FUNDAMENTAL GROUP IN O-MINIMAL STRUCTURES WITH DEFINABLE SKOLEM FUNCTIONS

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ABSTRACT. In this paper we work in an arbitrary o-minimal structure with definable Skolem functions and we prove that definably connected, locally definable manifolds are uniformly definably path connected, have an admissible cover by definable simply connected, open definable subsets and, definable paths and definable homotopies on such locally definable manifolds can be lifted to locally definable covering maps. These properties allows us to obtain the main properties of the general o-minimal fundamental group, including: invariance and comparison results; existence of universal locally definable covering maps; monodromy equivalence for locally constant o-minimal sheaves - from which one obtains, as in algebraic topology, classification results for locally definable covering maps, o-minimal Hurewicz and Seifert - van Kampen theorems.

1. INTRODUCTION

In this paper we work in an arbitrary o-minimal structure $\mathbb{M} = (M, <, (c)_{c \in \mathcal{C}}, (f)_{f \in \mathcal{F}}, (R)_{R \in \mathcal{R}})$ which we assume to have definable Skolem functions. We are interested in developing algebraic topology tools for objects definable in \mathbb{M} , more specifically a suitable fundamental group functor. One should point out here that, although the objects definable in \mathbb{M} have a topology induced by the order in M, if \mathbb{M} is non-archimedean, then all such objects (except the ones of dimension zero) are totally disconnected topological spaces and so the topological fundamental group is of no use.

In the case \mathbb{M} expands a real closed field $(M, <, 0, 1, +, \cdot)$, so this includes the semi-algebraic case, a suitable o-minimal fundamental group functor is already known and was studied in, for example, [7], [8], [3], [4], [18], [1] and [2], where its main properties were proved and applications to the theory of definable groups were obtained. These main properties include: (i) finite generation; (ii) invariance when going to o-minimal expansions of \mathbb{M} or to bigger models of the first order theory of \mathbb{M} and (iii) coincidence with the topological fundamental group when $M = \mathbb{R}$. Often, in this case, the proofs of these properties relied on the o-minimal triangulation theorem ([10]), a generalization of the semi-algebraic and of the semi-analytic triangulation theorems ([6], [25], [31] and [26]).

In the case \mathbb{M} expands an ordered group (M, <, 0, +) a suitable o-minimal fundamental group functor was studied in the papers [21], [22], [12] and [13], where

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besides the properties mentioned above, the following were also proved: (iv) the existence of universal locally definable covering maps; (v) monodromy equivalence for locally constant o-minimal sheaves - from which one obtains, as in algebraic topology: (vi) classification results for locally definable covering maps; (vii) o-minimal Hurewicz and Seifert - van Kampen theorems.

In the case \mathbb{M} is an arbitrary o-minimal structure with definable Skolem functions, following M. Mamino's ideas, a general o-minimal fundamental group was introduced in the paper [17], where Pillay's conjecture for definably compact groups was obtained in the general case. However, in that paper, the properties mentioned above were only proved for the o-minimal **J**-fundamental group which is the relativization of the general o-minimal fundamental group to a cartesian product $\mathbf{J} = \prod_{i=1}^{m} J_i$ of definable group-intervals $J_i = \langle (-ib_i, b_i), 0_i, +i, -i, < \rangle$. The goal of this paper is to show, as conjectured in [17], all of the above properties for the general o-minimal fundamental group in the more general setting of o-minimal structures with definable Skolem functions.

The main technical result of the paper, obtained in Subsection 3.1, says that definably connected, locally definable manifolds are uniformly definably path connected, have an admissible cover by definably simply connected, open definable subsets and, definable paths and definable homotopies on such locally definable manifolds can be lifted to locally definable covering maps. See Properties 3.1. The results (i), (ii), (iii) and (iv) mentioned above are obtained, in the general case of \mathbb{M} an arbitrary o-minimal structure with definable Skolem functions, in Subsection 3.2, and as explained in Subsection 3.1, they follow from Properties 3.1 in exactly the same way as in [13] for o-minimal expansions of ordered groups. In Subsection 3.2 we also show that if X is a definably connected locally definable manifold with definable charts in **J**, then the o-minimal **J**-fundamental group of X is isomorphic to the o-minimal fundamental group of X in \mathbb{M} . Result (v) and its consequences (vi) and (vii) are mentioned in Subsection 3.3 after some background is recalled.

It might be possible that part of this theory may be developed in a general ominimal structure without the definable Skolem functions assumption. However, in the applications that we have in mind, all the structures have definable Skolem functions. It is even possible that under the assumption of the existence of definable Skolem functions some arguments may be simplified. Indeed, due to the trichotomy theorem it might be possible that there is a definable family of groups covering all the points of the domain of the structure. In that case the arguments in Subsection 3.1 would be closer to those of the o-minimal **J** fundamental group treated in [17]. We leave these issues for future work.

Regarding the applications, consider an algebraically closed valued field with a nontrivial valuation and consider the definable sets induced in the value group with a point at infinity added (the valuation of 0), Γ_{∞} . Adding a copy of the value group Γ to Γ_{∞} we obtain an o-minimal structure Σ with definable Skolem functions. Understanding the definable topology/definable algebraic topology of definable subsets of $(\Gamma_{\infty})^n$ can have applications in non-Archimedean tame topology by the main theorem of Hrushovski and Loeser ([27, Theorem 11.1.1]). For instance, Theorem 3.4 below is used in an essential way in developing the cohomology in the non-Archimedean tame setting [20]. Moreover, we expect that the main results about the fundamental group developed here or adaptations of those might give further applications.

2. Preliminaries

In this section we recall the notion of locally definable manifolds and locally definable covering maps and the general o-minimal fundamental group from [17].

Before we start, recall that an o-minimal structure \mathbb{M} has definable Skolem functions if and only if for every uniformly definable family $\{X_t\}_{t\in T}$ of nonempty definable subsets of some M^k , there is a definable function $h: T \to M^k$ such that:

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$$h(t) \in X_t$$
 for all $t \in T$.

2.1. Locally definable manifolds and covering maps. Here we recall the definition of the category of locally definable manifolds with continuous locally definable maps and the notion of locally definable covering maps.

A locally definable manifold (of dimension n) is a triple $(S, (U_i, \theta_i)_{i < \kappa})$ where:

- $S = \bigcup_{i < \kappa} U_i;$
- each $\theta_i : U_i \to M^n$ is an injection such that $\theta_i(U_i)$ is an open definable subset of M^n ;
- for all $i, j, \theta_i(U_i \cap U_j)$ is an open definable subset of $\theta_i(U_i)$ and the transition maps $\theta_{ij}: \theta_i(U_i \cap U_j) \to \theta_j(U_i \cap U_j): x \mapsto \theta_j(\theta_i^{-1}(x))$ are definable homeomorphisms.

We call the (U_i, θ_i) 's the definable charts of S. If $\kappa < \aleph_0$ then S is a definable manifold.

A locally definable manifold S is equipped with the topology such that a subset U of S is open if and only if for each i, $\theta_i(U \cap U_i)$ is an open definable subset of $\theta_i(U_i)$.

We say that a subset A of S is *definable* if and only if there is a finite $I_0 \subseteq \kappa$ such that $A \subseteq \bigcup_{i \in I_0} U_i$ and for each $i \in I_0$, $\theta_i(A \cap U_i)$ is a definable subset of $\theta_i(U_i)$. A subset B of S is *locally definable* if and only if for each $i, B \cap U_i$ is a definable subset of S. We say that a locally definable manifold S is *definably connected* if it is not the disjoint union of two open and closed locally definable subsets.

If $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$ is a cover of S by open locally definable subsets, we say that \mathcal{U} is admissible if for each $i \leq \kappa$, the cover $\{U_{\alpha} \cap U_i\}_{\alpha \in I}$ of U_i admits a finite subcover. If $\mathcal{V} = \{V_{\beta}\}_{\beta \in J}$ is another cover of S by open locally definable subsets, we say that \mathcal{V} refines \mathcal{U} , denoted by $\mathcal{V} \leq \mathcal{U}$, if there is a map $\epsilon : J \to I$ such that $V_{\beta} \subseteq U_{\epsilon(\beta)}$ for all $\beta \in J$.

A map $f : X \to Y$ between locally definable manifolds with definable charts $(U_i, \theta_i)_{i \leq \kappa_X}$ and $(V_j, \delta_j)_{j \leq \kappa_Y}$ respectively is a locally definable map if for every finite $I \subseteq \kappa_X$ there is a finite $J \subseteq \kappa_Y$ such that:

- $f(\bigcup_{i\in I} U_i) \subseteq \bigcup_{j\in J} V_j;$
- the restriction $f_{|}: \bigcup_{i \in I} U_i \to \bigcup_{j \in J} V_j$ is a definable map between definable manifolds, i.e., for each $i \in I$ and every $j \in J$, $\delta_j \circ f \circ \theta_i^{-1}: \theta_i(U_i) \to \delta_j(V_j)$ is a definable map between definable sets.

Thus we have the category of locally definable manifolds with locally definable continuous maps. **Definition 2.1.** Given a definably connected locally definable manifold S, a locally definable manifold X and an admissible cover $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$ of S by open definable subsets, we say that a continuous surjective locally definable map $p_X : X \to S$ is a *locally definable covering map trivial over* $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$ if the following hold:

- $p_X^{-1}(U_\alpha) = \bigsqcup_{i < \lambda} U_\alpha^i$ a disjoint union of open definable subsets of X;
- each $p_{X|U_i^i}: U_{\alpha}^i \to U_{\alpha}$ is a definable homeomorphism.

A locally definable covering map $p_X : X \to S$ is a locally definable covering map trivial over some admissible cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of S by open definable subsets.

We say that two locally definable covering maps $p_X : X \to S$ and $p_Y : Y \to S$ are *locally definably homeomorphic* if there is a locally definable homeomorphism $F : X \to Y$ such that:

• $p_X = p_Y \circ F$.

A locally definable covering map $p_X : X \to S$ is *trivial* if it is locally definably homeomorphic to a locally definable covering map $S \times M \to S : (s, m) \mapsto s$ for some set M.

Let $p_Y : Y \to T$ be a locally definable covering map, X be a locally definable manifold and let $f : X \to T$ be a locally definable map. A *lifting of* f is a continuous map $\tilde{f} : X \to Y$ such that $p_Y \circ \tilde{f} = f$. Note that a lifting of a continuous locally definable map need not be a locally definable map. However, if X is definably connected, then any two continuous locally definable liftings which coincide in a point must be equal [13, Lemma 2.8].

2.2. A general o-minimal fundamental group. Here we recall the definition and the basic properties of the o-minimal fundamental group in arbitrary o-minimal structures from [17].

Definition 2.2. By a *basic d-interval*, short for *basic directed interval*, we mean a tuple

$$\mathcal{I} = \langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$$

where $a, b \in M$ with a < b and $\langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \in \{ \langle a, b \rangle, \langle b, a \rangle \}$. The *domain* of \mathcal{I} is [a, b] and the *direction* of \mathcal{I} is $\langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle$. The *opposite* of \mathcal{I} is the basic *d*-interval

$$\mathcal{I}^{\mathrm{op}} = \langle [a, b], \langle 0_{\mathcal{I}^{\mathrm{op}}}, 1_{\mathcal{I}^{\mathrm{op}}} \rangle \rangle$$

with the same domain and opposite direction $\langle 0_{\mathcal{I}^{\text{op}}}, 1_{\mathcal{I}^{\text{op}}} \rangle = \langle 1_{\mathcal{I}}, 0_{\mathcal{I}} \rangle$.

If $\mathcal{I}_i = \langle [a_i, b_i], \langle 0_{\mathcal{I}_i}, 1_{\mathcal{I}_i} \rangle \rangle$ are basic *d*-intervals, for $i = 1, \ldots, n$, we define the *d*-interval, short for directed interval, $\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n$, whose domain is the set

$$[a_1, b_1] \wedge \dots \wedge [a_n, b_n] := \bigsqcup_i \{c_i\} \times [a_i, b_i] / \sim$$

where c_1, \ldots, c_n are *n* distinct points of *M* and \sim is the equivalence relation defined by $(c_i, 1_{\mathcal{I}_i}) \sim (c_{i+1}, 0_{\mathcal{I}_{i+1}})$ for each $i = 1, \ldots, n-1$ and identity elsewhere. The *direction* of $\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n$ is $\langle 0_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n}, 1_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n} \rangle$ where $0_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n} = \langle c_1, 0_{\mathcal{I}_1} \rangle$ and $1_{\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n} = \langle c_n, 1_{\mathcal{I}_n} \rangle$.

