

Reinhardt cardinals and iterates of V

Farmer Schlutzenberg*

June 30, 2020

Abstract

Assume $\text{ZF}(j)$ and there is a Reinhardt cardinal, as witnessed by the elementary embedding $j : V \rightarrow V$. We investigate the linear iterates (N_α, j_α) of (V, j) , and their relationship to (V, j) , forcing and definability, including that for each infinite α , every set is set-generic over N_α , but N_α is not a set-ground.

Assume second order ZF. We prove that the existence of super Reinhardt cardinals and total Reinhardt cardinals is not affected by small forcing. And if $V[G]$ has a set of ordinals which is not in V , then $V[G]$ has no elementary embedding $j : V[G] \rightarrow M \subseteq V$ (even allowing M to be illfounded).^{1 2}

1 Introduction

A *Reinhardt cardinal*, introduced by William Reinhardt in [11] and [12], is the critical point of an elementary embedding $j : V \rightarrow V$. Kunen showed in [9] that if $V \models \text{ZFC}$ then there is no such j . So Reinhardt embeddings (that is, $j : V \rightarrow V$) are considered with background theory $\text{ZF}(j)$ (for the basic definitions §1.1 below), possibly augmented with fragments of AC. We work throughout in ZF or variants thereof such as $\text{ZF}(j)$ or ZF_2 , indicating any choice assumptions where they are adopted.

This paper primarily investigates the iterates of V under an elementary $j : V \rightarrow V$. Let $M_0 = (V, j)$. In Definition 3.2 we define the α th iterate $M_\alpha = (N_\alpha, j_\alpha)$ of M_0 for each $\alpha \in \text{OR}$. These are analogous to the iterates of $(L(V_{\lambda+1}), k)$ in the context of I_0 , investigated by Woodin in [20], and Mohammad Golshani posed a question of the nature of the M_α on the discussion board Mathoverflow.

In [7], many variants of and facts related to Kunen's ZFC inconsistency result are discussed. This paper makes a contribution toward understanding

*schlutze@uni-muenster.de

¹Keywords: Reinhardt cardinal, iterate, HOD, definability, forcing, Axiom of Choice

² This paper originates from a set of rather informal notes [16], published on arxiv.org (v1 of arXiv:2002.01215). The current paper is *vn* of (the same) arXiv:2002.01215, where $n > 1$. Some of the results from the original notes, regarding definability and constructibility of embeddings $j : V_\delta \rightarrow V_\delta$ and related facts, and those on extenders under ZF and definability of V -critical points from a proper class of weak Löwenheim Skolem cardinals, are to appear in the papers [6] and [14]. The original notes together with some errata can be seen at <https://sites.google.com/site/schlutzenberg/home-1>, but it is in general better to refer to the current paper and the two just cited. Of course, the original notes also remain available on arxiv.org, as mentioned above.

those issues under ZF, a question posed in [7]. The analysis of the iterates M_α will show that some of the conclusions proven in [7] fail in a strong way under $\text{ZF}(j) + j : V \rightarrow V$. On the other hand, we are able to establish a partial ZF generalization of Woodin's ZFC theorem [7, Theorem 5] excluding elementary $j : V[G] \rightarrow V$.

We prove the following facts. In Theorem 3.14, we show that every set is set-generic over N_α . The key to this is using a slight variant of a forcing due to Schindler (the latter being related to the Bukowsky forcing, and also to Vopenka and the extender algebra). In particular, for the iteration map $i_{0\alpha} : V \rightarrow N_\alpha$, $i_{0\alpha} \restriction V_\beta$ is set-generic over N_α , and this has the consequence that there are interesting generic extensions of V , including that $\text{cr}(j)$ is a *virtual Berkeley cardinal* (see Definition 3.15 and Corollary 3.16). In Theorem 3.19, we show that, however, for $\alpha \geq \omega$, N_α is not a set-ground of V (note $N_n = V$ for $n < \omega$). It is well known that every set is also set-generic over HOD. We show in Theorem 3.7 that $\text{HOD} \subseteq N_\omega$, and in fact $\text{Hull}^{(V, M_\omega)}(N_\omega) = N_\omega$ (where the hull is computed in V , with the predicate M_ω (including j_ω) from parameters in N_ω), but there are $\alpha \in \text{OR}$ with $\text{HOD} \not\subseteq N_\alpha$.

By [7, Theorem 12], under ZFC_2 , if $j : V \rightarrow M$ is elementary with M transitive then $V = \text{Hull}^V(\text{rg}(j))$. This fails trivially in the case that $(V, j) \models \text{ZFR}$ (where $M = V$), since then $\text{Hull}^V(\text{rg}(j)) = \text{rg}(j)$. However, because $\text{Hull}^V(N_\omega) = N_\omega$, we get a non-trivial failure, as we have the iteration map $i_{0\omega} : V \rightarrow N_\omega$, and $N_\omega \neq V$. We will also prove a stronger fact in Theorem 3.21, that for every set X , $V \neq \text{Hull}^{(V, M_\omega)}(N_\omega \cup X)$. At the same time we will establish a strengthening of Suzuki's theorem [17] that j is not definable from parameters (which was strengthened in other ways in [7], [6] and [14]): we show that $j \restriction \text{OR}$ is not definable from parameters and the predicate (N_ω, j_ω) .³

It follows in particular that if there is a Reinhardt, then for all sets X , $V \neq \text{HOD}(X)$. And if V is total Reinhardt or there is a Berkeley cardinal, a similar proof gives that $V \neq \text{HOD}_A(X)$ for any class A and set X . But in fact much more is now known; see Remark 3.22.

In §4 we prove some facts regarding small forcing and (very) large cardinals. Theorem 4.6 asserts that if $\mathbb{P} \in V$ and G is (V, \mathbb{P}) -generic and $(V[G], j) \models \text{ZFR} + \text{"}\mathbb{P} \text{ is wellorderable and } \text{card}(\mathbb{P})^+ \text{-DC holds"}$, then $(V, j \restriction V) \models \text{ZFR}$. Assume ZF_2 . By Theorem 4.8, if G is set-generic over V and $V[G]$ has a super Reinhardt cardinal, then so does V . A similar fact is proven for total Reinhardtness. Theorem 4.4 is a variant of the $\text{ZF} + \text{DC}_\delta$ ground definability result of Gitman-Johnstone [2], in which we assume some DC in the generic extension, but not (at least not explicitly) in the ground model (but we assume that the forcing in question is small).

Theorem 2.3 is a generalization of Woodin's ZFC_2 theorem [7, Theorem 4] that if G is set-generic over V then there is no elementary $j : V[G] \rightarrow V$. We prove assuming ZF_2 that if G is set-generic over V and $V[G]$ has some set of ordinals not in V , then there is no elementary $j : V[G] \rightarrow M \subseteq V$, even allowing M to be illfounded. This fact appears to be new even in the ZFC context; the results in [7] seem to rely on stationary set combinatorics which does not seem to be available in this generality.

³The fact that $j \restriction \text{OR}$ is not definable over V from parameters (but without any extra predicate) is already a consequence of [7, Theorem 35], by which no elementary $k : \text{HOD} \rightarrow \text{HOD}$ is definable from parameters (without assuming AC). In Remark 3.20 we also sketch an alternate proof of the latter fact.

There are many questions that are open, and some of these are mentioned throughout. We would like to thank Gabriel Goldberg and Toshimichi Usuba for their feedback regarding certain questions in the notes [16] (upon which the material here is based), and other helpful suggestions.

1.1 Basic definitions

OR denotes the class of ordinals and Lim the limit ordinals.

The language $\mathcal{L}_{A_1, \dots, A_n}$ of set theory with predicates A_1, \dots, A_n is the first order language with binary predicate symbol \in and predicate symbols A_1, \dots, A_n . The theory $\text{ZF}(\vec{A})$ is the theory in $\mathcal{L}_{\in, \vec{A}}$ with all ZF axioms, allowing all formulas of $\mathcal{L}_{\in, \vec{A}}$ in the Separation and Collection schemes (so each A_i represents a class).

ZFR (ZF+ Reinhardt) is the theory $\text{ZF}(\tilde{j}) + \text{"}\tilde{j} : V \rightarrow V \text{ is } \Sigma_1\text{-elementary"}$. And $\text{ZFR}(A)$ is the theory $\text{ZF}(\tilde{j}, A) + \text{"}\tilde{j} : (V, A) \rightarrow (V, A) \text{ is } \Sigma_1\text{-elementary"}$. Essentially by [8, Proposition 5.1], $\text{ZFR}(A)$ proves \tilde{j} is fully elementary (as a theorem scheme).

Second order set theory is denoted ZF_2 (see [1]). Models are of the form (V, \in, P) , where $(V, \in) \models \text{ZF}$ and P is a collection of classes/subsets of V , satisfying the ZF_2 axioms. Given a transitive W and $P \subseteq \mathcal{P}(W)$, we write $(W, P) \models \text{ZF}_2$ iff $(W, \in, P) \models \text{ZF}_2$. When we “work in ZF_2 ”, we mean that we work in such a model W , and all talk of proper classes refers to elements of P .

Let $\delta \in \text{Lim}$. The *cofinality* of δ , *regularity*, *singularity* are defined as usual (in terms of cofinal functions between ordinals). We say δ (or V_δ) is *inaccessible* iff there is no (γ, f) such that $\gamma < \delta$ and $f : V_\gamma \rightarrow \delta$ is cofinal in δ .

Work in ZF_2 . Then $\kappa \in \text{OR}$ is *Reinhardt* iff there is a class j such that $(V, j) \models \text{ZFR}$ and $\kappa = \text{cr}(j)$. Following [1], $\kappa \in \text{OR}$ is *super-Reinhardt* iff for every $\lambda \in \text{OR}$ there is a class j such that $(V, j) \models \text{ZFR}$ and $\text{cr}(j) = \kappa$ and $j(\kappa) \geq \lambda$. And OR is *total Reinhardt* if for every class A there is $\kappa \in \text{OR}$ such that κ is $(< \text{OR}, V, A)$ -*reflecting*, meaning that for every $\lambda \in \text{OR}$ there is a class j such that $(V, j, A) \models \text{ZFR}(A)$ and $\text{cr}(j) = \kappa$ and $j(\kappa) \geq \lambda$. And δ is *Berkeley* if for all transitive sets M with $\delta \in M$, and all $\eta < \delta$, there is an elementary $j : M \rightarrow M$ with $\eta < \text{cr}(j) < \delta$.

Let $j : V_\delta \rightarrow V_\delta$ or $j : V \rightarrow V$ be Σ_1 -elementary, for a limit δ or $\delta = \text{OR}$. We write $\kappa_0(j) = \text{cr}(j)$ and $\kappa_{n+1}(j) = j(\kappa_n(j))$ and $\kappa_\omega(j) = \sup_{n < \omega} \kappa_n$.

Let C be a class. Then $\text{OD}_C(X)$ denotes the class of all sets y such that y is definable from the predicate C and parameters in $\text{OR} \cup X$. And $\text{HOD}_C(X)$ denotes the class of all y such that the transitive closure of $\{y\}$ is $\subseteq \text{OD}_C(X)$. And $\text{OD}(X) = \text{OD}_\emptyset(X)$ and $\text{HOD}(X) = \text{HOD}_\emptyset(X)$. And for a structure M and $X \subseteq M$, $\text{Hull}^M(X)$ denotes the collection of elements $x \in M$ such that x is definable over M from (finitely many) parameters in X .

2 Generic $j : V[G] \rightarrow M \subseteq V$

The following fact, due to Woodin, is proved in [7, Theorem 5]:

Fact 2.1 (Woodin). *Assume ZFC. Let G be set-generic over V . Then there is no elementary $j : V[G] \rightarrow V$.*

Of course in the case that $G \in V$, this reduces to Kunen's inconsistency. Various generalizations of this result can also be seen in [7]; one in particular is that, by [7, Corollary 34], under ZF, if G is set-generic over V , then there is no elementary $j : V[G] \rightarrow V$ which is definable from parameters over $V[G]$.

In this section, we generalize Fact 2.1 in a few ways. First, we replace the assumption that ZFC holds in V with ZF plus the requirement that G adds a set of ordinals to V (which of course holds in the ZFC context, if $G \notin V$). Second, we only require that $M \subseteq V$, not that $M = V$ or M is eventually stationarily correct or anything further, and so in particular, we allow M to be illfounded. The proof also gives an alternate proof of Woodin's result, for the case that $G \notin V$; note that we do not use any stationary set combinatorics. The same kind of argument can be used in certain other places where the stationary set argument is traditionally used, for example, in Usuba's ZFC proof that the mantle coincides with the κ -mantle if κ is extendible.

Lemma 2.2. *Assume ZF. Let G be (V, \mathbb{P}) -generic for some forcing $\mathbb{P} \in V$ and let $\alpha \in \text{OR}$. Let $\theta \in \text{OR}$ be such that V has no surjection $\mathbb{P}^{<\omega} \times \alpha \rightarrow \theta$. Suppose (M, E, N, j) are such that $M, E \in V$ and $N, j \in V[G]$, N is transitive and rudimentarily closed, $\mathcal{H}_\theta^V \subseteq N$, and $j : (N, \in) \rightarrow (M, E)$ is elementary. (Note we allow (M, E) to be illfounded.) Then:*

1. $j \restriction \alpha \in V$,
2. $\mathcal{P}(\alpha) \cap N = \mathcal{P}(\alpha) \cap V$.

