ON THE STRUCTURE OF BOREL IDEALS IN-BETWEEN THE IDEALS $\mathcal{ED}$ AND $\mathrm{Fin}\otimes\mathrm{Fin}$ IN THE KATĚTOV ORDER

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Abstract. For a family $\mathcal{F}\subseteq\omega^\omega$ we define the ideal $\mathcal{I}(\mathcal{F})$ on $\omega\times\omega$ to be the ideal generated by the family

$$\{A\subseteq\omega\times\omega : \exists f\in\mathcal{F}\,\forall^\infty n\,(|\{k:(n,k)\in A\}|\leq f(n))\}.$$

Using ideals of the form $\mathcal{I}(\mathcal{F})$, we show that the structure of Borel ideals in-between two well known Borel ideals

$$\mathcal{ED}=\{A\subseteq\omega\times\omega : \exists m\,\forall^\infty n\,(|\{k:(n,k)\in A\}|<m)\}$$

and

$$\mathrm{Fin}\otimes\mathrm{Fin}=\{A\subseteq\omega\times\omega : \forall^\infty n\,(|\{k:(n,k)\in A\}|<\aleph_0)\}$$

in the Katětov order is fairly complicated. Namely, there is a copy of $\mathcal{P}(\omega)/\mathrm{Fin}$ in-between $\mathcal{ED}$ and $\mathrm{Fin}\otimes\mathrm{Fin}$, and consequently there are increasing and decreasing chains of length $\mathfrak{b}$ and antichains of size $\mathfrak{c}$.

Contents

1. Introduction

All the notions and notations used in the introduction are defined in Section 2.

Ideals (and the dual notion – filters) have traditionally been of much interest in set theory and on the other hand had interesting applications and usages in various parts of set theory, topology, real analysis and so on (see e.g. [1, 2, 4, 5, 9, 10, 12, 13, 18, 19, 23, 25, 26, 32, 36] to mention only a few recent publications).

Ideals can play an important role in characterizing other objects. For instance, Laczkovich and Recław [25], and independently Debs and Saint Raymond [12], used the ideal $\mathrm{Fin}\otimes\mathrm{Fin}$ to characterize the family of all functions of Baire class 1, whereas Hrušák [18] used the ideal $\mathcal{ED}$ to characterize selective ultrafilters. These

Date: August 21, 2023.
2010 *Mathematics Subject Classification.* Primary: 03E05, Secondary: 03E17, 03E15, 28A05, 40A35.
Key words and phrases. ideal, Borel ideal, F_σ ideal, Katětov order, Katětov-Blass order, cardinal characteristics of an ideal, cardinal characteristics of the continuum.

characterizations are expressed in terms of the Katětov order. Namely, in the first case, $\mathrm{Fin} \otimes \mathrm{Fin} \not\leq_K \mathcal{I} \iff \mathcal{I}$-limits of sequences of continuous functions are of the Baire class one, whereas, in the second case, $\mathcal{ED} \not\leq_K \mathcal{U}^* \iff \mathcal{U}$ is a selective ultrafilter.

Taking these into account, it should not come as a surprise that a lot of work has been done to examine the structures of ideals in the Katětov order so far (see e.g. [20, 29, 31, 16, 21, 15, 34]).

The purpose of this paper is to show that the structure of ideals in-between the ideals $\mathcal{ED}$ and $\mathrm{Fin}\otimes\mathrm{Fin}$ in the Katětov order is quite complicated. Namely, we show that there is a copy of $\mathcal{P}(\omega)/\mathrm{Fin}$ in-between $\mathcal{ED}$ and $\mathrm{Fin} \otimes \mathrm{Fin}$, and consequently there are increasing and decreasing chains of length $\mathfrak{b}$ and antichains of size $\mathfrak{c}$.

In order to obtain these results, we introduced a new class of ideals $\mathcal{I}(\mathcal{F})$ (parametrized by $\mathcal{F} \subseteq \omega^\omega$) generated by the family

$$\{A \subseteq \omega \times \omega : \exists f \in \mathcal{F}\, \forall^\infty n\, (|\{k : (n,k) \in A\}| \leq f(n))\}.$$

Beside the main results about the Katětov order, we also check other properties of the ideals of the form $\mathcal{I}(\mathcal{F})$. For instance, we examine the so called additive properties which can be utilized in the study of ideal convergence of double sequences and we also calculate the values of some well known cardinal characteristics for ideals of the form $\mathcal{I}(\mathcal{F})$.

2. Preliminaries

In the sequel we use ω to denote the set of all natural numbers, and we identify $n \in \omega$ with the set $\{0, 1, \ldots, n-1\}$ (in particular, $A \setminus n = A \setminus \{0, 1, \ldots, n-1\}$).

By $\mathbf{1}_A$ we denote the characteristic function of a set A. In the sequel we assume that $2^\omega = \{0,1\}^\omega$ and ω^ω are equipped with the product topology with the discrete topology on $\{0,1\}$ and ω, respectively. By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the topology of the space 2^ω.

For a formula $\phi(x)$ and a set X we write $\forall^\infty x \in X\, (\phi(x))$ (or simply $\forall^\infty x\, (\phi(x))$) to abbreviate that $\phi(x)$ holds for all but finitely many $x \in X$, and $\exists^\infty x \in X\, (\phi(x))$ (or simply $\exists^\infty x \in X\, (\phi(x))$) to abbreviate that $\phi(x)$ holds for infinitely many $x \in X$. For $f, g \in \omega^\omega$, we write $f \leq^* g$ if $\forall^\infty n\, (f(n) \leq g(n))$. For $\mathcal{F}, \mathcal{G} \subseteq \omega^\omega$, we say that $\mathcal{F}$ is *cofinal* (σ-*cofinal*, resp.) in $\mathcal{G}$ if for every $g \in \mathcal{G}$ (for every $g_0, g_1, \dots \in \mathcal{G}$, resp.) there is $f \in \mathcal{F}$ such that $g \leq^* f$ ($g_n \leq^* f$ for each n, resp.). By $\mathfrak{b}$ we denote the *bounding number* i.e. the smallest cardinality of an unbounded subset in $(\omega^\omega, \leq^*)$. By $\mathfrak{d}$ we denote the *dominating number* i.e. the smallest cardinality of a cofinal subset in $(\omega^\omega, \leq^*)$. By $\mathrm{non}(\mathcal{M})$ we denote the smallest cardinality of non-meager sets in ω^ω, whereas $\mathrm{cov}(\mathcal{M})$ denotes the smallest cardinality of families $\mathcal{F}$ of meager sets in ω^ω such that $\bigcup \mathcal{F} = \omega^\omega$.

We write $A \subseteq^* B$ if $A \setminus B$ is finite. For a set A and $n \in \omega$, we define $[A]^n = \{B \subseteq A : |B| = n\}$, $[A]^{<\omega} = \{B \subseteq A : |B| < \aleph_0\}$ and $[A]^\omega = \{B \subseteq A : |B| = \aleph_0\}$. For $\mathcal{F} \subseteq \omega^\omega$, we define $\mathrm{FS}(\mathcal{F}) = \{f_0 + \cdots + f_n : f_0, \ldots, f_n \in \mathcal{F} \wedge n \in \omega\}$. For $\mathcal{A} \subseteq \mathcal{P}(\omega)$, we define $\mathcal{F}_\mathcal{A} = \{f \cdot \mathbf{1}_A : A \in \mathcal{A} \wedge f \in \omega^\omega\}$.

2.1. **Ideals.** A family $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal on* X if it satisfies the following conditions:

1. if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$,
2. if $A \subseteq B$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$,
3. $\mathcal{I}$ contains all finite subsets of X,

(4) $X \notin \mathcal{I}$.

For an ideal $\mathcal{I}$ on X, we write $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$ and call it the *filter dual to* $\mathcal{I}$. The ideal of all finite subsets of an infinite set X is denoted by $\mathrm{Fin}(X)$ (or Fin for short).

For ideals $\mathcal{I}, \mathcal{J}$ on X and Y respectively, we write

(1) $\mathcal{I} \approx \mathcal{J}$ if there is a bijection $f : X \to Y$ such that $A \in \mathcal{I}$ if and only if $f[A] \in \mathcal{J}$ for every $A \subseteq X$ (we say that $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* in this case);
(2) $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $f : X \to Y$ such that $A \in \mathcal{I}$ if and only if $f[A] \in \mathcal{J}$ for every $A \subseteq X$ ($\leq_{RK}$ is called the *Rudin-Keisler order*);
(3) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ ($\leq_K$ is called the *Katětov order*);
(4) $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite to one function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ ($\leq_{KB}$ is called the *Katětov-Blass order*);
(5) $\mathcal{I} \sqsubseteq \mathcal{J}$ if there is a bijection $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ (in this case $\mathcal{J}$ contains an ideal isomorphic to $\mathcal{I}$).

For an ideal $\mathcal{I}$ on ω and $A \subseteq \omega$ we define $\mathcal{I} \restriction A = \{B \cap A : B \in \mathcal{I}\}$. It is easy to see that $\mathcal{I} \restriction A$ is an ideal on A if and only if $A \notin \mathcal{I}$.

An ideal $\mathcal{I}$ on X is

(1) *tall* if for every infinite $A \subseteq X$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$ (some authors use the name *dense ideal* in this case);
(2) *nowhere tall* if $\mathcal{I} \restriction A$ is not tall for every $A \notin \mathcal{I}$;
(3) a *P-ideal* (or *satisfies the condition (AP)*) if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B$ is finite for every $A \in \mathcal{A}$;
(4) a *Q-ideal* if for every countable partition $\mathcal{F}$ of X into finite sets there is $S \notin \mathcal{I}$ such that $|F \cap S| \leq 1$ for each $F \in \mathcal{F}$.

2.2. **Ideals on** $\omega \times \omega$**.** As this paper is about ideals on $\omega \times \omega$, before proceeding to introduce our ideals, let us have a quick look into some examples already existing in the literature.

For $A \subseteq \omega \times \omega$ and $n \in \omega$, we write $A_{(n)} = \{k \in \omega : (n,k) \in A\}$ and $A^{(n)} = \{k \in \omega : (k,n) \in A\}$ i.e. $A_{(n)}$ and $A^{(n)}$ are the vertical and horizontal (resp.) sections of A at the point n.