The *opposite* of $\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n$ is the *d*-interval

$$(\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n)^{\mathrm{op}} = \langle [a_1, b_1] \wedge \dots \wedge [a_n, b_n], \langle 0_{(\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n)^{\mathrm{op}}}, 1_{(\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n)^{\mathrm{op}}} \rangle \rangle$$

with the same domain and opposite direction

$$\langle 0_{(\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n)^{\mathrm{op}}}, 1_{(\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n)^{\mathrm{op}}} \rangle = \langle 1_{\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n}, 0_{\mathcal{I}_1 \wedge \dots \wedge \mathcal{I}_n} \rangle.$$

Fact 2.3. If $\mathcal{I}_i = \langle [a_i, b_i], \langle 0_{\mathcal{I}_i}, 1_{\mathcal{I}_i} \rangle \rangle$ are basic *d*-intervals, for $i = 1, \ldots, n$, then $(\mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n)^{\mathrm{op}} = \mathcal{I}_n^{\mathrm{op}} \wedge \cdots \wedge \mathcal{I}_1^{\mathrm{op}}$.

Below, for the notion of *definable space* we refer the reader to [10, page 156]. We have:

Fact 2.4. [17, Lemma 2.5] Let $\mathcal{I} = \langle I, \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$ be a d-interval. Then the domain I of \mathcal{I} is a Hausdorff, definably compact, definable space of dimension one which is equipped with a definable total order $\langle \mathcal{I} \rangle$.

Due to Fact 2.4, below we will identify a *d*-interval $\mathcal{I} = \langle I, \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$ with its domain equipped with the definable total order $\langle_{\mathcal{I}}$. In particular, since the domain I of \mathcal{I}^{op} is a definable space of dimension one which is equipped with the definable total order $\rangle_{\mathcal{I}}$, we have an order reversing definable homeomorphism (with respect to the topologies given by the orders)

$$o_{\mathcal{I}}: \mathcal{I} \to \mathcal{I}^{\mathrm{op}}$$

given by the identity on the domain.

Given two *d*-intervals $\mathcal{I} = \mathcal{I}_1 \wedge \cdots \wedge \mathcal{I}_n$ and $\mathcal{J} = \mathcal{J}_1 \wedge \cdots \wedge \mathcal{J}_m$, we define the *d*-interval

$$\mathcal{I} \land \mathcal{J} = \mathcal{I}_1 \land \dots \land \mathcal{I}_n \land \mathcal{J}_1 \land \dots \land \mathcal{J}_m$$

and we will regard \mathcal{I} and \mathcal{J} as definable subsets of $\mathcal{I} \wedge \mathcal{J}$.

We say that \mathcal{I} and \mathcal{J} are equal, denoted $\mathcal{I} = \mathcal{J}$, if n = m and $\mathcal{I}_i = \mathcal{J}_i$ for all i = 1, ..., n.

Remark 2.5. Let $\mathcal{I} = \langle I, \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$ be a *d*-interval. If $x, y \in I$ are such that $x \leq_{\mathcal{I}} y$ then the subset

$$[x,y]_{\mathcal{I}} = \{t \in I : x \leq_{\mathcal{I}} t \leq_{\mathcal{I}} y\}$$

of the elements of \mathcal{I} between x and y is itself a d-interval.

Indeed, let $\mathcal{I}_i = \langle [a_i, b_i], \langle 0_{\mathcal{I}_i}, 1_{\mathcal{I}_i} \rangle \rangle$ be basic d-intervals, for $i = 1, \ldots, n$, and suppose that $\mathcal{I} = \mathcal{I}_1 \land \cdots \land \mathcal{I}_n$. Let $i_x, i_y \in \{1, \ldots, n\}$ be such that $x \in [a_{i_x}, b_{i_x}]$ and $y \in [a_{i_y}, b_{i_y}]$. Note that $i_x \leq i_y, x \leq_{\mathcal{I}} 1_{\mathcal{I}_{i_x}}$ and $0_{\mathcal{I}_{i_y}} \leq_{\mathcal{I}} y$. If $i_x = i_y$ then

$$[x,y]_{\mathcal{I}} = \begin{cases} \langle [x,y], \langle x,y \rangle \rangle & \text{if } a_{i_x} \leq x < y \leq b_{i_x} \\ \\ \langle [y,x], \langle x,y \rangle \rangle & \text{if } a_{i_x} \leq y < x \leq b_{i_x}. \end{cases}$$

If $i_x < i_y$ then

$$[x,y]_{\mathcal{I}} = [x,1_{\mathcal{I}_{i_x}}]_{\mathcal{I}} \wedge \mathcal{I}_{i_x+1} \wedge \dots \wedge [0_{\mathcal{I}_{i_x}},y]_{\mathcal{I}}.$$

Below, if X is a locally definable manifold and Y is a definable space, we say that $h: Y \to X$ is a *definable continuous map* if for some (equivalently, for every) definable subspace U of X with $h(Y) \subseteq U$, the map $h: Y \to U$ is a definable continuous map between definable spaces.

Definition 2.6. Let X be a locally definable manifold. A definable path $\alpha : \mathcal{I} \to X$ is a continuous (with respect to the topology on \mathcal{I} given by the order) definable map from some d-interval \mathcal{I} to X. We define $\alpha_0 := \alpha(0_{\mathcal{I}})$ and $\alpha_1 := \alpha(1_{\mathcal{I}})$ and call them the endpoints of the definable path α .

A definable path $\alpha : \mathcal{I} \to X$ is *constant* if $\alpha_0 = \alpha(t)$ for all $t \in \mathcal{I}$. Below, given a *d*-interval \mathcal{I} and a point $x \in X$, we denote by $\mathbf{c}_{\mathcal{I}}^x$ the constant definable path in X with endpoints x.

A definable path $\alpha : \mathcal{I} \to X$ is a *definable loop* if $\alpha_0 = \alpha_1$. The *inverse* α^{-1} of a definable path $\alpha : \mathcal{I} \to X$ is the definable path

$$\alpha^{-1} := \alpha \circ o_{\mathcal{I}}^{-1} : \mathcal{I}^{\mathrm{op}} \to X.$$

A concatenation of two definable paths $\gamma : \mathcal{I} \to X$ and $\delta : \mathcal{J} \to X$ with $\gamma(1_{\mathcal{I}}) = \delta(0_{\mathcal{J}})$ is the definable path $\gamma \cdot \delta : \mathcal{I} \land \mathcal{J} \to X$ with:

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in \mathcal{I} \\ \delta(t) & \text{if } t \in \mathcal{J}. \end{cases}$$

We say that X is definably path connected if for every u, v in X there is a definable path $\alpha : \mathcal{I} \to X$ such that $\alpha_0 = u$ and $\alpha_1 = v$.

In the special case required for our applications we shall prove later, see Corollary 3.15 (1), that being definably connected is equivalent to being definably path connected.

Let X be a locally definable manifold and Y a definable space. Given two definable continuous maps $f, g: Y \to X$, we say that a definable continuous map $F(t,s): Y \times \mathcal{J} \to X$ is a definable homotopy between f and g if $f = F_0 := F_{0_{\mathcal{J}}}$ and $g = F_1 := F_{1_{\mathcal{J}}}$, where $F_s := F(\cdot, s)$ for all $s \in \mathcal{J}$. In this case we say that f and g are definably homotopic, denoted $f \sim g$.

Fact 2.7. [17, Remarks 2.8, 2.9 and 2.10] The definable homotopy \sim is an equivalence relation compatible with concatenation i.e., if $\gamma_i : \mathcal{I} \to X$ and $\delta_i : \mathcal{J} \to X$ (i = 1, 2) are definable paths with $(\gamma_i)_1 = (\delta_i)_0$ for i = 1, 2 and $\gamma_1 \sim \gamma_2$ and $\delta_1 \sim \delta_2$, then $\gamma_1 \cdot \delta_1 \sim \gamma_2 \cdot \delta_2$.

Moreover, if $\gamma : \mathcal{I} \to X$ is a definable path and \mathcal{J} is any d-interval, then

$$\mathbf{c}_{\mathcal{I}\wedge\mathcal{J}}^{\gamma_0}\cdot\gamma\sim\gamma\cdot\mathbf{c}_{\mathcal{J}\wedge\mathcal{I}}^{\gamma_1}.$$

Since definable paths need not have the same domain, the notion of homotopic definable paths is not contained in the notion of homotopic definable maps just defined:

Definition 2.8. Two definable paths $\gamma : \mathcal{I} \to X$, $\delta : \mathcal{J} \to X$, with $\gamma_0 = \delta_0$ and $\gamma_1 = \delta_1$, are called *definably homotopic*, denoted $\gamma \approx \delta$, if there are *d*-intervals \mathcal{I}' and \mathcal{J}' such that $\mathcal{J}' \wedge \mathcal{I} = \mathcal{J} \wedge \mathcal{I}'$, and there is a definable homotopy

$$\mathbf{c}_{\mathcal{T}'}^{\gamma_0} \cdot \gamma \sim \delta \cdot \mathbf{c}_{\mathcal{T}'}^{\delta_1}$$

fixing the end points (i.e., they are definably homotopic by a definable homotopy $F: \mathcal{K} \times \mathcal{A} \to X$, where $\mathcal{K} = \mathcal{J}' \wedge \mathcal{I} = \mathcal{J} \wedge \mathcal{I}'$, such that $F(0_{\mathcal{K}}, s) = \gamma_0 = \delta_0$ and $F(1_{\mathcal{K}}, s) = \gamma_1 = \delta_1$ for all $s \in \mathcal{A}$.)

We have:

Remark 2.9. [17, Remark 2.11] Let X be a locally definable manifold. If $\delta_i : \mathcal{J} \to X$ (i = 1, 2) are definable paths such that $\delta_1 \sim \delta_2$, then $\delta_1 \approx \delta_2$.

We also have:

Fact 2.10. [17, Proposition 2.13] Let X be a locally definable manifold and $x_0, x_1 \in X$. Let $\mathbb{P}(X, x_0, x_1)$ denote the set of all definable paths in X that start at x_0 and end at x_1 . Then the restriction of \approx to $\mathbb{P}(X, x_0, x_1) \times \mathbb{P}(X, x_0, x_1)$ is an equivalence relation on $\mathbb{P}(X, x_0, x_1)$. Moreover, if γ, γ', δ and δ' are definable paths in X such that $\gamma_1 = \delta_0, \gamma'_1 = \delta'_0, \gamma \approx \gamma'$ and $\delta \approx \delta'$, then $\gamma \cdot \delta \approx \gamma' \cdot \delta'$.

By [17, Lemmas 2.14 e 2.15] we have:

Definition 2.11. Let X be a locally definable manifold and $e_X \in X$. If $\mathbb{L}(X, e_X)$ denotes the set of all definable loops that start and end at a fixed element e_X of X (i.e. $\mathbb{L}(X, e_X) = \mathbb{P}(X, e_X, e_X)$). We define the *o-minimal fundamental group* $\pi_1(X, e_X)$ of X by

$$\pi_1(X, e_X) := \frac{\mathbb{L}(X, e_X)}{\approx}$$

with group operation given by $[\gamma][\delta] = [\gamma \cdot \delta]$, the inverse given by $[\gamma]^{-1} = [\gamma^{-1}]$ and identity the class of a constant loop at e_X .

If $f: X \to Y$ is a locally definable continuous map between two locally definable manifolds with $e_X \in X$ and $e_Y \in Y$ such that $f(e_X) = e_Y$, then we have an induced homomorphism $f_*: \pi_1(X, e_X) \to \pi_1(Y, e_Y) : [\sigma] \mapsto [f \circ \sigma]$ with the usual functorial properties.

Fact 2.12. [17, Corollary 2.18] Let X and Y be locally definable manifolds with $e_X \in X$ and $e_Y \in Y$. Then

(1) If X is definably path connected then $\pi_1(X, e_X) \simeq \pi_1(X, x)$ for every $x \in X$. (2) $\pi_1(X, e_X) \times \pi_1(Y, e_Y) \simeq \pi_1(X \times Y, (e_X, e_Y))$.

As usual for a definably path connected locally definable manifold X if there is no need to mention a base point $e_X \in X$, then by Fact 2.12 (1), we may denote $\pi_1(X, e_X)$ by $\pi_1(X)$.