Proof. Part 2: This follows directly from part 1. For $\mathcal{P}(\alpha) \cap V \subseteq \mathcal{H}_\theta^V \subseteq N$ by assumption, and supposing $X \in N \cap \mathcal{P}(\alpha)$, for each $\xi < \alpha$, we have

$$\xi \in X \iff N \models \text{"}\xi \in X\text{"} \iff (M, E) \models \text{"}j(\xi) \in j(X)\text{"},$$

but since $(M, E) \in V$ and $j \restriction \alpha \in V$ (by part 1), therefore $X \in V$.

Part 1: We prove a couple of claims.

Claim 1. $V[G] \models \text{"there is no surjection } f : \mathbb{P}^{<\omega} \times \alpha \rightarrow \theta\text{"}$.

Proof. Suppose otherwise, and fix $\tilde{f} \in V$ with $\tilde{f}_G = f$ and $p_0 \in G$ such that $p_0 \Vdash_{\mathbb{P}} \text{"}\tilde{f} : \mathbb{P}^{<\omega} \times \check{\alpha} \rightarrow \check{\theta} \text{ is surjective"}\text{"}$. We define a function $g \in V$. For $(p, \vec{q}, \xi, \gamma)$ in $\mathbb{P} \times \mathbb{P}^{<\omega} \times \alpha \times \theta$, set

$$g(p, \vec{q}, \xi) = \gamma \iff p \leq p_0 \text{ and } p \Vdash_{\mathbb{P}} \tilde{f}(\vec{q}, \check{\xi}) = \check{\gamma}.$$

Note $A = \text{dom}(g) \subseteq \mathbb{P} \times \mathbb{P}^{<\omega} \times \alpha$ and $g : A \rightarrow \theta$ is surjective, impossible. \square

In $V[G]$, we have $j \restriction \theta : \theta \rightarrow \text{OR}^M$. Fix $\tilde{k} \in V$ with $\tilde{k}_G = j \restriction \theta$ and $p_0 \in G$ with $p_0 \Vdash_{\mathbb{P}} \text{"}\tilde{k} : \check{\theta} \rightarrow \text{OR}^{\check{M}}\text{"}$.⁴ Working in V , for $p \in \mathbb{P}$ with $p \leq p_0$, define $X_p \subseteq \theta$ and $j_p : X_p \rightarrow \text{OR}^M$ the function giving all values of \tilde{k} decided by p ; that is, for $\xi < \theta$,

$$j_p(\xi) = x \iff x \in \text{OR}^M \text{ and } p \Vdash_{\mathbb{P}} \tilde{k}(\check{\xi}) = \check{x},$$

and $X_p = \text{dom}(j_p)$. So $X_p, j_p \in V$.

⁴Here $\text{OR}^M = \{x \in M \mid (M, E) \models \text{"}x \text{ is an ordinal"}\}$, and M might be illfounded.

Claim 2. *There is $p \in G$ such that $p \leq p_0$ and X_p has ordertype $\beta > \alpha$.⁵*

Proof. We have $\theta = \bigcup_{p \in G, p \leq p_0} X_p$. Suppose the claim fails. In $V[G]$, define a surjection $f : \mathbb{P} \times \alpha \rightarrow \theta$ by setting $f(p, \xi) =$ the ξ th element of X_p , in the case that X_p is defined and has ordertype $> \xi$, and $f(p, \xi) = 0$ otherwise. This contradicts Claim 1. \square

So fix p as in Claim 2. Let α' be the α th element of X_p . So $Y = X_p \cap \alpha'$ has ordertype α . Let $\pi : \alpha \rightarrow Y$ be the increasing enumeration of Y . Then $\alpha, Y, \pi \in \mathcal{H}_\theta^V \subseteq N$. Since N is transitive,

$$N \models \text{“}\pi : \alpha \rightarrow Y \text{ is the increasing enumeration of } Y\text{”},$$

so applying j ,

$$(M, E) \models \text{“}j(\pi) : j(\alpha) \rightarrow j(Y) \text{ is the increasing enumeration of } j(Y)\text{”}.$$

Let $j_Y = j \restriction Y \in V = j_p \restriction Y \in V$. Note that $j_Y(\pi(\xi)) = j(\pi(\xi)) = j(\pi)(j(\xi))$ for each $\xi < \alpha$,⁶ and so⁷ $j \restriction \alpha = j(\pi)^{-1} \circ j_Y \circ \pi$, and since $j_Y, \pi, j(\pi), (M, E) \in V$, we get $j \restriction \alpha \in V$. \square

From the lemma, we can deduce:

Theorem 2.3. *Assume $(V, P) \models \text{ZF}_2$. Let G be (V, \mathbb{P}) -generic for some $\mathbb{P} \in V$. Suppose there are $\alpha \in \text{OR}$ and $X \subseteq \alpha$ with $X \in V[G] \setminus V$.*

Then there is no $(j, M) \in P[G]$ with $M \subseteq V$ and $j : V[G] \rightarrow M$ elementary. Hence, there is no $j \in P[G]$ such that $j : V[G] \rightarrow V$ is elementary.

Question 2.4. Is it consistent that there is $j : V[G] \rightarrow M \subseteq V$ as in the statement of Theorem 2.3, if we instead have $\mathcal{P}(\alpha) \cap V[G] \subseteq V$ for each $\alpha \in \text{OR}$? Does it follow that $V^{<\text{OR}} \cap V[G] \subseteq V$? What about in the case that $M = V$?

What about the converse direction, i.e. an embedding $j : V \rightarrow V[G]$, where G is set-generic over $V \models \text{ZF}$?

In the next section we will see that if ZFR is consistent, then one *can* get $j : V \rightarrow M \subsetneq V$ where every *set* is (individually) set-generic over M ; this fact should be compared with [7, §4].

3 The iterates of (V, j)

3.1 Basic properties

Remark 3.1. As the reader familiar with inner model theory will see, the methods in this section are heavily based on analyses of HOD via direct limit systems of mice, and related methods from the study of Varsovian models. The present context actually provides a nice introduction to some of those ideas, which avoids various issues (iteration trees, fine structure, comparison).

⁵Note that the proof does not use AC.

⁶Here $j(\pi)(j(\xi))$ is a slight abuse of notation; it should really be $(j(\pi)(j(\xi)))^M$, i.e. the $x \in M$ such that $M \models \text{“}x = j(\pi)(j(\xi))\text{”}$.

⁷The notation $\text{“}j(\pi)^{-1}\text{”}$ involves a similar abuse.

We start with an easy observation. Assume $(V, j) \models \text{ZFR}$ and let $\lambda = \kappa_\omega(j)$. Then for each $X \in V_\lambda$ (i) there is no cofinal $f : \omega \rightarrow \lambda$ with $f \in \text{OD}_X$, and (ii) $j \upharpoonright \lambda \notin \text{OD}_X$.

For suppose f is OD_X . Then there is in fact some such f which is definable from (X, λ) , by minimizing on other ordinal parameters. Taking $n < \omega$ large enough, then $j^n(X) = X$, so $j^n(f) = f$, so $j^n(f(k)) = f(k)$ for each $k < \omega$. But $\text{rg}(f)$ is unbounded in λ , and λ is the least fixed point of j^n which is $> \text{cr}(j^n)$, a contradiction. And from $j \upharpoonright \lambda$ we can define $\langle \kappa_n \rangle_{n < \omega}$, which we have just seen is not OD_X , so $j \upharpoonright \lambda$ is not OD_X .

Now in the study of $L(V_{\lambda+1})$ under I_0 , (see [20], for example), the iterates of $(L(V_{\lambda+1}), j)$ are an important focus. And Mohammad Golshani asked, on the discussion board Mathoverflow, what properties might hold of the intersection of the models N_α which we now define.⁸

Definition 3.2 (Iterates M_α). Assume $(V, j) \models \text{ZFR}$. Set $M_0 = (V, j)$. We define, for $\alpha \leq \beta \in \text{OR}$, the α th iterate $M_\alpha = (N_\alpha, j_\alpha)$, where N_α is a transitive proper class and $j_\alpha : N_\alpha \rightarrow N_\alpha$ a Reinhardt embedding of N_α (so $(N_\alpha, j_\alpha) \models \text{ZFR}$) and $i_{\alpha\beta} : M_\alpha \rightarrow M_\beta$ is elementary (literally a class function $N_\alpha \rightarrow N_\beta$, but elementary with respect to the predicates j_α, j_β), and such that $i_{\alpha\gamma} = i_{\beta\gamma} \circ i_{\alpha\beta}$ for all $\alpha \leq \beta \leq \gamma$, and $i_{\alpha\alpha} = \text{id}$, as follows. Given M_β , set $M_{\beta+1} = (N_\beta, j_\beta(j_\beta))$ and $i_{\alpha, \beta+1} = j_\beta \circ i_{\alpha\beta}$. Given M_β for all $\beta < \eta$ where η is a limit, set M_η as the direct limit, and $i_{\alpha\eta}$ the direct limit map.

By elementarity, N_α is extensional. Assuming that N_α is wellfounded, we identify it with its transitive collapse.

We introduce some symbols to the ZFR language to express these notions in formal language. For example, \widetilde{M} represents the function $\alpha \mapsto M_\alpha$ defined under ZFR as above. So with the models above and $\alpha \in \text{OR}^V$, we have $(\widetilde{M}_\alpha)^{(V, j)} = M_\alpha$, and if M_α is wellfounded then $(\widetilde{M}_\beta)^{M_\alpha} = M_{\alpha+\beta}$. We write $\widetilde{N}_\alpha, \widetilde{j}_\alpha, \widetilde{i}_{\alpha\beta}$ similarly. (This literally overloads the symbol \widetilde{j} with two interpretations, but when we write \widetilde{j} without a subscript, it always refers to the original function \widetilde{j}_0 .) \dashv

We now work in a model $(V, j) \models \text{ZFR}$.

As pointed out by Hamkins on Mathoverflow (see Footnote 8), the usual proof of linear iterability with respect to a single normal measure gives the following (iterability also holds in the I_0 case and generalizations thereof, but the proof is finer there; see [20] and [15]):

Lemma 3.3. *For all $\alpha \in \text{OR}$, M_α is a wellfounded (we take it transitive), $M_\alpha = (N_\alpha, j_\alpha) \models \text{ZFR}$, and $i_{\beta\alpha} : M_\alpha \rightarrow M_\beta$ is elementary and \in -cofinal.⁹*

Proof. Everything is proved by induction. We get the elementarity because $M_0 = (N_0, j_0) \models \text{ZFR}$, so for each $n < \omega$ there is a club class of α such that $(V_\alpha^{N_0}, j \upharpoonright V_\alpha^{N_0}) \preceq_n (N_0, j)$, which easily gives that i_{01} is elementary with respect to M_0, M_1 , not just N_0, N_1 , and likewise for larger indices.

⁸For the discussion see <https://mathoverflow.net/questions/185253/reinhardt-cardinals-and-iterability>. Note there is a slight difference in notation; we are writing N_α for what he refers to as M_α there.

⁹Note that the elementarity is with respect to $M_\alpha = (N_\alpha, j_\alpha)$ and $M_\beta = (N_\beta, j_\beta)$, not just N_α, N_β .

Now suppose that $\eta \in \text{OR}$ and M_η is illfounded, and η is least such. Then OR^{M_η} is illfounded (consider ranks of elements). Since $M_{\alpha+1}$ and M_α have the same universe, η is a limit ordinal. Let $\lambda \in \text{OR}$ be least with $i_{0\eta}(\lambda)$ in the illfounded part of OR^{M_η} . There is $\alpha < \eta$ and $\lambda_1 \in \text{OR}$ such that $\lambda' = i_{0\alpha}(\lambda) > \lambda_1$ and $i_{\alpha\eta}(\lambda_1)$ is in the illfounded part of M_η . Let $\eta' = i_{0\alpha}(\eta) \geq \eta$. By elementarity of $i_{0\alpha}$, $M_\alpha \models \text{"}\eta' \text{ is the least } \gamma \in \text{OR} \text{ such that } \widetilde{M}_\gamma \text{ is illfounded"}\text{"}$. Since M_α is wellfounded, $(\widetilde{M}_\beta)^{M_\alpha} = M_{\alpha+\beta}$ and $\alpha + \eta' = \eta$, so $\eta' = \eta$ and $\widetilde{M}_\eta^{M_\alpha} = M_\eta$. By elementarity, $M_\alpha \models \text{"}\lambda' \text{ is the least } \xi \in \text{OR} \text{ with } \widetilde{i}_{0\eta}(\xi) \text{ in the illfounded part of } \widetilde{M}_\eta\text{"}$. But $\widetilde{i}_{0\eta}^{M_\alpha} = i_{\alpha\eta}$, and we chose α with $i_{\alpha\eta}(\lambda_1)$ in the illfounded part of M_η , a contradiction. \square

Lemma 3.4. *Let $x \in M_\omega$ and $m < \omega$ with $x \in \text{rg}(i_{m\omega})$. Then $i_{mn}(x) = x$ for all $n \in [m, \omega)$.*

Proof. Let \bar{x} be such that $i_{m\omega}(\bar{x}) = x$. Then $M_m \models \text{"}x = \widetilde{i}_{0\omega}(\bar{x})\text{"}$, so letting $(\bar{x}', x') = i_{mn}(\bar{x}, x)$, elementarity gives $M_n \models \text{"}x' = \widetilde{i}_{0\omega}(\bar{x}')\text{"}$, but $\widetilde{i}_{0\omega}^{M_n} = i_{n\omega}$, so by commutativity, $x' = i_{m\omega}(\bar{x}) = x$. \square

By the lemma, the following definition makes sense:

Definition 3.5. For $x \in M_\omega$ let $x^* = \lim_{n < \omega} i_{n\omega}(x) = i_{m\omega}(x)$ for any/all $m < \omega$ with $x \in \text{rg}(i_{m\omega})$. \dashv

Lemma 3.6. *For $x \in M_\omega$, we have $x^* = i_{\omega, \omega+\omega}(x) = \widetilde{i}_{0\omega}^{M_\omega}(x)$.*

Proof. Given x , let \bar{x} and $m < \omega$ with $i_{m\omega}(\bar{x}) = x$. Then $M_m \models \text{"}\widetilde{i}_{0\omega}(\bar{x}) = x\text{"}$, so since $x^* = i_{m\omega}(x)$, by elementarity we have $M_\omega \models \text{"}\widetilde{i}_{0\omega}(x) = x^*\text{"}$, but $\widetilde{i}_{0\omega}^{M_\omega} = i_{\omega, \omega+\omega}$, so $i_{\omega, \omega+\omega}(x) = x^*$. \square

We now establish that N_ω is closed under definability in V . Note that the notation $\text{Hull}^{(V, M_\omega)}(X)$ refers to definability in V with M_ω as a predicate, from parameters in X .

