CONSTRUCTIVE ACKERMANN'S INTERPRETATION

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This paper is dedicated to David C. McCarty.

ABSTRACT. The main goal of this paper is to formulate a constructive analogue of Ackermann's observation about finite set theory and arithmetic. We will see that Heyting arithmetic is bi-interpretable with $\mathsf{CZF}^{\mathsf{fin}}$, the finitary version of CZF . We also examine bi-interpretability between subtheories of finitary CZF and Heyting arithmetic based on the modification of Fleischmann's hierarchy of formulas, and the set of hereditarily finite sets over CZF , which turns out to be a model of $\mathsf{CZF}^{\mathsf{fin}}$ but not a model of finitary IZF.

1. INTRODUCTION

Ackermann [1] noticed in 1937 that ZF without Infinity is interpretable within PA. Kaye and Wong [8] improved Ackermann's result by showing that ZF^{fin} , a finitary version of ZF, is bi-interpretable with PA. We may ask the same question for HA: could we find a set theory that is bi-interpretable with HA? One possible solution is to expand the language of set theory and add some axioms that are related to arithmetic, like [12]. However, we want to stick our set-theoretic counterpart is closer to 'standard' set theories as possible.

Fortunately, we have a good start point: Aczel [2] characterized and gave a careful analysis of a weak subtheory ACST of constructive set theory CZF, which is able to interpret Heyting arithmetic. Moreover, Aczel observed that there is a faithful interpretation from HA to ACST. We may hope that we can find a set theory that is bi-interpretable with HA by extending ACST, as Kaye and Wong capture set-theoretic counterpart of PA by extending ZF – Infinity, which can be interpreted in PA.

It turns that our strategy works: we will see that HA is bi-interpretable with $ACST^{fin}$ + Set Induction, which is identical with CZF^{fin} , the finitary version of CZF. A bit surprisingly, Kaye and Wong's proof also works over constructive background, and we will see later how their proof can be implemented over a constructive setting.

Theorem. HA is bi-interpretable with CZF^{fin}.

Kaye and Wong also observed that their bi-interpretation is also a bi-interpretation between subtheories of PA and ZF^{fin} , which is called $I\Sigma_n$ and Σ_n -Sep respectively, where Σ_n -Sep is a theory of set theory that comprises Extensionality, Pairing, Empty set, Union, \neg Infintiy, Δ_0 -Collection, $\Sigma_1 \cup \Pi_1$ -Set Induction and Σ_n -Separation. These subtheories rely on the hierarchy of formulas, which is given by prenex normal form, and there is little known about the complexity of predicate formulas over intuitionistic predicate logic, which renders finding appropriate constructive counterparts seemingly hard. However, we will see that modifying Fleischmann's hierarchy of formulas [7] yields a hierarchy on formulas of intuitionistic predicate logic, and also provides a nice subtheories of HA and CZF^{fin}. As a result, we also have a constructive analogue of Kaye and Wong's analysis on subtheories:

Theorem. For each $n \ge 1$, there are subtheories of \mathbb{I}_n of HA and SI_n of $\mathsf{CZF}^{\mathsf{fin}}$ that are bi-interpretable with each other. Moreover, adding the full law of excluded middle into \mathbb{I}_n and SI_n yields I_n and Σ_n -Sep respectively.

We know that ZF proves ZF^{fin} has a natural model, namely, the set of all hereditarily finite sets. We may ask whether a similar result holds CZF. We will see the following theorem holds in Section 10:

Theorem. CZF proves the set of hereditarily finite sets HF exists and satisfies CZF^{fin}. However, HF need not be a model of IZF^{fin}.

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We develop facts that are necessary backgrounds in Section 2 to 6: we will define theories and interpretation between theories in Section 2 and Heyting arithmetic in Section 3 respectively. We will define and discuss ACST and finitary constructive set theories in Section 4. We will also provide a streamlined version \mathbb{T} of CZF^{fin} for further technical convenience. In Section 5, we define subtheories of Heyting arithmetic and set theory based on a modification of Fleischmann's hierarchy of formulas [7]. Section 7 and 8 explain the interpretation à la Kaye and Wong is also a bi-interpretation between HA and CZF^{fin}. It turns out in Section 9 that the interpretation can also interpret subtheories of HA and CZF^{fin} defined in Section 5. In Section 10, we will define and examine properties of the set of all hereditarily finite sets HF over CZF.

2. Theories and interpretations

The central notion of this article is interpretation. Kaye and Wong [8] and Aczel [2] adopted interpretations that are defined by Visser [18]. In this article, we will follow Enderton's definition of interpretation (See Section 2.7 of [6],) with some adoptation of notions from [8] and [2].

We will identify theories with a set of axioms. For given two theories T_0 and T_1 , we write $T_0 \vdash T_1$ if T_0 proves every axioms of T_1 . If both $T_0 \vdash T_1$ and $T_1 \vdash T_0$ holds, we write $T_0 \vdash T_1$ and say that T_0 and T_1 are *identical*.

To define an interpretation $\mathfrak{t}: T_0 \to T_1$, we need formulas $\pi_{\forall}(x), \pi_P(x_0, \cdots, x_{n-1})$ for *n*-ary predicate symbol *P* and $\pi_f(x_0, \cdots, x_{n-1}, y)$ for *n*-ary function symbol *f*. (We will regard constant symbols as nullary function symbols.) In addition, we assume that T_1 proves $\exists x \pi_{\forall}(x)$ and the functionality of π_f , that is,

(1)
$$T_1 \vdash \forall x_0 \cdots \forall x_{n-1} \exists ! y \pi_f(x_0, \cdots, x_{n-1}, y).$$

Then the interpretation \mathfrak{t} sends a formula of T_0 to a formula of T_1 as follows:

• Let $s_0, \dots, s_{n-1}, t_1, \dots, t_m$ be terms, f be a function symbol, and P be a predicate symbol or =. Then $(P(f(s_0, \dots, s_{n-1}), t_1, \dots, t_m))^t$ is

(2)
$$\exists x_0 \cdots \exists x_{n-1} \exists y \left[\bigwedge_{0 \le i < n} (x_i = s_i)^{\mathfrak{t}} \wedge \pi_f(x_0, \cdots, x_{n-1}, y) \wedge (P(y, t_1, \cdots, t_m))^{\mathfrak{t}} \right]$$

We will apply the similar procedure if $f(t_i)$ appears on another argument of P.

- $(P(x_0, \dots, x_{n-1}))^{\mathfrak{t}}$ is $\pi_P(x_0, \dots, x_{n-1})$, where each x_i is a variable.
- t respects logical connectives. For example, $(\phi \to \psi)^{t}$ is defined by $\phi^{t} \to \psi^{t}$.
- $(\forall x \phi(x))^{\mathfrak{t}}$ is $\forall x(\pi_{\forall}(x) \to \phi^{\mathfrak{t}}(x))$. $(\exists x \phi(x))^{\mathfrak{t}}$ is $\exists x(\pi_{\forall}(x) \land \phi^{\mathfrak{t}}(x))$.

Every theory T has the *identity interpretation* 1_T , which is defined by $\pi_{\forall}(x) \equiv (x = x), \pi_P \equiv P$ and $\pi_f(\vec{x}, y) \equiv (f(\vec{x}) = y)$. For two interpretation $\mathfrak{s}, \mathfrak{t}: T_0 \to T_1$, we say they are the *same* when $T_1 \vdash \forall \vec{x} [\phi^{\mathfrak{s}}(\vec{x}) \leftrightarrow \phi^{\mathfrak{t}}(\vec{x})]$ for all formulas $\phi(\vec{x})$. In that case, we write $\mathfrak{s} = \mathfrak{t}$. The composition of two interpretation is the result of application of the two interpretations.

We can see that if $\mathfrak{s}: T_0 \to T_1$ is an interpretation, then $T_0 \vdash \phi$ implies $T_1 \vdash \phi^{\mathfrak{s}}$. We call \mathfrak{s} is *faithful* if the converse also holds: that is, \mathfrak{s} is faithful if $T_0 \vdash \phi \iff T_1 \vdash \phi^{\mathfrak{s}}$. For $\mathfrak{s}: T_0 \to T_1$ and $\mathfrak{t}: T_1 \to T_0$, \mathfrak{s} and \mathfrak{t} are the inverses of each other if $\mathfrak{ts} = 1_{T_0}$ and $\mathfrak{st} = 1_{T_1}$. In that case, we call \mathfrak{s} a *bi-interpretation* between T_0 and T_1 .