Definition 2.13. Let X be a locally definable manifold. We define the *o-minimal* fundamental groupoid $\Pi_1(X)$ of X to be the small category $\Pi_1(X)$ given by

$$Ob(\Pi_1(X)) = X,$$

$$Hom_{\Pi_1(X)}(x_0, x_1) = \mathbb{P}(X, x_0, x_1) /_{\approx}$$

We set $[\gamma] :=$ the class of $\gamma \in \mathbb{P}(X, x_0, x_1)$. By Fact 2.10, the small category $\Pi_1(X)$ is indeed a groupoid with operations

 $\operatorname{Hom}_{\Pi_1(X)}(x_0, x_1) \times \operatorname{Hom}_{\Pi_1(X)}(x_1, x_2) \to \operatorname{Hom}_{\Pi_1(X)}(x_0, x_2)$

given by $[\delta] \circ [\gamma] = [\gamma \cdot \delta].$

If $f: X \to Y$ is a locally definable continuous map between locally definable manifolds, then we have an induced functor $f_*: \Pi_1(X) \to \Pi_1(Y)$ which is a morphism of groupoids sending the object $x \in X$ to the object $f(x) \in Y$ and a morphism $[\gamma]$ of $\Pi_1(X)$ to the morphism $[f \circ \gamma]$ of $\Pi_1(Y)$.

3. Main results

3.1. The main properties. The goal in this subsection is to prove the following properties:

Properties 3.1. Let \mathbf{P} be the full subcategory of locally definable spaces in \mathbb{M} whose objects are the locally definable manifolds. Then in the category \mathbf{P} the following hold:

- (P1) (a) every object of P which is definably connected is uniformly definably path connected;
 - (b) given a locally definable covering map p_X : X → S in P then: (i) every definable path γ in S has a unique lifting γ̃ which is a definable path in X with a given base point; (ii) every definable homotopy F between definable paths γ and σ in S has a unique lifting F̃ which is a definable homotopy between the definable paths γ̃ and σ̃ in X.
- (P2) Every object of P has admissible covers by definably simply connected, open definable subsets refining any admissible cover by open definable subsets.

It follows, as observed in the concluding remarks (Section 5) of the paper [13], that with (P1) and (P2) above one proves in exactly the same way all the main results of the paper [13], now in the more general context of arbitrary o-minimal structures with definable Skolem functions. These results include all those mentioned in the Introduction.

In fact, besides (P1) and (P2) (and their consequences) everything else that is required is, on the one hand, results from [15], which hold in arbitrary o-minimal structures (and for locally definable spaces as well), and on the other hand, [10, Chapter 6, (3.6)], which is used to notice that the domains of the "good" definable paths are definably normal. In our case here the domains of the definable paths are Hausdorff, definably compact definable spaces (Fact 2.4), definable in the o-minimal structure M with definable Skolem functions and so they are definably normal by [16, Theorem 2.11].

The fact that (P1) and (P2) are the only requirements needed to develop the theory presented in [13] is somewhat natural. Indeed in topology, where we have good notions of paths and homotopies with the lifting of paths and homotopies property, all one needs is existence of such nice open covers as in (P2). In the o-minimal context (here and in [13]), the role that (P1) (b) and (P2) play is similar to the role that the analogue properties play in topology. However, (P2) is often used in combination with the results from [15] mentioned above to get local definability. Also (P1) (a) is required essentially only once and to get local definability (see [13, Proposition 2.18]), the other places where it is used, it is used to replace definably connected by definably path connected.

Below let $\pi: M^{n+1} \to M^n$ be the projection onto the first *n* coordinates and let $\tau: M^{n+1} \to M: (x, y) \mapsto y$ be the projection onto the last coordinate.

We start by proving the existence of continuous definable sections for the projection of an open definable subset, over a finite cover by open definable subsets (Theorem 3.4 below). But first we recall a few facts.

The following is obtained from the definition of cells ([10, Chapter 3, $\S2$]):

Remark 3.2. Let $C \subseteq M^n$ be a *d*-dimensional cell. Then by the definition of cells, C is a (i_1, \ldots, i_n) -cell for some unique sequence (i_1, \ldots, i_n) of 0's and 1's. Moreover, if $\lambda(1) < \cdots < \lambda(d)$ are the indices $\lambda \in \{1, \ldots, n\}$ for which $i_{\lambda} = 1$ and

 $p_{\lambda(1),\dots,\lambda(d)}: M^n \to M^d: (x_1,\dots,x_n) \mapsto (x_{\lambda(1)},\dots,x_{\lambda(d)})$

is the projection, then $C' := p_{\lambda(1),\dots,\lambda(d)}(C)$ is an open *d*-dimensional cell in M^d and the restriction $p_C := p_{\lambda(1),\dots,\lambda(d)|C} : C \to C'$ is a definable homeomorphism ([10, Chapter 3, (2.7)]).

Let $\tau(1) < \cdots < \tau(n-d)$ be the indices $\tau \in \{1, \ldots, n\}$ for which $i_{\tau} = 0$. For each such τ , by the definition of cells, there is a definable continuous function $h_{\tau} : \pi_{\tau-1}(C) \subseteq M^{\tau-1} \to M$ where, for each $k = 1, \ldots, n, \pi_k : M^n \to M^k$ is the projection onto the first k-coordinates. Moreover we have $\pi_{\tau}(C) = \{(x, h_{\tau}(x)) : x \in \pi_{\tau-1}(C)\}.$

Let $f = (f_1, \ldots, f_{n-d}) : C' \to M^{n-d}$ be the definable continuous map where for each $l = 1, \ldots, n-d$ we set $f_l = h_{\tau(l)} \circ \pi_{\tau(l)-1} \circ p_C^{-1}$. Let $\sigma : M^n \to M^n :$ $(x_1, \ldots, x_n) \mapsto (x_{\lambda(1)}, \ldots, x_{\lambda(d)}, x_{\tau(1)}, \ldots, x_{\tau(n-d)})$. Then we clearly have

$$\sigma(C) = \{ (x, f(x)) : x \in C' \} \,.$$

Recall also the following fact:

Fact 3.3. [17, Theorem 2.2] Let U be an open definable subset of M^n . Then U is a finite union of open definable sets definably homeomorphic, by reordering of coordinates, to open cells.

Theorem 3.4. Let O be an open definable subset of M^{n+1} . Then there is a finite cover $\{U_i : i = 1, ..., m\}$ of $\pi(O)$ by open definable subsets such that for each i there is a continuous definable section $s_i : U_i \to O$ of π (i.e. $\pi \circ s_i = \mathrm{id}_{U_i}$).

Proof. By definable Skolem functions, let $s : \pi(O) \to O$ be a definable section, possibly discontinuous, of π . So $s = id \times t$ for some definable map $t : \pi(O) \to M$.

By the cell decomposition theorem ([10, Chapter 3, (2.11)]), let \mathcal{C} be a cell decomposition of $\pi(O)$ such that $s_{|C}$ is continuous for each cell $C \in \mathcal{C}$. Thus it is enough to show, by induction on d, that for any cell C of dimension d there are open definable subsets U_1, \ldots, U_m of $\pi(O)$ such that $C \subseteq \bigcup \{U_i : i = 1, \ldots, m\}$ and for each i there is a continuous definable map $s_i : U_i \to O$ such that $\pi \circ s_i = \operatorname{id}_{U_i}$.

The result for a zero dimensional cell (d = 0) is immediate. For the inductive step, by Remark 3.2, after a reordering of coordinates, we may assume that our cell C is of the form

$$C = \{ (x, f(x)) : x \in C' \}$$

where C' is a *d*-dimensional open cell in M^d , and $f = (f_1, \ldots, f_{n-d}) : C' \to M^{n-d}$ is continuous and definable.

By definable Skolem functions and the fact that O is open, for each i = 1, ..., n-d, there are definable functions $h_i, g_i : C' \to M$ such that for all $x \in C'$ and for all $y = (y_1, ..., y_{n-d}) \in M^{n-d}$ the following hold:

- for all *i* we have $h_i(x) < f_i(x) < g_i(x)$;

- if for all *i* it holds that $h_i(x) \le y_i \le g_i(x)$, then $(x, y, t(x, f(x))) \in O$.

Let $\rho: M^n \to M^d$ denote the projection onto the first d coordinates. By Remark 3.2, the restriction $\rho_{|C}: C \to C'$ to C is an homeomorphism onto C' whose inverse is $\operatorname{id} \times f: C' \to C$.

Let

$$K_C = \{(x, y) \in M^n : x \in C' \text{ and for all } i, h_i(x) \le y_i \le g_i(x)\}.$$

Then

$$s' := \mathrm{id} \times (t \circ (\mathrm{id} \times f) \circ \rho)$$

is a continuous definable section on K_C .

Let

 $S = \{x \in C' : \text{ for all } i, \text{ both } h_i \text{ and } g_i \text{ are continuous at } x\}.$

Then S is an open definable subset of C' and C' \ S has dimension smaller than d. Since $(id \times f)(S) \subseteq C$, $C \setminus (id \times f)(S) = (id \times f)(C' \setminus S)$ and $\dim C = \dim (\overline{C} \cap \pi(O))$, it follows that $(\overline{C} \cap \pi(O)) \setminus (id \times f)(S)$ has dimension smaller than d. (This follows all from properties of o-minimal dimension, see [10, Chapter 4, §1]).

By the induction hypothesis, $(\overline{C} \cap \pi(O)) \setminus (\operatorname{id} \times f)(S)$ can be covered by finitely many open definable subsets of $\pi(O)$ satisfying our requirements. Let W denote the union of these open definable subsets.

We still have to cover $C \setminus W$. If $\dim(C \setminus W) < d$ then we apply the induction hypothesis and we are done. So suppose that $\dim(C \setminus W) = d = \dim C$. Since $\rho_{|C}: C \to C'$ is a definable bijection, we also have $\dim \rho(C \setminus W) = \dim C'$.

Observe that:

Claim 3.5. $\rho(C \setminus W) \subseteq S$.

Proof. Since $(\operatorname{id} \times f)(C' \setminus S) \subseteq W \cap C$, have that $\rho(W \cap C)$ contains $C' \setminus S$. Since $\rho_{|C|}$ is a bijection, $\rho(C \setminus W) = C' \setminus \rho(W \cap C) \subseteq S$.

Let

 $K_{C\setminus W} = \{(x, y) \in M^n : x \in \rho(C \setminus W) \text{ and for all } i, h_i(x) \leq y_i \leq g_i(x)\}.$ Let V be the interior of $\rho(C \setminus W)$ and let

 $U = \{(x, y) \in M^n : x \in V \text{ and for all } i, h_i(x) < y_i < g_i(x)\}.$

Claim 3.6. Then $V \neq \emptyset$, U is open and $U \subseteq K_{C \setminus W}$.

Proof. The fact that $V \neq \emptyset$ follows from the fact that C' is open and dim $\rho(C \setminus W) = \dim C'$ (see [10, Chapter 4, (1.9)]). Since $V \subseteq \rho(C \setminus W)$, we have $U \subseteq K_{C \setminus W}$. Since $\rho(C \setminus W) \subseteq S$ (Claim 3.5), for all $i, h_i, g_i : V \to M$ are continuous definable maps. Therefore, U is open in $\rho^{-1}(V)$ and so it is open since V is open in C' and C' is open in M^d .

Since $U \subseteq K_{C \setminus W} \subseteq K_C$, we have that s' is continuous on U. We still have to cover $(C \setminus W) \setminus U$. However since C' is open and dim $\rho(C \setminus W) = \dim C'$ it follows that dim $(\rho(C \setminus W) \setminus V) < d$ (see [10, Chapter 4, (1.9)]). So dim $((C \setminus W) \setminus U) < d$

and we conclude by the induction hypothesis.

Let $Z \subseteq M^{n+1}$ be a definable set, X a definable space and let $f: X \to \pi(Z)$ be a definable map. A *lifting of* f is a continuous map $g: X \to Z$ such that $\pi \circ g = f$.

Lemma 3.7. Let $Z \subseteq M^{n+1}$ be a definable set. Let $\mathcal{I} = \langle I, \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$ be a basic dinterval, $\gamma : \mathcal{I} \to \pi(Z)$ be a definable path in $\pi(Z)$ and let $\alpha, \beta : \mathcal{I} \to Z$ be definable paths lifting γ . Suppose that for all $t \in I$ we have $\{(\gamma(t), t) : \min(\tau \circ \alpha(t), \tau \circ \beta(t)) \leq t \leq \max(\tau \circ \alpha(t), \tau \circ \beta(t))\} \subseteq Z$. Then $\alpha \sim \beta$. Moreover, if $\alpha_0 = \beta_0$ and $\alpha_1 = \beta_1$, then the definable homotopy $\alpha \sim \beta$ fixes the endpoints.