Theorem 3.7. *Let $\kappa_n = \text{cr}(j_n)$ and $\lambda = \kappa_\omega(j)$. Then:*

1. $\text{HOD} \subseteq \text{Hull}^V(N_\omega) = \text{Hull}^{(V, M_\omega)}(N_\omega) = N_\omega$,
2. $V_\lambda^{\text{HOD}} = V_\lambda^{\text{HOD}^{N_\omega}}$ and $V_{\lambda+1}^{\text{HOD}} \subseteq V_{\lambda+1}^{\text{HOD}^{N_\omega}}$.
3. λ is greatly Mahlo in HOD and weakly compact in HOD^{N_ω} .
4. λ is N_ω -Berkeley in V .¹⁰
5. $(\lambda^+)^{\text{HOD}} \leq (\lambda^+)^{N_\omega} < \lambda^+$
6. If κ_0^+ is regular or $\text{cof}(\kappa_0^+) = \kappa_0$ then $\text{cof}((\lambda^+)^{M_\omega}) = \omega$.
7. If $\text{cof}(\kappa_0^+) < \kappa_0$ then $\text{cof}((\lambda^+)^{M_\omega}) = \text{cof}(\kappa_0^+)$.

¹⁰Note this generalizes the (folklore?) fact that λ is HOD-Berkeley, mentioned in [6, §***].

Proof. Part 1: We just need to see that $\text{Hull}^{(V, M_\omega)}(N_\omega) \subseteq N_\omega$. We prove this by induction on rank of elements of $\text{Hull}^{(V, M_\omega)}(N_\omega)$. So let $X \in \text{Hull}^{(V, M_\omega)}(N_\omega)$ with $X \subseteq N_\omega$; we must see that $X \in N_\omega$. Let φ be a formula in $\mathcal{L}_{\dot{M}}$ and $p \in N_\omega$ be such that for $x \in N_\omega$, we have $x \in X$ iff $V \models \varphi(x, p; M_\omega)$, with M_ω interpreting the predicate symbol \dot{M} . Then for $n < \omega$, $x \in X$ iff $M_n \models \varphi(x, p; \widetilde{M}_\omega)$ iff $M_\omega \models \varphi(x^*, p^*; \widetilde{M}_\omega)$ iff $M_\omega \models \varphi(\tilde{i}_{0\omega}(x), \tilde{i}_{0\omega}(p), \widetilde{M}_\omega)$, so X is computed by M_ω , so $X \in N_\omega$, as desired.

Part 2: Because $\lim_{n < \omega} \text{cr}(i_{n\omega}) = \lambda$, we easily have $V_\lambda^{\text{HOD}} = V_\lambda^{\text{HOD}^{N_\omega}}$. If $A \in V_{\lambda+1}^{\text{HOD}}$ then just note that $A^* \in \text{HOD}^{N_\omega}$ and $A^* \cap V_\lambda = A$, so $A \in V_{\lambda+1}^{\text{HOD}^{N_\omega}}$.

Part 3: Since $\lambda = \text{cr}(j_\omega)$ and $j_\omega \upharpoonright \text{HOD}^{N_\omega} : \text{HOD}^{N_\omega} \rightarrow \text{HOD}^{N_\omega}$ is elementary, we easily get that λ is weakly compact in HOD^{N_ω} , hence greatly Mahlo in HOD^{N_ω} , which by part 2 implies λ is greatly Mahlo in HOD .

Part 4: By Lemma 3.4, for every $M \in N_\omega$, there is $n < \omega$ such that $j_m(M) = M$ for all $m \in [n, \omega)$, and since $\lambda = \sup_{m < \omega} \text{cr}(j_m)$, this suffices.

Parts 5–7: These facts now follow easily by considering the direct limit producing $(\lambda^+)^{M_\omega}$. We have $(\lambda^+)^{M_\omega} < \lambda^+$ because there is a surjection from $\omega \times \lambda \rightarrow (\lambda^+)^{M_\omega}$ computable directly from the direct limit. If κ_0^+ is regular or $\text{cof}(\kappa_0^+) = \kappa_0$ then j_{mn} (where $m \leq n < \omega$) is discontinuous at κ_m^+ , and since $(\lambda^+)^{M_\omega}$ is the direct limit of an ω -sequence of these, this ordinal has cofinality ω . In the other case, j_{mn} and $j_{m\omega}$ is continuous at (κ_0^+) . \square

Question 3.8. Is/can we have $V_{\lambda+1}^{\text{HOD}} = V_{\lambda+1}^{\text{HOD}^{N_\omega}}$? Or $V_{\lambda+1}^{\text{HOD}} = V_{\lambda+1}^{N_\omega}$? Is λ Woodin in HOD ? Weakly compact in HOD ? Reflecting in HOD ?¹¹ These properties seem (to the author) reasonable to ask about, since λ has them in M_ω and $\text{HOD} \subseteq M_\omega$. If $V_\lambda \subseteq \text{HOD}$ then λ is clearly Woodin in HOD , since it is Woodin (and more) in N_ω .

3.2 Every set is generic over N_α

By Vopenka, every set is set-generic over HOD . Moreover:

Lemma 3.9. *Every set in V is set-generic over M_ω .*

Proof. One way to prove this is to use a Vopenka argument, as done in [15]. But we want to generalize the theorem later to M_α for arbitrary α , and there we will not have $\text{HOD} \subseteq M_\alpha$. So we use a different argument, which seems to be more generalizable.

We will show directly that for each $\beta \in \text{OR}$ and each $X \subseteq V_\beta^{M_\omega}$, with $X \in V$, we can add X set-generically to M_ω . This suffices, because letting $\alpha \in \text{OR}$, we can apply this to $X = i_{0\omega} "V_\alpha$, and then $V_\alpha^{M_\omega[G]} = V_\alpha$ (and G adds $j \upharpoonright V_\alpha$ also).

The forcing \mathbb{P} we use is the natural adaptation of Schindler's forcing (to appear in [13]) to the current setting (and Schindler's forcing is related to the forcings of Vopenka, Bukowsky and Woodin's extender algebra; cf. 3.23).

Work in M_ω . Let \mathcal{L} be the class of all infinitary Boolean formulas φ in propositional symbols P_x , for each $x \in V_\beta$. That is, \mathcal{L} is the minimal class generated by the following rules:

1. For each $x \in V_\beta$, we have a corresponding propositional symbol $P_x \in \mathcal{L}$ (with $x \neq y \implies P_x \neq P_y$).

¹¹Recall that κ is a reflecting cardinal iff κ is inaccessible and for every stationary $A \subseteq \kappa$ there are stationarily many regular $\bar{\kappa} < \kappa$ such that $A \cap \bar{\kappa}$ is stationary in $\bar{\kappa}$.

2. If $\varphi \in \mathcal{L}$ then $\neg\varphi \in \mathcal{L}$.
3. If A is a set and $A \subseteq \mathcal{L}$ then $\bigvee A \in \mathcal{L}$ and $\bigwedge A \in \mathcal{L}$.

Working for a moment in a possibly larger universe, given $\varphi \in \mathcal{L}$ and a set $X \subseteq V_\beta^{M_\omega}$, we define satisfaction $X \models \varphi$ recursively in the obvious manner: $X \models P_x$ iff $x \in X$, $X \models \neg\varphi$ iff $\neg(X \models \varphi)$, $X \models \bigvee A$ iff $X \models \varphi$ for some $\varphi \in A$, and $X \models \bigwedge A$ iff $X \models \varphi$ for all $\varphi \in A$.

Recall $x^* = i_{\omega, \omega+\omega}(x) = \tilde{i}_{0\omega}^{M_\omega}(x)$ in M_0 . Work again in M_ω . For $\varphi \in \mathcal{L}$, let

$$\mathcal{E}_\varphi = \{X \subseteq V_{\beta^*} \mid X \models \varphi^*\}.$$

Let \mathcal{L}' be the class of formulas $\varphi \in \mathcal{L}$ such that $\mathcal{E}_\varphi \neq \emptyset$. Given $\varphi, \psi \in \mathcal{L}'$, set $\varphi \equiv \psi$ iff $\mathcal{E}_\varphi = \mathcal{E}_\psi$. So there are only set-many (in fact at most V_{β^*+2} -many) equivalence classes. Let \mathbb{P} be the partial order whose conditions are the equivalence classes $[\varphi]$, and with $[\varphi] \leq [\psi] \iff \mathcal{E}_\varphi \subseteq \mathcal{E}_\psi$. This is a partial order (forcing equivalent to a partial order $\subseteq V_{\beta^*+2}$).

Now work in V and let $X \subseteq V_\beta$. Let

$$G_X = \{[\varphi] \in \mathbb{P} \mid \varphi \in \mathcal{L}' \text{ and } X \models \varphi\}.$$

Here if $\varphi, \psi \in \mathcal{L}'$ and $\varphi \equiv \psi$ then $X \models \varphi \iff X \models \psi$. For suppose that $X \models \varphi \wedge \neg\psi$. Let $n < \omega$ be large enough that $i_{nm}(\varphi \wedge \neg\psi) = \varphi \wedge \neg\psi$ and $i_{nm}(\beta) = \beta$ for all $m \in [n, \omega)$. Then note that

$$V \models \text{“There is } X' \subseteq V_\beta \text{ such that } X' \models \varphi \wedge \neg\psi\text{”}$$

(as witnessed by X). Therefore, applying $i_{n\omega}$, we have

$$M_\omega \models \text{“There is } X' \subseteq V_{\beta^*} \text{ such that } X' \models (\varphi \wedge \neg\psi)^*\text{”}.$$

But then for any such X' , we have $X' \models \varphi^*$ and $\neg(X' \models \psi^*)$ (as $(\varphi \wedge \neg\psi)^* = \varphi^* \wedge \neg(\psi^*)$), so $\mathcal{E}_\varphi \neq \mathcal{E}_\psi$, a contradiction.

Now we claim that G_X is (M_ω, \mathbb{P}) -generic; and then it is easy to see that $M_\omega[G_X] = M_\omega[X]$. For G_X is easily a filter, so we just need to verify genericity. So let $D \in \mathcal{P}(\mathbb{P}) \cap M_\omega$ be dense. We must see that $G_X \cap D \neq \emptyset$. Note that since D is a set, for each $\xi \in \text{OR}$, M_ω has the set

$$E = \{\varphi \in \mathcal{L}' \cap V_\xi \mid [\varphi] \in D\},$$

and taking ξ large enough, $D = \{[\varphi] \mid \varphi \in E\}$. Now $\bigvee E \in \mathcal{L}$. We must see that $X \models \bigvee E$, because then $X \models \varphi$ for some $\varphi \in E$, and therefore $G_X \cap D \neq \emptyset$.

So suppose $X \models \neg \bigvee E$. Let $n < \omega$ with $i_{n, n+1}(E, \beta) = (E, \beta)$. Then

$$V \models \text{“There is } X' \subseteq V_\beta \text{ such that } X' \models \neg \bigvee E\text{”},$$

$$M_\omega \models \text{“There is } X' \subseteq V_{\beta^*} \text{ such that } X' \models (\neg \bigvee E)^*\text{”}.$$

So $\neg \bigvee E \in \mathcal{L}'$. By the density of D , there is $\varphi \in E$ such that $[\varphi]$ is compatible with $[\neg \bigvee E]$. So let $\psi \in \mathcal{L}'$ with $[\psi] \leq [\varphi]$ and $[\psi] \leq [\neg \bigvee E]$. Then $\emptyset \neq \mathcal{E}_\psi \subseteq \mathcal{E}_\varphi \cap \mathcal{E}_{\neg \bigvee E}$. So let $Y \in M_\omega$, with $Y \subseteq V_{\beta^*}$, witness that $\mathcal{E}_\varphi \cap \mathcal{E}_{\neg \bigvee E} \neq \emptyset$. Then $Y \models \varphi^*$ and $Y \models (\neg \bigvee E)^*$, but note $(\neg \bigvee E)^* = \neg \bigvee (E^*) \equiv \bigwedge \{\neg \varrho \mid \varrho \in E^*\}$ and $\varphi \in E$, so $\varphi^* \in E^*$, so $Y \models \varphi^*$ and $Y \models \neg \varphi^*$, contradiction. \square

We can immediately deduce:

Corollary 3.10. *Suppose $(V, j) \models \text{ZFR}$. Let $\alpha < \omega^2$. Then every set in V is set-generic over M_α .*

Proof. We have this for $\alpha < \omega + \omega$, by Theorem 3.9. Consider then $\alpha = \omega + \omega$. By Theorem 3.9, applied in M_ω , we get that every set in M_ω is set-generic over $M_{\omega+\omega}$. Therefore given any $\alpha \in \text{OR}$, we can find a set-generic extension $M_{\omega+\omega}[G]$ of $M_{\omega+\omega}$ such that $V_\alpha^{M_{\omega+\omega}[G]} = V_\alpha^{M_\omega}$. But then for any set $X \in V$, by taking α high enough, we get that X is set-generic over $M_{\omega+\omega}[G]$. So we have added X by a 2-step iteration over $M_{\omega+\omega}$, which suffices. Clearly this generalizes to all $\alpha < \omega + \omega$. \square

We next generalize these initial facts to arbitrary iterates.