Example 2.1 ([27])**.** The ideal

$$\mathcal{I}_2 = \left\{A \subseteq \omega \times \omega : \exists k\, \forall^\infty n \, \left(A_{(n)} \cup A^{(n)} \subseteq k\right)\right\}$$

is called the *Pringsheim's ideal*.

A double sequence $x = (x_{m,n})$ of real numbers is said to be *convergent in Pringsheim's sense* to $L \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists $N \in \omega$ such that $|x_{m,n} - L| < \varepsilon$ whenever $m, n \geq N$ ([33]). It is easy to see that the convergence in Pringsheim's sense is equivalent to $\mathcal{I}_2$-convergence (i.e. ideal convergence with respect to the ideal $\mathcal{I}_2$) ([27]).

An ideal $\mathcal{I}$ on $\omega \times \omega$ *satisfies the condition (AP2)* if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B \in \mathcal{I}_2$ for each $A \in \mathcal{A}$ ([11]).

It is known ([22]) that $\mathcal{I}$-convergence of a double sequence $(x_{n,m})$ can be reduced to the ordinary convergence of a subsequence $(x_{n,m})_{(n,m)\in F}$ with some $F \in \mathcal{I}^*$ if and only if the ideal $\mathcal{I}$ satisfies the condition (AP). Moreover, one can show ([11, 27])

that $\mathcal{I}$-convergence of a double sequence $(x_{n,m})$ can be reduced to the convergence in Pringsheim's sense of a subsequence $(x_{n,m})_{(n,m)\in F}$ with some $F \in \mathcal{I}^*$ if and only if $\mathcal{I}$ satisfies the condition (AP2).

Example 2.2 ([30])**.** For a set $A \subseteq \omega \times \omega$ we write

$$\delta_2(A) = \lim_{m,n\to\infty} \frac{|A \cap (m \times n)|}{mn}$$

if the considered limit exists in the Pringsheim's sense, and then we say that $\delta_2(A)$ is the *double natural density* of A. The ideal $\mathcal{I}_{\delta_2} = \{A \subseteq \omega \times \omega : \delta_2(A) = 0\}$ is called the *ideal of sets of the double natural density zero.*

Example 2.3. The following families are ideals on $\omega \times \omega$ (see e.g. [18]).

(1) $\{\emptyset\} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall n \, (|A_{(n)}| < \aleph_0)\}$.
(2) $\mathrm{Fin} \otimes \mathrm{Fin} = \{A \subseteq \omega \times \omega : \forall^\infty n \, (|A_{(n)}| < \aleph_0)\}$.
(3) $\mathrm{Fin} \otimes \{\emptyset\} = \{A \subseteq \omega \times \omega : \forall^\infty n \, (A_{(n)} = \emptyset)\}$.
(4) $\mathcal{ED} = \{A \subseteq \omega \times \omega : \exists m \, \forall^\infty n \, (|A_{(n)}| < m)\}$.

It is easy to see that all but the last ideals from Example 2.3 can also be defined using the following general notion. The *Fubini product* of families $\mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(\omega)$ is defined by $\mathcal{B} \otimes \mathcal{C} = \{A \subseteq \omega \times \omega : \{n \in \omega : A_{(n)} \notin \mathcal{C}\} \in \mathcal{B}\}$.

3. Ideals defined with the aid of infinite sequences of integers

3.1. **Definition and basic properties.** We first introduce our main object for investigation.

Definition 3.1. For a nonempty family $\mathcal{F} \subseteq \omega^\omega$ we define the ideal $\mathcal{I}(\mathcal{F})$ on $\omega \times \omega$ to be the ideal generated by the family

$$\mathcal{A} = \left\{A \subseteq \omega \times \omega : \exists f \in \mathcal{F} \, \forall^\infty n \, (|A_{(n)}| \leq f(n))\right\}$$

i.e. $A \in \mathcal{I}(\mathcal{F}) \iff A \subseteq A_1 \cup \ldots \cup A_n$ for some $A_1, \ldots, A_n \in \mathcal{A}$.

It is easy to see that

$$\mathrm{Fin} \otimes \{\emptyset\} \subseteq \mathcal{I}(\mathcal{F}) \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$$

for every nonempty family $\mathcal{F}$. Moreover, some well known ideals on $\omega \times \omega$ are of the form $\mathcal{I}(\mathcal{F})$, namely, $\mathrm{Fin} \otimes \{\emptyset\} = \mathcal{I}(\{\langle 0, 0, \ldots\rangle\})$, $\mathcal{ED} = \mathcal{I}(\{f \in \omega^\omega : f \text{ is constant}\})$ and $\mathrm{Fin} \otimes \mathrm{Fin} = \mathcal{I}(\omega^\omega)$.

It is easy to see that the Pringsheim's ideal $\mathcal{I}_2$ is not equal to any ideal of the form $\mathcal{I}(\mathcal{F})$, though there are some inclusions (for instance $\mathrm{Fin} \otimes \{\emptyset\} \subseteq \mathcal{I}_2 \subseteq \mathcal{ED}$).

The ideal $\mathcal{I}_{\delta_2}$ of sets of the double natural density zero is not equal to (even not contained in) any ideal of the form $\mathcal{I}(\mathcal{F})$ as it is not difficult to see that $\mathcal{I}_{\delta_2}$ contains a set having infinite number of infinite vertical sections (for instance $A = \{(m^3, n) : m, n \in \omega\} \in \mathcal{I}_{\delta_2}$). On the other hand, it happens that ideals of the form $\mathcal{I}(\mathcal{F})$ are contained in $\mathcal{I}_{\delta_2}$ (for instance $\mathrm{Fin} \otimes \{\emptyset\} \subseteq \mathcal{I}_{\delta_2}$), but not all of them (for instance $\mathrm{Fin} \otimes \mathrm{Fin} \not\subseteq \mathcal{I}_{\delta_2}$ as $B = \{(m, n) : n \leq m\} \in \mathrm{Fin} \otimes \mathrm{Fin}$ but evidently $\delta_2(B) > 0$).

Proposition 3.2. *The following conditions are equivalent.*

(1) *The family* $\mathcal{A} = \left\{A \subseteq \omega \times \omega : \exists f \in \mathcal{F} \, \forall^\infty n \, (|A_{(n)}| \leq f(n))\right\}$ *is an ideal on* $\omega \times \omega$ *(equivalently* $\mathcal{A} = \mathcal{I}(\mathcal{F})$*).*
(2) $\forall f, g \in \mathcal{F} \, \exists h \in \mathcal{F} \, (f + g \leq^* h)$.

Proof. (1) $\implies$ (2) Let $f,g\in\mathcal{F}$. Let $A=\{(n,k):k\leq f(n)\}$ and $B=\{(n,k):f(n)<k\leq f(n)+g(n)\}$. Since $A,B\in\mathcal{A}$, we get $A\cup B\in\mathcal{A}$. Hence there is $h\in\mathcal{F}$ such that $|(A\cup B)_{(n)}|\leq h(n)$ for all but finitely many n. Since $|(A\cup B)_{(n)}|=|A_{(n)}|+|B_{(n)}|$ in this case, we obtain $f+g\leq^* h$.

(2) $\implies$ (1) First note that it is obvious that $\emptyset\in\mathcal{A}$, $\omega\times\omega\notin\mathcal{A}$ and $\mathcal{A}$ contains all finite subsets of $\omega\times\omega$.

Let $A\in\mathcal{A}$ and $B\subseteq A$. Let $f\in\mathcal{F}$ be such that $|A_{(n)}|\leq f(n)$ for all but finitely many n. Since $|B_{(n)}|\leq|A_{(n)}|$ for all n, we have $B\in\mathcal{A}$.

Let $A,B\in\mathcal{A}$. Let $f,g\in\mathcal{F}$ be such that $|A_{(n)}|\leq f(n)$ and $|B_{(n)}|\leq g(n)$ for all but finitely many n. Take $h\in\mathcal{F}$ with $f+g\leq^* h$. Then $|(A\cup B)_{(n)}|\leq|A_{(n)}|+|B_{(n)}|\leq h(n)$ for all but finitely many n and $h\in\mathcal{F}$, so $A\cup B\in\mathcal{A}$. $\square$

Proposition 3.3.
$$\mathcal{I}(\mathcal{F})=\mathcal{I}(\mathrm{FS}(\mathcal{F}))=\left\{A\subseteq\omega\times\omega:\exists f\in\mathrm{FS}(\mathcal{F})\,\forall^\infty n\,(|A_{(n)}|\leq f(n))\right\},$$
where $\mathrm{FS}(\mathcal{F})=\{f_0+\dots+f_n:f_0,\dots,f_n\in\mathcal{F}\wedge n\in\omega\}$.

Proof. The second equality follows from Proposition 3.2. The inclusion $\mathcal{I}(\mathcal{F})\subseteq\mathcal{I}(\mathrm{FS}(\mathcal{F}))$ follows from $\mathcal{F}\subseteq\mathrm{FS}(\mathcal{F})$. To show $\mathcal{I}(\mathcal{F})\supseteq\mathcal{I}(\mathrm{FS}(\mathcal{F}))$, we take $A\in\mathcal{I}(\mathrm{FS}(\mathcal{F}))$ and, using the second equality, choose $f\in\mathrm{FS}(\mathcal{F})$ such that $|A_{(n)}|\leq f(n)$ for all but finitely many n. Let $f_1,\dots,f_m\in\mathcal{F}$ be such that $f=f_1+\dots+f_m$. We partition the set A into sets $B_1,\dots,B_m$ such that $|(B_i)_{(n)}|\leq f_i(n)$ for all but finitely many n and all $i=1,\dots,m$. Then $B_1,\dots,B_m\in\mathcal{I}(\mathcal{F})$, so $A=B_1\cup\dots\cup B_m$ belongs to the ideal $\mathcal{I}(\mathcal{F})$. $\square$

The following easy proposition provides a characterization (in terms of $\mathcal{F}$) showing when some well known ideals on $\omega\times\omega$ are of the form $\mathcal{I}(\mathcal{F})$.

Proposition 3.4.