3. Heyting arithmetic

Heyting arithmetic HA is the constructive counterpart of Peano arithmetic. There are various possible formulations of Heyting arithmetic: for example, we may take the language of arithmetic as the set of all primitive recursive functions and add axioms that define each primitive recursive functions. Since we want to analyze the relation between classical interpretation and constructive one, we choose the form given over the language $\mathcal{L} = \{0, S, +, \cdot\}$ with the following axioms:

- (1) $\forall x, y(Sx = Sy \rightarrow x = y),$
- (2) $\forall x(x=0 \lor \exists y(x=Sy)),$
- $(3) \quad \forall x(x+0=x),$
- (4) $\forall x, y(x + Sy = S(x + y)),$
- (5) $\forall x(x \cdot 0 = 0),$
- (6) $\forall x, y(x \cdot Sy = x \cdot y + y),$
- (7) For each formula $\phi(x), \phi(0) \land \forall x [\phi(x) \to \phi(S(x))] \to \forall x \phi(x).$

The last axiom is called the induction scheme. These set of axioms are strong enough to define primitive recursive functions and show they are provably total. Especially, we are interested in the totality of the exponential function, and we postulate it as an axiom Exp. It is known that HA proves Exp, but Exp could not be provable from a weaker subtheory of HA.

Heyting arithmetic does not include the law of excluded middle, but it proves the law of excluded middle for bounded formulas, that is, formulas whose quantifiers are of the form $\forall (x < y)$ or $\exists (x < y)$:

Proposition 3.1. Let ϕ be a bounded formula of HA. Then HA proves $\phi \lor \neg \phi$.

4. Constructive finitary set theories

Aczel [2] defines an arithmetical version of constructive set theory ACST to analyze finite sets over constructive set theory CZF. We clarify some notions to define what ACST is. A formula $\phi(x)$ of set theory is Δ_0 if every quantifier in the formula is bounded, that is, every quantifier is of the form $\forall x (x \in a \to \cdots)$ or $\exists x (x \in a \land \cdots)$. We will abbreviate previous formulas into $\forall x \in a(\cdots)$ and $\exists x \in a(\cdots)$ respectively.

Aczel defined RCST before defining ACST.

Definition 4.1. RCST is the theory consisting of Extensionality, Empty set, Binary Intersection, Pairing and *Global Union-Replacement Rule* GURR: for each formula $\phi(u, v)$,

(3)
$$\forall u \exists ! v \phi(u, v) \to \forall x \exists y \forall z [z \in y \leftrightarrow \exists v (z \in v \land \exists u \in x \phi(u, v))].$$

Intuitively, GURR states if F is a class function and x is a set, then $\bigcup F''[x]$ is also a set. Hence GURR implies Union and Replacement. Some readers might wonder why RCST does not include a form of Separation. However, the following well-known result shows that RCST proves Separation for Δ_0 -formulas:

Lemma 4.2. (Corollary 9.5.7 of [3]) BCST without Δ_0 -Separation proves that Δ_0 -Separation is equivalent to Binary Intersection.

Here BCST comprises Extensionality, Pairing, Replacement, Union, Empty set, and Δ_0 -Separation. Since RCST proves every axiom of BCST except for Δ_0 -Separation, we can apply the previous lemma to RCST.

ACST is obtained by adding the induction scheme on natural numbers. We need to define what natural numbers are:

Definition 4.3. The class of natural numbers ω is defined as follows:

(4)
$$\omega = \{ \alpha \in \operatorname{Ord} \mid \alpha^+ \subseteq \{0\} \cup \{\gamma^+ \mid \gamma \in \operatorname{Ord} \} \}$$

where $\alpha^+ := \alpha \cup \{\alpha\}$ and Ord is the class of all ordinals, that is, transitive sets whose elements are also transitive.

Definition 4.4. Mathematical Induction Axiom Scheme $MathInd(\omega)$ is the following statement: for every definable class X, the following holds:

(5)
$$\operatorname{Ind}(X) \to \omega \subseteq X,$$

where Ind(X) states X is an inductive class:

(6)
$$\operatorname{Ind}(X) \equiv (0 \in X \land \forall x \in X(x^+ \in X)).$$

ACST is the theory obtained by adding $MathInd(\omega)$ to RCST.

ACST is strong enough to do finitary mathematics. For example, the following theorem is a consequence of ACST.

Lemma 4.5 (Primitive Recursion). (ACST) Let A and B be classes and let $F_0: B \to A, F_1: B \times \omega \times A \to A$ be class functions. Then there is $H: B \times \omega \to A$ such that $H(b, 0) = F_0(b)$ and $H(b, k+1) = F_1(b, k, H(b, k))$ for all $k \in \omega$.

Proof. The proof of the lemma is available at Theorem 10.6 of [2]. We will give a direct proof that works over ACST for later analysis.

A function f is partially given under b up to m if $f: m^+ \to A$ such that $f(0) = F_0(b)$ and $f(k^+) = F_1(b, k, f(k))$ for all $k \in m$. Take $\psi(b, k, f)$ if and only if f is partially given under b up to m and $k \in \text{dom } f$.

We claim that $\forall b, m \exists f \psi(b, m, f)$ holds. We will use induction on m. The case m = 0 is obvious. Suppose that $\exists g \psi(b, m, g)$ holds. Take $u_0 = F_1(b, k, g(m))$ and let $f = g \cup \{(m^+, u_0)\}$. Then f witnesses $\psi(b, m^+, f)$.

Now assume that f_0 and f_1 are partially given under b, up to m_0 and m_1 respectively. We can see that $k < \min(m_0, m_1) \rightarrow f_0(k) = f_1(k)$ by induction on k. Finally let

(7)
$$H(b,m) = x \iff \exists f[f \text{ is partially given under } b \text{ up to } m \text{ and } f(m) = x].$$

By Primitive recursion, we can see that ACST is able to define addition and multiplication over ω . We will see later, however, that ACST does not suffice to be bi-interpretable with HA. For example, ACST does not prove every set is finite, while the set theory simulated by HA seems should do, as PA does.

Definition 4.6. Let Fin be the class of all finite sets, that is,

(8) Fin = {
$$x \mid \exists n \in \omega \exists f : n \to x(f \text{ is a bijection between } n \text{ and } x)$$
}

The assertion V = Fin is that every set is finite, i.e., for each set x there is $n \in \omega$ such that x is a bijective image of n. One can show that ACST proves, for every finite set x, there is a unique such a natural number n. The readers can find its proof in Section 8.2 of [3]. We call this natural number n the *cardinality* of x.

Let $ACST^{fin}$ be the theory ACST + (V = Fin). It is easy to see that $ACST^{fin}$ proves the axiom of choice

(9)
$$\forall a [\forall x \in a \exists y \phi(x, y) \to \exists f \in {}^{a}V \forall x \in a \phi(x, f(x))]$$

and Δ_0 -excluded middle. Finally, let us consider the theory $\mathsf{ACST}^{\mathsf{fin}}$ + Set Induction. We will verify that the interpreted axiom is valid, but the verification takes effort per each axiom. Hence we prefer an axiom system as simple as possible. We will see that the following system \mathbb{T} is a streamlined version of $\mathsf{ACST}^{\mathsf{fin}}$ + Set Induction:

Definition 4.7. The theory \mathbb{T} comprises the following axioms: Extensionality, Pairing, Union, Binary Intersection, Set Induction and V = Fin.

Obviously $\mathbb T$ is a subsystem of $\mathsf{ACST}^\mathsf{fin} + \mathrm{Set}$ Induction. Moreover, we have

Proposition 4.8. \mathbb{T} and $ACST^{fin} + Set$ Induction are identical.

Proof. It suffices to show that $MathInd(\omega)$ and GURR is derivable from \mathbb{T} .

(1) MathInd(ω): Let X be an inductive class. Applying Set induction to the formula $x \in \omega \to x \in X$ yields the result. However, we will give an alternative proof for later analysis.

Assume that $\alpha \in \omega$. Then we have $\alpha = 0$ or $\alpha = \gamma^+$ for some ordinal γ . The former obviously implies $\alpha \in X$. In the latter case, apply the set induction to $x \in \gamma^{++} \to x \in A$. Then we have $\alpha = \gamma^+ \in A$.

(2) GURR: Let F be a class function. We will use the induction on the cardinality of x: assume that $\bigcup F^{"}[x]$ exists for sets x of cardinality x. Then $\bigcup F^{"}[x \cup \{y\}] = (\bigcup F^{"}[x]) \cup F(y)$, whose existence follows from the listed axioms.

Aczel showed that $ACST^{fin}$ + Set Induction is identical with CZF^{fin} , the finitary CZF, which is obtained by replacing the axiom of Infinity in CZF to V = Fin. The proof is direct, and it uses induction on the size of sets. We will give a proof of axiom of Strong collection from \mathbb{T} for later analysis:

Proposition 4.9. \mathbb{T} proves the axiom of Strong Collection.

Proof. Assume that $\forall x \in a \cup \{c\} \exists y \phi(x, y)$, where a is a set of size n and $c \notin a$. Assume inductively that Strong Collection holds for sets of size n, so we have b such that $\forall x \in a \exists y \in b \phi(x, y)$ and $\forall y \in b \exists x \in a \phi(x, y)$. Take d such that $\phi(c, d)$, then $b \cup \{d\}$ witnesses Strong Collection for ϕ and $a \cup \{c\}$.