Definition 3.11. For a limit α , say x is α -eventually stable (α -es) iff $x \in \bigcap_{\xi < \alpha} M_\xi$ and there is $\xi < \alpha$ such that $i_{\xi\gamma}(x) = x$ for all $\gamma \in [\xi, \alpha)$. Say that x is *hereditarily α -eventually stable* (α -hes) iff every $y \in \text{tranc}(\{x\})$ is α -es. For α -es x , let $x_\alpha^* = \lim_{\xi < \alpha} i_{\xi\alpha}(x)$. Note here that $i_{\xi\alpha}(x) = i_{\gamma\alpha}(x)$ whenever $\xi \leq \gamma < \alpha$ and ξ witnesses the α -eventual stability of x . \dashv

Definition 3.12. Given an ordinal $\alpha > 0$, the *eventual ordertype* $\text{eot}(\alpha)$ of α is the least χ such that $\alpha = \xi + \chi$ for some $\xi < \alpha$. \dashv

Note that (i) α is a successor iff $\text{eot}(\alpha) = 1$ iff $\text{eot}(\alpha)$ is a successor; (ii) $\text{eot}(\alpha) = \text{eot}(\text{eot}(\alpha))$; (iii) $\alpha = \xi + \text{eot}(\alpha)$ for all sufficiently large $\xi < \alpha$. Clearly $\eta = \text{eot}(\alpha) \leq \alpha \leq \kappa_\alpha$, and if $\eta < \kappa_\alpha$ then $\eta_\alpha^* = \eta$, whereas if $\eta = \alpha = \kappa_\alpha$ then $\alpha < \alpha_\alpha^* = \kappa_{\alpha_\alpha^*}$.

Lemma 3.13. *Suppose $(V, j) \models \text{ZFR}$. Let α be a limit ordinal. Then:*

1. $N_\alpha = \{x \mid x \text{ is } \alpha\text{-hes}\}$
2. $x_\alpha^* = i_{\alpha, \alpha + \eta_\alpha^*}(x) = \tilde{i}_{0\eta_\alpha^*}^{M_\alpha}(x)$ where $\eta = \text{eot}(\alpha)$.

Proof. First note that all ordinals are α -es, since M_α is wellfounded.

Now let $x \in M_\alpha$; we show that x is α -hes. We have $M_\alpha \subseteq M_\xi$ for each $\xi < \alpha$. Assume by induction that every $y \in \text{tranc}(x)$ is α -es. So we need to see x is α -es. Let $\xi < \alpha$ be such that $x \in \text{rg}(i_{\xi\alpha})$ and $\xi + \eta = \alpha$ where $\eta = \text{eot}(\alpha)$, and $i_{\xi\gamma}(\eta) = \eta$ for all $\gamma \in [\xi, \alpha)$. We claim that $i_{\xi\gamma}(x) = x$ for all $\gamma \in [\xi, \alpha)$. For $x = i_{\xi\alpha}(\bar{x}) = \tilde{i}_{0\eta}^{M_\xi}(\bar{x})$ for some \bar{x} , so $i_{\xi\gamma}(x) = i_{\xi\gamma}(\tilde{i}_{0\eta}^{M_\xi}(\bar{x})) = \tilde{i}_{0\eta}^{M_\gamma}(i_{\xi\gamma}(\bar{x}))$, but $\gamma + \eta = \alpha$, so $\tilde{i}_{0\eta}^{M_\gamma}(i_{\xi\gamma}(\bar{x})) = i_{\gamma\alpha}(i_{\xi\gamma}(\bar{x})) = i_{\xi\alpha}(\bar{x}) = x$, as desired.

Now let $x \in M_\alpha$ and $\eta = \text{eot}(\alpha)$. We must see that $x_\alpha^* = i_{\alpha, \alpha + \eta_\alpha^*}(x)$. Let $\xi < \alpha$ be such that (i) $\xi + \eta = \alpha$, (ii) $x \in \text{rg}(i_{\xi\alpha})$, and (iii) $i_{\xi\gamma}(x, \eta) = (x, \eta)$ for all $\gamma \in [\xi, \alpha)$. Then $(x_\alpha^*, \eta_\alpha^*) = i_{\xi\alpha}(x, \eta)$; let \bar{x} be such that $i_{\xi\alpha}(\bar{x}) = x$. Since $M_\xi \models "x = \tilde{i}_{0\eta}(\bar{x})"$, applying $i_{\xi\alpha}$, we get

$$M_\alpha \models "x_\alpha^* = \tilde{i}_{0\eta_\alpha^*}(x)" ,$$

so $i_{\alpha, \alpha + \eta_\alpha^*}(x) = x_\alpha^*$, as desired. \square

Theorem 3.14. *Suppose $(V, j) \models \text{ZFR}$ and let $\tau \in \text{OR}$. Then every set in V is set-generic over M_τ .*

Proof. By induction on τ . Successor steps are trivial, so let τ be a limit and $\chi = \text{eot}(\tau)$. We write $x^* = x_\tau^* = i_{\tau, \chi_\tau^+}(x)$ throughout (as computed in (V, j)).

Let $\alpha \in \text{OR}$ and $\beta = \sup i_{0\tau} \alpha$. Let $X = i_{0\tau} V_\alpha$. It suffices to see that X is set-generic over M_τ . We will just use the fact that $X \subseteq V_\beta^{M_\tau}$.

Now X is set-generic over M_ξ for each $\xi < \tau$. Fix $\delta \in \text{OR}$ such that for each $\xi < \tau$, there is an $(M_\xi, \text{Col}(\omega, V_\delta^{M_\xi}))$ -generic G such that $X \in M_\xi[G]$.

Work in M_τ . Let \mathcal{L} be as before (with P_x for each $x \in V_\beta$). Let $\mathbb{Q} = \text{Col}(\omega, V_\delta)$. For $\varphi \in \mathcal{L}$, let \mathcal{E}_φ be the \mathbb{Q} -name for the set of all $X \in \mathcal{P}(V_\beta^{M_\tau}) \cap V^\mathbb{Q}$ such that $X \models \varphi^*$. Let \mathcal{L}' be the class of all $\varphi \in \mathcal{L}$ such that $\mathbb{Q} \Vdash \mathcal{E}_\varphi \neq \emptyset$ (equivalently, there is $p \in \mathbb{Q}$ such that $p \Vdash \mathcal{E}_\varphi \neq \emptyset$). Given $\varphi, \psi \in \mathcal{L}'$, let $\varphi \equiv \psi$ iff $\mathbb{Q} \Vdash \mathcal{E}_\varphi = \mathcal{E}_\psi$. Let $[\varphi]$ be the equivalence class of $\varphi \in \mathcal{L}'$. Let \mathbb{P} be the partial order whose conditions are the equivalence classes $[\varphi]$, with $[\varphi] \leq [\psi]$ iff $\mathbb{Q} \Vdash \mathcal{E}_\varphi \subseteq \mathcal{E}_\psi$.

Now work in V . Let $G_X = \{\varphi \in \mathcal{L}' \mid X \models \varphi\}$. We claim that G is (M_τ, \mathbb{P}) -generic; it follows that $M_\tau[G_X] = M_\tau[X]$. This is proven like before. Let $D \in \mathcal{P}(\mathbb{P}) \cap M_\tau$ be dense. Let $E \in M_\tau$ be such that $D = \{[\varphi] \mid \varphi \in E\}$. We need to see that $X \models \bigvee E$, so suppose otherwise. Then for all sufficiently large $\xi < \tau$,

$$M_\xi \models \text{let } Y = V_\beta^{M_\tau}; \text{ then } \text{Col}(\omega, V_\delta) \Vdash \exists X' \subseteq \check{Y} \text{ such that } X' \models \neg \bigvee E,$$

(as witnessed by X , by induction and choice of δ), which gives that

$$M_\tau \models \text{“let } Y = V_\beta^{M_\tau^*}; \text{ then } \mathbb{Q} \Vdash \exists X' \subseteq \check{Y} \text{ such that } X' \models (\neg \bigvee E)^* \text{”}.$$

So $(\neg \bigvee E) \in \mathcal{L}'$, so $(\neg \bigvee E) \in \mathbb{P}$.

Work in M_τ . By the density of D and since $(\neg \bigvee E) \in \mathbb{P}$, there is $\varphi \in E$ such that $[\varphi]$ is compatible with $(\neg \bigvee E)$. Let $[\psi] \in \mathbb{P}$ with $[\psi] \leq [\varphi]$ and $[\psi] \leq (\neg \bigvee E)$. Let G be $\text{Col}(\omega, V_\delta)$ -generic. Then in $V[G]$, there is X' such that $X' \models \psi^*$, and by the definition of \leq , therefore $X' \models \varphi^*$ and $X' \models (\neg \bigvee E)^*$, so $X' \models \bigwedge_{\varphi \in E} \neg \varphi$, so $X' \models \neg \varphi^*$, a contradiction. \square

We deduce the existence of some interesting forcing extensions under ZFR. Recall the notion of *virtual large cardinals* from [3].

Definition 3.15. δ is *virtually Berkeley* iff for every transitive M with $\delta \in M$ and every $\eta < \delta$, there is a forcing \mathbb{P} which forces the existence of an elementary $j : M \rightarrow M$ with $\eta < \text{cr}(j) < \delta$. Say that δ is *V_α -preserving-virtually Berkeley* iff δ is virtually Berkeley, as witnessed by forcings \mathbb{P} which preserve V_α . \dashv

Corollary 3.16. Suppose $(V, j) \models \text{ZFR}$. Let $\kappa = \text{cr}(j)$ and $\lambda = \kappa_\omega(j)$. Then there are partial orders $\mathbb{P}_0, \mathbb{P}_1 \in V$ such that:

1. κ is V_κ -preserving-virtually Berkeley.
2. $\mathbb{P}_0 \Vdash V_\kappa = \check{V}_\kappa$ and there is an elementary $k : V_\kappa \rightarrow V_\kappa$ such that
 - (a) $\kappa_\omega(k) = \check{\kappa}$ (hence $\text{cof}(\check{\kappa}) = \omega$) and
 - (b) $(\check{V}_\lambda, \check{j} \restriction \check{V}_\lambda)$ is the ω th iterate of (\check{V}_κ, k) .

3. for each $\beta < \kappa$ there is $\bar{\lambda} \in (\beta, \kappa)$ such that \mathbb{P}_1 forces that $V_{\bar{\lambda}} = \check{V}_{\bar{\lambda}}$ and $(\bar{\lambda}^+) = \check{\kappa}$ and there is a rank-into-rank embedding $k : V_{\bar{\lambda}} \rightarrow V_{\check{\lambda}}$ such that $(\check{V}_{\bar{\lambda}}, \check{j} \upharpoonright \check{V}_{\bar{\lambda}})$ is the $\check{\kappa}$ th iterate of $(V_{\bar{\lambda}}, k)$.

Proof. The corresponding things are forceable over M_ω (for parts 1 and 2) and over $M_{(\lambda^+)}$ (for part 3) with respect to λ , respectively. So the elementarity of $i_{0\omega}$ and $i_{0(\lambda^+)}$ pulls them back to $M_0 = (V, j)$. \square

3.3 Intersections of iterates

We consider now intersections of the form $\bigcap_{\xi < \tau} N_\xi$.

Definition 3.17. Suppose $(V, j) \models \text{ZFR}$ and let τ be a limit ordinal and $\beta < \tau$. Let $x \in \bigcap_{\xi < \tau} N_\xi$. Then x is (β, τ) -stable iff $i_{\beta\gamma}(x) = x$ for all $\gamma \in [\beta, \tau)$. \dashv

Theorem 3.18. Suppose $(V, j) \models \text{ZFR}$ and let τ be a limit ordinal and $\chi = \text{eot}(\tau)$. Let $J = \bigcap_{\xi < \tau} N_\xi$. Let $\xi < \tau$ be such that χ is (ξ, τ) -stable and either

- (i) $\chi < \text{cr}(j_\xi)$, or
- (ii) $\kappa_\xi < \chi = \tau = \kappa_\tau = \sup_{\gamma < \tau} \kappa_\gamma$.

Let $\mu = \text{cof}^{M_\xi}(\tau) = \text{cof}^{M_\xi}(\chi)$, so $i_{\xi\gamma}(\mu) = \text{cof}^{M_\gamma}(\chi)$ for all $\gamma \in [\xi, \tau)$. Then:

1. J is transitive, proper class with $N_\tau \subseteq J$ and $(J, M_\tau) \models \text{ZF}$.
2. if $\mu > \omega$ then $N_\tau = J$.
3. if $\mu = \omega$ then $N_\tau \subsetneq J \models \text{“cof}(\gamma) = \omega \text{ for all } N_\tau\text{-regular } \gamma \in [\kappa_\tau, \kappa_{\tau+\omega})\text{”}$.

Proof. Part 1: J is clearly transitive with $N_\tau \subseteq J$. And (J, M_τ) is a class of $M_\xi = (N_\xi, j_\xi)$ for each $\xi < \tau$. So J satisfies Powerset: if $X \in J$ and $\xi < \tau$ then because J is a class of M_ξ , we have $\mathcal{P}(X) \cap J \in M_\xi$; therefore $\mathcal{P}(X) \cap J \in J$. Likewise, for each $\alpha \in \text{OR}$, we have $V_\alpha \cap J \in J$ and $\langle V_\beta \cap J \rangle_{\beta < \alpha} \in J$. Separation (with respect to the predicate M_τ) is similar to Powerset, and Collection follows from the preceding remarks.