(1) $\mathcal{I}(\mathcal{F})=\mathrm{Fin}\otimes\mathrm{Fin}\iff\mathrm{FS}(\mathcal{F})$ *is cofinal in* ω^ω.
(2) $\mathcal{I}(\mathcal{F})=\mathrm{Fin}\otimes\{\emptyset\}\iff\forall f\in\mathcal{F}\,\forall^\infty n\,(f(n)=0)$.
(3) $\mathcal{ED}\subseteq\mathcal{I}(\mathcal{F})\iff\exists f\in\mathrm{FS}(\mathcal{F})\,\forall^\infty n\,(f(n)\neq 0)$.
(4) $\mathcal{I}(\mathcal{F})=\mathcal{ED}\iff$
 (a) $\forall f\in\mathcal{F}\,\exists k\,\forall^\infty n\,(f(n)\leq k)$,
 (b) $\exists f\in\mathrm{FS}(\mathcal{F})\,\forall^\infty n\,(f(n)\neq 0)$.

3.2. **Additive properties.**

Proposition 3.5. *The ideal* $\mathcal{I}(\mathcal{F})$ *is not a P-ideal (i.e. does not satisfy the condition (AP)).*

Proof. Let $A_n=\{n\}\times\omega$ and note that $A_n\in\mathcal{I}(\mathcal{F})$ for every n. Let $B\subseteq\omega\times\omega$ be such that $A_n\setminus B$ is finite for every n. Then $B_{(n)}$ is infinite for every n, hence $B\notin\mathcal{I}(\mathcal{F})$. $\square$

Proposition 3.6. *If* $\mathrm{FS}(\mathcal{F})$ *is* σ*-cofinal in* $\mathrm{FS}(\mathcal{F})$*, then the ideal* $\mathcal{I}(\mathcal{F})$ *satisfies the condition (AP2).*

Proof. Let $A_0,A_1,\dots\in\mathcal{I}(\mathcal{F})$. Let $f_i\in\mathrm{FS}(\mathcal{F})$ and $K_i\in\omega$ be such that $|(A_i)_{(n)}|\leq f_i(n)$ for all $n>K_i$.

Since $\mathrm{FS}(\mathcal{F})$ is σ-cofinal in $\mathrm{FS}(\mathcal{F})$, there is $f\in\mathrm{FS}(\mathcal{F})$ such that for each $i\in\omega$ there is $L_i\in\omega$ with $f_0(n)+\dots+f_i(n)\leq f(n)$ for each $n>L_i$.

Let $M_i\in\omega$ be such that $M_0<M_1<\dots$ and $M_i>\max(K_i,L_i)$ for each $i\in\omega$.

Let
$$B = \bigcup_{i\in\omega} A_i \setminus (M_i \times \omega).$$

Then $B \in \mathcal{I}(\mathcal{F})$ as $|B_{(n)}| \leq f_0(n) + \cdots + f_i(n) \leq f(n)$ for each $M_i \leq n < M_{i+1}$ and $i \in \omega$. Moreover, $A_i \setminus B \subseteq M_i \times \omega \in \mathcal{I}_2$ for each $i \in \omega$. □

The ideal $\mathrm{Fin}\otimes\mathrm{Fin} = \mathcal{I}(\omega^\omega)$ satisfies the condition (AP2), as the family $\mathcal{F} = \omega^\omega$ is σ-cofinal in $\mathrm{FS}(\omega^\omega)$. On the other hand, $\mathcal{ED}$ is an ideal of the form $\mathcal{I}(\mathcal{F})$ that does not satisfy the condition (AP2). Indeed, for each $n \in \omega$ take $A_n = \{(k, nk) : k \in \omega\} \in \mathcal{ED}$. Then for every $B \in \mathcal{ED}$, there exist $M \in \omega$ such that for all but finitely many k we have $|B_{(k)}| \leq M$. Now, one can notice that since A_n are pairwise disjoint, there are at most M sets A_n such that $A_n \setminus B$ is finite, thus there exists n such that $A_n \setminus B$ is infinite. However, any infinite subset of $\{(k, nk) : k \in \omega\}$ does not belong to $\mathcal{I}_2$, hence $A_n \setminus B \notin \mathcal{I}_2$.

3.3. **Tallness.**

Proposition 3.7. *The ideal $\mathcal{I}(\mathcal{F})$ is tall $\iff \forall A \in [\omega]^\omega\, \exists B \in [A]^\omega\, \exists f \in \mathcal{F}\, \forall n \in B\, (f(n) > 0)$.*

Proof. ($\implies$) Let $A \in [\omega]^\omega$. Since $C = A \times \{0\}$ is infinite, there is an infinite set $D \in \mathcal{I}(\mathcal{F})$ such that $D \subseteq C$. Let $f \in \mathrm{FS}(\mathcal{F})$ and $n_0 \in \omega$ be such that $|D_{(n)}| \leq f(n)$ for every $n \geq n_0$. Then $B = \{n \geq n_0 : (n, 0) \in D\}$ is infinite and $B \subseteq A$. Moreover, $f(n) \geq |D_{(n)}| = 1 > 0$ for every $n \in B$. Since $f = f_1 + \ldots + f_k$ for some $f_1, \ldots, f_k \in \mathcal{F}$, there is $i \leq k$ such that $f_i(n) \geq 1$ for infinitely many $n \in B$.

($\impliedby$) Let $C \subseteq \omega \times \omega$ be infinite. If there is n such that $C_{(n)}$ is infinite then $D = \{n\} \times C_{(n)} \in \mathcal{I}(\mathcal{F})$ and $D \subseteq C$. Assume now that $C_{(n)}$ is finite for every n. Since C is infinite, the set $A = \{n : C_{(n)} \neq \emptyset\}$ is infinite. Let $B \in [A]^\omega$ and $f \in \mathcal{F}$ be such that $f(n) > 0$ for every $n \in B$. Then for every $n \in B$ there is k_n such that $(n, k_n) \in C$. Observe that $D = \{(n, k_n) : n \in B\}$ is infinite, belongs to $\mathcal{I}(\mathcal{F})$ and $D \subseteq C$. □

Corollary 3.8. *If there exists $f \in \mathrm{FS}(\mathcal{F})$ such that $f(n) \neq 0$ for all but finitely many n, then the ideal $\mathcal{I}(\mathcal{F})$ is tall.*

Proposition 3.9. *The following conditions are equivalent.*

1. *$\mathcal{I}(\mathcal{F})$ is nowhere tall (i.e. $\mathcal{I}(\mathcal{F}) \restriction B$ is not a tall ideal for any $B \notin \mathcal{I}(\mathcal{F})$).*
2. *$\mathcal{I}(\mathcal{F}) \restriction (A \times \omega)$ is not a tall ideal for any $A \in [\omega]^\omega$.*
3. *$\mathcal{I}(\mathcal{F}) = \mathrm{Fin} \otimes \{\emptyset\}$.*

Proof. (1) $\implies$ (2) If $A \in [\omega]^\omega$, then $B = A \times \omega \notin \mathcal{I}(\mathcal{F})$. Thus $\mathcal{I}(\mathcal{F}) \restriction (A \times \omega)$ is not tall.

(2) $\implies$ (3) By Proposition 3.4(2) it is enough to show that for every $f \in \mathcal{F}$ we have $f(n) = 0$ for all but finitely many n. Suppose, to the contrary, that $C = \{n : f(n) \neq 0\}$ is infinite for some $f \in \mathcal{F}$. Then, by Proposition 3.7, $\mathcal{I}(\mathcal{F}) \restriction (C \times \omega)$ is tall, a contradiction.

(3) $\implies$ (1) Let $B \notin \mathcal{I}(\mathcal{F}) = \mathrm{Fin} \otimes \{\emptyset\}$. Let $A = \{n : B_{(n)} \neq \emptyset\}$ and $k_n \in B_{(n)}$ for every $n \in A$. Let $C = \{(n, k_n) : n \in A\}$. Then $C \subseteq B$ and $\mathcal{I}(\mathcal{F}) \restriction C = \mathrm{Fin}$. Hence there is no infinite subset of C that belongs to $\mathcal{I}(\mathcal{F}) \restriction B$. □

4. Topological complexity

4.1. Baire property.

Proposition 4.1. *The ideal $\mathcal{I}(\mathcal{F})$ has the Baire property.*

Proof. For every $i \in \omega$ we define $F_i = \{(n,k) \in \omega\times\omega : n+k = i\}$. Then $\{F_i : i \in \omega\}$ is a partition of $\omega \times \omega$ into finite sets. Moreover, it is not difficult to see that if $A \subseteq \omega\times\omega$ contains infinitely many sets F_i then all vertical sections of A are infinite, so $A \notin \mathcal{I}(\mathcal{F})$. Thus, by Talagrand characterization [37, Théorème 21] (see also [3, Theorem 4.1.2]), the ideal $\mathcal{I}(\mathcal{F})$ has the Baire property. $\square$

Proposition 4.1 shows that even if $\mathcal{F}$ does not have the Baire property in ω^ω, the ideal $\mathcal{I}(\mathcal{F})$ has the Baire property. A natural question arises whether there is a set $\mathcal{F}$ without the Baire property such that $\mathcal{F}$ is cofinal in $\mathrm{FS}(\mathcal{F})$. Below we show that the answer is positive.

Let X be a topological space. A set $B \subseteq X$ is a *Bernstein set* if both B and $X \setminus B$ are totally imperfect (i.e. neither B nor $X \setminus B$ contain a nonempty perfect set). It is known that Bernstein sets exist in the space ω^ω and they do not have the Baire property (see e.g. [7, Theorem 7.20 and 7.22]).

Proposition 4.2. *Every Bernstein set $B \subseteq \omega^\omega$ is cofinal in ω^ω. In particular, B is cofinal in* $\mathrm{FS}(B)$.

Proof. Let $f \in \omega^\omega$. Let $A_n = \{f(n)+1, f(n)+2\}$ for every n. Since $P = \prod_{n\in\omega} A_n$ is a nonempty and perfect subset of ω^ω, $B \cap P \neq \emptyset$. Moreover, $f \leq g$ for any $g \in B \cap P$. $\square$

Now a natural question arises whether there is a set $\mathcal{F}$ without the Baire property such that $\mathcal{F}$ is cofinal in $\mathrm{FS}(\mathcal{F})$ but $\mathcal{F}$ is not cofinal in ω^ω. Below we show that the answer is positive.