Kaye and Wong [8] included the existence of transitive closure into their set-theoretic counterpart of PA. Thus it is natural to ask whether our \mathbb{T} should contain the existence of transitive closure as an axiom. The following lemma ensures it is provable from \mathbb{T} , so adding it is unnecessary:

Lemma 4.10. T proves every set has a transitive closure; that is, for each x there is a transitive set y such that $x \subseteq y$ and for each transitive z such that $x \subseteq z$, we have $y \subseteq z$. Furthermore, the class function TC(x) is definable.

Proof. We will show first that the class TC(x) is uniformly definable. Consider $F_0(x) = x$ and $F_1(x, y) = x \cup \bigcup y$. By Lemma 4.5, there is H such that H(x, 0) = x and $H(x, n + 1) = x \cup \bigcup H(x, n)$. Now take $TC(x) = \bigcup_{n \in \omega} H(x, n)$.

We can show that $H(x,n) \subseteq y$ for all transitive y such that $x \subseteq y$ by induction on n. Therefore TC(x) is the least transitive class that contains x. However, we do not know TC(x) is a set yet. We will show the following statement by induction on x:

(10)
$$\exists u(u \text{ is transitive } \land x \subseteq u \subseteq \mathrm{TC}(x))$$

Assume that (10) holds for all $y \in x$, i.e., for each $y \in x$ we can find a transitive v such that $v \subseteq TC(y)$. By Strong collection, we can find c such that

(11)
$$[\forall y \in x \exists v \in c(v \text{ is transitive } \land y \subseteq v \subseteq \mathrm{TC}(y))]$$

 $\wedge [\forall v \in c \exists y \in x(v \text{ is transitive and } \land y \subseteq v \subseteq \mathrm{TC}(y))].$

Now let $u = x \cup \bigcup c$. We claim that u witnesses (10). Since u is a union of transitive sets, u is transitive. $u \subseteq TC(x)$ follows from $y \in x \to TC(y) \subseteq TC(x)$.

We work over \mathbb{T} in the remaining part of the section unless specified. We can see that the recursion theorem on sets holds since \mathbb{T} proves every axiom of CZF except for Infinity.

Lemma 4.11 (Set Recursion). Let G be a total (k+2)-ary class function. Then there is a total (k+1)-ary class function F such that

(12)
$$\forall \vec{x} \forall y [F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) \mid z \in y \rangle)].$$

Proof. We will follow the proof of Proposition 19.2.1 of [3]. We present the whole proof for the sake of completeness and later analysis.

Call f be partially given under \vec{x} if f is a function of a transitive domain and $\forall y \in \text{dom } f[f(y) = G(\vec{x}, y, f \upharpoonright y)]$. Take

(13)
$$\psi(\vec{x}, y, f) \equiv (f \text{ is partially given under } \vec{x}) \land y \in \text{dom } f.$$

We will see that for given \vec{x} , $\forall y \exists f \psi(\vec{x}, y, f)$ holds. We appeal to Set Induction on y: assume that $\forall u \in y \exists g \psi(\vec{x}, u, g)$ holds. By Strong Collection, we have a set A such that $\forall u \in y \exists g \in A \psi(\vec{x}, u, g)$ and $\forall g \in A \exists u \in y \psi(\vec{x}, u, g)$. Let $f_0 = \bigcup A$ and $u_0 = G(\vec{x}, y, \langle f_0(u) \mid u \in y \rangle)$. Take $f = f_0 \cup \{(y, u_0)\}$.

We want to claim that f is a function. We need to ensure the following statement which can be shown easily by applying Set Induction: for any $g_0, g_1 \in A$ and $x \in \text{dom}(g_0) \cap \text{dom}(g_1)$, we have $g_0(x) = g_1(x)$. Its upshot is that f_0 is a function. It is easy to see that the dom f_0 is transitive, $y \subseteq \text{dom } f_0$ and $\forall u \in$ dom $f_0[f(y) = G(\vec{x}, y, f_0 \upharpoonright y)]$. Hence $\forall u \in \text{dom } f_0 \psi(\vec{x}, u, f_0)$.

Now assume that $(a, b), (a, c) \in f$. Then ether both of them is a member of f_0 or one of them is a member of $\{(y, u_0)\}$. In the latter case, we can see that b = c by applying $\psi(\vec{x}, u, f_0)$ for all $u \in \text{dom } f_0$ or the definition of u_0 . Hence f is a function. Checking the remaining conditions of $\psi(\vec{x}, y, f)$ is direct, so we omit it.

A class Φ is an inductive definition if it is a class of pairs. Every inductive class Φ is associated with a class of consequences $\Gamma_{\Phi}(A) = \{x \mid (A, x) \in \Phi\}$. A class C is Γ_{Φ} -closed if $\Gamma_{\Phi}(C) \subseteq C$.

Proposition 4.12. (Class Inductive Definition Theorem) Let Φ be an inductive definition. Then there is a least Γ_{Φ} -closed class.

See Theorem 12.1.1 of [3] for the proof of Class Inductive Definition Theorem. Note that the proof in [3] works over BCST with Strong collection and Set induction, which are provable from \mathbb{T} .

We will conclude this section with the following lemma, which asserts the composition of a Δ_0 -formula and a class function is still decidable.

Lemma 4.13. Let F be a class function and $\phi(x)$ be a Δ_0 -formula. Then $\phi(F(x)) \vee \neg \phi(F(x))$ holds.

Proof. It follows from the equivalence between $\phi(F(x)) \vee \neg \phi(F(x))$ and

(14)
$$\forall y[(y = F(x)) \to (\phi(y) \lor \neg \phi(y))].$$

5. Complexity of formulas and subtheories

We will analyze the interpretation over subtheories of \mathbb{T} and HA, as Kaye and Wong scrutinize subtheories of PA and finitary ZFC. Kaye and Wong analyzes these theories along the complexity of Induction schemes and Separation. It calls into a need for a new hierarchy for arithmetical formulas, so we introduce a variation of Fleischmann's hierarchy of formulas [7].

Definition 5.1. Let Φ and Ψ be sets of formulas over the language of arithmetic or set theory. Then the set $\mathcal{E}(\Phi)$ is defined as the closure of Φ under \wedge , \vee , bounded quantifications and \exists . The set $\mathcal{U}(\Phi, \Psi)$ is defined as the smallest set such that

- (1) $\Phi \subseteq \mathcal{U}(\Phi, \Psi),$
- (2) $\mathcal{U}(\Phi, \Psi)$ is closed under \land , \lor , \forall , and bounded quantifications, and
- (3) if $\psi \in \Psi$ and $\phi \in \mathcal{U}(\Phi, \Psi)$ then $\psi \to \phi$ is in $\mathcal{U}(\Phi, \Psi)$.

Definition 5.2. Let $\mathcal{E}_0 = \mathcal{U}_0$ be the collection of all bounded formulas. For each $n \ge 1$, define $\mathcal{E}_n := \mathcal{E}(\mathcal{U}_{n-1})$ and $\mathcal{U}_n := \mathcal{U}(\mathcal{E}_{n-1}, \mathcal{E}_{n-1})$.

Then \mathcal{E}_n and \mathcal{U}_n form increasing sequences of sets. Moreover, we have the following fact by modifying the proof of Theorem 3.10 of [7]:

Proposition 5.3. $\bigcup_{n < \omega} \mathcal{E}_n = \bigcup_{n < \omega} \mathcal{U}_n$ and they are equal to the set of all formulas. Moreover, \mathcal{E}_n and \mathcal{U}_n are subsets of \mathcal{E}_{n+1} and \mathcal{U}_{n+1} respectively.

We will confuse \mathcal{E}_n and \mathcal{U}_n with a family of formulas that are *provably* equivalent to an \mathcal{E}_n and \mathcal{U}_n formula respectively. Note that \mathcal{E}_1 -formulas over the language of set theory are also known as Σ -formulas (cf. Definition 19.1.2 of [3].) Σ -reflection principle over IKP (Theorem 19.1.4 of [3]) shows every \mathcal{E}_1 -formula over IKP is in fact Σ -formula. Moreover, \mathcal{E}_n and \mathcal{U}_n classes are classically equivalent to Σ_n and Π_n classes respectively.

We will analyze some definitions and theorems under the mentioned hierarchy.

Definition 5.4. Let $|\mathcal{E}_n$ be a subtheory of HA where the full induction scheme is weakened to the induction scheme for \mathcal{E}_n -formulas. Sl \mathcal{E}_n be a subtheory of \mathbb{T} that restricts Set Induction schemes to \mathcal{E}_n -formulas.

 $I\mathcal{E}_1$ is still strong enough to show that every primitive recursive function is definable and total. Especially, $I\mathcal{E}_1$ proves Exp. On the set-theoretic side, we can see that Δ_0 -LEM, the axiom of power set are still provable from $SI\mathcal{E}_1$. Moreover, we have the following results:

Lemma 5.5 (Mathematical Induction for \mathcal{E}_n -formulas). Let $n \ge 1$. Then $Sl\mathcal{E}_n$ proves $Ind(X) \to \omega \subseteq X$ for each class X that is given by an \mathcal{E}_n -formula.