Part 2: Suppose $\mu > \omega$. Let $x \in J$. By Lemma 3.13 and transitivity, it suffices to see that x is τ -es. Suppose not. Then for each $\gamma < \tau$ there is $\gamma' \in (\gamma, \tau)$ such that $i_{\gamma\gamma'}(x) \neq x$.

Work in M_ξ , where $\text{cof}(\chi) = \mu > \omega$. We have $\xi + \chi = \tau$. Define a sequence $\langle \xi_n \rangle_{n < \omega}$ as follows: Set $\xi_0 = 0$. Given ξ_n , let ξ_{n+1} be the least $\xi' \in (\xi_n, \eta)$ such that $i_{\xi_n \xi'}^{M_\xi}(x) \neq x$. Then since $\text{cof}(\chi) > \omega$, we have $(\sup_{n < \omega} \xi_n) = \xi_\omega < \chi$. So $x \in \tilde{N}_{\xi_\omega}$, so by Lemma 3.13, x is ξ_ω -es, a contradiction.

Part 3: Suppose $\mu = \omega$. Work in M_ξ . Let $\langle \xi_n \rangle_{n < \omega}$ with $\sup_{n < \omega} \xi_n = \chi$ and $\xi_0 = 0$. Define $\langle \xi'_n \rangle_{1 \leq n < \omega}$ by $i_{\xi_n \xi'_n}^{M_\xi}(\xi_n)$ and $\xi'_0 = 0$. Note $\langle \xi'_n \rangle_{n > m} \in M_{\xi'_m}$ (by commutativity of the maps), so in fact $\langle \xi'_n \rangle_{n < \omega} \in J$. Note $\xi_n \leq \xi'_n < \xi'_{n+1} < \chi$ (since χ is (ξ, τ) -stable) and $\sup_{n < \omega} \xi'_n = \sup_{n < \omega} \xi_n = \chi$.

Renaming, we may assume $\langle \xi_n \rangle_{n < \omega} \in J$. Now let $\gamma \in [\kappa_\tau, \kappa_{\tau+\omega})$ be regular in N_τ . We need $J \models \text{“cof}(\gamma) = \omega\text{”}$. We may assume $\gamma \in \text{rg}(i_{\xi\tau})$; let $i_{\xi\tau}(\bar{\gamma}) = \gamma$. So $M_\xi \models \text{“}\bar{\gamma} \text{ is regular”}$. Let $\bar{\gamma}_n = i_{\xi, \xi+\xi_n}(\bar{\gamma})$, so $i_{\xi+\xi_n, \tau}(\bar{\gamma}_n) = \gamma$. Let

$$\bar{\zeta}_{n+1} = \sup i_{\xi+\xi_n, \xi+\xi_{n+1}} \text{“}\bar{\gamma}_n\text{”}.$$

Now $\bar{\zeta}_{n+1} < \bar{\gamma}_{n+1}$. For $j_{\xi+\xi_n}(\gamma') > \gamma'$, for all $\gamma' \in [\kappa_{\xi+\xi_n}, \kappa_{\xi+\xi_n+\omega})$. But $j_{\xi+\xi_n} : M_{\xi+\xi_n} \rightarrow M_{\xi+\xi_n}$, so by the regularity of $j_{\xi+\xi_n}(\bar{\gamma}_n)$ in $M_{\xi+\xi_n}$, and since $j_{\xi+\xi_n} \upharpoonright \bar{\gamma}_n \in M_{\xi+\xi_n}$, we get

$$(\sup j_{\xi+\xi_n} \text{ ``}\bar{\gamma}_n\text{''}) < j_{\xi+\xi_n}(\bar{\gamma}_n),$$

and by commutativity, therefore $\bar{\zeta}_{n+1} < \bar{\gamma}_{n+1}$. Let $\zeta_{n+1} = i_{\xi+\xi_{n+1}, \tau}(\bar{\zeta}_{n+1})$. Then $\sup_{n < \omega} \zeta_{n+1} = \gamma$, and note $\langle \zeta_{n+1} \rangle_{n < \omega} \in J$, using $\langle \xi_n \rangle_{n < \omega} \in J$. \square

3.4 The (infinite) iterates N_τ are not set grounds of V

In contrast to the fact that every set is generic over N_τ , we have:

Theorem 3.19. *Suppose $(V, j) \models \text{ZFR}$. Then:*

1. *For all $\tau \in \text{OR}$ with $\tau \geq \omega$, N_τ is not a set ground of V .*
2. *For $\xi < \tau \in \text{OR}$, we have:*

- (a) $i_{\xi\tau} \upharpoonright V_\alpha^{N_\xi} \in N_\tau$ for all $\alpha < \kappa_{\xi+\omega}$.
- (b) *If $\xi + \omega \leq \tau$ then $i_{\xi\tau} \upharpoonright \kappa_{\xi+\omega} \notin N_\tau$.*

Proof. Part 2a: If $\tau < \xi + \omega$ this is immediate because $N_\tau = N_\xi$. So suppose $\xi + \omega \leq \tau$. Let $n < \omega$ be such that $\alpha < \kappa_{\xi+n}$. Then $k = i_{\xi, \xi+n} \upharpoonright V_\alpha^{N_\xi} \in N_{\xi+n} = N_\xi$, and note $i_{\xi\tau} \upharpoonright V_\alpha^{N_\xi} = i_{\xi+n, \tau} \circ k = i_{\xi+n, \tau}(k) \in N_\tau$.

Part 2b: Suppose first $\tau = \xi + \omega$. Then $i_{\xi, \xi+\omega}$ “ $\kappa_{\xi+\omega}$ is cofinal in $\kappa_{\xi+\omega+\omega}$, and $\text{cof}^{N_{\xi+\omega}}(\kappa_{\xi+\omega}) = \kappa_{\xi+\omega} = \text{cr}(j_{\xi+\omega})$. But $\text{cof}^{N_{\xi+\omega}}(\kappa_{\xi+\omega+\omega}) = \omega$, because $\kappa_{\xi+\omega+\omega} = \tilde{\kappa}_\omega^{M_{\xi+\omega}}$. Therefore $i_{\xi, \xi+\omega} \upharpoonright \kappa_{\xi+\omega} \notin N_{\xi+\omega}$.

Now suppose $\tau > \xi + \omega$. Since $i_{\xi+\omega, \tau}$ is computed by $M_{\xi+\omega}$, and by commutativity, if $i_{\xi\tau} \upharpoonright \kappa_{\xi+\omega} \in N_\tau \subseteq N_{\xi+\omega}$, then $i_{\xi, \xi+\omega} \upharpoonright \kappa_{\xi+\omega} \in N_{\xi+\omega}$, a contradiction.

Part 1: By part 2, $i_{0\tau} \upharpoonright \kappa_\omega$ is a set of ordinals in $V \setminus N_\tau$, and $i_{0\tau} : V \rightarrow N_\tau$ is elementary. So by Theorem 2.3, N_τ is not a set-ground of V . \square

3.5 Non-definability of $j \upharpoonright \text{OR}$

Remark 3.20. Recall Suzuki [17] showed that no elementary $j : V \rightarrow V$ is definable from parameters. Variants of this are proved in [7], [6], [14]. In particular, [7, Theorem 35] establishes the stronger fact that no elementary $k : \text{HOD} \rightarrow \text{HOD}$ is definable from parameters (hence if $j : V \rightarrow V$ is elementary then $j \upharpoonright \text{OR}$ is not definable from parameters). We sketch now an alternate proof of this fact, closer to the methods of this paper. Suppose $k : \text{HOD} \rightarrow \text{HOD}$ is definable from parameter x . Using Vopenka (or maybe a slight adaptation thereof to ZF, if it is not standard; see Fact 3.24), there is a HOD-set-generic filter G_x such that $\text{HOD}[G_x] = \text{HOD}_x$.¹² So $k \upharpoonright \text{HOD}$ is amenable to $\text{HOD}[G_x]$. But this is impossible by [7, Corollary 9]; or alternatively, using Lemma 2.2, we get that $k \upharpoonright \text{HOD}$ is amenable to HOD , contradicting Kunen.

In this section we strengthen the fact that $j \upharpoonright \text{OR}$ is not definable from parameters, if $j : V \rightarrow V$ is elementary, showing $j \upharpoonright \text{OR}$ cannot be definable from parameters over the structure (V, M_ω) , that is, allowing M_ω as a predicate.

¹²We cannot expect that $x \in \text{HOD}[G_x]$ in general, since $\text{HOD}[G_x] \models \text{ZFC}$, and x is arbitrary. However, the usual arguments still give $\text{HOD}[G_x] = \text{HOD}_x$.

Also, it is shown in [7, Theorem 12] under ZFC_2 that if $j : V \rightarrow M$ is an elementary with M wellfounded, then $V = \text{Hull}^V(\text{rg}(j))$. But the proof relies strongly on AC. It trivially fails with respect to j when $(V, j) \models \text{ZFR}$ (we have $\text{rg}(j) = \text{Hull}^V(\text{rg}(j))$ by elementarity). It also fails with respect to $j = i_{0\omega}$, as shown in the following theorem (but since $N_\omega \neq V$, this time it is not trivial). Note that $\text{Hull}^M(X)$ was defined in §1.1.

Theorem 3.21. *Assume $(V, j) \models \text{ZFR}$. Then:*

1. *For no set X is $V = \text{Hull}^{(V, M_\omega)}(N_\omega \cup X)$.*
2. *$j \restriction \text{OR}$ and $i_{0\omega} \restriction \text{OR}$ are not definable from parameters over (V, M_ω) .*

Proof. Part 1: Suppose $X \in V$ and $V = \text{Hull}^{(V, M_\omega)}(N_\omega \cup X)$. Let $k_n = i_{n\omega} \restriction X^{<\omega}$, for $n < \omega$. So $\text{rg}(k_n) \subseteq N_\omega$. Let $\vec{k} = \langle k_n \rangle_{n < \omega}$. Then $(X, \vec{k}) \in V$, so by Theorem 3.9, we have $(X, \vec{k}) \in N_\omega[G]$ for some set-generic extension $N_\omega[G]$ of N_ω . We will show that $N_\omega[G] = V$, contradicting Theorem 3.19.

Let $Y \in V$. By our contradictory assumption, there is $\alpha \in \text{OR}$ and $n < \omega$ such that Y is encoded into $t = \text{Th}_{\Sigma_n}^{(V, M_\omega)}((N_\omega \cap V_\alpha) \cup X)$. So it suffices to see $t \in N_\omega[G]$. We have $(N_\omega \cap V_\alpha) \cup X \in N_\omega[G]$, so we just need to verify the appropriate instance of Separation.

So let φ be Σ_n and $\vec{y} \in (N_\omega \cap V_\alpha)^{<\omega}$ and $\vec{x} \in X^{<\omega}$. For each $m < \omega$, note

$$[(V, M_\omega) = (N_m, M_\omega) \models \varphi(\vec{y}, \vec{x})] \iff [(N_\omega, M_{\omega+\omega}) \models \varphi(i_{m\omega}(\vec{y}), i_{m\omega}(\vec{x}))].$$

But $i_{m\omega}(\vec{x}) = k_m(\vec{x})$ for all $m < \omega$, and $i_{m\omega}(\vec{y}) = \vec{y}^* = i_{\omega, \omega+\omega}(\vec{y})$ for all sufficiently large $m < \omega$. Therefore

$$[(V, M_\omega) \models \varphi(\vec{y}, \vec{x})] \iff \exists \ell < \omega \forall m \geq \ell [(N_\omega, M_{\omega+\omega}) \models \varphi(\vec{y}^*, k_m(\vec{x}))].$$

Since $\langle k_m \rangle_{m < \omega} \in N_\omega[G]$ and $i_{\omega, \omega+\omega} \restriction (N_\omega \cap V_\alpha) \in N_\omega$, we get $t \in N_\omega[G]$.¹³

Part 2: Arguing as above, note that for each $x \in V$, letting $x_n = i_{n\omega}(x)$, and letting $N_\omega[G]$ be a set-generic extension such that $\langle x_n \rangle_{n < \omega} \in N_\omega[G]$, we get that every set $A \subseteq N_\omega$ which is definable over (V, M_ω) from parameters in $N_\omega \cup \{x\}$, is in $N_\omega[G]$.

Now suppose $i_{0\omega} \restriction \text{OR}$ is definable over (V, M_ω) from elements of $N_\omega \cup \{x\}$. Then by the previous paragraph, $i_{0\omega} \restriction \text{OR}$ is amenable to $M_\omega[G]$. But then much as in the proof of Lemma 2.2, we can show that $i_{0\omega} \restriction \text{OR}$ is amenable to M_ω , contradicting Theorem 3.19. That is, fix $\alpha \in \text{OR}$. We can find an $X \in M_\omega$ and $\beta \in \text{OR}$ with $X \subseteq \beta$ and X of ordertype α , such that $i_{0\omega} \restriction \alpha \in M_\omega$. Let $\pi : \alpha \rightarrow X$ be the increasing enumeration. Then $i_{0\omega}(\pi), i_{0\omega}(X) \in N_{\omega+\omega} \subseteq N_\omega$, and $i_{0\omega}(\pi) : i_{0\omega}(\alpha) \rightarrow i_{0\omega}(X)$ is the increasing enumeration of $i_{0\omega}(X)$. Since $i_{0\omega} \circ \pi = i_{0\omega}(\pi) \circ i_{0\omega} \restriction \alpha$, and $i_{0\omega} \restriction X = \text{rg}(\pi) \in N_\omega$, we therefore get $i_{0\omega} \restriction \alpha \in N_\omega$, as desired.