Proof. Let $\mu : \mathcal{I} \to C$ be the definable path given by

$$\mu(t) = (\gamma(t), \min\left(\tau \circ \alpha(t), \tau \circ \beta(t)\right)).$$

Let $a = \max\{\max(\tau \circ \alpha(t), \tau \circ \beta(t)) : t \in I\}$ and let $b = \min\{\min(\tau \circ \alpha(t), \tau \circ \beta(t)) : t \in I\}$. If a = b then $\alpha = \beta$, so we may assume that b < a. Consider the basic *d*-interval $\mathcal{K} = \langle [b, a], \langle a, b \rangle \rangle$. Let $F : \mathcal{I} \times \mathcal{K} \to C$ be the continuous definable map given by

$$F(t,r) = (\gamma(t), \max(\tau \circ \mu(t), \min(\tau \circ \alpha(t), r))).$$

Then $F_0 = \alpha$ and $F_1 = \mu$. Therefore, $\alpha \sim \mu$. Similarly, $\beta \sim \mu$. Hence $\alpha \sim \beta$. \Box

We will want to show that a given definable path is definably homotopic to a second definable path, but we cannot do it because their domains are not the same. For this we will modify the first definable path by patching to it appropriate constants that allow it to "wait" for the second definable path. That "waiting period" can occur either before or after (or both) appropriate basic *d*-intervals that composes the *d*-interval of the domain of the first path.

We say that a definable path $\gamma' : \mathcal{I}' \to X$ is obtained from a definable path $\gamma : \mathcal{I} \to X$ by modifying with constants if there are $0_{\mathcal{I}} = t_0 <_{\mathcal{I}} t_1 <_{\mathcal{I}} \cdots <_{\mathcal{I}} t_r = 1_{\mathcal{I}}$ and there are *d*-intervals $\mathcal{J}_0, \ldots, \mathcal{J}_r$ such that, if $\gamma^i := \gamma_{|[t_i, t_{i+1}]_{\mathcal{I}}}$ for $i = 0, 1, \ldots, r-1$, then one of the following three cases holds:

(1)
$$\mathcal{I}' = \mathcal{J}_0 \wedge [t_0, t_1]_{\mathcal{I}} \wedge \ldots \wedge \mathcal{J}_{r-1} \wedge [t_{r-1}, t_r]_{\mathcal{I}} \text{ and } \gamma' = (\mathbf{c}_{\mathcal{J}_0}^{\gamma_0^0} \cdot \gamma^0) \cdots (\mathbf{c}_{\mathcal{J}_{r-1}}^{\gamma_0^{n-1}} \cdot \gamma^{r-1}).$$

(2) $\mathcal{I}' = \mathcal{J}_0 \wedge [t_0, t_1]_{\mathcal{I}} \wedge \ldots \wedge \mathcal{J}_{r-1} \wedge [t_{r-1}, t_r]_{\mathcal{I}} \wedge \mathcal{J}_r \text{ and } \gamma' = (\mathbf{c}_{\mathcal{J}_0}^{\gamma_0^0} \cdot \gamma^0) \cdots (\mathbf{c}_{\mathcal{J}_{r-1}}^{\gamma_{r-1}^{n-1}} \cdot \gamma^{r-1}) \cdot \mathbf{c}_{\mathcal{J}_{r}}^{\gamma_{r-1}^{n-1}}.$
(3) $\mathcal{I}' = [t_0, t_1]_{\mathcal{I}} \wedge \mathcal{J}_1 \wedge [t_1, t_2]_{\mathcal{I}} \wedge \ldots \wedge \mathcal{J}_{r-1} \wedge [t_{r-1}, t_r]_{\mathcal{I}} \wedge \mathcal{J}_r \text{ and } \gamma' = \gamma^0 \cdot (\mathbf{c}_{\mathcal{J}_{1}}^{\gamma_{1}^{n-1}} \cdot \gamma_{1}) \cdots (\mathbf{c}_{\mathcal{J}_{r-1}}^{\gamma_{r-1}^{n-1}} \cdot \gamma^{r-1}) \cdot \mathbf{c}_{\mathcal{J}_{r}}^{\gamma_{r-1}^{n-1}}.$

Lemma 3.8. If a definable path $\gamma' : \mathcal{I}' \to X$ is obtained from a definable path $\gamma : \mathcal{I} \to X$ by modifying with constants, then $\gamma \approx \gamma'$.

Proof. Consider the first case. By Fact 2.7,

$$\mathbf{c}_{[t_i,t_{i+1}]_{\mathcal{I}}}^{(\mathbf{c}_{\mathcal{J}_i}^{\gamma_0^i}\cdot\gamma^i)_0} \cdot (\mathbf{c}_{\mathcal{J}_i}^{\gamma_0^i}\cdot\gamma^i) = \mathbf{c}_{[t_i,t_{i+1}]_{\mathcal{I}}\wedge\mathcal{J}_i} \cdot \gamma^i \sim \gamma^i \cdot \mathbf{c}_{\mathcal{J}_i\wedge[t_i,t_{i+1}]_{\mathcal{I}}}^{\gamma_1^i}$$

and so $(\mathbf{c}_{\mathcal{J}_i}^{\gamma_0^i} \cdot \gamma^i) \approx \gamma^i$. Since $\gamma = \gamma^0 \cdot \gamma^1 \cdots \gamma^{r-1}$, we conclude by Fact 2.10 that $\gamma \approx \gamma'$.

The third case is similar and the second case follows from the first two cases by transitivity of \approx .

Remark 3.9. Let $I = \prod_{i=1}^{n} [t_{i-1}, t_i] \subseteq M^n$ be a product of intervals. By [32, Theorem 2.1], I is a Hausdorff, definably compact definable space. Since \mathbb{M} has definable Skolem functions, it follows that I is definably normal ([16, Theorem 2.11]).

Lemma 3.10. Let $C \subseteq M^{n+1}$ be an open cell. Then the following hold.

- (1) Let $\gamma : \mathcal{I} \to \pi(C)$ be a definable path in $\pi(C)$. Let $(x, y) \in C$ be such that $x = \gamma_0$. Then there is a definable path $\gamma' : \mathcal{I}' \to \pi(C)$ obtained from γ by modifying with constants and there is a lifting $\beta : \mathcal{I}' \to C$ of γ' with $\beta_0 = (x, y)$.
- (2) Suppose that $F: \mathcal{I} \times \mathcal{J} \to \pi(C)$ is a definable homotopy between the definable paths $\gamma, \sigma: \mathcal{I} \to \pi(C)$ in $\pi(C)$. Then there are definable paths $\gamma', \sigma': \mathcal{I}' \to \pi(C)$ obtained by modifying with constants γ and σ respectively, and there are liftings $\beta, \tau: \mathcal{I}' \to C$ of γ' and σ' respectively such that $\beta \sim \tau$.

Proof. By Theorem 3.4, there is a finite cover $\{U_i : i = 1, ..., m\}$ of $\pi(C)$ by open definable subsets such that for each *i* there is a continuous definable section $s_i : U_i \to C$ of the projection π (i.e. $\pi \circ s_i = id_{U_i}$).

(1) First we assume that \mathcal{I} is a basic *d*-interval $\langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$. We may also assume that the definable total order $\langle_{\mathcal{I}}$ on the domain [a, b] of \mathcal{I} is \langle . If not, the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

Let $L = \{l : \gamma([a, b]) \cap U_l \neq \emptyset\}$. Then $[a, b] \subseteq \bigcup_{l \in L} \gamma^{-1}(U_l)$, with the $\gamma^{-1}(U_l)$'s open in [a, b]. So by Remark 3.9 and the shrinking lemma ([16, Corollary 2.12], [10, Chapter 6, (3.6)]), for each $l \in L$ there is $W_l \subset [a, b]$, open in [a, b] such that $W_l \subset \overline{W_l} \subset \gamma^{-1}(U_l)$ and $[a, b] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $a = t_0 <$ $t_1 < \cdots < t_r = b$ such that for each $i = 0, \ldots, r-1$ we have $\gamma([t_i, t_{i+1}]) \subset U_{l(i)}$ (and $\gamma(t_{i+1}) \in U_{l(i)} \cap U_{l(i+1)}$). For each $i = 0, \ldots, r-1$ let $\gamma^i := \gamma_{\mid [t_i, t_{i+1}]}$. Then $s_{l(i)} \circ \gamma^i : [t_i, t_{i+1}] \to C$ is a definable path joining $s_{l(i)}(\gamma_0^i)$ to $s_{l(i)}(\gamma_1^i)$. Now, observe that the points $s_{l(i)}(\gamma_1^i)$ and $s_{l(i+1)}(\gamma_0^{i+1})$ differ just by the y coordinate, and the same happens with the points (x, y) and $s_{l(0)}(\gamma_0^0)$. Such pairs of points are clearly connected by (definable) vertical paths. Let $\nu^0 : \mathcal{J}_0 \to C$ be the vertical path with $\nu_0^0 = (x, y)$ and $\nu_1^0 = s_{l(0)}(\gamma_0^0)$ and for $i = 1, \ldots r-1$ let $\nu^i : \mathcal{J}_i \to C$ the vertical path with $\nu_0^i = s_{l(i-1)}(\gamma_1^{i-1})$ and $\nu_1^i = s_{l(i)}(\gamma_0^i)$. Let $\mathcal{I}' = \mathcal{J}_0 \land [t_0, t_1] \land \ldots \land \mathcal{J}_{r-1} \land [t_{r-1}, t_r]$ and let $\beta = \nu^0 \cdot (s_{l(0)} \circ \gamma^0) \cdots \nu^{r-1} \cdot (s_{l(r-1)} \circ \gamma^{r-1})$. Let $\gamma' : \mathcal{I}' \to \pi(C)$ be given by $\gamma' = (\mathbf{c}_{\mathcal{J}_0}^{\gamma_0^0} \cdot \gamma^0) \cdots (\mathbf{c}_{\mathcal{J}_{r-1}}^{\gamma_{r-1}} \cdot \gamma^{r-1})$. Then the definable path $\beta : \mathcal{I}' \to C$ in C is a lifting of γ' such that $\beta_0 = (x, y)$.

Now if $\mathcal{I} = \mathcal{I}_1 \wedge \ldots \wedge \mathcal{I}_k$ with each \mathcal{I}_i a basic *d*-interval apply the previous process to $\gamma_{|\mathcal{I}_1|}$ to get β^1 , γ'^1 with $\beta^1(0_{\mathcal{I}_1}) = (x, y)$ and repeat the process for each $\gamma_{|\mathcal{I}_{i+1}}$ with $\beta^i(1_{\mathcal{I}_i})$ instead of (x, y). Patch these together to obtain β and γ' .

(2) Here we cannot just apply (1) to γ and σ since the liftings β and τ of the corresponding modifications by constants need not be definably homotopic.

Instead we need to use the definable homotopy F to build a sequence of definable paths which are "close" enough to guarantee that the liftings of the corresponding modifications by constants are definably homotopic. The sequence of definable paths will be obtained from a cell decomposition of the domain of the definable homotopy F compatible with the pull backs of the open definable subsets of $\pi(C)$ on which we have continuous definable sections (Theorem 3.4) and the liftings of modifications by constants of these definable paths will be obtained using the continuous definable sections.

First assume that \mathcal{J} is a basic *d*-interval $\langle [c, d], \langle 0_{\mathcal{J}}, 1_{\mathcal{J}} \rangle \rangle$. We may also assume that the definable total order $\langle_{\mathcal{J}}$ on the domain [c, d] of \mathcal{J} is \langle . If not, the argument is similar, one just has to construct the lifting from top to bottom instead of from bottom to top.

To proceed we also assume that \mathcal{I} is a basic *d*-interval $\langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$. We may furthermore assume that the definable total order $\langle_{\mathcal{I}}$ on the domain [a, b] of \mathcal{I} is \langle . If not the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

Let $L = \{l : F([a, b] \times [c, d]) \cap U_l \neq \emptyset\}$. Then $[a, b] \times [c, d] \subseteq \bigcup_{l \in L} F^{-1}(U_l)$, with the $F^{-1}(U_l)$'s open in $[a, b] \times [c, d]$. So by Remark 3.9 and the shrinking lemma ([16, Corollary 2.12], [10, Chapter 6, (3.6)]), we have that for each $l \in L$ there is $W_l \subset [a, b] \times [c, d]$, open in $[a, b] \times [a, d]$ such that $W_l \subset \overline{W_l} \subset F^{-1}(U_l)$ and $[a, b] \times [c, d] \subseteq \bigcup_{l \in L} W_l$. Now take a cell decomposition of $[a, b] \times [c, d]$ compatible with the W_l 's. This cell decomposition is given by a decomposition $a = t_0 < t_1 < \cdots < t_r = b$ of [a, b] together with definable continuous functions $f_{i,j} : [t_i, t_{i+1}] \rightarrow$ [c, d] for $i = 0, \ldots, r - 1$ and $j = 0, \ldots, k_i$ such that: (i) $f_{i,0} < f_{i,1} < \ldots < f_{i,k_i}$ for $i = 0, \ldots, r-1$; (ii) $f_{i,0} = c$ and $f_{i,k_i} = d$ for $i = 0, \ldots, r-1$; (iii) the two-dimensional cells are of the form $C_{i,j} = (f_{i,j}, f_{i,j+1})_{(t_i,t_{i+1})}$. For each two-dimensional cell $C_{i,j}$ and each l(i, j) such that $C_{i,j} \subset W_{l(i,j)}$, we have $F(\overline{C_{i,j}} \cap \overline{C_{i',j'}}) \subset U_{l(i,j)} \cap U_{l(i',j')}$.