Proof. Note that we cannot apply Set Induction to $x \in \omega \to x \in X$ since the complexity of $x \in \omega$ is \mathcal{E}_1 , so the complexity of the whole formula could not be \mathcal{E}_n . Therefore, we instead apply the alternative proof of Proposition 4.8: observe that $x \in \gamma^{++} \to x \in A$ is \mathcal{E}_n if A is an \mathcal{E}_n -class since it is equivalent to $\exists z \in \gamma^{++} (z = x \land z \in A)$, so the previous proof works.

The following results can be obtained by modifying the proofs in Section 4:

Lemma 5.6. Let $n \geq 1$. Then the axiom of Strong Collection for \mathcal{E}_n -formulas are provable from $Sl\mathcal{E}_n$. \Box

Lemma 5.7 (Primitive recursion for \mathcal{E}_n -formulas). Sl \mathcal{E}_n proves the following: let $n \ge 1$ and $m \ge n$. Assume that A and B are \mathcal{E}_m -definable classes and take \mathcal{E}_m -definable class functions $F_0: B \to A$ and $F_1: B \times \omega \times A \to A$. Then there is a unique $H: B \times \omega \to A$ such that $H(b,0) = F_0(b)$ and $H(b,k+1) = F_1(b,k,H(b,k))$.

Note that Lemma 5.7 shows the addition and multiplication on ω is still definable over SI \mathcal{E}_n .

Lemma 5.8. $SI\mathcal{E}_1$ proves the existence and \mathcal{E}_1 -definability of TC(x).

Proof. The proof of Lemma 4.10 still works over $Sl\mathcal{E}_1$. Moreover, F_0 and F_1 in the proof of Lemma 4.10 is \mathcal{E}_1 , so H and TC is also \mathcal{E}_1 by Lemma 5.7.

Hence we can carry on the usual proof of Set Recursion theorem and Class Inductive Definition Theorem for \mathcal{E}_n -formulas over $\mathsf{Sl}\mathcal{E}_n$ for $n \geq 1$. The readers could consult with Section 9.3 and Chapter 12 of [3] for the usual proof of Set Recursion Theorem and Class Inductive Definition Theorem.

 $\overline{7}$

Lemma 5.9 (Set Recursion for \mathcal{E}_n -formulas). Assume $n \ge 1$. Then the following statement is provable from $Sl\mathcal{E}_n$: Let G be a total (k+2)-ary class function of complexity \mathcal{E}_m for $m \ge n$. Then there is a total (k+1)-ary class function of complexity \mathcal{E}_m such that $\forall \vec{x} \forall y [F(\vec{x}, y) = G(\vec{x}, y, \langle F(\vec{x}, z) | z \in y \rangle)]$.

Corollary 5.10 (Class Inductive Definition Theorem for \mathcal{E}_n -definitions). Assume $n \ge 1$ and $m \le n$. Let Φ be an inductive definition of complexity \mathcal{E}_m . Then $\mathsf{Sl}\mathcal{E}_n$ proves there is a least Γ_{Φ} -closed class, whose complexity is \mathcal{E}_m .

6. Ordinals over $SI\mathcal{E}_1$

We will work over $Sl\mathcal{E}_1$ in this section unless specified. The aim of this section is to prove the following theorem:

Theorem 6.1. The following statements hold over $SI\mathcal{E}_1$:

- (1) $\operatorname{Ord} = \omega$,
- (2) There is a bijection from V to Ord.

We defined ordinals as transitive sets whose elements are also transitive, and Ord as the class of all ordinals. For example, every member of ω is an ordinal:

Lemma 6.2. Every member of ω is an ordinal.

Proof. The proof can be done by induction on n.

It is known that if every ordinal satisfies \in -least principle, then Δ_0 -LEM holds. We will show the converse: that is, we will prove that every ordinal satisfies \in -least principle. In fact, we will see that some classical properties of ordinals hold. The proof is not difficult, but we present every detail of it. The reader should bear in mind that $Sl\mathcal{E}_1$ proves the law of excluded middle for Δ_0 -formulas, so we may use it freely.

Lemma 6.3. If α is an ordinal and A is an inhabited subset of α , then A has an \in -minimal element.

Proof. Observe that the assertion 'There is an \in -minimal element $a \in A$ ' is a Δ_0 statement. Hence either A has an \in -minimal element or every element of A is not \in -minimal. Assume that the latter holds. We will prove $\forall a \in A (a \notin A)$ from the assumption by appealing to Set induction. Let a be a set such that $b \notin A$ for all $b \in a$. If $a \in A$, then a is an \in -minimal element of A, contradicting with the assumption on A. Therefore $a \notin A$. By Set induction, $a \notin A$ holds for all a. This contradicts with that A is inhabited.

Lemma 6.4. If α and β are an ordinals, then exactly one of $\alpha \in \beta$, $\alpha = \beta$ or $\alpha \ni \beta$ holds.

Proof. We will use set induction on α and β simultaneously. That is, assume that either $\gamma \in \delta$, $\gamma = \delta$ or $\gamma \ni \delta$ holds for all $\gamma \in \alpha$ and $\delta \in \beta$ if γ and δ are ordinals.

Assume that α and β are ordinals. By Δ_0 -excluded middle, we have $\alpha = \beta$ or $\alpha \neq \beta$. If the latter holds, then

(15)
$$\neg(\forall \gamma \in \alpha (\gamma \in \beta) \land \forall \gamma \in \beta (\gamma \in \alpha)),$$

so either $\alpha \setminus \beta$ or $\beta \setminus \gamma$ has an element. Without loss of generality assume that γ is an \in -minimal element of $\alpha \setminus \beta$. We want to show $\gamma = \beta$.

Assume the contrary that $\gamma \neq \beta$ holds. Then one of $\gamma \setminus \beta$ or $\beta \setminus \gamma$ is inhabited. If $\delta \in \gamma \setminus \beta$, then $\delta \in \alpha \setminus \beta$ since $\delta \in \gamma \in \alpha$, contradicting with the minimality of γ . If δ is an \in -minimal element of $\beta \setminus \gamma$, then either $\gamma = \delta$, $\gamma \in \delta$ or $\gamma \ni \delta$ by inductive hypothesis. The two former hypotheses implies $\gamma \in \beta$, which contradicts with $\gamma \notin \beta$. The latter one contradicts with $\delta \in \beta \setminus \gamma$. In total, we have a contradiction. Therefore $\neg(\gamma \neq \beta)$, so we have $\gamma = \beta$ due to Δ_0 -excluded middle.

Proof of Theorem 6.1. We first claim that $\operatorname{Ord} = \omega$ holds. We will show by induction on n that if there is an injection from n to an ordinal α , then $n \subseteq \alpha$. The case n = 0 is trivial. Let assume the inductive hypothesis holds for n and there is an injection $f: n+1 \to \alpha$. Since we have an injection from n to $\alpha, n \subseteq \alpha$. Moreover, there is $\beta \in \alpha$ which is different from members of n. Since both n and β are ordinals, we have $n \in \beta, n = \beta$ or $n \ni \beta$ holds. However, the latter case never happens, and the remaining two implies $n \in \alpha$.

We will use Σ and \mathfrak{p} functions defined by [8] to show $\operatorname{Ord} \cong V$. By Lemma 4.5, we can define $\hat{\Sigma}$: $\omega \times \mathcal{P}(\omega) \to \omega$ recursively as follows: $\hat{\Sigma}(0, x) = 0$ and

(16)
$$\hat{\Sigma}(c+1,x) = \begin{cases} \hat{\Sigma}(c,x), & \text{if } c+1 \notin x, \\ \hat{\Sigma}(c,x) + (c+1), & \text{if } c+1 \in x, \end{cases}$$

for all $c \in \omega$ and $x \in \mathcal{P}(\omega)$. Now take $\Sigma(x) = \hat{\Sigma}(\bigcup x, x)$. Finally, let

(17)
$$\mathfrak{p}(x) = \Sigma(\{2^{\mathfrak{p}(y)} \mid y \in x\}),$$

where the exponentiation is a natural number exponentiation. We can see that $\hat{\Sigma}$ is \mathcal{E}_1 , thus so does Σ by Lemma 5.7. Lemma 5.9 ensures \mathfrak{p} is well-defined and is \mathcal{E}_1 . Moreover, we can show that $\mathfrak{p} : V \to \omega$ is a bijection as [8] did: injectivity uses Set induction, and surjectivity uses induction on ω .

7. The interpretation

The main theorem of this paper is as follows:

Theorem 7.1. HA and \mathbb{T} are bi-interpretable.

We first define an interpretation from \mathbb{T} to HA. Work over HA, and define a primitive recursive binary relation E as follows:

(18)
$$a \to b \iff \exists r < 2^a \exists m[b = (2m+1) \cdot 2^a + r]$$

Intuitively, $a \ge b$ means the *a*th digit of the binary representation of *b* is 1 as classically did. Since *m* in the above formula satisfies m < b, $a \ge b$ is equivalent to a decidable formula.

Define an interpretation $\mathfrak{a}: T \to \mathsf{HA}$ as follows: the domain of the interpretation is the whole natural numbers. Take $(a \in b)^{\mathfrak{a}} \equiv (a \in b)$. We will show that every axiom of \mathbb{T} is valid under \mathfrak{a} . The proof for V =Fin takes more effort, so we will postpone the proof of its validity.