Lemma 4.3. *Let $\phi : \omega^\omega \times \mathcal{P}(\omega) \to \omega^\omega$ be given by $\phi(f, A) = (f + 1) \cdot \mathbf{1}_A$.*

1. *$\phi[\omega^\omega \times \mathcal{A}] = \mathcal{F}_\mathcal{A}$ (recall that $\mathcal{F}_\mathcal{A} = \{f \cdot \mathbf{1}_A : A \in \mathcal{A} \wedge f \in \omega^\omega\}$).*
2. *ϕ is continuous.*
3. *The inverse images under ϕ of nowhere dense sets are nowhere dense.*
4. *The inverse images under ϕ of sets with the Baire property have the Baire property.*

Proof. (1) Let $f \in \mathcal{F}_\mathcal{A}$. Take $A = \omega \setminus f^{-1}[0]$, $g(n) = f(n) - 1$ for $n \in A$ and $g(n) = 0$ otherwise. Then $A \in \mathcal{A}$ and $\phi(g, A) = f$.

(2) Continuity of ϕ easily follows from the following simple observation. If $(f, A), (g, B) \in \omega^\omega \times \mathcal{P}(\omega)$ are such that $f \restriction n = g \restriction n$ and $A \cap n = B \cap n$ for some $n \in \omega$, then $\phi(f, A) \restriction n = \phi(g, B) \restriction n$.

(3) Let $U \subseteq \omega^\omega$ be open and dense. Then $\phi^{-1}[U]$ is open as ϕ is continuous. If we show that $\phi^{-1}[U]$ is also dense, the proof will be finished.

Let $V = \{(f, A) \in \omega^\omega \times \mathcal{P}(\omega) : s = f \restriction n \wedge t = A \cap n\}$ with $s \in \omega^n$, $t \in [\omega]^n$ and $n \in \omega$ be a basic open set. Take any $(f, A) \in V$. Since U is dense, there is $g \in U \cap \{h \in \omega^\omega : ((f + 1) \cdot \mathbf{1}_A) \restriction n = h \restriction n\}$.

Let $C = \{i \geq n : g(i) \neq 0\}$ and $D = \{i \geq n : g(i) = 0\}$.

We define $B = (A \cap n) \cup C$ and $h = (f \restriction n) \cup ((g - 1) \restriction C) \cup (g \restriction D)$. Then $(h, B) \in V$ and $\phi(h, B) = (h + 1) \cdot \mathbf{1}_B = g \in U$. Thus $V \cap \phi^{-1}[U] \neq \emptyset$.

(4) It follows from (2) and (3) as every set with the Baire property is a symmetric difference of an open set and a countable union of nowhere dense sets. $\square$

Proposition 4.4. *Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an ideal on ω.*

1. $\mathrm{FS}(\mathcal{F}_\mathcal{A}) = \mathcal{F}_\mathcal{A}$. *In particular, $\mathcal{F}_\mathcal{A}$ is cofinal in* $\mathrm{FS}(\mathcal{F}_\mathcal{A})$.
2. *$\mathcal{F}_\mathcal{A}$ is not cofinal in ω^ω.*
3. $\mathcal{I}(\mathcal{F}_\mathcal{A}) = (\mathrm{Fin} \otimes \mathrm{Fin}) \cap (\mathcal{A} \otimes \{\emptyset\})$.
4. *If $\mathcal{A}$ is a nonmeager ideal, then $\mathcal{F}_\mathcal{A}$ does not have the Baire property.*

Proof. (1) easily follows from the fact that $\mathcal{A}$ is closed under taking finite unions of its elements. (2) easily follows from the fact that $\mathcal{A}$ does not contain any cofinite subsets of ω. (3) is straightforward. (Note that ideals defined as an intersection of this form were used in [24].)

(4) Suppose, to the contrary, that $\mathcal{F}_\mathcal{A}$ has the Baire property. Let $\phi : \omega^\omega \times \mathcal{P}(\omega) \to \omega^\omega$ be given by $\phi(f, A) = (f + 1) \cdot \mathbf{1}_A$. Then, by Lemma 4.3, $\phi^{-1}[\mathcal{F}_\mathcal{A}]$ has the Baire property. Since $\phi^{-1}[\mathcal{F}_\mathcal{A}] = \omega^\omega \times \mathcal{A}$, by using Kuratowski-Ulam theorem (a topological counterpart of Fubini's theorem) we obtain that $\mathcal{A}$ has the Baire property. But $\mathcal{A}$ is a nonmeager ideal, so it does not have the Baire property (see e.g. [3, Theorem 4.1.1]), a contradiction. □

Corollary 4.5. *There is a set $\mathcal{F} \subseteq \omega^\omega$ without the Baire property such that $\mathcal{F}$ is cofinal in* $\mathrm{FS}(\mathcal{F})$, *but $\mathcal{F}$ is not cofinal in ω^ω.*

4.2. **Borel complexity.**

Proposition 4.6.

1. *If $\mathcal{F}$ is σ-compact, then $\mathcal{I}(\mathcal{F})$ is a σ-compact (hence, F_σ) ideal.*
2. *If $\mathcal{F}$ is countable, then $\mathcal{I}(\mathcal{F})$ is an F_σ ideal.*
3. *If $\mathcal{F}$ is bounded in $(\omega^\omega, \leq^*)$, then $\mathcal{I}(\mathcal{F})$ is contained in a σ-compact (hence, F_σ) ideal.*
4. *If $\mathcal{F}$ is of cardinality less than $\mathfrak{b}$, then $\mathcal{I}(\mathcal{F})$ is contained in an F_σ ideal.*
5. *If $\mathcal{F}$ is a Borel (or even analytic) set, then $\mathcal{I}(\mathcal{F})$ is an analytic ideal.*

Proof. First observe that the set

$$B = \{(A, f) \in \mathcal{P}(\omega \times \omega) \times \omega^\omega : \forall^\infty n \in \omega\, |A_{(n)}| \leq f(n)\}$$

is F_σ, because $B = \bigcup_{m \in \omega} \bigcap_{n > m} B_n$, where $B_n = \{(A, f) \in \mathcal{P}(\omega \times \omega) \times \omega^\omega : |A_{(n)}| \leq f(n)\}$ are closed. Second observe that $\mathcal{I}(\mathcal{F})$ is equal to the the projection of the set $X = (\mathcal{P}(\omega \times \omega) \times \mathrm{FS}(\mathcal{F})) \cap B$ onto the first coordinate.

(1) If $\mathcal{F}$ is σ-compact, then X is σ-compact (as the intersection of the σ-compact set $\mathcal{P}(\omega \times \omega) \times \mathrm{FS}(\mathcal{F})$ and the F_σ set B). Consequently $\mathcal{I}(\mathcal{F})$ is σ-compact as a continuous image of X.

(2) It follows from (1) because every countable family is σ-compact.

(3) It follows from (1) as every bounded set in $(\omega^\omega, \leq^*)$ is contained in a σ-compact set.

(4) It follows from (3) because every family of cardinality less than $\mathfrak{b}$ is bounded in $(\omega^\omega, \leq^*)$.

(5) If $\mathcal{F}$ is analytic, then X is analytic as the intersection of the analytic set $\mathcal{P}(\omega \times \omega) \times \mathrm{FS}(\mathcal{F})$ and the F_σ set B. □

Lemma 4.7. *Let $\Phi : \mathcal{P}(\omega) \to \mathcal{P}(\omega \times \omega)$ be given by $\Phi(A) = A \times \{0\}$.*

1. *Φ is continuous.*
2. *If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an ideal on ω, then $\Phi^{-1}[\mathcal{I}(\mathcal{F}_\mathcal{A})] = \mathcal{A}$.*

Proof. Since continuity of Φ is straightforward, we only verify (2).

($\subseteq$) Let $B \in \Phi^{-1}[\mathcal{I}(\mathcal{F}_{\mathcal{A}})]$. Then $\Phi(B) \in \mathcal{I}(\mathcal{F}_{\mathcal{A}})$, so there is $A \in \mathcal{A}$ and $f \in \omega^\omega$ such that $|(\Phi(B))_{(n)}| \leq f \cdot \mathbf{1}_A(n)$ for all but finitely many n. On the other hand, $|(\Phi(B))_{(n)}| = |(B \times \{0\})_{(n)}| = \mathbf{1}_B(n)$ for every n. Thus, $\mathbf{1}_B(n) \leq \mathbf{1}_A(n)$ for all but finitely many n. Hence $B \subseteq^* A$. Since $A \in \mathcal{A}$ and $\mathcal{A}$ is an ideal, $B \in \mathcal{A}$.

($\supseteq$) Let $A \in \mathcal{A}$. Then $|(\Phi(A))_{(n)}| = |(A \times \{0\})_{(n)}| = \mathbf{1}_A(n)$ for every n and $\mathbf{1}_A \in \mathcal{F}_{\mathcal{A}}$. Thus $\Phi(A) \in \mathcal{I}(\mathcal{F}_{\mathcal{A}})$. □

Proposition 4.8. *Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an ideal on ω.*

1. *If $\mathcal{A}$ is a Borel ideal, then $\mathcal{I}(\mathcal{F}_{\mathcal{A}})$ is a Borel ideal.*
2. *There are Borel ideals of the form $\mathcal{I}(\mathcal{F})$ of arbitrarily high Borel complexity i.e. for every $\alpha < \omega_1$ there exists $\mathcal{F} \subseteq \omega^\omega$ such that the ideal $\mathcal{I}(\mathcal{F})$ is Borel but not in $\mathbf{\Sigma}^0_\alpha$.*
3. *There exists $\mathcal{F} \subseteq \omega^\omega$ such that the ideal $\mathcal{I}(\mathcal{F})$ is not Borel.*

Proof. (1) Since $\mathcal{A} \in \mathbf{\Sigma}^0_\alpha$ for some $\alpha < \omega_1$ and $\{\emptyset\} \in \mathbf{\Pi}^0_1$, we obtain $\mathcal{A} \otimes \{\emptyset\} \in \mathbf{\Sigma}^0_{1+\alpha}$ (see e.g. [28, Proposition 1.6.16]). Since Fin $\otimes$ Fin $\in \mathbf{\Sigma}^0_4$, we have $\mathcal{I}(\mathcal{F}_{\mathcal{A}}) = (\text{Fin} \otimes \text{Fin}) \cap (\mathcal{A} \otimes \{\emptyset\}) \in \mathbf{\Sigma}^0_\beta$, where $\beta = \max\{4, 1+\alpha\}$. All in all, $\mathcal{I}(\mathcal{F}_{\mathcal{A}})$ is a Borel ideal.