Fix a sequence $\overline{j} = (j_0, j_1, \dots, j_{r-1})$ with $j_i \in \{0, 1, \dots, n_i\}$ for each $i = 0, \dots, r-1$. For each $i = 1, \dots, r$ let $r^{\overline{j},i} : \mathcal{R}_{\overline{j},i} \to [a,b] \times [c,d]$ be the vertical path in $[a,b] \times [c,d]$ with $r_0^{\overline{j},i} = (t_i, f_{i-1,j_{i-1}}(t_i))$ and $r_1^{\overline{j},i} = (t_i, f_{i-1,\min(j_{i-1}+1,n_{i-1})}(t_i))$. This is the vertical path, going up on the right hand side of the "cell"

$$[f_{i-1,j_{i-1}}, f_{i-1,\min(j_{i-1}+1,n_{i-1})}]_{(t_{i-1},t_i)}.$$

For each $i = 0, \ldots, r-1$ let $l^{\overline{j},i} : \mathcal{L}_{\overline{j},i} \to [a,b] \times [c,d]$ be the vertical path in $[a,b] \times [c,d]$ with $l_0^{\overline{j},i} = (t_i, f_{i,j_i}(t_i))$ and $l_1^{\overline{j},i} = (t_i, f_{i,\min(j_i+1,n_i)}(t_i))$. This is the vertical path, going up on the left hand side of the "cell"

$$[f_{i,j_i}, f_{i,\min(j_i+1,n_i)}]_{(t_i,t_{i+1})}.$$

For i = 0, 1, ..., r - 1 let

$$a^{j,i}: [t_i, t_{i+1}] \rightarrow [a, b] \times [c, d]$$

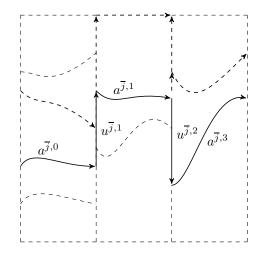
be the definable path given by

$$a^{j,i}(t) = (t, f_{i,j_i}(t)).$$

For each $i = 1, \ldots, r-1$ let $u^{\overline{j},i} : \mathcal{U}_{\overline{j},i} \to [a,b] \times [c,d]$ be the vertical path in $[a,b] \times [c,d] \text{ with } u_0^{\overline{j},i} = (t_i, f_{i-1,j_{i-1}}(t_i)) = a_1^{\overline{j},i-1} \text{ and } u_1^{\overline{j},i} = (t_i, f_{i,j_i}(t_i)) = a_0^{\overline{j},i}.$ Let

 $a^{\overline{j}}: [t_0, t_1] \land \mathcal{U}_{\overline{j}, 1} \dots \land \mathcal{U}_{\overline{j}, r-1} \land [t_{r-1}, t_r] \to [a, b] \times [c, d]$ be the definable path given by

 $a^{\overline{j}} = a^{\overline{j},0} \cdot u^{\overline{j},1} \cdot a^{\overline{j},1} \cdot \cdots \cdot u^{\overline{j},r-1} \cdot a^{\overline{j},r-1}$



For $k = 0, 1, \ldots, r - 1$ let $\overline{j}[k]$ the sequence which is equal to \overline{j} except in position k where it is $\min(j_k + 1, n_k)$.

By Lemma 3.7, we have:

Remark 3.11. If y is a point on the right hand side of C_{k,j_k} let $y^- : \mathcal{Y}^- \to$

 $\begin{array}{l} [a,b] \times [c,d] \ be \ the \ vertical \ path \ such \ that \ y_0^- = a_1^{\overline{j}[k],k} \ and \ y_1^- = y \ and \ let \ y^+ : \\ \mathcal{Y}^+ \to [a,b] \times [c,d] \ be \ the \ vertical \ path \ such \ that \ y_0^- = a_1^{\overline{j}[k],k} \ and \ y_1^- = y. \\ If \ x \ is \ a \ point \ on \ the \ left \ hand \ side \ of \ C_{k,j_k} \ let \ x^+ : \mathcal{X}^+ \to [a,b] \times [c,d] \ be \ the \\ vertical \ path \ such \ that \ x_0^+ = x \ and \ x_1^+ = a_0^{\overline{j}[k],k} \ and \ let \ x^- : \mathcal{X}^- \to [a,b] \times [c,d] \ be \\ the \ vertical \ path \ such \ that \ x_0^- = x \ and \ x_1^- = a_0^{\overline{j},k}. \end{array}$ Then

(i)

$$\begin{aligned} a^{\overline{j}[0],0} \cdot y^{-} \cdot \mathbf{c}_{\mathcal{Y}^{+}}^{y} &\sim a^{\overline{j},0} \cdot \mathbf{c}_{\mathcal{Y}^{-}}^{a_{1}^{\overline{j},0}} \cdot y^{+} \\ and \ a^{\overline{j}[0],i} &= a^{\overline{j},i} \ for \ i \neq 0. \end{aligned}$$
(ii)
$$\begin{aligned} x^{+} \cdot \mathbf{c}_{\mathcal{X}^{-}}^{x_{1}^{+}} \cdot a^{\overline{j}[k],k} \cdot y^{-} \cdot \mathbf{c}_{\mathcal{Y}^{+}}^{y} &\sim \mathbf{c}_{\mathcal{X}^{+}}^{x} \cdot x^{-} \cdot a^{\overline{j},k} \cdot \mathbf{c}_{\mathcal{Y}^{-}}^{a_{1}^{\overline{j},k}} \cdot y^{+} \\ and \ a^{\overline{j}[k],i} &= a^{\overline{j},i} \ for \ i \neq k. \end{aligned}$$

$$\begin{split} x^+ \cdot \mathbf{c}_{\mathcal{X}^-}^{x_1^+} \cdot a^{\overline{j}[r-1],r-1} \sim \mathbf{c}_{\mathcal{X}^+}^x \cdot x^- \cdot a^{\overline{j},r-1} \\ and \ a^{\overline{j}[r-1],i} = a^{\overline{j},i} \ for \ i \neq r-1. \end{split}$$

On the other hand we also have, see the picture on page 14:

Remark 3.12. We have:

- (i) $u^{\overline{j}[0],1} = (r^{\overline{j},1})^{-1} \cdot u^{\overline{j},1}$ or $u^{\overline{j},1} = r^{\overline{j},1} \cdot u^{\overline{j}[0],1}$ or $u_1^{\overline{j}[0],1} = u_1^{\overline{j},1}$ is on the right hand side of C_{0,j_0} . Furthermore, $u^{\overline{j}[0],i} = u^{\overline{j},i}$ for $i \neq 1$.
- (ii) $u^{\overline{j}[k],k} = u^{\overline{j},k} \cdot l^{\overline{j},k}$ or $u^{\overline{j},k} = u^{\overline{j}[k],k} \cdot (l^{\overline{j},k})^{-1}$ or $u^{\overline{j}[k],k} = u^{\overline{j},k}_{0}$ is on the left hand side of C_{k,j_k} . $u^{\overline{j}[k],k+1} = (r^{\overline{j},k+1})^{-1} \cdot u^{\overline{j},k+1}$ or $u^{\overline{j},k+1} = r^{\overline{j},k+1} \cdot u^{\overline{j}[k],k+1}$ or $u^{\overline{j}[k],k+1} = u^{\overline{j},k+1}_{1}$ is on the right hand side of C_{k,j_k} . Furthermore, $u^{\overline{j}[k],i} = u^{\overline{j},i}$ for $i \neq k, k+1$.
- (ii) $u^{\overline{j}[r-1],r-1} = u^{\overline{j},r-1} \cdot l^{\overline{j},r-1} \text{ or } u^{\overline{j},r-1} = u^{\overline{j}[r-1],r-1} \cdot (l^{\overline{j},r-1})^{-1} \text{ or } u^{\overline{j}[r-1],r-1} = u^{\overline{j},r-1} \text{ is on the left hand side of } C_{r-1,j_{r-1}}.$ Furthermore, $u^{\overline{j}[r-1],i} = u^{\overline{j},i} \text{ for } i \neq r-1.$

Let $\mu^{\overline{j},i} = F \circ u^{\overline{j},i} : \mathcal{U}_{\overline{j},i} \to \pi(C), \ \rho^{\overline{j},i} = F \circ r^{\overline{j},i} : \mathcal{R}_{\overline{j},i} \to \pi(C), \ \lambda^{\overline{j},i} = F \circ l^{\overline{j},i} : \mathcal{L}_{\overline{j},i} \to \pi(C), \ \alpha^{\overline{j},i} = F \circ a^{\overline{j},i} : [t_i, t_{i+1}] \to \pi(C) \text{ and}$

$$\alpha^{\overline{j}} = F \circ a^{\overline{j}} : [t_0, t_1] \wedge \mathcal{U}_{\overline{j}, 1} \dots \wedge \mathcal{U}_{\overline{j}, r-1} \wedge [t_{r-1}, t_r] \to \pi(C).$$

Then

$$\alpha^{\overline{j}} = \alpha^{\overline{j},0} \cdot \mu^{\overline{j},1} \cdot a^{\overline{j},1} \cdot \cdots \cdot \mu^{\overline{j},r-1} \cdot \alpha^{\overline{j},r-1},$$

 $\alpha^{\overline{0}} = \gamma$ and $\alpha^{\overline{n}} = \sigma$ where $\overline{0} = (0, 0, \dots, 0)$ and $\overline{n} = (n_0, n_1, \dots, n_{r-1})$.

Applying F to Remarks 3.11 and 3.12 we obtain:

Claim 3.13. For every \overline{j} and every k, after modifying with constants, that we ignore for simplicity,

 $\alpha^{\overline{j}} \sim \alpha^{\overline{j}[k]}.$

Moreover, since for every $\overline{j'}$ there are k_1, \ldots, k_m such that $\overline{j'} = \overline{j}[k_1] \ldots [k_m]$ we also have, by transitivity of ~ (Fact 2.7), ignoring modifications by constants,

$$\alpha^{\overline{j}} \sim \alpha^{\overline{j'}}.$$

Proof. Suppose that k = 0. Then $\alpha^{\overline{j}} = \alpha^{\overline{j},0} \cdot \mu^{\overline{j},1} \cdot \alpha'$ and $\alpha^{\overline{j}[0]} = \alpha^{\overline{j}[0],0} \cdot \mu^{\overline{j}[0],1} \cdot \alpha'$. By Remark 3.12 (i) we have three cases. Suppose that $u^{\overline{j}[0],1} = (r^{\overline{j},1})^{-1} \cdot u^{\overline{j},1}$. Then $\mu^{\overline{j}[0],1} = (\rho^{\overline{j},1})^{-1} \cdot \mu^{\overline{j},1}$. Therefore, $\alpha^{\overline{j}[0]} = \alpha^{\overline{j}[0],0} \cdot (\rho^{\overline{j},1})^{-1} \cdot \mu^{\overline{j},1} \cdot \alpha'$. Since $a^{\overline{j}[0],0} \cdot (r^{\overline{j},1})^{-1} \sim a^{\overline{j},0}$ by Remark 3.11, we get $\alpha^{\overline{j}[0],0} \cdot (\rho^{\overline{j},1})^{-1} \sim \alpha^{\overline{j},0}$ and the result follows. The other cases are similar.

Suppose that $k = 1, \ldots, r-2$. Then $\alpha^{\overline{j}} = \alpha'' \cdot \mu^{\overline{j},k} \cdot \alpha^{\overline{j},k} \cdot \mu^{\overline{j},k+1} \cdot \alpha'$ and $\alpha^{\overline{j}[k]} = \alpha'' \cdot \mu^{\overline{j}[k],k} \cdot \alpha^{\overline{j}[k],k} \cdot \mu^{\overline{j}[k],k+1} \cdot \alpha'$. By Remark 3.12 (i) we have several cases. Suppose that $u^{\overline{j},k} = u^{\overline{j}[k],k} \cdot (l^{\overline{j},k})^{-1}$ and $u_1^{\overline{j}[k],k+1} = u_1^{\overline{j},k+1}$ is on the right hand side of C_{k,j_k} .