Finally suppose $j \restriction \text{OR}$ is definable over (V, M_ω) from elements of $N_\omega \cup \{x\}$.¹⁴ Note $j \restriction \kappa_\omega \notin N_\omega$, since otherwise $\text{cof}^{N_\omega}(\kappa_\omega) = \omega$. But $j \restriction N_\omega : M_\omega \rightarrow M_\omega$ is

¹³Of course, in $N_\omega[G]$ we can compute the relevant truth of $(N_\omega, M_{\omega+\omega})$, just noting that whenever $(V_\xi, V_\xi \cap M_\omega) \preceq_n (V, M_\omega)$, then $(V_{i_{0\omega}(\xi)}^{N_\omega}, V_{i_{0\omega}(\xi)}^{N_\omega} \cap M_{\omega+\omega}) \preceq_n (N_\omega, M_{\omega+\omega})$. (In the notes [16], at this point it was claimed that we could alternatively use the definability of set-grounds here to allow $N_\omega[G]$ to compute N_ω , but this was an oversight, since AC fails, and it is not clear to the author whether one can get around that failure here.)

¹⁴Note this doesn't immediately contradict the previous paragraph, because from $j \restriction \text{OR}$ alone it doesn't seem that one can directly compute $i_{0\omega} \restriction \text{OR}$. We can compute $i_{0n} \restriction \text{OR}$ for $n < \omega$, since this is the same as $j \circ \dots \circ j \restriction \text{OR}$. But for $i_{0\omega} \restriction \text{OR}$ it seems we need the direct limit of all $j_n \restriction \text{OR}$.

elementary. So we can just use essentially the same argument as in the previous paragraph for a contradiction (but slightly easier). \square

Remark 3.22. An immediate corollary is that there is no $X \in V$ such that $V = \text{HOD}(X)$. And if $(V_\delta, V_{\delta+1}) \models \text{ZF}_2 + \text{“}\delta \text{ is total Reinhardt”}$, then there is no $A \subseteq V_\delta$ and $X \in V_\delta$ such that $V_\delta = (\text{HOD}_A(X))^{(V_\delta, A)}$. For just let $j : (V_\delta, (A, X)) \rightarrow (V_\delta, (A, X))$ be elementary, and run the obvious variant of the preceding proof.

But suppose $(V, j) \models \text{ZFR}$. In the notes [16, v1 on arxiv.org], the author asked whether possibly $V = \text{Hull}^V(\text{OR} \cup \text{rg}(j))$, or $V = \text{Hull}^V(M_\omega \cup \text{rg}(j))$, and whether AC can be set-forceable over V . These questions have since been answered: Goldberg [5] and Usuba [19] have independently shown that AC is not set-forceable (not by considering the iterates M_α , but by a more direct combinatorial method). In particular, there is no class A and set X such that $V = \text{HOD}_A(X)$. Goldberg’s proof also gives that if there is a Berkeley cardinal then AC is not set-forceable, and he also showed that (i) $V = \text{Hull}^V(M_\omega \cup \text{rg}(j))$, but (ii) $V \neq \text{Hull}^V(\text{OR} \cup \text{rg}(j))$.

By an easier argument than that for (i), we have $\text{Hull}^{(V, j)}(N_\omega) = V$, in contrast to the fact that $\text{Hull}^{(V, M_\omega)}(N_\omega) = N_\omega$. For $i_{0\omega}$ is computable from j , and given $x \in V$, we have $x = (i_{0\omega})^{-1}(x')$ where $x' = i_{0\omega}(x) \in N_\omega$.

Remark 3.23. Assume ZF. Let $x \subseteq V_\alpha$. We recall the definition of the *Vopenka algebra* Vop for extending HOD generically to $\text{HOD}_x = \text{HOD}(\{x\})$. Define Vop' in V as follows. Conditions are non-empty OD subsets of $V_{\alpha+1}$; the ordering is $A \leq B \iff A \subseteq B$. Then $\text{Vop} \in \text{HOD}$ is the natural coding of Vop' as a set of ordinals. Let $\pi : \text{Vop} \rightarrow \text{Vop}'$ be the natural isomorphism. Now (with $x \subseteq V_\alpha$), define the filter $G_x = \{p \in \text{Vop} \mid x \in \pi(p)\}$.

Then by the ZFC arguments, we have:

Fact 3.24. G_x is (HOD, Vop) -generic and $\text{HOD}[G_x] = \text{HOD}_x$.¹⁵

Although every set is set-generic over N_τ for every $\tau < \text{OR}$, this fails at $\tau = \text{OR}$, and although $\text{HOD} \subseteq N_\omega$, there is $\tau < \text{OR}$ with $\text{HOD} \not\subseteq N_\tau$:

Theorem 3.25. Suppose $(V, j) \models \text{ZFR}$. Let $J = \bigcap_{\alpha \in \text{OR}} N_\alpha$. Then:

1. J is a transitive proper class and $V_{\kappa_\tau}^{N_\tau} \preceq V_{\kappa_\tau + \omega}^{N_\tau} \preceq J$ for each $\tau \in \text{OR}$.
2. For all $\alpha \leq \beta < \text{OR}$, $i_{\alpha\beta} \restriction J : J \rightarrow J$ is elementary.
3. $j \restriction V_{\kappa_\omega}^J : V_{\kappa_\omega}^J \rightarrow V_{\kappa_\omega}^J$ is not set-generic over J .
4. $\text{HOD} \not\subseteq J$.

Proof. Part 1 is straightforward, noting in particular that $V_{\kappa_\tau + \omega}^{N_\tau} = V_{\kappa_\tau + \omega}^{N_{\tau+\omega}} \subseteq J$.

Part 2: Note that J is defined in the same manner over $M_\alpha = (N_\alpha, j_\alpha)$ as over M_β , so $i_{\alpha\beta}(J) = J$.

¹⁵ The fact that $x \in \text{HOD}[G_x]$, in the case that $x \subseteq \gamma \in \text{OR}$, is of course due to Vopenka. The author is not sure who first proved the entire result. Let φ be a formula, $\eta, \gamma \in \text{OR}$ and $A = \{\xi < \gamma \mid \varphi(x, \xi, \eta)\}$. We need $A \in \text{HOD}[G_x]$. For $\xi < \gamma$ let $t'_\xi = \{y \mid y \in V_{\alpha+1} \text{ and } \varphi(y, \xi, \eta)\}$. So $t'_\xi \in \text{Vop}'$. Let $\pi(t_\xi) = t'_\xi$. Note $\langle t_\xi \rangle_{\xi < \gamma} \in \text{HOD}$, and $t_\xi \in G_x$ iff $x \in t'_\xi$ iff $\xi \in X$, so $X \in \text{HOD}[G_x]$.

Part 3: Let $R = (V_{\kappa_\omega}, j \upharpoonright V_{\kappa_\omega})$. Note R is iterable with iterates $R_\alpha = (V_{\kappa_{\alpha+\omega}}^{M_\alpha}, j_\alpha \upharpoonright V_{\kappa_{\alpha+\omega}}^{M_\alpha}) = (V_{\kappa_{\alpha+\omega}}^J, j_\alpha \upharpoonright V_{\kappa_{\alpha+\omega}}^J)$. But then if $R \in J[G]$, we can form the iteration $\langle R_\alpha \rangle_{\alpha \in \text{OR}}$ there, and considering some sufficiently large limit α , this collapses all cardinals in $(\kappa_\alpha, \kappa_{\alpha+\omega}]$ in $J[G]$, although each $\kappa_{\alpha+n+1}$ is inaccessible in J and hence also in $J[G]$, a contradiction.

Part 4: Let G be $(V, \text{Col}(\omega, V_{\kappa_\omega}))$ -generic. Let $H = \text{HOD}^{V[G]}$. By the homogeneity of the collapse, $H \subseteq \text{HOD}$. Let $x \in V[G]$ be a real coding R . Then by Fact 3.24, $R \in H[G_x] = \text{HOD}_x^{V[G]}$ is a set-generic extension of H . Let $\xi \in \text{OR}$ with $\text{Vop}^{V[G]} \subseteq \xi$. So $(\xi^{+n+1})^{H[G_x]} = (\xi^{+n+1})^H$. Since $R \in \text{HC}^{H[G_x]}$, in $H[G_x]$ we can iterate R , and R_{ξ^+H} has cardinality ξ^{+H} there, and $\xi^{+H} = \kappa_{\xi^+H}$. So $\mathcal{P}(\xi^{+H}) \cap J$ also has cardinality ξ^{+H} there. So $H \not\subseteq J$. \square

3.6 No Berkeley cardinal in $\text{HOD}_A(X)$

We now adapt the proof of the previous section to show that if there is a Berkeley cardinal (see [1]) then there is no class A and set X such that $V = \text{HOD}_A(X)$. However, as mentioned in Remark 3.22, one can prove much more.

Theorem 3.26. *Let $\Omega \in \text{OR}$ be such that $V_\Omega \models \text{ZF} + \text{“There is a Berkeley cardinal”}$. Then there is no $A \subseteq V_\Omega$ such that $(V_\Omega, A) \models \text{ZF}(A) + \text{“}V = \text{HOD}_A(X)\text{”}$ for some set X .*

Proof sketch. Suppose otherwise and work in (V_Ω, A) . Let β be Berkeley. Let $\lambda \in \text{OR}$, $\lambda > \beta$, be such that $(V_\lambda, A \cap V_\lambda) \preceq_{10} (V, A)$ and $\text{cof}(\lambda) > \omega$ (note β is a limit of inaccessibles, so there are certainly ordinals of cofinality $> \omega$). Let

$$j : (V_\lambda, A \cap V_\lambda) \rightarrow (V_\lambda, A \cap V_\lambda)$$

be elementary with $\text{cr}(j) < \beta$. By elementarity, there is $X \in V_\lambda$ such that $(V_\lambda, A \cap V_\lambda) \models \text{“}V = \text{HOD}_A(X)\text{”}$, where here $\text{HOD}_A(X)$ is defined locally from A ; that is, for every $x \in V_\lambda$, there is $\xi < V_\lambda$ such that x is definable over $(V_\xi, A \cap V_\xi)$ from parameters in $X \cup \xi$.

Now we adapt the proof in the Reinhardt cardinal context to $(V_\lambda, A \cap V_\lambda, j)$. We have $j \upharpoonright V_\alpha \in V_\lambda$ for each $\alpha < \lambda$, and $j \upharpoonright V_\lambda$ is \in -cofinal in V_λ . Consider the structure $M_0 = (V_\lambda, A \cap V_\lambda, j_0)$ where $j = j_0$. Define $M_1 = j_0^+(M_0)$ as usual, and note $M_1 = (V_\lambda, A \cap V_\lambda, j_1)$, where $j_1 = j_0^+(j_0)$. There are unboundedly many $\alpha < \lambda$ of cofinality ω which are fixed by j . Fix such an α . Note that $k = j_0 \upharpoonright V_\alpha$ is fully elementary as a map

$$k : (V_\alpha, A \cap V_\alpha, j_0 \upharpoonright V_\alpha) \rightarrow (V_\alpha, A \cap V_\alpha, j_0(j_0 \upharpoonright V_\alpha)).$$

Therefore $j : M_0 \rightarrow M_1$ is cofinal Σ_1 -elementary.

Also since these bounded fragments of j_0 are fully elementary, these facts are lifted by j_0 , and so iterating, $j_1 : M_1 \rightarrow M_2$ is also cofinal Σ_1 -elementary, etc. (In particular, j_1 also coheres A .) This proceeds through all $n < \omega$. Let $i_{mn} : M_m \rightarrow M_n$ be the iteration map.

Given $\alpha < \lambda$, let $\alpha' = \sup_{n < \omega} i_{0n}(\alpha)$. Then $\alpha' < \lambda$, so note that the sequence $\vec{j}_\alpha = \langle i_{0n}(j \upharpoonright V_\alpha) \rangle_{n < \omega} \in V_\lambda$. The map $\alpha \mapsto \vec{j}_\alpha$ is Σ_1 -definable over M_0 . It is now easy to adapt the usual proof to see that M_ω and the maps $i_{n\omega} : M_n \rightarrow M_\omega$ are uniformly Σ_1 -definable over M_n , and M_ω is wellfounded, with $\text{OR}^{M_\omega} = \lambda$.

We can now run the rest of the Reinhardt proof for a contradiction (of course using $\text{HOD}_A(X)$ in place of $\text{HOD}(X)$, which works because each j_n coheres A). \square

4 Reinhardts and small forcing

In this section we consider a different question regarding Reinhardt cardinals:

Question 4.1. Suppose G is (V, \mathbb{P}) -generic for some \mathbb{P} in V , and $(V[G], j) \models \text{ZFR}$. Is there a k such that $(V, k) \models \text{ZFR}$? Does $j \restriction V$ work? What if we assume that $\mathbb{P} \in V_{\text{cr}(j)}$? Note that the iterates N_α don't seem to help here, since V is not a proper set-generic extension of N_α .

We don't know the answer to this question in general, but we establish some related facts here. The forcing arguments in this section relate to Laver [10].

Definition 4.2. For a set X , $X\text{-AC}$ is the statement that for every function f with domain X , there is a choice function for f . \dashv

Remark 4.3. The following is a straightforward adaptation of the standard ground definability arguments of Laver, Woodin and Hamkins. Gitman and Johnstone also proved a related fact in [2, Main Theorem 1]. They show in particular that $(*)$ if $V \models \text{ZF} + \delta\text{-DC}$, where δ is a cardinal, $\text{card}(\mathbb{P}) < \delta$, and G is (V, \mathbb{P}) -generic, then V is definable over $V[G]$ from the parameter $\mathcal{P}(\delta)^V$. Actually, they prove something more general, in that they only assume that \mathbb{P} admits a gap at δ , not actually that $\text{card}(\mathbb{P}) < \delta$. We will prove below, however, a different generalization of $(*)$. The argument is very similar to both of those mentioned above.