Theorem 7.2. If σ is an axiom of \mathbb{T} except for V = Fin, then $\mathsf{HA} \vdash \sigma^{\mathfrak{a}}$.

- Proof. (1) Extensionality: The interpreted Extensionality states the following: if two natural numbers have the same binary representation, then they are the same. Since the existence and uniqueness of binary representation only relies on division algorithm and induction, which are still valid over HA, the interpreted Extensionality is valid.
 - (2) Pairing: Consider the primitive recursive function pair(a, b) defined by $pair(a, a) = 2^a$, and $pair(a, b) = 2^a + 2^b$ if $a \neq b$. It is easy to see that if c = pair(a, b), then $(\{a,b\} = c)^a$ holds.
 - (3) Union: Consider the following primitive recursive functions: define binunion(a, b) as binunion(a, 0) = 0and

 $\mathsf{binunion}(a, 2^c + b') = \begin{cases} a & \text{if } b' = 0 \text{ and } c \to a, \\ a + 2^c & \text{if } b' = 0 \text{ and } \neg c \to a, \\ \mathsf{binunion}(\mathsf{binunion}(a, 2^c), \mathsf{binunion}(a, b')) & \text{otherwise.} \end{cases}$

for $b = 2^c + b'$, $b' < 2^c$. Now let union(0) = 0 and union(a) = binunion(c, union(a')) for $a = 2^c + a'$, $a' < 2^c$. Then we can see that union(pair(a, b)) = binunion(a, b) holds. Moreover, if c = union(a) then $(\bigcup a = c)^{\mathfrak{a}}$ holds.

(4) Binary Intersection: Define $\mathsf{bininter}(a, b)$ by primitive recursion on b as follows: $\mathsf{bininter}(a, 0) = 0$ and

(20)
$$\operatorname{bininter}(a, 2^{c} + b') = \begin{cases} 2^{c} & \text{if } b' = 0 \text{ and } c \to a, \\ 0 & \text{if } b' = 0 \text{ and } \neg c \to a, \\ \operatorname{binunion}(\operatorname{bininter}(a, 2^{c}), \operatorname{bininter}(a, b')) & \text{otherwise.} \end{cases}$$

for $b = 2^c + b'$ and $b' < 2^c$. Then $\mathsf{bininter}(a, b)$ witnesses the intersection of a and b.

(5) Set induction: It directly follows from the induction of HA and the fact $a \ge b \rightarrow a < b$ for all a and b.

The case for V = Fin needs some preparation. We need an interpreted version of various notions to describe the interpreted axiom, so we define them. The ordered pair op(a, b) of a and b is pair(pair(a, a), pair(a, b)). The von Neumann ordinal is defined recursively as follows: v(0) = 0 and

(21)
$$v(n+1) = \text{binunion}(v(n), 2^{v(n)}).$$

Lemma 7.3. If $(a \in \omega)^{\mathfrak{a}}$, then a = v(n) for some n.

Proof. We will use induction on a. If a = 0, then take n = 0. Assume that our theorem holds for all c < a, and a satisfies $(a \in \omega)^{\mathfrak{a}}$. Since a > 0, there is γ such that

(22)
$$\exists c [(c \in \operatorname{Ord})^{\mathfrak{a}} \land a = \mathsf{binunion}(c, 2^{c})]$$

We can see that $a = \text{binunion}(c, 2^c)$ implies c < a. By the inductive hypothesis, c = v(n) for some n. By definition of v, we have a = v(n + 1).

Theorem 7.4. HA proves $(V = Fin)^{\mathfrak{a}}$.

Proof. Note that the word 'function' in this proof means a binary relation with the definining condition of a function. Before to describe the proof, let me define a size $\sigma(a)$ of a natural number: $\sigma(0) = 0$ and $\sigma(a) = 1 + \sigma(a')$ if $a = 2^c + a'$ for some c < a and, $a' < 2^c$.

We will use induction on a. If a = 0, then 0 witnesses the bijection between 0 and v(0) = 0. Let $a = 2^c + a'$ for $a' < 2^c$. Assume inductively that we have a function f' from a' to $v(\sigma(a'))$, and the inverse g' of f'. We claim that the relation

(23)
$$f = \text{binunion}(f', 2^{(\text{op}(c, v(\sigma(a'))))})$$

is a function from a and $v(\sigma(a)) = v(\sigma(a') + 1)$ with the inverse function $g = \mathsf{binunion}(g', 2^{\mathsf{op}(v(\sigma(a')), c)})$

Since f' is a function of domain a' and $a' < 2^c$, the domain of f' does not contain c. Hence f is a function. It is obvious that the domain of g', namely $v(\sigma(a'))$, does not contain $v(\sigma(a'))$. It shows g is a function. It remains to show that f and g are inverses of each other, but it is clear from the properties of f' and g' and the definition of f and g.

In summary, we have

Corollary 7.5. \mathfrak{a} is an interpretation from \mathbb{T} to HA.

8. The inverse interpretation

We follow the inverse interpretation given by [8]: we first define the ordinal interpretation \mathfrak{o} from HA to \mathbb{T} , and compose it with \mathfrak{p} .

Definition 8.1. The ordinal interpretation \mathfrak{o} is defined as follows: $0^{\mathfrak{o}}$ is the empty set, $S^{\mathfrak{o}}(x) = x \cup \{x\}$. Interpretation of addition and multiplication employs the corresponding operation on ordinals.

Then we have

Theorem 8.2. \mathfrak{o} is an interpretation from HA to \mathbb{T} .

Proof. The only remaining axiom we need to check its validity is the induction scheme, and it follows from induction on ω .

 \mathfrak{o} is not an inverse interpretation of \mathfrak{a} because of the interpretation of quantifiers. The formula $\forall x \phi(x)$ over HA is interpreted into $\forall x \in \omega \phi^{\mathfrak{o}}(x)$ under \mathfrak{o} , and its interpretation under \mathfrak{a} is $\forall x[(x \in \omega)^{\mathfrak{a}} \to \phi^{\mathfrak{ao}}(x)]$, which is not equivalent to the original formula even if $\phi(x)$ is atomic. Kaye and Wong resolved this problem by relying on the bijection $\mathfrak{p}: V \cong \omega$.

Definition 8.3. The interpretation \mathfrak{b} is defined as follows: the domain of the interpretation is x = x. If $t(\vec{x})$ is a term of HA whose variables are all expressed, then $t^{\mathfrak{b}}$ is defined as $t^{\mathfrak{o}}(\mathfrak{p}(\vec{x}))$.

Then \mathfrak{b} is an interpretation from HA to \mathbb{T} . Moreover, \mathfrak{a} and \mathfrak{b} are inverses of each other:

Theorem 8.4. $\mathfrak{ab} = 1_{HA}$ and $\mathfrak{ba} = 1_{\mathbb{T}}$.

Proof. Both \mathfrak{ab} and \mathfrak{ba} respects equality and domain, so it suffices to see that both interpretations preserve atomic symbols.

To prove \mathfrak{ba} is the identity, it is sufficient to see that $(x \in y)^{\mathfrak{ba}}$ is equivalent to $x \in y$. Note that $(x \in y)^{\mathfrak{ba}}$ is equivalent to $(x \in y)^{\mathfrak{b}}$, in other words,

(24)
$$\exists r < 2^{\mathfrak{p}(x)} \exists m \in \omega[\mathfrak{p}(y) = (2m+1)2^{\mathfrak{p}(x)} + r]$$

The above formulation $(x \in y)^{\mathfrak{ba}}$ is not a Δ_0 -formula for the following reasons: First, we do not know ω is a set, and it could be a proper class. Second, the definition of \mathfrak{p} is not Δ_0 . Despite that, $(x \in y)^{\mathfrak{ba}}$ is a decidable formula by the fact that $x \in y$ is equivalent to a decidable formula and Lemma 4.13.

Let $y = \{z_k \mid k < n\}$ be an enumeration of y such that $\mathfrak{p}(z_0) > \cdots > \mathfrak{p}(z_{n-1})$. We can see that for each k < n, there are m and $r < \mathfrak{p}(z_k)$ such that $\mathfrak{p}(y) = (2m+1)\mathfrak{p}(z_k) + r$ by induction on k and Euclidean division algorithm. If $x \in y$, then $x = z_k$ for some k, thus it satisfies (24). If $x \notin y$, then $\mathfrak{p}(x)$ is equal to none of the $\mathfrak{p}(z_k)$. By dividing cases, we can see that $\mathfrak{p}(y) = 2m2^{\mathfrak{p}(x)} + r$ for some m and $r < 2^{\mathfrak{p}(x)}$. By uniqueness of the remainder and quotient, we have the negation of (24). Since both $x \in y$ and (24) are decidable, we have the equivalence of these two formulas.

