambda}\right)^{+}, \omega}\right)
$$

is $\left(<\lambda, \lambda^{+k-1}\right)$-tame, $\left(<\lambda, \lambda^{+k-1}\right)$-typeshort over models of size $\lambda^{+k-2}$, and that

1. for each $\mathfrak{m} \leq k-1$ there is a frame $\mathfrak{s}^{*}(\lambda, k)_{\mathfrak{m}}$ that is type-full and $\lambda^{+m}$-good on $\operatorname{Mod}\left(\psi_{\mathrm{k}}^{\lambda}\right)$,
2. The (type-full and $\lambda^{+k-1}$-good) frame $\mathfrak{s}^{*}(\lambda, k)_{k-1}$ is not weakly successful.

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[^1]:    ${ }^{4}$ here, of course, we are abusing notation - by $(\bar{x}, \bar{y}, \bar{z})_{s} \subset(\bar{x}, \bar{y}, \bar{z})_{\mathrm{t}}$ we mean $\bar{x}_{s} \subset \bar{x}_{\mathrm{t}}, \bar{y}_{\mathrm{s}} \subset \bar{y}_{\mathrm{t}}$ and $\bar{z}_{s} \subset \bar{z}_{\mathrm{t}}$.

[^2]:    ${ }^{5}$ Here $I(\lambda, \psi)$ is counted by the group of correction functions, derived from some $g_{1}, g_{2}, g_{3}$ :

    $$
    I\left(\lambda, \psi_{k}^{\lambda}\right)=\left\{f \text { a correction function } \mid f(u, s)=\sum_{\ell<k} g_{1}\left(u_{\ell}, s\right)-g_{2}\left(u_{0}, s\right)-g_{3}(u)\right\} .
    $$