(iii)

Then $\mu^{\overline{j},k} = \mu^{\overline{j}[k],k} \cdot (\lambda^{\overline{j},k})^{-1}$ and so $\alpha^{\overline{j}} = \alpha'' \cdot \mu^{\overline{j}[k],k} \cdot (\lambda^{\overline{j},k})^{-1} \cdot \alpha^{\overline{j},k} \cdot \mu^{\overline{j},k+1} \cdot \alpha'.$ Since by Remark 3.11, $(l^{\overline{j},k})^{-1} \cdot a^{\overline{j},k} \cdot u^{\overline{j},k+1} \sim a^{\overline{j}[k],k} \cdot u^{\overline{j}[k],k+1}$ we obtain $(\lambda^{\overline{j},k})^{-1} \cdot a^{\overline{j},k} \cdot u^{\overline{j},k+1} \cdot \alpha'.$ $\alpha^{\overline{j},k} \cdot \mu^{\overline{j},k+1} \sim \alpha^{\overline{j}[k],k} \cdot \mu^{\overline{j}[k],k+1}$ and the result follows. The other cases are similar. For k = r - 1 the argument is the same.

By Lemmas 3.7 and 3.8 and the transitivity of \sim (Fact 2.7), to finish the proof it is enough to show that after modifying with constants, $\alpha^{\overline{j}}$ and $\alpha^{\overline{j}[k]}$ have liftings $\beta^{\overline{j}}$ and $\beta^{\overline{j}[k]}$ respectively such that

$$\beta^{\overline{j}} \sim \beta^{\overline{j}[k]},$$

ignoring modifications by constants.

Suppose that k = 0. Then $\alpha^{\overline{j}} = \alpha^{\overline{j},0} \cdot \mu^{\overline{j},1} \cdot \alpha'$ and $\alpha^{\overline{j}[0]} = \alpha^{\overline{j}[0],0} \cdot \mu^{\overline{j}[0],1} \cdot \alpha'$. By Remark 3.12 (i) we have three cases. Suppose that $u^{\overline{j}[0],1} = (r^{\overline{j},1})^{-1} \cdot u^{\overline{j},1}$. Then $\mu^{\overline{j}[0],1} = (\rho^{\overline{j},1})^{-1} \cdot \mu^{\overline{j},1}. \text{ Therefore, } \alpha^{\overline{j}[0]} = \alpha^{\overline{j}[0],0} \cdot (\rho^{\overline{j},1})^{-1} \cdot \mu^{\overline{j},1} \cdot \alpha'.$

By (1) let μ be a lifting of (a modification by constants of) $\mu^{\overline{j},1}$ and let β' be a lifting of (a modification by constants of) α' such that $\beta'_0 = \mu_1$. Recall that $F(\overline{C_{0,j_0}}) \subseteq U_{l(0,j_0)}. \text{ Thus } \rho^{\overline{j},1} = F \circ r^{\overline{j},1} : \mathcal{R}_{\overline{j},1} \to U_{l(0,j_0)} \subseteq \pi(C), \ \alpha^{\overline{j},0} = F \circ a^{\overline{j},0} : [t_0,t_1] \to U_{l(0,j_0)} \subseteq \pi(C) \text{ and } \alpha^{\overline{j}[0],0} = F \circ a^{\overline{j}[0],0} : [t_0,t_1] \to U_{l(0,j_0)} \subseteq \pi(C). \text{ Let } \beta^{\overline{j},0} = s_{l(0,j_0)} \circ \alpha^{\overline{j},0}, \ \beta^{\overline{j}[0],0} = s_{l(0,j_0)} \circ \alpha^{\overline{j}[0],0} \text{ and } \rho = s_{l(0,j_0)} \circ \rho^{\overline{j},1}. \text{ Let } \delta : \mathcal{D} \to C$ be the vertical path such that $\delta_0 = \beta_1^{\overline{j},0}$ and $\delta_1 = \mu_0$. Let $\beta^{\overline{j}} = \beta^{\overline{j},0} \cdot \delta \cdot \mu \cdot \beta'$ and $\beta^{\overline{j}[0]} = \beta^{\overline{j}[0],0} \cdot \rho^{-1} \cdot \delta \cdot \mu \cdot \beta'$. Then $\beta^{\overline{j}}$ and $\beta^{\overline{j}[0]}$ are

lifting of $\alpha^{\overline{j}}$ and $\alpha^{\overline{j}[k]}$ respectively after being modified by constants. Since $a^{\overline{j}[0],0} \cdot (r^{\overline{j},1})^{-1} \sim a^{\overline{j},0}$ by Remark 3.11, applying F we get $\alpha^{\overline{j}[0],0} \cdot (\rho^{\overline{j},1})^{-1} \sim a^{\overline{j},0}$ $\alpha^{\overline{j},0}$ and applying $s_{l(0,j_0)}$ we get $\beta^{\overline{j}[0],0} \cdot \rho^{-1} \sim \beta^{\overline{j},0}$. Therefore, $\beta^{\overline{j}} \sim \beta^{\overline{j}[0]}$.

For all the other cases the argument is similar.

Now if $\mathcal{I} = \mathcal{I}_1 \land \ldots \land \mathcal{I}_k$ with each \mathcal{I}_i a basic *d*-interval apply the previous process to each $F_{|\mathcal{I}_i \times [c,d]}$, $\gamma_{|\mathcal{I}_i}$ and $\sigma_{|\mathcal{I}_i}$ to get γ'^i , σ'^i , β^i and τ^i such that $\beta^i \sim \tau^i$. If needed use (1) to replace the β^{i} 's so that they patch together and similarly for the τ^{i} 's. Patch all these to get γ' , σ' , β and τ and the result follows from transitivity and compatibility with concatenation of \sim (Fact 2.7).

Now if $\mathcal{J} = \mathcal{J}_1 \wedge \ldots \wedge \mathcal{J}_k$ with each \mathcal{J}_j a basic *d*-interval apply the previous process to each $F_{|\mathcal{I}\times\mathcal{J}_j}$ and conclude by the transitivity of \sim .

The main consequence of Lemma 3.10 is the following:

Lemma 3.14. Let $C \subseteq M^n$ be a cell. Then:

- (1) C is definably path connected. In fact there is a uniformly definable family of definable paths connecting a given fixed point in C to any other point in C.
- (2) C is definably simply connected, i.e $\pi_1(C) = 1$.

Proof. (1) The proof is by complete induction on the dimension n of the ambient space. If n = 0, the space is reduced to a single point and the result follows trivially. Assume that the result holds for every k < n. In order to show that it still holds for *n* we proceed by a new induction on the definition of cells. The zero-dimensional case is immediate. If $C = \Gamma(f) \subseteq M^{n+1}$ is the graph of a definable continuous function $f: B \to M$, where $B \subseteq M^n$ is a cell, then the projection of C onto B is a definable homeomorphism and the result follows by the induction hypothesis. It remains to be considered the following case

$$C = \{ (x, y) \in B \times M : f(x) < y < g(x) \}$$

where $f, g: B \to M$ are definable continuous maps, $B \subseteq M^n$ is a cell and f < g.

If $C \subseteq M^{n+1}$ is not open, then dim C < n + 1. By Remark 3.2, C is definably homeomorphic to an open cell $D \subseteq M^{\dim C}$ and the result follows by the main induction hypothesis.

Assume that C is an open cell. By Theorem 3.4, there is a finite cover $\{U_i : i = 1, \ldots, m\}$ of B by open definable subsets such that for each i there is a continuous definable section $s_i : U_i \to C$ of the projection $\pi : M^{n+1} \to M^n$ onto the first n coordinates (i.e. $\pi \circ s_i = \mathrm{id}_{U_i}$). By Fact 3.3, after replacing the U_i 's if needed, we may assume that each U_i is definably homeomorphic to an open cell in M^n . So by the induction hypothesis, each U_i is definably path connected, in fact there is a uniformly definable family of definable paths connecting a given fixed point in U_i to any other point in U_i .

For each *i* fix $u_i \in U_i$. Since *B* is definably path connected, for each *i*, *j*, let $\gamma^{i,j}$ be a definable path in *B* such that $\gamma_0^{i,j} = u_i$ and $\gamma_1^{i,j} = u_j$. If needed modify each $\gamma^{i,j}$ by constants and take by Lemma 3.10 (1) a lifting $\beta^{i,j}$ such that $\beta_0^{i,j} = s_i(u_i)$ and $\beta_1^{i,j} = s_j(u_j)$. Then there is a uniformly definable family of definable paths in *C* connecting any point of $s_i(U_i)$ to any point of $s_j(U_j)$. Since vertical paths are uniformly definable, the same holds for $(\pi_{|C})^{-1}(U_i)$ and $(\pi_{|C})^{-1}(U_j)$. Since $C = (\pi_{|C})^{-1}(U_1) \cup \ldots \cup (\pi_{|C})^{-1}(U_m)$ the result follows.

(2) The proof is again by complete induction on the dimension n of the ambient space. If n = 0, the space is reduced to a single point and the result follows trivially. Assume that the result holds for every k < n. In order to show that it still holds for n we proceed by a new induction on the definition of cells. The only case to be examined is the case

$$C = \{ (x, y) \in B \times M : f(x) < y < g(x) \},\$$

where $f, g: B \to M$ are definable continuous maps, $B \subseteq M^n$ is a cell and f < g. (The zero-dimensional case is immediate; if $C = \Gamma(f) \subseteq M^{n+1}$ is the graph of a definable continuous function $f: B \to M$, where $B \subseteq M^n$ is a cell, then the projection of C onto B is a definable homeomorphism and the result follows by the induction hypothesis.)

If $C \subseteq M^{n+1}$ is not an open set, then dim C < n + 1. By Remark 3.2, we have that C is definably homeomorphic to an open cell $D \subseteq M^{\dim C}$ and the result follows by the main induction hypothesis.

Assume that *C* is an open set. Let $\alpha: \mathcal{I} \to C$ be a definable loop at $p \in C$. We want to show that $\alpha \approx \mathbf{c}_{\mathcal{I}}^p$. By the induction hypothesis, $\pi \circ \alpha \approx \mathbf{c}_{\mathcal{J}}^{\pi(p)}$. After modifying by constants, by Lemmas 3.8 and 3.10 (2) there are liftings β and τ of $\pi \circ \alpha$ and $\mathbf{c}_{\mathcal{J}}^{\pi(p)}$ such that $\beta \sim \tau$. Since after modifying by constants, α and $\mathbf{c}_{\mathcal{I}}^p$ are also liftings of $\pi \circ \alpha$ and $\mathbf{c}_{\mathcal{I}}^{\pi(p)}$, by Lemmas 3.7 and 3.8 and the transitivity of \sim we obtain $\alpha \approx \mathbf{c}_{\mathcal{T}}^p$.

By Fact 3.3 and Lemma 3.14 we have the following which shows (P1) (a) and (P2). Compare with the corresponding results [13, Lemma 2.9 and Proposition 3.1] in o-minimal expansions of ordered groups.

Corollary 3.15. Let X be a definable manifold of dimension n. Then the following hold:

- (1) X is definably connected if and only if X is definably path connected. In fact, for any definably connected definable subset D of X there is a uniformly definable family of definable paths in D connecting a given fixed point in D to any other point in D.
- (2) X has an admissible cover $\{O_s\}_{s\in S}$ by open definably connected definable subsets such that:
 - $\{O_s\}_{s\in S}$ refines the definable charts of X;
 - for each $s \in S$, O_s is definably homeomorphic to a cell of dimension n, in particular, the o-minimal fundamental group $\pi_1(O_s)$ is trivial.

Finally we show (P1) (b). For analogues compare with [18, Section 2] in ominimal expansions of fields or with [13, Lemma 2.13] in o-minimal expansions of ordered groups. In all three cases the proofs are the same, they only use the fact that the domains of the corresponding definable paths and definable homotopies are definably normal.

Lemma 3.16. Let X and S be locally definable manifolds with definable charts. Suppose that $p_X : X \to S$ is a locally definable covering map. Then the following hold.

- (1) Let $\gamma : \mathcal{I} \to S$ be a definable path in S. Let $x \in X$ be such that $p_X(x) = \gamma_0$. Then there exists a unique definable path $\tilde{\gamma} : \mathcal{I} \to X$ in X lifting γ such that $\tilde{\gamma}_0 = x$.
- (2) Suppose that F: I × J → S is a definable homotopy between the definable paths γ and σ in S. Let γ be a definable path in X lifting γ. Then there exists a definable path σ in X lifting σ and there exists is a unique definable lifting F : I × J → X of F, which is a definable homotopy between γ and σ.

Proof. Let $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$ be an admissible cover of S by open definable subsets over which $p_X : X \to S$ is trivial. We may assume that $\mathcal{U} = \{U_{\alpha}\}_{\alpha \in I}$ refines the definable charts of S witnessing the fact that S is a locally definable manifold with definable charts.