It remains to show that \mathfrak{ab} is the identity. It requires a sequel of lemmas on interpreted notions of \mathbb{T} . We can see that $(a \in \omega)^{\mathfrak{a}}$ if and only if a = v(n) for some n by Lemma 7.3 and an easy inductive argument. Moreover, we can show the following fact by induction on y:

Lemma 8.5. For each x and y, we have
$$(S^{\mathfrak{a}}(v(y)) = v(y+1), v(x) + {}^{\mathfrak{a}}v(y) = v(x+y), v(x) \cdot {}^{\mathfrak{a}}v(y) = v(x \cdot y),$$

and $(v(x)^{v(y)})^{\mathfrak{a}} = v(x^y).$

Here the functions under \mathfrak{a} are set-theoretic functions for von Neumann ordinals, and the functions appearing in the argument of v are functions of the language of HA. From this lemma, we have

Lemma 8.6.
$$\mathfrak{p}^{\mathfrak{a}}(x) = v(x)$$
; that is, $(\mathfrak{p}(x) = y)^{\mathfrak{a}}$ if and only if $v(x) = y$.

The proof uses induction on x: assume that the desired equality holds for all y < x. Let $x = 2^c + x'$ for $x < 2^c$. Then $2^c + x' = \text{binunion}(2^c, x')$, so $(x = \{c\} \cup x')^{\mathfrak{a}}$. Hence $(\mathfrak{p}(x) = \mathfrak{p}(x' \cup \{c\}) = 2^{v(c)} + v(x'))^{\mathfrak{a}}$, and the previous lemma ensures $\mathfrak{p}^{\mathfrak{a}}(x) = v(x)$.

It remains to check that function symbols S, + and \cdot are preserved under \mathfrak{ab} . We will only see the proof for S, as the remaining cases are analogous: $(S(x) = y)^{\mathfrak{ab}}$ if and only if $(\mathfrak{p}(x) \cup \{\mathfrak{p}(x)\} = \mathfrak{p}(y))^{\mathfrak{a}}$, which turns out to be equivalent to $\mathfrak{binunion}(v(x), 2^{v(x)}) = v(y)$. Hence v(S(x)) = v(y), which is equivalent to S(x) = y.

We will conclude this section by correcting a result of Aczel. Aczel stated in Section 11 of [2] that the ordinal interpretation $\mathfrak{o} \colon \mathsf{HA} \to T$ is a faithful interpretation if T is a subtheory of $\mathsf{IZF}^{\mathsf{fin}}$ that contains ACST. We know that $\mathsf{IZF}^{\mathsf{fin}}$ is a classical theory, so it proves the interpreted version of semi-classical principles like WLEM. However, we know that HA does not prove WLEM. Hence there is no faithful interpretation from HA to $\mathsf{IZF}^{\mathsf{fin}}$. The result holds, fortunately, if we correct $\mathsf{IZF}^{\mathsf{fin}}$ to $\mathsf{CZF}^{\mathsf{fin}}$:

Proposition 8.7. Let T be a subtheory of $\mathsf{CZF}^{\mathsf{fin}}$ that contains ACST as a subtheory. Then $\mathfrak{o} \colon \mathsf{HA} \to T$ is a faithful interpretation.

Proof. We can see that ACST is capable of defining \mathfrak{o} , hence $\mathfrak{o} \colon \mathsf{HA} \to T$ is an interpretation. We will see that the composition $\mathfrak{ao} \colon \mathsf{HA} \to \mathsf{HA}$ is faithful: Assume that $\mathsf{HA} \vdash \phi^{\mathfrak{ao}}$. By applying inverse interpretation \mathfrak{b} , we have $\mathsf{CZF}^{\mathsf{fin}} \vdash \phi^{\mathfrak{o}}$. We can see that every quantifier of $\phi^{\mathfrak{o}}$ is of the form $\forall x \in \omega$ or $\exists x \in \omega$. We can see that the formula is still provable if we replace every variable x to $\mathfrak{p}(x)$ and omit $\in \omega$ from quantifiers since \mathfrak{p} is a definable bijection between V to ω . Hence $\mathsf{CZF}^{\mathsf{fin}} \vdash \phi^{\mathfrak{b}}$, and we have $\mathsf{HA} \vdash \phi$. The main result follows directly from the previous argument.

9. INTERPRETATING SUBTHEORIES OF HA

Kaye and Wong gave not only a bi-interpretation between PA and its set-theoretic counterpart, but also a bi-interpretation between their subtheories. It can be given by asserting the previous proof works over a subtheory of PA and finitary set theory. We can do a similar work over HA and \mathbb{T} under the \mathcal{E}_n -hierarchy of formulas.

Let us analyze the definitional complexity of \mathfrak{a} . As [8] pointed out, the definition of \mathfrak{a} only involves bounded formulas except for exponentiation function. However, the exponentiation function is definable by a \mathcal{E}_1 -formula with the help of \mathcal{E}_1 -induction. In fact, we can show the following general theorem: **Proposition 9.1.** Every primitive recursive function is definable by an \mathcal{E}_1 -formula over $|\mathcal{E}_1$.

The proof of Proposition 9.1 is the same with the classical counterpart of the theorem: namely, every primitive recursive function is Σ_1 -definable over $|\Sigma_1$. As a special case of Proposition 9.1, we can see $a \ge b$ is definable by a \mathcal{E}_1 -formula. Hence \mathfrak{a} sends each \mathcal{E}_n (or \mathcal{U}_n) formulas of set theory to \mathcal{E}_n (or \mathcal{U}_n) formulas of arithmetic. From this, we can infer the following theorem:

Theorem 9.2. Let $n \ge 1$. Then \mathfrak{a} is an interpretation from $\mathsf{Sl}\mathcal{E}_n$ to $\mathsf{l}\mathcal{E}_n$.

The case of \mathfrak{b} involves with the complexity of \mathfrak{o} and \mathfrak{p} . We can see that the definitional complexity of Σ , \mathfrak{p} and primitive recursive functions are \mathcal{E}_1 by Lemma 5.7 and Lemma 5.9. As the definition of \mathfrak{b} employs \mathfrak{p} , \mathfrak{b} could increase the complexity of formulas. Fortunately, we can see that \mathcal{E}_n -classes are stable under this substitution since $\phi(F(x), y)$ is equivalent to $\exists z(z = F(x) \land \phi(z, y))$:

Lemma 9.3. (BCST) Let $\phi(x, \vec{y})$ be an \mathcal{E}_n -formula for $n \ge 1$ and F(x) be a \mathcal{E}_1 -class function. Then $\phi(F(x), \vec{y})$ is an \mathcal{E}_n -formula.

Hence we have the following theorem:

Theorem 9.4. \mathfrak{b} is an interpretation from $|\mathcal{E}_n|$ to $S|\mathcal{E}_n$.

We can see that the proof of Theorem 8.4 works over $|\mathcal{E}_n|$ and $S|\mathcal{E}_n$. Therefore, we have the following:

Corollary 9.5. Let $n \ge 1$. Then the interpretations $\mathfrak{a}: Sl\mathcal{E}_n \to l\mathcal{E}_n$ and $\mathfrak{b}: l\mathcal{E}_n \to Sl\mathcal{E}_n$ are inverses of each others.

We will finish this section by showing that the previous results are exactly the constructive counterpart of [8]. Kaye and Wong proved that \mathfrak{a} and \mathfrak{b} is bi-interpretations between $I\Sigma_n$ and Σ_n -Sep. It is easy to see that $I\mathcal{E}_n$ with the full law of excluded middle is $I\Sigma_n$. The following theorem shows $SI\mathcal{E}_n$ is the constructive counterpart of Σ_n -Sep:

Theorem 9.6. Let $n \geq 1$. Then $Sl\mathcal{E}_n$ with the full law of excluded middle is identical with Σ_n -Sep.

Proof. Let $Sl\mathcal{E}_n^c$ be $Sl\mathcal{E}_n$ with the full law of excluded middle. Since we have the full law of excluded middle, \mathcal{E}_n and \mathcal{U}_n -classes coincide with Σ_n and Π_n -classes respectively. We will divide the proof into two parts:

Lemma 9.7. SI $\mathcal{E}_n^c \vdash \Sigma_n$ -Sep.

Proof. Note that $\mathsf{Sl}\mathcal{E}_n^c \vdash \Sigma_n$ -Strong Collection by the proof of Proposition 4.9. We can see that Σ_n -Strong Collection with the full law of excluded middle proves Σ_n -Separation. Proving Δ_0 -Collection and \neg Infinity from $\mathsf{Sl}\mathcal{E}_n^c$ are easy. In sum, we have shown that $\mathsf{Sl}\mathcal{E}_n^c$ proves every axioms of $\mathsf{I}\Sigma_n$ except for $\Sigma_1 \cup \Pi_1$ -Induction.