(2) Let $\alpha < \omega_1$. Let $\mathcal{A}$ be a Borel ideal on ω such that $\mathcal{A} \notin \mathbf{\Sigma}^0_\alpha$ (for the existence of such ideals see e.g. [8], [39] or [38]). By (1), $\mathcal{I}(\mathcal{F}_{\mathcal{A}})$ is a Borel ideal, but Lemma 4.7 implies that $\mathcal{I}(\mathcal{F}_{\mathcal{A}}) \notin \mathbf{\Sigma}^0_\alpha$.

(3) If $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is an ideal which is not Borel (for instance a maximal ideal), then $\mathcal{I}(\mathcal{F}_{\mathcal{A}})$ is not Borel by Lemma 4.7. □

5. How many ideals are there?

Lemma 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be ideals on ω.*

1. *If $\mathcal{I}(\mathcal{F}_{\mathcal{B}}) \leq_K \mathcal{I}(\mathcal{F}_{\mathcal{A}})$, then $\mathcal{B} \leq_{KB} \mathcal{A}$.*
2. *If $\mathcal{B} \leq_{KB} \mathcal{A}$, then $\mathcal{I}(\mathcal{F}_{\mathcal{B}}) \leq_{KB} \mathcal{I}(\mathcal{F}_{\mathcal{A}})$.*
3. *If $\mathcal{B} \leq_K \mathcal{A}$ and $\mathcal{A}$ is a P-ideal, then $\mathcal{I}(\mathcal{F}_{\mathcal{B}}) \leq_{KB} \mathcal{I}(\mathcal{F}_{\mathcal{A}})$.*
4. *$\mathcal{I}(\mathcal{F}_{\mathcal{B}}) \leq_K \mathcal{I}(\mathcal{F}_{\mathcal{A}})$ if and only if $\mathcal{I}(\mathcal{F}_{\mathcal{B}}) \leq_{KB} \mathcal{I}(\mathcal{F}_{\mathcal{A}})$.*

Proof. (1) Let $\Phi : \omega \times \omega \to \omega \times \omega$ be a function such that $\Phi^{-1}[B] \in \mathcal{I}(\mathcal{F}_{\mathcal{A}})$ for every $B \in \mathcal{I}(\mathcal{F}_{\mathcal{B}})$. Below we define a function $\Psi : \omega \to \omega$ which will witness $\mathcal{B} \leq_{KB} \mathcal{A}$.

Let $i_0 = 0$ and define $m_0 = \min\{m \in \omega : \Phi[\{i_0\} \times \omega] \cap (\{m\} \times \omega) \neq \emptyset\}$. Since $\Phi^{-1}[\{m_0\} \times \omega] \in \mathcal{I}(\mathcal{F}_{\mathcal{A}})$, the set $F_0 = \{i_0\} \cup \{i \in \omega : \Phi[\{i\} \times \omega] \subseteq \{m_0\} \times \omega\}$ is finite. For every $i \in F_0$, we define $\Psi(i) = m_0$.

We proceed inductively. Let $i_n = \min\left(\omega \setminus \bigcup_{k<n} F_k\right)$ and define

$$m_n = \min\left\{m \in \omega \setminus \{m_k : k < n\} : \Phi\left[\{i_n\} \times \omega\right] \cap (\{m\} \times \omega) \neq \emptyset\right\}.$$

Since $\Phi^{-1}[\{G\} \times \omega] \in \mathcal{I}(\mathcal{F}_{\mathcal{A}})$ for every finite set $G \subseteq \omega$, the set

$$F_n = \{i_n\} \cup \left\{i \in \omega \setminus \bigcup_{k<n} F_k : \Phi[\{i\} \times \omega] \subseteq \{m_k : k \leq n\} \times \omega\right\}$$

is finite. For every $i \in F_n$, we define $\Psi(i) = m_n$.

Clearly, Ψ is a finite-to-one function. Once we show that $\Psi^{-1}[B] \in \mathcal{A}$ for every $B \in \mathcal{B}$, the proof will be finished.

Take $B \in \mathcal{B}$. For every $i \in F_n$, $n \in \omega$, we pick $(m_n, c_i) \in \Phi[\{i\} \times \omega] \cap (\{m_n\} \times \omega)$. Let $C = \{(m_n, c_i) : n \in \omega, i \in F_n\} \cap (B \times \omega)$. Since $C_{(m)}$ is finite for every $m \in B$ and empty for each $m \in \omega \setminus B$, there exists a function $f \in \omega^\omega$ such that $|C_{(m)}| \leq$

$(f\cdot \mathbf{1}_B)(m)$ for every $m\in\omega$. Thus, $C\in\mathcal{I}(\mathcal{F}_\mathcal{B})$, so $\Phi^{-1}[C]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$. It follows that there is a function $g\in\omega^\omega$ and a set $A\in\mathcal{A}$ such that $|(\Phi^{-1}[C])_{(i)}|\leq (g\cdot\mathbf{1}_A)(i)$ for all but finitely many $i\in\omega$. Then it is not difficult to see that

$$\Psi^{-1}[B]=\bigcup_{m_n\in B}F_n=\{i\in\omega:(\Phi^{-1}[C])_{(i)}\neq\emptyset\}\subseteq^* A,$$

so $\Psi^{-1}[B]\in\mathcal{A}$.

(2) Let $\Psi:\omega\to\omega$ be a finite-to-one function such that $\Psi^{-1}[B]\in\mathcal{A}$ for every $B\in\mathcal{B}$. Below we define a function $\Phi:\omega\times\omega\to\omega\times\omega$ which will witness $\mathcal{I}(\mathcal{F}_\mathcal{B})\leq_{KB}\mathcal{I}(\mathcal{F}_\mathcal{A})$.

Let $\operatorname{ran}(\Psi)=\{m_n:n\in\omega\}$ and $F_n=\Psi^{-1}[\{m_n\}]$ for $n\in\omega$.

Let $\Phi:\omega\times\omega\to\omega\times\omega$ be a one-to-one function such that $\Phi[F_n\times\omega]=\{m_n\}\times\omega$ for each $n\in\omega$.

Let $C\in\mathcal{I}(\mathcal{F}_\mathcal{B})$, and take $B\in\mathcal{B}$, $f\in\omega^\omega$ and $M\in\omega$ such that $|(C)_{(m)}|\leq(f\cdot\mathbf{1}_B)(m)$ for all $m\geq M$.

Let $A=\bigcup\{F_n:m_n\in B\setminus M\}=\Psi^{-1}[B\setminus M]\in\mathcal{A}$ and $F=\bigcup\{F_n:m_n\in B\cap M\}\in\mathrm{Fin}$.

Since $(\Phi^{-1}[C])_{(i)}$ is finite for all $i\in A$ and $(\Phi^{-1}[C])_{(i)}=\emptyset$ for all $i\in\omega\setminus(A\cup F)$, there is $g\in\omega^\omega$ with $|(\Phi^{-1}[C])_{(i)}|\leq(g\cdot\mathbf{1}_A)$ for all $i\in\omega\setminus F$, and consequently $\Phi^{-1}[C]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$.

(3) It follows from (2) and the fact that if $\mathcal{A}$ is a P-ideal, then $\mathcal{B}\leq_K\mathcal{A}$ implies $\mathcal{B}\leq_{KB}\mathcal{A}$.

(4) It follows from (1) and (2). □

From the above lemma we easily obtain the following theorem which says, in particular, that the structure of the Katětov-Blass order among the ideals of the form $\mathcal{I}(\mathcal{F})$ is as complicated as the structure of the Katětov-Blass order among all ideals on ω.

Theorem 5.2. *If $\mathcal{A}$ and $\mathcal{B}$ are ideals on ω, then*

$$\mathcal{B}\leq_{KB}\mathcal{A}\iff\mathcal{I}(\mathcal{F}_\mathcal{B})\leq_{KB}\mathcal{I}(\mathcal{F}_\mathcal{A})$$

i.e. there is an order embedding of the family of all ideals, ordered by the Katětov-Blass order, into the family of ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov-Blass order.

Using the above theorem we prove the following theorem which says, among others, that there are as many as possible pairwise nonisomorphic ideals of the form $\mathcal{I}(\mathcal{F})$.

Theorem 5.3. *There are $2^\mathfrak{c}$ pairwise $\leq_K$-incomparable ideals of the form $\mathcal{I}(\mathcal{F})$ (i.e. there is $\leq_K$-antichain of cardinality $2^\mathfrak{c}$ of ideals of the form $\mathcal{I}(\mathcal{F})$). In particular, there are $2^\mathfrak{c}$ pairwise nonisomorphic ideals of the form $\mathcal{I}(\mathcal{F})$.*

Proof. First, we notice that if $\mathcal{A}$ and $\mathcal{B}$ are $\leq_{RK}$-incomparable *maximal* ideals, then $\mathcal{I}(\mathcal{F}_\mathcal{A})$ and $\mathcal{I}(\mathcal{F}_\mathcal{B})$ are $\leq_K$-incomparable. Indeed, suppose that $\mathcal{I}(\mathcal{F}_\mathcal{A})$ and $\mathcal{I}(\mathcal{F}_\mathcal{B})$ are $\leq_K$-comparable i.e. $\mathcal{I}(\mathcal{F}_\mathcal{A})\leq_K\mathcal{I}(\mathcal{F}_\mathcal{B})$ or $\mathcal{I}(\mathcal{F}_\mathcal{B})\leq_K\mathcal{I}(\mathcal{F}_\mathcal{A})$. Then, by Lemma 5.1, $\mathcal{A}\leq_{KB}\mathcal{B}$ or $\mathcal{B}\leq_{KB}\mathcal{A}$. Now, using maximality of $\mathcal{A}$ and $\mathcal{B}$, it is easy to see that $\mathcal{A}\leq_{RK}\mathcal{B}$ or $\mathcal{B}\leq_{RK}\mathcal{A}$, so $\mathcal{A}$ and $\mathcal{B}$ are $\leq_{RK}$-comparable.