(1) First we assume that \mathcal{I} is a basic *d*-interval $\langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$. We may also assume that the definable total order $\langle_{\mathcal{I}}$ on the domain [a, b] of \mathcal{I} is \langle . If not, the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

Let $L \subseteq I$ be a finite subset such that $\gamma([a, b]) \subseteq \bigcup_{l \in L} U_l$. Then $[a, b] \subseteq \bigcup_{l \in L} \gamma^{-1}(U_l)$, with the $\gamma^{-1}(U_l)$'s open in [a, b]. So by Remark 3.9 and the shrinking lemma ([16, Corollary 2.12], [10, Chapter 6, (3.6)]), for each $l \in L$ there is $W_l \subset [a, b]$, open in [a, b] such that $W_l \subset \overline{W_l} \subset \gamma^{-1}(U_l)$ and $[a, b] \subseteq \bigcup_{l \in L} W_l$. Therefore, there are $a = t_0 < t_1 < \cdots < t_r = b$ such that for each $i = 0, \ldots, r-1$ we have $\gamma([t_i, t_{i+1}]) \subset U_{l(i)}$ (and $\gamma(t_{i+1}) \in U_{l(i)} \cap U_{l(i+1)})$. Lift $\gamma_1 = \gamma_{|[a,t_1]}$ to $\widetilde{\gamma_1} = (p_{|U_{l(0)}^{i_0}})^{-1} \circ \gamma_{|[a,t_1]}$, with $\widetilde{\gamma_{10}} = x$, using the definable

Lift $\gamma_1 = \gamma_{|[a,t_1]}$ to $\tilde{\gamma_1} = (p_{|U_{l(0)}^{i_0}})^{-1} \circ \gamma_{|[a,t_1]}$, with $\tilde{\gamma_{10}} = x$, using the definable homeomorphism $p_{|U_{l(0)}^{i_0}} : U_{l(0)}^{i_0} \to U_{l(0)}$, where $U_{l(0)}^{i_0}$ is the definable connected component of $p^{-1}(U_{l(0)})$ in which x lays. Repeat the process for each $\gamma_{i+1} = \gamma_{|[t_i,t_{i+1}]}$ with $\tilde{\gamma_i}(t_i)$ instead of x. Patch the liftings together to obtain $\tilde{\gamma}$.

Now if $\mathcal{I} = \mathcal{I}_1 \wedge \ldots \wedge \mathcal{I}_k$ with each \mathcal{I}_i a basic *d*-interval apply the previous process to lift $\gamma_1 = \gamma_{|\mathcal{I}_1|}$ to $\tilde{\gamma_1}$, with $\tilde{\gamma_{10}} = x$ and repeat the process for each $\gamma_{i+1} = \gamma_{|\mathcal{I}_{i+1}|}$ with $\tilde{\gamma_i}(1_{\mathcal{I}_i})$ instead of *x*. Patch the liftings together to obtain $\tilde{\gamma}$.

Uniqueness follows (in each step) from [13, Lemma 2.8].

(2) First assume that \mathcal{J} is a basic *d*-interval $\langle [c, d], \langle 0_{\mathcal{J}}, 1_{\mathcal{J}} \rangle \rangle$. We may also assume that the definable total order $\langle_{\mathcal{J}}$ on the domain [c, d] of \mathcal{J} is \langle . If not, the argument is similar, one just has to construct the lifting from top to bottom instead of from bottom to top.

To proceed we also assume that \mathcal{I} is a basic *d*-interval $\langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$. We may furthermore assume that the definable total order $<_{\mathcal{I}}$ on the domain [a, b] of \mathcal{I} is <. If not the argument is similar, one just has to construct the lifting from right to left instead of from left to right.

Let $L \subseteq I$ be a finite subset such that $F([a, b] \times [c, d]) \subseteq \bigcup_{l \in L} U_l$. Then $[a, b] \times [c, d] \subseteq \bigcup_{l \in L} F^{-1}(U_l)$, with the $F^{-1}(U_l)$'s open in $[a, b] \times [c, d]$. So by Remark 3.9 and the shrinking lemma ([16, Corollary 2.12], [10, Chapter 6, (3.6)]), we have that for each $l \in L$ there is $W_l \subset [a, b] \times [c, d]$, open in $[a, b] \times [a, d]$ such that $W_l \subset \overline{W_l} \subset F^{-1}(U_l)$ and $[a, b] \times [c, d] \subseteq \bigcup_{l \in L} W_l$. Now take a cell decomposition of $[a, b] \times [c, d]$ compatible with the W_l 's. This cell decomposition is given by a decomposition $a = t_0 < t_1 < \cdots < t_r = b$ of [a, b] together with definable continuous functions $f_{i,j} : [t_i, t_{i+1}] \to [c, d]$ for $i = 0, \ldots, r-1$ and $j = 0, \ldots, k_i$ such that: (i) $f_{i,0} < f_{i,1} < \ldots < f_{i,k_i}$ for $i = 0, \ldots, r-1$; (ii) $\Gamma(f_{i,0}) = [t_i, t_{i+1}] \times \{c\}$ and $\Gamma(f_{i,k_i}) = [t_i, t_{i+1}] \times \{d\}$ for $i = 0, \ldots, r-1$; (iii) the two-dimensional cells are of form $C_{i,j} = (f_{i,j}, f_{i,j+1})_{(t_i,t_{i+1})}$. For each two-dimensional cell $C_{i,j}$ and each l(i,j) such that $C_{i,j} \subset W_{l(i,j)}$, we have $F(\overline{C_{i,j}}) \subset U_{l(i,j)}$ and for any two-dimensional cells $C_{i,j}$ and $C_{i',j'} \subset W_{l(i',j')}$ we also have $F(\overline{C_{i,j}} \cap \overline{C_{i',j'}}) \subset U_{l(i,j)} \cap U_{l(i',j')}$.

Lift $F_{0,1} = F_{|\overline{C}_{0,1}}$ to $\widetilde{F}_{0,1} = (p_{|U_{l(0,1)}^{i_{0,1}}})^{-1} \circ F_{|\overline{C}_{0,1}}$, using the definable homeomorphism $p_{|U_{l(0,1)}^{i_{0,1}}} : U_{l(0,1)}^{i_{0,1}} \to U_{l(0,1)}$, where $U_{l(0,1)}^{i_{0,1}}$ is the definable connected component of $p^{-1}(U_{l(0,1)})$ in which $\widetilde{\gamma}([t_0, t_1])$ lays. Repeat the process for each $F_{0,j+1} = F_{|\overline{C}_{0,j+1}}$ with $\widetilde{F}_{0,j}(\Gamma(f_{0,j}))$ instead of $\widetilde{\gamma}([t_0, t_1])$. Patch the liftings together to obtain $\widetilde{F}_0 : [t_0, t_1] \times [c, d] \to X$ a definable lifting of $F_{|[t_0, t_1] \times [c, d]}$ which is a definable homotopy between $\widetilde{\gamma}_{|[t_0, t_1]}$ and $\widetilde{\sigma}_{|[t_0, t_1]}$. Repeat the above process again but now for each $i = 1, \ldots, r-1$, starting in each case with $\widetilde{\gamma}([t_i, t_{i+1}])$ and obtain the liftings $\widetilde{F}_i : [t_i, t_{i+1}] \times [c, d] \to X$ a definable lifting of $F_{|[t_i, t_{i+1}] \times [c, d]}$ which is a definable homotopy between $\widetilde{\gamma}_{|[t_i, t_{i+1}]}$ and $\widetilde{\sigma}_{|[t_i, t_{i+1}]}$. These liftings patch together to give a definable lifting $\widetilde{F} : [a, b] \times [c, d] \to X$ of F which is a definable homotopy between $\widetilde{\gamma}_{i}$ and $\widetilde{\sigma}$.

Now if $\mathcal{I} = \mathcal{I}_1 \land \ldots \land \mathcal{I}_k$ with each \mathcal{I}_i a basic *d*-interval apply the previous process to lift $F_1 = F_{|\mathcal{I}_1 \times [c,d]}$ to $\widetilde{F_1}$, with $\widetilde{F_1}(\mathcal{I}_1, c) = \widetilde{\gamma}(\mathcal{I}_1)$ and repeat the process for each $F_{i+1} = F_{|\mathcal{I}_{i+1} \times [c,d]}$ with $\widetilde{\gamma}(\mathcal{I}_{i+1})$ instead of $\widetilde{\gamma}(\mathcal{I}_1)$. Then patch these liftings together to obtain a definable lifting $\widetilde{F} : \mathcal{I} \times \mathcal{J} \to X$ of F which is a definable homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$.

Now if $\mathcal{J} = \mathcal{J}_1 \wedge \ldots \wedge \mathcal{J}_k$ with each \mathcal{J}_j a basic *d*-interval apply the previous process to lift $F_1 = F_{|\mathcal{I} \times \mathcal{J}_1}$ to $\widetilde{F_1}$, with $\widetilde{F_1}(\mathcal{I}, 0_{\mathcal{J}_1}) = \widetilde{\gamma}(\mathcal{I})$ and repeat the process for each $F_{j+1} = F_{|\mathcal{I} \times \mathcal{J}_{j+1}}$ with $\widetilde{F_j}(\mathcal{I}, 1_{\mathcal{J}_j})$ instead of $\widetilde{F_1}(\mathcal{I}, 0_{\mathcal{J}_1})$. To finish patch these liftings together to obtain a definable lifting $\widetilde{F} : \mathcal{I} \times [c, d] \to X$ of F which is a definable homotopy between $\widetilde{\gamma}$ and $\widetilde{\sigma}$.

As above, uniqueness follows from [13, Lemma 2.8].

3.2. Universal covering maps and fundamental groups. As explained in Subsection 3.1, from the main properties of definable paths and definable homotopies (Properties 3.1) we obtain, in arbitrary o-minimal structures with definable Skolem functions, in exactly the same way as in [13] for o-minimal expansions of ordered groups, all of the results stated below.

Theorem 3.17. Let X be a definably connected locally definable manifold. Then:

- (1) there exists a universal locally definable covering map $u : U \to X$. Moreover, if X is Lindelöf (resp. paracompact), then U is also Lindelöf (resp. paracompact).
- (2) If X is Lindelöf, then the o-minimal fundamental group $\pi_1(X)$ of X is countable. In fact, if X is definable, then $\pi_1(X)$ is finitely generated.

For similar previously known results in special cases see [3], [2], [7], [12], [21] and [22].

Theorem 3.18. Let \mathbb{J} be an elementary extension of \mathbb{M} or an o-minimal expansion of \mathbb{M} . Let X be a definably connected locally definable manifold. Then the following hold:

- (1) A universal locally \mathbb{J} -definable covering map of X is \mathbb{J} -definably homeomorphic to a universal locally definable covering map of X.
- (2) The o-minimal fundamental group of X in J is isomorphic to the o-minimal fundamental group of X in M.

Similarly, we have:

Theorem 3.19. Suppose that \mathbb{M} is an o-minimal expansion of the ordered set of real numbers. Let X be a definably connected locally definable manifold. Then the following hold:

- (1) A topological universal covering map of X is topologically homeomorphic to the o-minimal universal locally definable covering map of X.
- (2) The topological fundamental group of X is isomorphic to the o-minimal fundamental group of X.

For previously known analogues of these invariance results in special cases see [3], [1], [2]), [15], [7] and [8].

Remark 3.20. By Theorem 3.19 when \mathbb{M} is an o-minimal expansion of the ordered set of real numbers, for definably connected locally definable manifolds, the theory developed in this paper coincides with the classical theory of topological covering maps ([24]). However, one should point out that, in an arbitrary o-minimal structures \mathbb{M} , the theory of topological covering maps is in some sense useless. In fact in that situation, if \mathbb{M} is non-archimedean, then all definably connected locally definable manifolds are, with their natural topology, totally disconnected spaces and so have no non-trivial covering spaces. Our Theorem 3.17 shows that it is possible to find a suitable replacement of the theory of topological covering maps which in the archimedean case coincides with the classical theory and moreover it is preserved under elementary extensions (Theorem 3.18).

In the paper [17] it was convenient to introduced the o-minimal J-fundamental group which is the relativization of the general o-minimal fundamental group to a product of definable group-intervals. Our next goal is to show that these two kinds of o-minimal fundamental groups are isomorphic.

First we recall a couple of definitions. See [17, Definition 3.1, 3.7, 3.18 and 3.19] (see also [23, Definition 3.1]).

Definition 3.21. A definable group-interval $J = \langle (-b, b), 0, +, < \rangle$ is an open interval $(-b, b) \subseteq M$, with -b < b in $M \cup \{-\infty, +\infty\}$, together with a binary partial continuous definable operation $+: J^2 \to J$ and an element $0 \in J$, such that:

- (i) x + y = y + x when defined; (x + y) + z = x + (y + z) when defined; if x < y and x + z and y + z are defined then x + z < y + z;
- (ii) for every $x \in J$, if x > 0, then the set $\{y \in J : x + y \text{ is defined}\}$ is an interval of the form (-b, r(x));
- (iii) for every $x \in J$, we have $\lim_{z\to 0}(z+x) = x$ and if x > 0 we have also $\lim_{z\to r(x)^-}(x+z) = b$;
- (iv) for every $x \in J$ there exists $-x \in J$ such that x + (-x) = 0.