Proving the induction scheme is trivial if $n \geq 2$, since every Π_1 -formula is Σ_2 . However, there is another way to prove Π_1 -induction from the remaining axioms, which works even if we only have \mathcal{E}_1 -induction: Let $\phi(x)$ be a Π_1 -formula. Assume the contrary that $\phi(x)$ does not satisfy set induction, in other words, $(\forall y \in x\phi(y)) \rightarrow \phi(x)$ holds for all x, but $\neg \phi(a)$ also holds for some a. Let n be a cardinality of $\mathrm{TC}(a)$, and consider the set

(25)
$$X = \{ m \in n^+ \mid \exists x [\neg \phi(x) \land \exists f \colon m \to \mathrm{TC}(x) \text{ such that } f \text{ is a bijection.}] \}$$

 $(\Sigma_1$ -Separation is needed to define X.) Then X is an inhabited set of ordinals, so it has a minimal element m. Let x be a set such that $\neg \phi(x)$ and the cardinality of $\operatorname{TC}(x)$ is m. For each $y \in \operatorname{TC}(x)$, we have $|\operatorname{TC}(y)| < |\operatorname{TC}(x)|$, so $\phi(y)$. Therefore $\phi(y)$ for all $y \in x$, and it implies $\phi(x)$, a contradiction.

Lemma 9.8. Σ_n -Sep \vdash Sl \mathcal{E}_n^c .

Proof. Observe that Δ_0 -Set Induction proves the axiom of Regularity. We will see that Σ_n -induction scheme holds over $I\Sigma_n$.

Assume that $\phi(x)$ is a Σ_n -formula that violates Set Induction, so that $\forall y \in x\phi(x) \to \phi(x)$ for all x, but there is a such that $\neg \phi(a)$. Now consider the set $X = \{x \in a \mid \neg \phi(x)\}$. If X is empty, then $\phi(a)$. If Xis not empty, choose an \in -minimal element x of X, which would satisfy $\phi(x)$. In either case, we have a contradiction. Therefore $\phi(x)$ follows Set Induction scheme.

Finally, the negation of Infinity implies V = Fin classically: Assume that $V \neq$ Fin holds. Let *a* be a set that is not bijectable with any $n \in \omega$. Since we have Σ_1 -Set Induction, we can apply Lemma 5.9 to define rank function as follows:

(26)
$$\operatorname{rank} x = \bigcup \{\operatorname{rank} y + 1 \mid y \in x\}.$$

Note that we proved Lemma 5.9 over $Sl\mathcal{E}_1$, so there is a possibility that we are using V = Fin in the proof of Lemma 5.9. However, we do not need V = Fin in this proof of Lemma 4.10 and Lemma 5.9 if we have Σ_1 -Collection, which is provable from Σ_1 -Sep.

We can show that if every element of x has a rank smaller than n, then x is finite: it follows from that V_n exists for each natural number n, which is a theorem of Σ_1 -Sep. Hence the set $X = \{ \operatorname{rank} y \mid y \in a \}$ is infinite. Since $\omega \subseteq \bigcup X$, ω is a set. Therefore, we have the axiom of Infinity. \Box

Combining these two lemmas, we have Σ_n -Sep $\vdash \dashv Sl\mathcal{E}_n^c$

Note that the only properties of \mathcal{E}_n used in this section for proving \mathfrak{a} and \mathfrak{b} are well-defined and biinterpretations of each other are that \mathcal{E}_n contains bounded formulas and $\mathcal{E}_n = \mathcal{E}(\mathcal{E}_n)$. Thus we can extend our argument to any class of formulas with certain conditions. We state it without proof:

Proposition 9.9. Let Γ and Γ' be collection of formulas over set theory and arithmetic respectively, such that both of Γ and Γ' contains Δ_0 . Assume that Γ and Γ' satisfies $\mathcal{E}(\Gamma) = \Gamma$ and $\mathcal{E}(\Gamma') = \Gamma'$. Furthermore, assume that \mathfrak{a} sends Γ -formulas to Γ' -formulas and vice versa for \mathfrak{b} .

Then \mathfrak{a} is a bi-interpretation between SI Γ , a theory obtained by restricting Set Induction in \mathbb{T} to Γ -formulas, and I Γ' , which is obtained by restricting induction scheme of HA to Γ' -formulas.

10. A natural model of $\mathsf{CZF}^{\mathsf{fin}}$: the set of hereditarily finite sets

In this section, we will work over CZF unless stated otherwise.

ZF proves that the set of sets of all finite rank V_{ω} is a model of ZF^{fin}. Moreover, we may regard V_{ω} as a natural model of ZF^{fin} since it is countable and every set given by Ackermann's intepretation falls into V_{ω} . We may ask CZF can prove the existence of a natural model of CZF^{fin}. One candidate is V_{ω} , but we will not consider it for the following reasons: first, V_{ω} could not be a set in CZF. (In fact, even the power set of 1 need not be a set in CZF.) Second, we cannot ensure V_{ω} need not be countable even if we assume the axiom of power set. In fact, McCarty showed that $V[\mathcal{K}_0]$, the model given by Kleene realizability satisfies IZF with ' $\mathcal{P}(1)$ is not subcountable.' (See Corollary 3.8.3 of [9].) Aczel [2] also introduced the class of hereditarily finite sets HF, and provide some properties of it. We have to add some remarks on finiteness and related concepts over CZF before we can say about what HF is.

Finite sets are sets that have a bijection with a von Neumann natural number. However, this notion of finite sets over constructive set theory is not well-behaved unlike classical finite sets. For example, a subset of a finite set need not be finite. It does not mean we have to take another definition of finite sets. Instead, we divide notions of finiteness:

Definition 10.1. A set x is *finite* if there is a bijection from $n \in \omega$ to x. x is *finitely enumerable* if there is a surjection from $n \in \omega$ to x. x is *subfinite* if x is a subset of a finite set.

The following condition has a essential role to characterize which subsets of a finite is again finite:

Definition 10.2. Let A be a set. A subset $B \subseteq A$ is *decidable* if $x \in B \lor x \notin B$ for all $x \in A$. A is *discrete* if the equality relation is a decidable subset of $A \times A$.

Proposition 10.3. (Proposition 8.1.11 of [3], CZF) A set x is finite if and only if x is finitely enumerable and discrete. \Box

Lemma 10.4. Every decidable subset of a finite set is finite.

Proof. It suffices to show that every decidable subset of $n \in \omega$ is finite. We can show by induction on n that if $\phi(m)$ is decidable on n then either $\exists m \in n\phi(m)$ or $\forall m \in n\neg\phi(m)$. Hence if $x \subseteq n$ is decidable, then x is empty or inhabited. If x is inhabited and $m_0 \in x$, then the function

(27)
$$f(m) = \begin{cases} m & \text{if } m \in x, \\ m_0 & \text{otherwise} \end{cases}$$

enumerates elements of x. Hence x is finitely enumerable. Since $x \subseteq n$ is discrete, x is finite.

Definition 10.5. Let Φ be an inductive definition given as $\Phi_{\text{fin}} := \{(\{a\}, a) \mid a \text{ is finite}\}$. Then HF is the least $\Gamma_{\Phi_{\text{fin}}}$ -closed class.

Accel showed that we have the same HF if we replace Φ_{fin} to $\Phi_{\text{f.e.}} = \{(\{a\}, a) \mid a \text{ is finitely enumerable}\}$ or $\Phi_{\text{adj}} = \{(\{a, b\}, a \cup \{b\}) \mid a, b \in V\}$. Moreover, Accel showed the following facts:

Proposition 10.6. (1) HF is transitive.

- (2) If $\forall x \in \mathrm{HF}[(\forall y \in x\phi(y)) \to \phi(x)]$, then $\forall x \in \mathrm{HF}\phi(x)$.
- (3) If $\forall x, y \in \mathrm{HF}[(\forall u \in x \forall v \in y \phi(u, v)) \to \phi(x, y)]$, then $\forall x, y \in \mathrm{HF}\phi(x, y)$.

Proposition 10.7. = and \in is discrete over HF, that is, $\forall x, y \in \text{HF}(x = y \lor x \neq y)$ and $\forall x, y \in \text{HF}(x \in y \lor x \notin y)$.

See Proposition 10.9 and 10.10 of [2] for the proof.

We may ask whether HF is a set, as ZF proves V_{ω} is a set. The answer is affirmative, and we will prove it by constructing a hierarchy of HF:

Definition 10.8. Let Dec(A) be a set of all decidable subsets of A:

(28)
$$\operatorname{Dec}(A) = \{ B \subseteq A \mid \forall x \in A [x \in B \lor x \notin B] \}$$

Define D_n recursively as follows: $D_0 = \emptyset$ and $D_{n+1} = \text{Dec}(D_n)$.

Lemma 10.9. $\langle D_n \mid n \in \omega \rangle$ is a strictly increasing sequence of transitive discrete sets of HF.

Proof. It is easy to see that $D_n \subsetneq D_{n+1}$ by induction on n. For transitivity, observe that $x \in D_{n+1}$ implies $x \subseteq D_n \subseteq D_{n+1}$. Moreover, Dec(X) and X_2 have the same cardinality, so we can see each D_n is finite.

For discreteness of D_n , assume that $x, y \in D_{n+1} = \text{Dec}(D_n)$. Then the formula $z \in x$ and $z \in y$ is decidable over D_n . Hence the formula $z \in x \leftrightarrow z \in y$ is also decidable over D_n . Since D_n is finite, we have

(29)
$$\exists z \in D_n \neg (z \in x \leftrightarrow z \in y) \lor \neg [\exists z \in D_n \neg (z \in x \leftrightarrow z \in y)],$$

which implies $x \neq y \lor x = y$.