Second, we take $2^\mathfrak{c}$ pairwise $\leq_{RK}$-incomparable maximal ideals (see [35]) and obtain $2^\mathfrak{c}$ pairwise $\leq_K$-incomparable ideals of the form $\mathcal{I}(\mathcal{F}_\mathcal{A})$. □

5.1. **Borel ideals.**

Lemma 5.4. *If $\mathcal{A}$ is an ideal on ω which is* not *a Q-ideal, then $\mathcal{ED}\sqsubseteq\mathcal{I}(\mathcal{F}_\mathcal{A})$ (in particular, $\mathcal{ED}\leq_{KB}\mathcal{I}(\mathcal{F}_\mathcal{A})$).*

Proof. Since $\mathcal{A}$ is not a Q-ideal, there is a partition $\{F_n : n\in\omega\}$ of ω into finite sets such that for every $S\notin\mathcal{A}$ there is $n\in\omega$ with $|F_n\cap S|\geq 2$.

Let $G:\omega\times\omega\to\omega\times\omega$ be a bijection such that $G[F_n\times\omega]=\{n\}\times\omega$ for each n.

We claim that G witnesses $\mathcal{ED}\sqsubseteq\mathcal{I}(\mathcal{F}_\mathcal{A})$ i.e. $G^{-1}[B]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$ for every $B\in\mathcal{ED}$.

Since the ideal $\mathcal{ED}$ is generated by vertical lines $\{n\}\times\omega$ and functions $f\in\omega^\omega$ it is enough to check that $G^{-1}[\{n\}\times\omega]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$ and $G^{-1}[f]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$ for every $n\in\omega$ and $f\in\omega^\omega$.

For any $n\in\omega$, we have $G^{-1}[\{n\}\times\omega]=F_n\times\omega\in\mathcal{I}(\mathcal{F}_\mathcal{A})$.

Take any $f\in\omega^\omega$. Let A be the projection of the set $G^{-1}[f]$ onto the first coordinate. Then it is not difficult to see that $|A\cap F_n|=1$ for each $n\in\omega$, so $A\in\mathcal{A}$. Since $|(G^{-1}[f])_{(n)}|\leq\mathbf{1}_A(n)$ for each n, we get $G^{-1}[f]\in\mathcal{I}(\mathcal{F}_\mathcal{A})$ □

Theorem 5.5. *There is an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of Borel (in fact $\mathbf{\Sigma}^0_4$) ideals of the form $\mathcal{I}(\mathcal{F})$ which are in-between the ideals $\mathcal{ED}$ and $\mathrm{Fin}\otimes\mathrm{Fin}$, ordered by the Katětov (or equivalently Katětov-Blass) order. In particular,*

1. *there is a $\leq_K$-antichain of cardinality $\mathfrak{c}$ of Borel ideals of the form $\mathcal{I}(\mathcal{F})$ (in particular, there are $\mathfrak{c}$ pairwise nonisomorphic Borel ideals of the form $\mathcal{I}(\mathcal{F})$);*
2. *there are increasing and decreasing $\leq_{KB}$-chains of length $\mathfrak{b}$ of Borel ideals of the form $\mathcal{I}(\mathcal{F})$.*

Proof. In [16, 31], the authors proved that there is an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of tall F_σ P-ideals, ordered by the Katětov order. Since $\mathcal{I}\leq_K\mathcal{J}\iff\mathcal{I}\leq_{KB}\mathcal{J}$ for P-ideals, we see that there is also an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of tall F_σ P-ideals, ordered by the Katětov-Blass order. Now using Theorem 5.2, we obtain an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov-Blass order. Since the Katětov order is equivalent to the Katětov-Blass order in the realm of ideals of the form $\mathcal{I}(\mathcal{F})$ (see Lemma 5.1(4)), we also obtain an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov order.

If $\mathcal{A}$ is an F_σ-ideal, then by Proposition 4.8 the ideal $\mathcal{I}(\mathcal{F}_\mathcal{A})$ is Borel (from the proof of Proposition 4.8 it follows that the ideal $\mathcal{I}(\mathcal{F}_\mathcal{A})$ is in fact $\mathbf{\Sigma}^0_4$).

All ideals of the form $\mathcal{I}(\mathcal{F})$ are contained in $\mathrm{Fin}\otimes\mathrm{Fin}$, so they are $\leq_{KB}$-below $\mathrm{Fin}\otimes\mathrm{Fin}$.

In [14], the authors proved that if $\mathcal{A}$ is a tall F_σ-ideal, then $\mathcal{A}$ is not a Q-ideal, so Lemma 5.4 gives $\mathcal{ED}\leq_{KB}\mathcal{I}(\mathcal{F}_\mathcal{A})$.

(1) It follows from the fact that there are $\subseteq^*$-antichains of cardinality $\mathfrak{c}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$.

(2) It follows from the fact that there are increasing and decreasing $\subseteq^*$-chains of length $\mathfrak{b}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$. □

5.2. **F_σ-ideals.** For $f\in\omega^\omega$ we write $\mathcal{F}_f=\{k\cdot f : k\in\omega\}$. By Proposition 4.6(2), the ideal $\mathcal{I}(\mathcal{F}_f)$ is F_σ for any $f\in\omega^\omega$.

Lemma 5.6. *Let $f, g \in \omega^\omega$. If $g \neq^* 0$ and*

$$\forall M\, \exists N\, \forall n, k > N \left(\frac{f(n)}{g(k)} > M \vee \frac{g(k)}{f(n)} > M\right),$$

then $\mathcal{I}(\mathcal{F}_g)$ and $\mathcal{I}(\mathcal{F}_f)$ are not isomorphic.

Proof. Suppose for the sake of contradiction that $\mathcal{I}(\mathcal{F}_g)$ and $\mathcal{I}(\mathcal{F}_f)$ are isomorphic. Let $\Phi : \omega \times \omega \to \omega \times \omega$ be a bijection such that $B \in \mathcal{I}(\mathcal{F}_g) \iff \Phi^{-1}[B] \in \mathcal{I}(\mathcal{F}_f)$.

Claim. $\Phi[\{i\} \times \omega] \in \mathrm{Fin} \otimes \{\emptyset\}$ for all but finitely many $i \in \omega$.

Proof of Claim. Suppose for the sake of contradiction that $\Phi[\{i\} \times \omega] \notin \mathrm{Fin} \otimes \{\emptyset\}$ for infinitely many i. Let $i_1 < i_2 < \ldots$ be such that $\Phi[\{i_n\} \times \omega] \notin \mathrm{Fin} \otimes \{\emptyset\}$ for each n. Now, we inductively pick, using the diagonal argument, pairwise distinct elements a_k^n and some b_k^n for $n, k \in \omega$ such that $(a_k^n, b_k^n) \in \Phi[\{i_n\} \times \omega]$.

If $B = \{(a_k^n, b_k^n) : n, k \in \omega\}$, then $|B_{(i)}| \leq 1 \leq g(i)$ for all but finitely many i. Thus $B \in \mathcal{I}(\mathcal{F}_g)$, and consequently $\Phi^{-1}[B] \in \mathcal{I}(\mathcal{F}_f)$.

On the other hand, $\Phi^{-1}[\{(a_k^n, b_k^n) : k \in \omega\}] \subseteq \{i_n\} \times \omega$ for each n, so $(\Phi^{-1}[B])_{(i_n)}$ is infinite for each n. This means that $\Phi^{-1}[B] \notin \mathcal{I}(\mathcal{F}_f)$, a contradiction. □

Claim. There are $a_0 < a_1 < \ldots$ and $b_0 < b_1 < \ldots$ such that $(\Phi[\{a_n\} \times \omega])_{(b_n)}$ is infinite for each $n \in \omega$.

Proof of Claim. By the previous Claim, there is a_0 such that $\Phi[\{a\} \times \omega] \in \mathrm{Fin} \otimes \{\emptyset\}$ for each $a \geq a_0$. Take b_0 such that $(\Phi[\{a_0\} \times \omega])_{(b_0)}$ is infinite.

There is $a_1 > a_0$ such that $\Phi[\{a\} \times \omega] \setminus \{0, 1, \ldots, b_0\} \times \omega$ is infinite for each $a \geq a_1$ (otherwise the set $(\Phi^{-1}[\{0, 1, \ldots, b_0\} \times \omega])_{(a)}$ would be cofinite for infinitely many $a \geq a_0$, and consequently $\Phi^{-1}[\{0, 1, \ldots, b_0\} \times \omega] \notin \mathcal{I}(\mathcal{F}_f)$, a contradiction). Take $b_1 > b_0$ such that $(\Phi[\{a_1\} \times \omega])_{(b_1)}$ is infinite.

It is not difficult to see that proceeding by induction we obtain the required sequences (a_n) and (b_n) □

Now, we are ready to finish the proof of the lemma. Let (a_n) and (b_n) be as in the last Claim, and take a sequence $k_1 < k_2 < \ldots$ such that either

$$\lim_{n\to\infty} \frac{f(a_{k_n})}{g(b_{k_n})} = \infty \quad \text{or} \quad \lim_{n\to\infty} \frac{g(b_{k_n})}{f(a_{k_n})} = \infty.$$

In the former case, we pick a set $A \subseteq \{a_{k_n} : n \in \omega\} \times \omega$ such that $|A_{(a_{k_n})}| = |(\Phi[A])_{(b_{k_n})}| = f(a_{k_n})$ for each n. Then $A \in \mathcal{I}(\mathcal{F}_f)$ and consequently $\Phi[A] \in \mathcal{I}(\mathcal{F}_g)$, so there is k with $|(\Phi[A])_{(i)}| \leq k \cdot g(i)$ for all but finitely many i. It means that $f(a_{k_n}) \leq k \cdot g(b_{k_n})$ for all but finitely many n, a contradiction with $\lim_n f(a_{k_n})/g(b_{k_n}) = \infty$.