For the rest of this subsection let $\mathbf{J} = \prod_{i=1}^{m} J_i$ be a fixed cartesian product of definable group-intervals $J_i = \langle (-ib_i, b_i), 0_i, +_i, -_i, < \rangle$.

Definition 3.22.

- We say that a set X is a **J**-set if $X \subseteq \mathbf{J}$; in particular, a **J**-cell is a cell which is a **J**-set.
- We say that X is a (locally) definable manifold with definable **J**-charts if X has definable charts $\{(U_l, \phi_l)\}_{l < \kappa}$ with each $\phi_l(U_l)$ a definable **J**-set.
- We say that a set X is a **J**-bounded set if $X \subseteq \prod_{i=1}^{m} [-ic_i, c_i]$ for some $c_i > 0_i$ in J_i ; in particular, a **J**-bounded cell is a cell which is a **J**-bounded set.
- We say that X is a (locally) definable manifold with definable **J**-bounded charts if X has definable charts $\{(U_l, \phi_l)\}_{l \le \kappa}$ with each $\phi_l(U_l)$ a definable **J**-bounded set.

Definition 3.23.

• A basic *d*-**J**-interval is a basic *d*-interval $\mathcal{I} = \langle [a, b], \langle 0_{\mathcal{I}}, 1_{\mathcal{I}} \rangle \rangle$ with $[a, b] \subseteq J_l$ for some $l \in \{1, \ldots, m\}$; a *d*-**J**-interval is a *d*-interval $\mathcal{I} = \mathcal{I}_1 \land \cdots \land \mathcal{I}_n$ with each \mathcal{I}_i a basic *d*-**J**-interval. Note that the \mathcal{I}_i 's can be in different J_l 's.

- If X is a locally definable manifold with definable **J**-charts, then a definable **J**-path (resp. constant definable **J**-path, or definable **J**-loop) is a definable path (resp. constant definable path or definable loop) $\alpha : \mathcal{I} \to X$ with \mathcal{I} a *d*-**J**-interval; X is definably **J**-path connected if for every u, v in X there is a definable **J**-path $\alpha : \mathcal{I} \to X$ such that $\alpha_0 = u$ and $\alpha_1 = v$.
- If X and Y are locally definable manifolds with definable J-charts, then two definable continuous maps $f, g: Y \to X$ are definably J-homotopic, denoted $f \sim_{\mathbf{J}} g$, if there is a definable homotopy $F(t,s): Y \times \mathcal{J} \to X$ between f and g with \mathcal{J} a d-J-interval; two definable J-paths $\gamma: \mathcal{I} \to X$, $\delta: \mathcal{J} \to X$, with $\gamma_0 = \delta_0$ and $\gamma_1 = \delta_1$, are definably J-homotopic, denoted $\gamma \approx_{\mathbf{J}} \delta$, if there are d-J-intervals \mathcal{I}' and \mathcal{J}' such that $\mathcal{J}' \wedge \mathcal{I} = \mathcal{J} \wedge \mathcal{I}'$, and there is a definable J-homotopy

$$\mathbf{c}_{\mathcal{T}'}^{\gamma_0} \cdot \gamma \sim_{\mathbf{J}} \delta \cdot \mathbf{c}_{\mathcal{T}'}^{\delta_1}$$

fixing the end points.

The results mentioned in Subsection 2.2 for the relations \sim and \approx hold also for $\sim_{\mathbf{J}}$ and $\approx_{\mathbf{J}}$ respectively.

Definition 3.24. Let X be a locally definable manifolds with definable **J**-charts, $e_X \in X$ and $x_0, x_1 \in X$. Let $\mathbb{P}^{\mathbf{J}}(X, x_0, x_1)$ denote the set of all definable **J**-paths in X that start at x_0 and end at x_1 and let $\mathbb{L}^{\mathbf{J}}(X, e_X)$ denotes the set of all definable **J**-loops that start and end at a fixed element e_X of X (i.e. $\mathbb{L}^{\mathbf{J}}(X, e_X) = \mathbb{P}^{\mathbf{J}}(X, e_X, e_X)$). Then the restriction of $\approx_{\mathbf{J}}$ to $\mathbb{P}^{\mathbf{J}}(X, x_0, x_1) \times \mathbb{P}^{\mathbf{J}}(X, x_0, x_1)$ is an equivalence relation on $\mathbb{P}^{\mathbf{J}}(X, x_0, x_1)$ and

$$\pi_1^{\mathbf{J}}(X, e_X) := \frac{\mathbb{L}^{\mathbf{J}}(X, e_X)}{\approx_{\mathbf{J}}} / \approx_{\mathbf{J}}$$

is a group, the *o-minimal* **J**-fundamental group $\pi_1^{\mathbf{J}}(X, e_X)$ of X, with group operation given by $[\gamma][\delta] = [\gamma \cdot \delta]$ and identity the class a of constant **J**-loop at e_X . Moreover, if $f: X \to Y$ is a locally definable continuous map between two locally definable manifolds with definable **J**-charts with $e_X \in X$ and $e_Y \in Y$ such that $f(e_X) = e_Y$, then we have an induced homomorphism $f_*: \pi_1^{\mathbf{J}}(X, e_X) \to \pi_1^{\mathbf{J}}(Y, e_Y):$ $[\sigma] \mapsto [f \circ \sigma]$ with the usual functorial properties.

As usual for a definably **J**-path connected locally definable manifold X with definable **J**-charts if there is no need to mention a base point $e_X \in X$, then by Fact 2.12 (1), we may denote $\pi_1^{\mathbf{J}}(X, e_X)$ by $\pi_1^{\mathbf{J}}(X)$.

For **J**-definable paths and the o-minimal **J**-fundamental group we also have the corresponding properties (P1) and (P2). See [17, Corollary 3.21] for (P1) (a) and (P2) and see [17, Lemma 3.23] for (P1) (b). Thus, just like in Theorems 3.18 and 3.19, we can use these properties in the two setting to prove the following, which was conjectured in the paper [17]:

Theorem 3.25. Let $\mathbf{J} = \prod_{i=1}^{m} J_i$ be a cartesian product of definable group-intervals. Let X be a definably connected locally definable manifold with definable \mathbf{J} -charts. Then the following hold:

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- (1) there exists a universal locally definable covering map $w : W \to X$ where W is a locally definable manifold with definable **J**-charts. Moreover this locally definable covering is definably homeomorphic to a universal locally definable covering map of X.
- (2) The o-minimal J-fundamental group of X is isomorphic to the o-minimal fundamental group of X in M.

Proof. By properties (P1) and (P2) in **J** and the proof of [13, Theorem 1.2], X has a universal locally definable covering map $w : W \to X$ where W is a definably connected, locally definable manifold with definable **J**-charts, $e_W \in W$ and $w(e_W) = e_X$. In particular, by [13, Remark 3.8], we have $\pi_1^{\mathbf{J}}(W, e_W) = 1$.

By Theorem 3.17, let $u: U \to X$ be a universal locally definable map with U definably connected, $e_U \in U$ and $u(e_U) = e_X$. By [13, Remark 3.8], $\pi_1(U, e_U) = 1$. Also there exists a locally definable covering map $q: U \to W$ such that



is a commutative diagram and $q(e_U) = e_W$.

Since X has definable **J**-charts, the same holds for U (we can refine the charts of U using the admissible cover given by (P2) in **J**). Therefore, $u: U \to X$ and $q: U \to W$ are also locally definable covering maps in **J**. Since $\pi_1^{\mathbf{J}}(U, e_U) \simeq q_*(\pi_1^{\mathbf{J}}(U, e_U)) \le \pi_1^{\mathbf{J}}(W, e_W) = 1$ ([13, Corollary 2.17]), by [13, Remark 3.8], $u: U \to X$ is a universal locally definable covering map in **J**. Therefore, $u: U \to X$ and $w: W \to X$ are locally definably homeomorphic (actually in **J**) as required.

Now note that the group $\operatorname{Aut}(U/X)$ of locally definable homeomorphisms ϕ : $U \to U$ such that $u = u \circ \phi$, is the same in \mathbb{M} and in \mathbf{J} . By [13, Theorem 3.9] in \mathbb{M} and in \mathbf{J} respectively, we have $\pi_1(X, e_X) \simeq \operatorname{Aut}(U/X)$ and $\pi_1^{\mathbf{J}}(X, e_X) \simeq \operatorname{Aut}(U/X)$. Therefore, $\pi_1(X, e_X) \simeq \pi_1^{\mathbf{J}}(X, e_X)$.

3.3. The monodromy. Recall that if X is a locally definable manifold, then X is equipped with the o-minimal site X_{def} given by: (i) the category $Op(X_{def})$ of open definable subsets of X with morphisms being inclusions; (ii) the Grothendieck topology such that for $U \in Op(X_{def})$, a collection $\{U_j\}_{j \in J}$ of objects of $Op(X_{def})$ is an admissible cover of U if it admits a finite subcover.

If C is any category admitting projective and inductive limits and satisfying the IPC property (see [30, Definition 3.1.10] for more details), then the category of C-pre-sheaves on the o-minimal site X_{def} , denoted $Psh_{C}(X_{def})$, is the category $Fct(Op(X_{def})^{op}, C)$ of contravariant functors

$$\begin{aligned} \mathcal{F} : \mathrm{Op}(X_{\mathrm{def}}) &\to \mathsf{C} \\ U &\mapsto \mathcal{F}(U) \\ (V \subset U) &\mapsto (\mathcal{F}(U) \to \mathcal{F}(V)) \\ s &\mapsto s_{|V} \end{aligned}$$

from $Op(X_{def})$ to C with morphisms being natural transformations of such functors. The category of C-sheaves on the o-minimal site X_{def} , denote $Sh_C(X_{def})$, is the full subcategory of $Psh_C(X_{def})$ whose objects satisfy the following gluing conditions: for every $U \in Op(X_{def})$ and every admissible cover $\{U_j\}_{j \in J}$ of U we have:

- if $s, t \in \mathcal{F}(U)$ and $s_{|U_j|} = t_{|U_j|}$ for each j, then s = t;
- if $s_j \in \mathcal{F}(U_j)$ are such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \mathcal{F}(U)$ (i.e. $s_{|U_j|} = s_j$).

If $V \in \operatorname{Op}(X_{\operatorname{def}})$, a C-sheaf \mathcal{F} on V_{def} is constant if it is isomorphic to the C-sheaf C_V on V_{def} associated to the C-pre-sheaf sending every $W \in \operatorname{Op}(V_{\operatorname{def}})$ to a fixed $C \in \operatorname{ObC}$. We denote by $\operatorname{CSh}_{\mathsf{C}}(X_{\operatorname{def}})$ the category of constant C-sheaves on the o-minimal site X_{def} on X. We denote by $\operatorname{LCSh}_{\mathsf{C}}(X_{\operatorname{def}})$ the category of locally constant C-sheaves on the o-minimal site X_{def} on X. By definition, this means that, $\mathcal{F} \in \operatorname{ObLCSh}_{\mathsf{C}}(X_{\operatorname{def}})$ if there exists an admissible cover $\{U_j\}_{j\in J}$ of X by open definable subsets such that the restriction $\mathcal{F}_{|U_j|}$ is a constant C-sheaf on U_j_{def} for each $j \in J$. (For further details on the theory of o-minimal sheaves we refer to, for example, [14] and [19]).

Just like in [13], from Properties 3.1 we obtain the monodromy equivalence for locally constant o-minimal sheaves:

Theorem 3.26. Then the monodromy functor

$$\mu : \mathrm{LCSh}_{\mathsf{C}}(X_{\mathrm{def}}) \to \mathrm{Fct}(\Pi_1(X), \mathsf{C})$$

is an equivalence between the category of locally constant C-sheaves on the o-minimal site X_{def} on X and the category of representations of the o-minimal fundamental groupoid $\Pi_1(X)$ of X in C.

Note that when X is definably connected and $x \in X$, then $Fct(\Pi_1(X), C)$ is the category of representations of the o-minimal fundamental group $\pi_1(X, x)$ of X in C.

Taking for C the category of $\pi_1(X, x)$ -sets or of G-torsors, we obtain from Theorem 3.26 classification results for locally definable covering maps, the o-minimal Hurewicz and Seifert - van Kampen theorems just like in the case of o-minimal expansions of ordered groups in [13, Subsection 4.3]. Analogues of the o-minimal Hurewicz and Seifert - van Kampen theorems for definable sets in o-minimal expansions of fields were proved before in [18] and [3] respectively.

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