It remains to show that $D_n \in \text{HF}$. It follows from finiteness of D_n and $D_n \subseteq \text{HF}$ that will be shown by induction on n: if $D_n \subseteq \text{HF}$ and $x \in D_{n+1}$, then $x \subseteq \text{HF}$ is finite by Lemma 10.4. Hence $x \in \text{HF}$ and we have $D_{n+1} \subseteq \text{HF}$.

Theorem 10.10. HF = $\bigcup_{n \in \omega} D_n$. Especially, HF is a set provided if ω is a set.

Proof. Since $D_n \in \text{HF}$ for all n and HF is transitive, we have $\bigcup_{n \in \omega} D_n \subseteq \text{HF}$. We will show that $\forall x \in \text{HF}(x \in \bigcup_{n \in \omega} D_n)$ by applying Proposition 10.6 to obtain the remaining inclusion. Assume that $x \subseteq \bigcup_{n \in \omega} D_n$ holds. Since x is finite, we can find n such that $x \subseteq D_n$. Choose a bijection $f: m \to x$ for some $m \in \omega$. We can see that for $y \in D_n$, $y \in x$ if and only if $\exists k < m(f(k) = y)$, which is a decidable formula since the quantifier is bounded by a natural number and D_n is discrete. Hence $x \in \text{Dec}(D_n) = D_{n+1}$.

In classical world, V_{ω} is a model of finitary ZF, which is bi-interpretable with PA. Since CZF^{fin} is biinterpretable with HA and the classical V_{ω} is the set of all hereditarily finite sets, we may ask HF is a model of finitary CZF. The following theorem shows the answer is affirmative:

Theorem 10.11. HF satisfies CZF^{fin}, that is, if σ is an axiom of CZF^{fin}, then the relativization σ^{HF} of σ holds.

Proof. It suffices to show that σ^{HF} holds for all axioms of T. Extensionality and Set Induction follow from Proposition 10.6. Moreover, HF is closed under the operation $x, y \mapsto x \cup \{y\}$, and this proves HF satisfies Pairing and Binary union. (For Binary union, we need to use the induction on the size of sets.) Since every $x \in \text{HF}$ is finite, we can see $\bigcup x \in \text{HF}$ by induction on the size of x. It shows HF satisfies Union.

For Binary Intersection, let $x, y \in HF$. Take $n \in \omega$ such that $x, y \in D_n$. By Proposition 10.7, both $z \in x$ and $z \in y$ is decidable over HF. Hence the set $x \cap y = \{z \in D_n \mid z \in x \land z \in y\}$ is also decidable, so $x \cap y \in D_{n+1}$.

It remains to show that V = Fin is valid in HF. We know that for each $x \in \text{HF}$ there is $n \in \omega$ and a bijection $f: n \to x$. We can see that f is a finite set and $f \subseteq \text{HF}$. Hence $f \in \text{HF}$, and f witnesses finiteness of x in HF.

We may further ask HF satisfies finitary IZF , which is identical with $\mathsf{ZF}^{\mathsf{fin}}$. We will see that the answer is negative in general.

Theorem 10.12. Working over $V[\mathcal{K}_0]$, the model of Kleene realizability, HF does not satisfy the full law of excluded middle.

Proof. It is known by [9] and [14] that $V[\mathcal{K}_0]$ satisfies *Church's thesis* CT_0 . By Proposition 4.3.4 of [17], CT_0 implies the following instance of the negation of weak excluded middle WLEM holds:

(30)
$$\neg \forall x \in \omega [\neg \exists y \in \omega T(x, x, y) \lor \neg \neg \exists y \in \omega T(x, x, y)],$$

where T is Kleene's T-predicate, which has a \mathcal{E}_1 -definition over ω . Observe that the function \mathfrak{p} defined in Section 6 yields a definable bijection from ω to HF, which is accessible inside HF. Replacing all x and y of (30) to $\mathfrak{p}^{-1}(x)$ and $\mathfrak{p}^{-1}(y)$ provides the formula of the form

(31)
$$\neg \forall x \in \mathrm{HF}[\neg \exists y \in \mathrm{HF}\phi(x,y) \lor \neg \neg \exists y \in \mathrm{HF}\phi(x,y)],$$

where $\phi(x, y)$ is a formula whose quantifiers inside this formula is bounded by HF. Hence we may regard $\phi(x, y)$ as a relativization $\psi^{\text{HF}}(x, y)$ of some formula $\psi(x, y)$. Therefore, HF satisfies an instance of the negation of WLEM.

11. Remarks and Questions

We will finish this article with a philosophical remark and some questions. We pointed out that \mathbb{T} is bi-interpretable with HA, and the bi-interpretation also captures bi-interpretability between subtheories of \mathbb{T} and HA. Moreover, \mathbb{T} is CZF^{fin}, and the set of all hereditarily finite sets HF is a model of \mathbb{T} . On the other hand, the finitary IZF is just ZF^{fin}, which is not bi-interpretable with HA. Moreover, Theorem 10.12 shows HF may not be a model of finitary IZF, even though the background universe satisfies IZF: In Theorem 10.12, the background universe is $V[\mathcal{K}_0]$. If V satisfies IZF, then so does $V[\mathcal{K}_0]$. These two facts could bolster the viewpoint that CZF is a more natural constructive counterpart of ZF than IZF.

Classically, the negation of the axiom of Infinity proves V = Fin. We do not know whether this is possible constructively. Instead, we postulate V = Fin as an axiom. It is natural to ask whether the negation of Infinity proves V = Fin. Since \mathbb{T} is an extension of Tharp's quasi-intuitionistic set theory [16] without the axiom of Infinity, and it contains the principle Ord-Im that states every set is an image of an ordinal. Since V = Fin proves Ord-Im, we may also ask the question whether we can obtain an implication under Ord-Im:

Question 11.1. Does the negation of the axiom of Infinity prove V = Fin? Can we prove it with Ord-Im?

We gave a bi-interpretation between $I\mathcal{E}_n$ and $SI\mathcal{E}_n$, which are subtheories of HA and \mathbb{T} respectively. Unfortunately, we do not know the set-theoretic counterpart of constructive $I\Delta_0 + Exp$. Pettigrew [11] characterized the set-theoretic equivalent of $I\Delta_0 + Exp$, which is derived from Mayberry's set theory called EA. Unfortunately, the auther does not know how to characterize the constructive counterpart of Pettigrew's set theory, hence the following question is still open:

Question 11.2. Can we identify a constructive set theory that is bi-interpretable with constructive $|\Delta_0 + \mathsf{Exp}\rangle$?

The author only noticed after finishing the draft that McCarty and Shapiro had planned to gave an online talk on the Logic Supergroup (the detail will appear in [10]), and they independently achieved some of the author's result. The author was aware of their talk by chance on 21 September, only five days before their talk. The author contacted McCarty to inform the author's research, and he kindly responded with their slide, the main source the author can check the detail of their research. They worked with Heyting arithmetic with symbols for primitive recursion functions and another constructive set theory called SST, which comprises Extensionality, the existence of adjunction $x + y := x \cup \{y\}$, and adjunction induction

$$[\phi(0) \land \forall x \forall y (y \notin x \land \phi(x) \land \phi(y) \to \phi(x+y))] \to \forall x \phi(x)$$

what they called Set Induction. They proved that the expanded HA is bi-interpretable, or *definitionally* equivalent according to McCarty and Shapiro, with SST expanded by adding function symbols for primitive recursive functions. They also present a variant of Heyting arithmetic called HA_{BIT} , and showed that it is bi-interpretable with SST and the extended HA. Therefore, the extended HA and SST are bi-interpretable with each other.

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Moreover, we can show that SST is identical with CZF^{fin}. The hardest part is deriving all axioms of CZF^{fin} with Collection instead of Strong Collection from SST, and this is done by [10]. Hence we can see the author's bi-interpretability result for HA coincides with that of McCarty and Shapiro up to the difference of the language and choice of axioms of theories.

The reviewer pointed out that Rathjen [15] provided a fragment of CZF, which has the same proof-theoretic strength with HA. Rathjen called this theory CZF⁻, and is obtained by discarding Set Induction from CZF and employing Strong Infinity instead of usual Infinity. Especially, CZF⁻ proves Mathematical Induction Axiom Scheme for Δ_0 -formulas.

Rathjen proved in [15] that CZF^- is Π_2^0 -conservative over HA, and he used a mixture of type-theoretic interpretation of set theory and realizability interpretation of type theory over a saturated model of PA. Rathjen also claimed that one can establish a similar synthetic translation into the theory PA_{Ω}^r , which is a conservative extension of PA. However, Rathjen's translation of CZF^- to PA_{Ω}^r is not an interpretation in our sense because his translation remolds quantifiers and logical connectives. Thus we have the following question:

Question 11.3. Is there an interpretation from CZF^- to HA (or equivalently, CZF_{fin})?

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