In the latter case, we pick a set $B \subseteq \{b_{k_n} : n \in \omega\} \times \omega$ such that $|B_{(b_{k_n})}| = |(\Phi^{-1}[B])_{(a_{k_n})}| = g(b_{k_n})$ for each n. Then $B \in \mathcal{I}(\mathcal{F}_g)$ and consequently $\Phi^{-1}[B] \in \mathcal{I}(\mathcal{F}_f)$, so there is k with $|(\Phi^{-1}[B])_{(i)}| \leq k \cdot f(i)$ for all but finitely many i. It means that $g(b_{k_n}) \leq k \cdot f(a_{k_n})$ for all but finitely many n, a contradiction with $\lim_n g(b_{k_n})/f(a_{k_n}) = \infty$. □

Theorem 5.7. *There are $\mathfrak{c}$ pairwise nonisomorphic F_σ ideals of the form $\mathcal{I}(\mathcal{F})$.*

Proof. Let $\mathcal{A}$ be a family of $\mathfrak{c}$ pairwise almost disjoint infinite subsets of ω. Let $c_0^A < c_1^A < \ldots$ be the increasing enumeration of $A \in \mathcal{A}$.

For each $A \in \mathcal{A}$ we define $f_A : \omega \to \omega$ by $f_A(n) = (c_n^A)!$ for each n.

If we show that for distinct $A, B \in \mathcal{A}$ the functions $f = f_A$ and $g = f_B$ satisfy the assumption from Lemma 5.6, the proof of the theorem will be finished.

Let M be fixed. Since $A \cap B$ is finite, there is $N > M$ such that $c_n^A \neq c_k^B$ for each $n, k > N$. Taking $n, k > N$ we have either $c_n^A < c_k^B$ or $c_n^A > c_k^B$, so either

$$\frac{f_B(k)}{f_A(n)} = \frac{(c_k^B)!}{(c_n^A)!} \geq c_k^B \geq k > N > M \quad \text{or} \quad \frac{f_A(n)}{f_B(k)} = \frac{(c_n^A)!}{(c_k^B)!} \geq c_n^A \geq n > N > M.$$

□

Question 1. Is there an order embedding of $\mathcal{P}(\omega)/\mathrm{Fin}$, ordered by $\subseteq^*$, into the family of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$, ordered by the Katětov order? Are there uncountable increasing (decreasing, resp.) $\leq_K$-chains of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$? Are there uncountable $\leq_K$-antichains of F_σ ideals of the form $\mathcal{I}(\mathcal{F})$?

6. Cardinal characteristics of ideals

Some properties of ideals can be described by cardinal characteristics associated with them. There are four well known cardinal characteristics called additivity, covering, uniformity and cofinality defined in the following way for an ideal $\mathcal{I}$ on X (see e.g. [3]): $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}$, $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\}$, $\mathrm{non}(\mathcal{I}) = \min\{|A| : A \notin \mathcal{I}\}$, $\mathrm{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\, (B \subseteq A)\}$. These characteristics are useful in the case of ideals on an uncountable set X (for instance in the case of the σ-ideal of all meager sets and the σ-ideal of all Lebesgue null sets). However, they are (but cof) useless in the case of ideals on ω, since $\mathrm{add}(\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \aleph_0$ for every ideal $\mathcal{I}$ on ω. Fortunately, Hernández and Hrušák introduced in [17] (see also [18]) certain versions of these characteristics more suitable for tall ideals on ω. Namely, for each tall ideal $\mathcal{I}$ they define:

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg\exists B \in \mathcal{I}\, \forall A \in \mathcal{A}\, (A \subseteq^* B)\},$$
$$\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \neg\exists B \in \mathcal{P}(\omega) \setminus \mathrm{Fin}^*\, \forall A \in \mathcal{A}\, (A \subseteq^* B)\},$$
$$\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \setminus \mathrm{Fin}^* \wedge \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\, (B \subseteq^* A)\},$$
$$\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I}\, \exists A \in \mathcal{A}\, (B \subseteq^* A)\}.$$

If $\mathcal{U}$ is a free ultrafilter on ω and $\mathcal{I} = \mathcal{U}^*$ is the dual ideal, the above mentioned characteristics were earlier introduced by Brendle and Shelah in [6] were the authors used the notations $\mathfrak{p}(\mathcal{I})$, $\pi\mathfrak{p}(\mathcal{I})$, $\pi\chi(\mathcal{I})$ and $\chi(\mathcal{I})$ for $\mathrm{add}^*(\mathcal{I})$, $\mathrm{cov}^*(\mathcal{I})$, $\mathrm{non}^*(\mathcal{I})$ and $\mathrm{cof}^*(\mathcal{I})$ respectively.

If an ideal $\mathcal{I}$ is not tall then $\mathrm{cov}^*(\mathcal{I})$ is not well-defined (as the min would be taken from the empty set) and $\mathrm{non}^*(\mathcal{I}) = 1$.

There are some inequalities holding among these characteristics for all tall ideals (see e.g. [18, p. 578]), namely: $\aleph_0 \leq \mathrm{add}^*(\mathcal{I}) \leq \mathrm{cov}^*(\mathcal{I}) \leq \mathrm{cof}^*(\mathcal{I}) \leq \mathfrak{c}$ and $\aleph_0 \leq \mathrm{add}^*(\mathcal{I}) \leq \mathrm{non}^*(\mathcal{I}) \leq \mathrm{cof}^*(\mathcal{I}) \leq \mathfrak{c}$.

It is easy to check (see [28, Theorems 1.6.4 and 1.6.19]) that $\mathrm{add}^*(\mathcal{ED}) = \mathrm{add}^*(\mathrm{Fin} \otimes \mathrm{Fin}) = \aleph_0$, $\mathrm{non}^*(\mathcal{ED}) = \mathrm{non}^*(\mathrm{Fin} \otimes \mathrm{Fin}) = \aleph_0$. Using almost the same argument one can easily show the following.

Proposition 6.1. $\mathrm{add}^*(\mathcal{I}(\mathcal{F})) = \aleph_0$ *and* $\mathrm{non}^*(\mathcal{I}(\mathcal{F})) = \aleph_0$ *for all tall ideals* $\mathcal{I}(\mathcal{F})$.

6.1. **Cofinality.** It is known that $\mathrm{cof}^*(\mathrm{Fin}\otimes\mathrm{Fin}) = \mathfrak{d}$ (see [28, Theorem 1.6.19]). Below we show that $\mathfrak{d}$ is a lower bound of the cardinal cof^* for every ideal of the form $\mathcal{I}(\mathcal{F})$ but $\mathrm{Fin}\otimes\{\emptyset\}$.

Proposition 6.2. $\mathrm{cof}^*(\mathcal{I}(\mathcal{F})) \geq \mathfrak{d}$ *if and only if* $\mathcal{I}(\mathcal{F}) \neq \mathrm{Fin}\otimes\{\emptyset\}$.

Proof. ($\Longrightarrow$) It follows from the easy fact that $\mathrm{cof}^*(\mathrm{Fin}\otimes\{\emptyset\}) = \aleph_0 < \mathfrak{d}$.

($\Longleftarrow$) By Proposition 3.4(2) there is $f \in \mathcal{F}$ such that the set $A = \{n \in \omega : f(n) \neq 0\}$ is infinite. Suppose, to the contrary, that there is $\mathcal{B} \subseteq \mathcal{I}(\mathcal{F})$ such that $|\mathcal{B}| < \mathfrak{d}$ and for each $A \in \mathcal{I}(\mathcal{F})$ there is $B \in \mathcal{B}$ with $A \subseteq^* B$. Let $\mathcal{D} \subseteq \omega^\omega$ be cofinal in ω^ω and $|\mathcal{D}| = \mathfrak{d}$.

For each $g \in \mathcal{D}$ we put $A^g = \{(n, g(n)) : n \in A\}$. Since $|(A^g)_{(n)}| \leq f(n)$ for every $n \in \omega$, $A^g \in \mathcal{I}(\mathcal{F})$. Hence there is $B^g \in \mathcal{B}$ with $A^g \subseteq B^g$. Let $n_g \in \omega$ be such that $(B^g)_{(n)}$ is finite for all $n \geq n_g$. We define $h_g : A \to \omega$ by $h_g(n) = \max((B^g)_{(n)})$ for $n \geq n_g$ and $h_g(n) = 0$ otherwise.

Let $\mathcal{H} = \{h_g : g \in \mathcal{D}\}$. Since $\mathcal{H}$ is cofinal in $\mathcal{D} \restriction A = \{g \restriction A : g \in \mathcal{D}\}$ and $\mathcal{D} \restriction A$ is cofinal in ω^A, $\mathcal{H}$ is cofinal in ω^A. Thus, $|\mathcal{H}| \geq \mathfrak{d}$. On the other hand, $|\mathcal{H}| \leq |\mathcal{B}| < \mathfrak{d}$, a contradiction. □

It is known (see e.g. [18, Proposition 6.13]) that $\mathrm{cof}^*(\mathcal{I}) \geq \mathrm{cov}(\mathcal{M})$ for analytic ideals $\mathcal{I}$ that are not countably generated. Since $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$, Proposition 6.2 gives a better bound for $\mathrm{cof}^*(\mathcal{I}(\mathcal{F}))$ in the case of analytic ideals of the form $\mathcal{I}(\mathcal{F})$.

It is known that $\mathrm{cof}^*(\mathcal{ED}) = \mathfrak{c}$ (see [28, Theorem 1.6.4]). Below we show that the same holds for ideals "generated" by one essentially nonzero function.

Recall that $\mathcal{F}_f = \{k \cdot f : k \in \omega\}$ for $f \in \omega^\omega$.

Theorem 6.3. $\mathrm{cof}^*(\mathcal{I}(\mathcal{F}_f)) = \mathfrak{c}$ *for each* $f \in \omega^\omega$ *such that* $f \neq^* 0$.

Proof. Suppose, to the contrary, that $\mathrm{cof}^*(\mathcal{I}(\mathcal{F}_f)) < \mathfrak{c}$. Then there is $\mathcal{B} \subseteq \mathcal{I}(\mathcal{F}_f)$ such that $|\mathcal{B}| < \mathfrak{c}$ and for every $A \in \mathcal{I}(\mathcal{F}_f)$ there is $B \in \mathcal{B}$ with $A \subseteq B$.

We define $I_{n,0} = \{k \in \omega : 0 \leq k < f(n)\}$ and $I_{n,i} = \{k \in \omega : f(n)\cdot i \leq k < f(n)\cdot(i+1)\}$ for every $n, i \in \omega$, $i \geq 1$.