Theorem 4.4. Let $M \models \text{ZF}$ and $\mathbb{P} \in M$. Let G be (M, \mathbb{P}) -generic, and C be the set of conditions of \mathbb{P} . Suppose C is wellorderable in $M[G]$ and $M[G] \models \gamma^+\text{-DC}$ where $\gamma = \text{card}^{M[G]}(C)$. Then M is definable over $M[G]$ from the parameter $\mathcal{P}(C^2)^M$.¹⁶

This improves on $(*)$ above, in that we do not assume that $M \models \gamma^+\text{-DC}$. (But of course, [2] only demands that \mathbb{P} admits a gap at δ , not that $\text{card}(\mathbb{P}) < \delta$.)

Proof. Work in $M[G]$. Let $\lambda > \eta$ be a cardinal such that there is no surjection $f : V_\alpha \rightarrow \lambda$ with $\alpha < \lambda$. The following claim completes the proof:

Claim 3. $V_\lambda \cap M$ is the unique transitive $N \subseteq V_\lambda$ such that (i) $\lambda = \text{OR} \cap N$, (ii) $V_\alpha \cap N \in N$ for each $\alpha < \lambda$, (iii) $N \models \text{Separation, Pairing, Union, Powerset}$, (iv) $C \in N$ and $\mathcal{P}(C^4)^N = \mathcal{P}(C^4)^M$, (v) There is a partial order $\mathbb{Q} \in \mathcal{P}(C^2)^N$ and an (N, \mathbb{Q}) -generic H such that $V_\lambda = N[H]$.

Proof. We show by induction on ξ that $V_\xi^N = V_\xi^M$. This is trivial by induction if ξ is a limit. So assume it holds at $\xi \in [\omega, \lambda)$. Given $A \subseteq N$ and $B \in N$, we say that A is B -amenable to N iff $A \cap \text{rg}(\pi) \in N$ for every function $\pi \in N$ such that $\pi : B \rightarrow N$.

Let $A \subseteq V_\xi$. We must show that $A \in M$ iff $A \in N$. We do this by proving:

1. $A \in M$ iff A is C^2 -amenable to M ,
2. $A \in N$ iff A is C^2 -amenable to N ,
3. for all $\pi : C^2 \rightarrow V_\xi^M = V_\xi^N$, we have $\pi \in M$ iff $\pi \in N$.

¹⁶The author does not know whether $\mathcal{P}(C)^M$ suffices instead of $\mathcal{P}(C^2)^M$.

Putting these things together, we get $A \in M$ iff $A \in N$, as desired.

Note first that we have the Forcing Theorem with respect to Σ_0 formulas, and definability of the Forcing Relation for Σ_0 formulas, over V_λ^M and N ; this follows from Separation and that λ is closed enough.

Parts 1,2: It suffices to consider part 2. For the non-trivial direction, suppose that $A \notin N$; we show A is not C^2 -amenable to N . We have $V_\lambda = N[H]$. So let $\dot{A} \in N$ be such that $\dot{A}_H = A$. Since $A \subseteq V_\xi^N$ and $A \notin N$, there is $p_0 \in H$ such that $N \models p_0 \Vdash_{\mathbb{Q}} \text{"}\dot{A} \subseteq \check{V}_\xi \text{ but } \dot{A} \notin \check{V}\text{"}$.

Now in V , where $\mathbb{Q} \subseteq C$ is wellordered and $|C|^+\text{-AC}$ holds, let $f : C \rightarrow V_\xi^N$ be such that for each $p \leq^{\mathbb{Q}} p_0$, $f(p)$ is some x such that

$$N \models p \text{ does not decide (w.r.t. } \mathbb{Q}) \text{ whether } \check{x} \in \dot{A}.$$

Let $\dot{f} \in N$ be such that $\dot{f}_H = f$. Let $p_1 \in H$, $p_1 \leq p_0$, such that $N \models p_1 \Vdash_{\mathbb{Q}} \text{"}\dot{f} : \check{C} \rightarrow \check{V}_\xi \text{, and for each } p \leq^{\mathbb{Q}} p_0, \check{V} \text{ thinks that } p \text{ does not decide whether } \dot{f}(p) \in \dot{A}\text{"}$.

In N , define $f_* : C^2 \rightarrow N$, where for $(p, q) \in C^2$, if $q \leq^{\mathbb{Q}} p \leq^{\mathbb{Q}} p_1$ then $f_*(p, q)$ is the unique x such that $q \Vdash \dot{f}(\check{p}) = \check{x}$, if there is such an x (and say (p, q) is *good*), and $f_*(p, q) = \emptyset$ otherwise.

Let $R = \text{rg}(f_*) \in N$. We claim that $R \cap A \notin N$, which completes the proof of part 2. So let $S \in \mathcal{P}(R)^N$. For each $p \leq p_1$ there is $q \leq p$ such that (p, q) is good, and so $x = f_*(p, q) \in R$; but then there are also $p' \leq p$ and $p'' \leq p$ such that $p' \Vdash \check{x} \in \dot{A}$ and $p'' \Vdash \check{x} \notin \dot{A}$. So by density, there will be some such x such that $x \in A$ iff $x \notin S$. So $R \cap A \neq S$, as desired.

Part 3: Let $\pi^N : C^2 \rightarrow V_\xi^M = V_\xi^N$, with $\pi^N \in N$. Let $\pi^M : C^2 \rightarrow V_\xi^M$, with $\pi^M \in M$. We show that $\pi^N \in M$ and $\pi^M \in N$.

Work in $M[G]$, where $|C| = |C^2| = \gamma$ and $\gamma^+\text{-DC}$ holds. We construct a continuous increasing sequence $\langle X_\alpha \rangle_{\alpha < \gamma^+}$ of sets such that $X_0 = \text{rg}(\pi^N)$, $X_0 \cup \text{rg}(\pi^M) \subseteq X_1$, $X_{\omega\alpha+2n+1} \in M$, $X_{\omega\alpha+2n+2} \in N$, $X_\alpha \subseteq V_\xi^M = V_\xi^N$, $|X_\alpha| \leq \gamma$ and X_α is extensional for each limit α .

Given $X = X_{\omega\alpha+2n}$, first let $Y \subseteq V_\xi^M$ be some set with $X \subseteq Y$ and $|Y| = \gamma$ and such that for all $x, y \in X$, if $x \neq y$ then there is $z \in Y$ such that $z \in x$ iff $z \notin y$. Then let $X_{\omega\alpha+2n+1} \in M$ be some Y' such that $Y \subseteq Y' \subseteq V_\xi^M$ and Y' is the surjective image of W in M ; this exists because $V_\lambda = V_\lambda^M[G]$, so M can find a small covering set. For $X_{\omega\alpha+2n+2}$ it is likewise. Make the sequence continuous. By $\gamma^+\text{-DC}$, we can proceed through γ^+ sets; note that $\gamma\text{-DC}$ ensures here that at limit stages α , we still have $|X_\alpha| = \gamma$.

Now since λ is large, the sequence is in V_λ . So let $\dot{X}^N \in N$ and $\dot{X}^M \in V_\lambda^M$ be names for it. Then we can find $p \in G$ such that for cofinally many $\alpha < \gamma^+$, in M , p decides the value of \dot{X}_α , and forces that the sequence is continuous. Let $D^M \in M$ be the cofinal set on which p decides this. Then $\langle X_\alpha \rangle_{\alpha \in D^M} \in M$. Note that D^M is closed. We get a similar club $D^N \in N$. Let $\alpha \in D^M \cap D^N$ be a limit ordinal. Then $X_\alpha \in M \cap N$, X_α is extensional, and M, N both have surjections $C^2 \rightarrow X_\alpha$ (since $M[G]$ has a surjection $C \rightarrow X_{\alpha+1}$, M has one $C^2 \rightarrow X_{\alpha+1}$, and likewise N has one $C^2 \rightarrow X_{\alpha+2}$).

Since $X = X_\alpha$ is extensional, we can let \bar{X} be the transitive collapse of X , and $\pi : \bar{X} \rightarrow X$ the uncollapse map. Then $\bar{X}, \pi \in M \cap N$ (note the recursion defining the collapse maps take less than λ many steps, by choice of λ). Note that N, M have surjections from C^2 to \bar{X} . Since \bar{X} is transitive, it follows

that \bar{X} is coded by subsets of C^4 in M, N . But $\mathcal{P}(C^4)^M = \mathcal{P}(C^4)^N$. So let $\sigma : C^2 \rightarrow \bar{X}$ be a common surjection. Then $\pi \circ \sigma : C^2 \rightarrow X$ is a common surjection. Now $\pi^M \in M$, so $(\pi \circ \sigma)^{-1} \circ \pi^M \in \mathcal{P}(C^4)^M = \mathcal{P}(C^4)^N$, so $\pi^M \in N$. Similarly $\pi^N \in M$. \square

This completes the proof of the theorem. \square

Theorem 4.5. *Let $\eta \in \text{OR}$ and $\mathbb{P} \subseteq V_\eta$ and G be (V, \mathbb{P}) -generic. Suppose $(V[G], j) \models \text{ZFR}$ where $\eta < \kappa = \text{cr}(j)$, and $V[G] \models \text{"X-AC"}$ where $X = V_\eta$. Suppose $j \restriction V : V \rightarrow V$. Then j is amenable to V .*

Proof. Suppose not and let $\alpha \in \text{OR}$ be such that $j \restriction V_\alpha \notin V$. Let $\beta = j(\alpha)$. Let \tilde{k} be such that $\tilde{k}_G = j \restriction V_{\alpha+5}$. We may assume that \mathbb{P} forces " $\tilde{k} : V_{\alpha+5} \rightarrow V_{\beta+5}$ is elementary and $\tilde{k} \restriction V_\alpha \notin \check{V}$ ".

In $V[G]$, fix a function $f : \mathbb{P} \rightarrow V_\alpha^V$ such that $f(p)$ is some x such that $V \models \neg \exists y [p \Vdash_{\mathbb{P}} \tilde{k}(\check{x}) = \check{y}]$; such f exists by X-AC in $V[G]$. Let $\tilde{f} \in V$ be such that $\tilde{f}_G = f$ and $V \models \mathbb{P} \Vdash \tilde{f} : \check{\mathbb{P}} \rightarrow \check{V}_\alpha$.

In V , define the partial function $f_* :_{\mathbb{P}} \mathbb{P} \times \mathbb{P} \rightarrow V_\alpha$ by

$$f_*(p, q) = \text{unique } x \text{ such that } q \Vdash_{\mathbb{P}} \tilde{f}(\check{p}) = \check{x},$$

if $q \leq p$ and there is such an $x \in V_\alpha$, and $f_*(p, q)$ is undefined otherwise. Let $D = \text{dom}(f_*)$.

Let $g_* = j(f_*) \in V$ and let $p_0 \in \mathbb{P}$ be such that $V \models p_0 \Vdash_{\mathbb{P}} \tilde{k}(\check{f}_*) = \check{g}_*$. Since $\mathbb{P} \in V_\kappa$ and by elementarity, $g_* : D \rightarrow V_{j(\alpha)}$ and $j \circ f_* = g_*$. Let $p_1 \leq p_0$ with $V \models p_1 \Vdash_{\mathbb{P}} \tilde{k} \circ \check{f}_* = \check{g}_*$. Fix $q \leq p_1$ with $(p_1, q) \in D$. Let $x = f_*(p_1, q)$. So by choice of f, x , $V \models \neg \exists y [p_1 \Vdash_{\mathbb{P}} \tilde{k}(\check{x}) = \check{y}]$. But then letting $y = g_*(p_1, q)$ we have $V \models p_1 \Vdash_{\mathbb{P}} \tilde{k}(\check{x}) = \check{y}$, contradiction. \square

We can now deduce that one cannot force a “new” Reinhardt cardinal κ such that κ -DC holds in the generic extension with small forcing:

Theorem 4.6. *Let $\mathbb{P} \in V$ and G be (V, \mathbb{P}) -generic and C the set of \mathbb{P} -conditions. Suppose $(V[G], j) \models \text{ZFR} + \text{"}C \text{ is wellordered and } \gamma^+\text{-DC holds where } \text{card}(C) = \gamma\text{"}$. Then $(V, j \restriction V) \models \text{ZFR}$.*

Proof. By 4.4, V is definable over $V[G]$ from $V_{\eta+1}^V$, so $j \restriction V : V \rightarrow V$. So by 4.5, j is amenable to V . And $(V, j \restriction V)$ satisfies the ZF axioms because $(V[G], j)$ does and $(V, j \restriction V)$ is definable from parameters over $V[G]$. \square

Remark 4.7. Note that the converse direction is routine: If $(V, j) \models \text{ZFR}$ and $\mathbb{P} \in V_\kappa$ where $\kappa = \text{cr}(j)$ and G is (V, \mathbb{P}) -generic, then $(V[G], j^+) \models \text{ZFR}$ where $j^+ : V[G] \rightarrow V[G]$ extends j with $j(G) = G$. It follows under $(V, P) \models \text{ZF}_2 + \text{"there is a super Reinhardt cardinal"}$ then (since then there is a proper class of such) $(V[G], P[G])$ satisfies the same, and an easy variant gives that if $(V, P) \models \text{"OR is total Reinhardt"}$, so does $(V[G], P[G])$. The other direction is less trivial.