Let $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ be an almost disjoint family on ω (i.e. $A_\alpha \cap A_\beta$ is finite for $\alpha \neq \beta$ and $A_\alpha \in [\omega]^\omega$).

By $\phi_\alpha : \omega \to A_\alpha$ we denote the increasing enumeration of the set A_α.

For every $\alpha < \mathfrak{c}$, we define

$$C_\alpha = \bigcup_{n\in\omega} \{n\} \times I_{n,\phi_\alpha(n)}$$

and note that $C_\alpha \in \mathcal{I}(\mathcal{F}_f)$ for every $\alpha < \mathfrak{c}$.

Now for every $B \in \mathcal{B}$ let $C_B = \{\alpha < \mathfrak{c} : C_\alpha \subseteq B\}$. Since $C_\alpha \in \mathcal{I}(\mathcal{F}_f)$, $\bigcup_{B\in\mathcal{B}} C_B = \mathfrak{c}$.

If we show that C_B is finite for every $B \in \mathcal{B}$ then $|\bigcup_{B\in\mathcal{B}} C_B| \leq \omega \cdot |\mathcal{B}| < \mathfrak{c}$ and we obtain a contradiction that finishes the proof.

Let $B \in \mathcal{B}$ and $k \in \omega$ be such that $|B_{(n)}| \leq kf(n)$ for all but finitely many n. Then $|C_B| \leq k$. Indeed, if $|C_B| > k$ then there is $D \in [\mathfrak{c}]^{k+1}$ such that $C_\alpha \subseteq B$ for every $\alpha \in D$. Since $\{A_\alpha : \alpha \in D\}$ is almost disjoint, the elements $\phi_\alpha(n)$, where $\alpha \in D$, are pairwise distinct for all but finitely many n. Then the sets $(C_\alpha)_n$ where $\alpha \in D$ are pairwise disjoint for all but finitely many n. Hence $|B_{(n)}| \geq |D| \cdot |(C_\alpha)_{(n)}| = (k+1)\cdot f(n)$ for all but finitely many n.

Since $f(n) \neq 0$ for infinitely many n, there are infinitely many n with $|B_{(n)}| > kf(n)$, a contradiction. $\square$

Question 2. Does $\mathrm{cof}^*(\mathcal{I}(\mathcal{F})) = \mathfrak{c}$ for each countable family $\mathcal{F}$ such that $\mathcal{I}(\mathcal{F}) \neq \mathrm{Fin} \otimes \{\emptyset\}$?

6.2. **Covering.** It is known that $\mathrm{cov}^*(\mathrm{Fin} \otimes \mathrm{Fin}) = \mathfrak{b}$ and $\mathrm{cov}^*(\mathcal{ED}) = \mathrm{non}(\mathcal{M})$ (see [28, Theorems 1.6.19 and 1.6.4]). Below we observe that $\mathfrak{b}$ is a lower bound and $\mathrm{non}(\mathcal{M})$ is an upper bound of the cardinal cov^* for every tall ideal of the form $\mathcal{I}(\mathcal{F})$. In the proofs we will use the following lemma.

Lemma 6.4 ([17, Proposition 3.1])**.** *Let $\mathcal{I}, \mathcal{J}$ be tall ideals on ω.*

1. *If $\mathcal{I} \leq_K \mathcal{J}$ then $\mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cov}^*(\mathcal{J})$.*
2. *If $\mathcal{I} \subseteq \mathcal{J}$ then $\mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cov}^*(\mathcal{J})$.*
3. *If $X \notin \mathcal{I}$ then $\mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cov}^*(\mathcal{I} \restriction X)$.*

Proposition 6.5. $\mathfrak{b} \leq \mathrm{cov}^*(\mathcal{I}(\mathcal{F})) \leq \mathrm{non}(\mathcal{M})$ *for each tall ideal $\mathcal{I}(\mathcal{F})$.*

Proof. Using the inclusion $\mathcal{I}(\mathcal{F}) \subseteq \mathrm{Fin} \otimes \mathrm{Fin}$ and Proposition 6.4(2) we obtain $\mathrm{cov}^*(\mathcal{I}(\mathcal{F})) \geq \mathrm{cov}^*(\mathrm{Fin} \otimes \mathrm{Fin}) = \mathfrak{b}$.

Since $\mathcal{I}(\mathcal{F})$ is tall, there is $B \in [\omega]^\omega$ and $f \in \mathcal{F}$ such that $f(n) \neq 0$ for every $n \in B$ (see Proposition 3.7). Then $X = B \times \omega \notin \mathcal{I}(\mathcal{F})$ and by Proposition 3.4(3) we have $\mathcal{ED} \restriction X \subseteq \mathcal{I}(\mathcal{F}) \restriction X$.

Since $\mathcal{ED} \restriction X$ and $\mathcal{ED}$ are isomorphic, $\mathrm{cov}^*(\mathcal{ED} \restriction X) = \mathrm{cov}^*(\mathcal{ED}) = \mathrm{non}(\mathcal{M})$.

By Lemma 6.4(2) we have $\mathrm{non}(\mathcal{M}) = \mathrm{cov}^*(\mathcal{ED} \restriction X) \geq \mathrm{cov}^*(\mathcal{I}(\mathcal{F}) \restriction X)$ and by Lemma 6.4(3) we have $\mathrm{cov}^*(\mathcal{I}(\mathcal{F}) \restriction X) \geq \mathrm{cov}^*(\mathcal{I}(\mathcal{F}))$. Thus, $\mathrm{cov}^*(\mathcal{I}(\mathcal{F})) \leq \mathrm{non}(\mathcal{M})$. $\square$

It is easy to see that $\mathrm{cov}^*(\mathcal{I}) \geq \mathfrak{p}$ for every tall ideal $\mathcal{I}$, where $\mathfrak{p}$ is the pseudo-intersection number. Since $\mathfrak{p} \leq \mathfrak{b}$, Proposition 6.5 gives a better lower bound for $\mathrm{cov}^*(\mathcal{I}(\mathcal{F}))$ in the case of tall ideals of the form $\mathcal{I}(\mathcal{F})$.

In the sequel we will use the following characterization of the cardinal $\mathrm{non}(\mathcal{M})$.

Lemma 6.6 (see e.g. [3, Lemma 2.4.8])**.**

$$\mathrm{non}(\mathcal{M}) = \min\{|\mathcal{S}| : \mathcal{S} \subseteq \mathcal{C} \wedge \forall f \in \omega^\omega\, \exists S \in \mathcal{S}\, \exists^\infty n \in \omega\, (f(n) \in S(n))\},$$

where

$$\mathcal{C} = \left\{ S \in \left([\omega]^{<\omega}\right)^\omega : \sum_{n\in\omega} \frac{|S(n)|}{(n+1)^2} < \infty \right\}.$$

Proposition 6.7. *Let $\mathcal{I}(\mathcal{F})$ be a tall ideal. If there is an increasing sequence $n_0 < n_1 < \ldots$ such that*

$$\sum_{i\in\omega} \frac{f(i)}{(n_i+1)^2} < \infty$$

for every $f \in \mathcal{F}$ then $\mathrm{cov}^*(\mathcal{I}(\mathcal{F})) = \mathrm{non}(\mathcal{M})$.

Proof. ($\leq$) It follows from Proposition 6.5.

($\geq$) Let $\mathcal{A} \subseteq \mathcal{I}(\mathcal{F})$ be a family of cardinality less than $\mathrm{non}(\mathcal{M})$. For every $A \in \mathcal{A}$ there is a function $f_A \in \mathcal{F}$ and $n_A \in \omega$ such that $|A_{(n)}| \leq f_A(n)$ for every $n \geq n_A$. For every $A \in \mathcal{A}$ we define the function $S_A : \omega \to [\omega]^{<\omega}$ by

$$S_A(n) = \begin{cases} A_{(i)} & \text{for } n = n_i \wedge i \geq n_A, \\ \emptyset & \text{otherwise.} \end{cases}$$

Note that

$$\sum_{n\in\omega}\frac{|S(n)|}{(n+1)^2}=\sum_{i\geq n_A}\frac{|S(n_i)|}{(n_i+1)^2}=\sum_{i\geq n_A}\frac{|A_{(i)}|}{(n_i+1)^2}\leq\sum_{i\geq n_A}\frac{f_A(i)}{(n_i+1)^2}<\infty$$

and $|\{S_A : A\in\mathcal{A}\}|\leq|\mathcal{A}|<\mathrm{non}(\mathcal{M})$. By Lemma 6.6 there is $g\in\omega^\omega$ such that for every $A\in\mathcal{A}$ and almost all $n\in\omega$ we have $g(n)\notin S_A(n)$.

Let $h:\omega\to\omega$ be given by $h(k)=g(n_k)$ for every $k\in\omega$.

Then $B=(\omega\times\omega)\setminus h\in\mathcal{P}(\omega\times\omega)\setminus\mathrm{Fin}^*$ and for every $A\in\mathcal{A}$ we have $A\subseteq^* B$. □

Proposition 6.8. *Let $\mathcal{I}(\mathcal{F})$ be a tall ideal. If $|\mathcal{F}|<\mathfrak{b}$, then $\mathrm{cov}^*(\mathcal{I}(\mathcal{F}))=\mathrm{non}(\mathcal{M})$.*

Proof. It is not difficult to see that for every $f\in\mathcal{F}$ there is an increasing sequence $\langle n_i^f\rangle$ such that $\sum_{i\in\omega}f(i)/(n_i^f+1)^2<\infty$. Since $|\mathcal{F}|<\mathfrak{b}$, there is an increasing sequence $\langle n_i\rangle$ such that $\langle n_i^f\rangle\leq^*(n_i)$ for every $f\in\mathcal{F}$. Let $n^f\in\omega$ be such that $n_i^f\leq n_i$ for every $i\geq n^f$. Then

$$\sum_{i\in\omega}\frac{f(i)}{(n_i+1)^2}\leq\sum_{i<n^f}\frac{f(i)}{(n_i+1)^2}+\sum_{i\geq n^f}\frac{f(i)}{(n_i^f+1)^2}<\infty,$$

so Proposition 6.7 finishes the proof. □

Acknowledgments. The authors would like to thank the reviewers for their comments that helped to improve the manuscript.

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