Theorem 4.8. *Assume $(V, P) \models \text{ZF}_2$. Let G be (V, \mathbb{P}) -generic where $\mathbb{P} \in V$. Then:*

1. If κ is super Reinhardt in $(V[G], P[G])$ then there is $\kappa' \in \text{OR}^V$ which is super Reinhardt in (V, P) .
2. If $\mathbb{P} \in V_\delta$ and $(V_\delta^{V[G]}, V_{\delta+1}^{V[G]})$ is total Reinhardt, then $(V_\delta, V_{\delta+1})$ is total Reinhardt.

Proof. Part 2: Let $\delta \in \text{OR}$ with $\mathbb{P} \in V_\delta$ and G be (V, \mathbb{P}) -generic and suppose that $(V_\delta^{V[G]}, V_{\delta+1}^{V[G]})$ is total Reinhardt. In V , fix $A \subseteq V_\delta$. In $V[G]$, let $\kappa < \delta$ be $(< \delta, (V_\delta, A))$ -reflecting with $\mathbb{P} \in V_\kappa$. Let

$$j : (V_\delta[G], (V_\delta, A)) \rightarrow (V_\delta[G], (V_\delta, A))$$

be elementary with $\text{cr}(j) = \kappa$. We claim that $j \restriction V_\delta \in V$ and hence, j witnesses what we need in V . It suffices to see that for each $\alpha < \delta$, we have $j \restriction V_\alpha \in V$, because we have a name $\dot{j} \in V$ for j , and V_δ is inaccessible, which gives $j \in V$ as usual.

So fix $\alpha < \delta$ with $\alpha = j(\alpha)$ and let $\dot{k} \in V_\delta$ be such that $\dot{k}_G = j \restriction V_\alpha$. We may assume $\alpha > \kappa$. We may assume that \mathbb{P} forces “ $\dot{k} : V_\alpha \rightarrow V_\alpha$ is elementary”. Let $\xi \in (\alpha + \omega, \delta)$ with $\dot{k} \in V_\xi$. Note that the forcing facts in the following claim are defined from the relevant parameters over V_ξ :

Claim 1. *There are $Y, \sigma, \eta \in V$, with $Y \subseteq V_\alpha$ and such that (i) for each $p \in \mathbb{P}$, if there is $x \in V_\alpha$ such that p does not decide (with respect to \mathbb{P}) the value of $k(\check{x})$ then there is $x \in Y$ with this property, (ii) $\eta < \kappa$, and (iii) $\sigma : V_\eta \rightarrow Y$ is a surjection.*

Proof. (The first part here is motivated by Usuba’s discussion of Löwenheim-Skolem cardinals in [18].) In $V[G]$, there is some $X \preceq V_\xi^{17}$ with $\mathbb{P}, \dot{k} \in X$ and such that the transitive collapse \bar{X} of X is in $V_\kappa^{V[G]}$. For in $V[G]$, let $\ell : V_\delta^{V[G]} \rightarrow V_\delta^{V[G]}$ be elementary with $\text{cr}(\ell) = \kappa$ and $\ell(\kappa) > \xi$ and $\ell(V_\delta) = V_\delta$. Then $\ell \restriction V_\xi \preceq \ell(V_\xi) = V_{\ell(\xi)}$, and $\ell \restriction V_\xi \in V_\delta[G]$, and $\ell \restriction V_\xi$ has transitive collapse $V_\xi \in V_{j(\kappa)}^{V[G]}$. So $V_\delta[G] \models$ “There is $X \preceq V_{\ell(\xi)}$ with $\ell(\mathbb{P}, \dot{k}) \in X$ and such that the transitive collapse \bar{X} of X is in $V_{\ell(\kappa)}^{V[G]}$ ”. So pulling back, in $V[G]$ we get an $X \preceq V_\xi$ with $\mathbb{P}, \dot{k} \in X$ and such that the transitive collapse \bar{X} of X is in $V_\kappa^{V[G]}$. Let $\dot{X} \in V$ be a name for \bar{X} ; we may take $\dot{X} \in V_\kappa$. Let $\pi : \bar{X} \rightarrow X$ be the uncollapse map. Let $\dot{\pi} \in V$ be a name for π .

Now we may assume that \mathbb{P} forces “ $\dot{\pi} : \dot{X} \rightarrow \check{V}_\xi$ ”. Using the forcing relation in V , looking at all outputs forceable for $\dot{\pi}$, we can find some $Y \subseteq V_\alpha$ such that $X \cap V_\alpha \subseteq Y$ and $\eta < \kappa$ and a surjection $\sigma : V_\eta \rightarrow Y$ which works. \square

Let $\tau = j(\sigma) \in V$, so $\tau = j \circ \sigma$. Let $p \in G$ force “ $\dot{j} : \check{V}_\delta \rightarrow \check{V}_\delta$ is elementary with $\text{cr}(\dot{j}) = \check{\kappa}$ and $\dot{j} \restriction \check{V}_\alpha = \dot{k}$ and $\check{\tau} = \dot{j}(\check{\sigma}) = \dot{j} \circ \check{\sigma}$ ”. Note that p forces the value of $k(\check{x}) = \dot{j}(\check{x})$ for every $x \in \text{rg}(\sigma)$. But then by the claim, p forces the value of $\dot{k}(\check{x})$ for every $x \in V_\alpha$. Therefore $j \restriction V_\alpha \in V$, as desired.

Part 1: Let $V[G]$ be a set-generic extension such that κ is super-Reinhardt in $V[G]$. Note then that if $j : V[G] \rightarrow V[G]$ is elementary with $\text{cr}(j) = \kappa$, then $j(\kappa)$ is also super-Reinhardt in $V[G]$ (consider $j(k)$ for various k). And super Reinhardt easily implies supercompact. So (see [18]) $V[G] \models$ “There is

¹⁷We write V_ξ for rank segments of V and $V_\xi^{V[G]}$ for those of $V[G]$.

a proper class of supercompact cardinals”, so $V[G] \models$ “There is a proper class of Löwenheim-Skolem cardinals”, so all set-grounds of $V[G]$ are definable from parameters over $V[G]$. Taking κ' super-Reinhardt then above the defining parameter, and $j' : V[G] \rightarrow V[G]$ with $\text{cr}(j') = \kappa'$, it follows that $j' \upharpoonright V : V \rightarrow V$. But then essentially the same argument as above shows that j' is amenable to V , as required. \square

Remark 4.9. Goldberg has shown [4, Theorem 6.12] that assuming $\text{ZF} + \text{DC}_{V_{\lambda+1}}$ and $j : V_{\lambda+3} \rightarrow V_{\lambda+3}$ is Σ_1 -elementary, one can prove $\text{Con}(\text{ZFC} + I_0)$. It is not known at present, however, whether ZFR proves $\text{Con}(\text{ZFC} + I_0)$.

Assume $\text{ZF} + k : V \rightarrow M$ is elementary with M transitive, $\kappa = \text{cr}(k)$, $\kappa_1 = k(\kappa)$, and $k \upharpoonright V_{\kappa_1} \in M$, and

$$j : V_{\lambda+3} \rightarrow V_{\lambda+3} \text{ is } \Sigma_1\text{-elementary}$$

with $\kappa < \text{cr}(j) < \kappa_\omega(j) \leq \lambda < \kappa_1$. Then there is $\mathbb{P} \subseteq V_\kappa$ forcing $\text{DC}_{V_{\lambda+1}}$, and j extends Σ_1 -elementarily to $j^+ : V_{\lambda+3}^{V[G]} \rightarrow V_{\lambda+3}^{V[G]}$, so we can apply Goldberg’s result above. To force $\text{DC}_{V_{\lambda+1}}$, one can use a kind of argument used by Woodin to force DC from supercompacts. That is, let \mathbb{P} be the finite support product of $\text{Col}(\omega, V_\alpha)$, over all $\alpha < \kappa$. Let G be (V, \mathbb{P}) -generic. We claim $V[G] \models \text{DC}_{V_\alpha}$ for all $\alpha < \kappa_1$, which suffices.

For first note that for every $x \in V_{\kappa_1}$, there is $\kappa' < \kappa_1$ and a Σ_1 -elementary $\pi : V_\kappa \rightarrow V_{\kappa'}$ with $x \in \text{rg}(\pi)$ and $\pi(\text{cr}(\pi)) = \kappa$. For $\pi' = k \upharpoonright V_{\kappa_1} : V_{\kappa_1}^M \rightarrow V_{\kappa_1}^M$ is Σ_1 -elementary where $\kappa' = \sup k \text{“} \kappa_1$, and $k(x) \in \text{rg}(\pi')$, and $\pi'(\text{cr}(\pi')) = \kappa_1$, and $\pi' \in M$. So the existence of such a map π' pulls back under k , yielding a π as desired.

Now let $R \in V_{\kappa_1}^{V[G]}$ be a finitely extendible relation, and $\dot{R} \in V_{\kappa_1}$ with $\dot{R}_G = R$. Let $\pi : V_\kappa \rightarrow V_{\kappa'}$ be Σ_1 -elementary with $\dot{R}_G \in \text{rg}(\pi)$ and $\pi(\bar{\kappa}) = \kappa$ where $\bar{\kappa} = \text{cr}(\pi)$. Let $\pi(\bar{\mathbb{P}}, \dot{R}) = (\mathbb{P}, \dot{R})$ and $\bar{G} = G \cap V_{\bar{\kappa}}$. Note that every dense subset of $\bar{\mathbb{P}}$ in V_κ is a pre-dense subset of \mathbb{P} , so \bar{G} is $(V_\kappa, \bar{\mathbb{P}})$ -generic. Note that π extends to a Σ_1 -elementary $\pi^+ : V_\kappa[\bar{G}] \rightarrow V_{\kappa'}[G]$ with $\pi^+(\bar{G}) = G$. Let $\bar{R} = \dot{R}_{\bar{G}}$. So $\pi^+(\bar{R}) = R$. But \bar{R} is countable in $V[G]$, so $V[G]$ has an infinite branch b through \bar{R} , and $\pi \text{“} b$ is a branch through R .

Recall that by [1], $\text{ZF} + \text{“}\delta_0 \text{ is the least Berkeley cardinal”}$ implies the failure of $\text{cof}(\delta_0)\text{-DC}$ fails (in fact of $\text{cof}(\delta_0)\text{-AC}$). If $j : V \rightarrow V$ is elementary then as mentioned in [6] and §3.2, $\kappa_\omega(j)$ is somewhat Berkeley. These and the other considerations in this section lead to the following conjecture:

Conjecture 4.10. $\text{ZFR} + \text{DC}$ has consistency strength strictly greater than that of ZFR .

5 Competing Interests

The author has no competing interests.

6 Acknowledgements

Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044-390685587, Mathematics Münster: Dynamics-Geometry-Structure.

The author thanks Gabriel Goldberg and Toshimichi Usuba for their answers to certain questions and helpful feedback.

References

- [1] Joan Bagaria, Peter Koellner, and W. Hugh Woodin. Large cardinals beyond choice. *Bulletin of Symbolic Logic*, 25, 2019.
- [2] Victoria Gitman and Thomas A. Johnstone. On ground model definability. In *Infinity, Computability, and Metamathematics: Festschrift in honour of the 60th birthdays of Peter Koepke and Philip Welch*, Series: Tributes. College publications, London, GB, 2014. arXiv:1311.6789.
- [3] Victoria Gitman and Ralf Schindler. Virtual large cardinals. *Annals of Pure and Applied Logic*, 169(12), 2018.
- [4] Gabriel Goldberg. Even ordinals and the Kunen inconsistency. arXiv:2006.01084.
- [5] Gabriel Goldberg. Personal communication, 2020.
- [6] Gabriel Goldberg and Farmer Schlutzenberg. Periodicity in the cumulative hierarchy. arXiv: 2006.01103.
- [7] Joel David Hamkins, Greg Kirmayer, and Norman Lewis Perlmutter. Generalizations of the kunen inconsistency. *Annals of Pure and Applied Logic*, 163(12), 2012.
- [8] Akihiro Kanamori. *The higher infinite: large cardinals in set theory from their beginnings*. Springer monographs in mathematics. Springer-Verlag, second edition, 2005.
- [9] Kenneth Kunen. Elementary embeddings and infinitary combinatorics. *Journal of Symbolic Logic*, 36(3), 1971.
- [10] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Annals of Pure and Applied Logic*, 2007.
- [11] W. N. Reinhardt. *Topics in the metamathematics of set theory*. PhD thesis, UC Berkeley, 1967.
- [12] W. N. Reinhardt. Remarks on reflection principles, large cardinals, and elementary embeddings. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 189–205. Amer. Math. Soc., Providence, R. I., 1974.
- [13] Grigor Sargsyan, Ralf Schindler, and Farmer Schlutzenberg. Varsovian models II. In preparation.
- [14] Farmer Schlutzenberg. Extenders under ZF and constructibility of rank-into-rank embeddings. arXiv: 2006.10574.
- [15] Farmer Schlutzenberg. On the consistency of ZF with an elementary embedding from $V_{\lambda+2}$ into $V_{\lambda+2}$. arXiv: 2006.01077.

- [16] Farmer Schlutzenberg. Reinhardt cardinals and non-definability (draft 1). arXiv: 2002.01215v1, notes.
- [17] Akira Suzuki. No elementary embedding from V into V is definable from parameters. *Journal of Symbolic Logic*, 64(4), 1999.
- [18] Toshimichi Usuba. Choiceless Löwenheim-Skolem property and uniform definability of grounds. arXiv: 1904.00895, 2019.
- [19] Toshimichi Usuba. Personal communication, 2020.
- [20] W. Hugh Woodin. Suitable extender models II: Beyond ω -huge. *Journal of Mathematical Logic*, 11(2), 2011.