

INDUCED AND HIGHER-DIMENSIONAL STABLE INDEPENDENCE

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ABSTRACT. We provide several crucial technical extensions of the theory of stable independence notions in accessible categories. In particular, we describe circumstances under which a stable independence notion can be transferred from a subcategory to a category as a whole, and examine a number of applications to categories of groups and modules, extending results of [MA21a]. We prove, too, that under the hypotheses of [LRV], a stable independence notion immediately yields higher-dimensional independence as in [SV].

1. INTRODUCTION

We here concern ourselves with stable independence in the context of accessible categories. This notion has its origins in the model-theoretic concept of stable nonforking, which can be thought of on one hand as a freeness property of type extensions and, on the other, as a notion of freeness or independence for amalgams of models. The latter perspective, taken to its logical conclusion, leads to a formulation of stable independence as a property of commutative squares in a general category, described by a family of purely category-theoretic axioms, cf. [LRV19a]. This generalization is of practical value: often in mathematics we begin with a nice category of objects \mathcal{K} , then restrict to a particular family of desirable morphisms \mathcal{M} , obtaining a subcategory, $\mathcal{K}_{\mathcal{M}}$, which loses much of the useful structure of \mathcal{K} . Even if \mathcal{K} is locally finitely presentable (say, $\mathcal{K} = \mathbf{Ab}$, the category of abelian groups and homomorphisms), if we take \mathcal{M} to be a family of monomorphisms (say, \mathcal{M} consists of the pure monomorphisms in \mathbf{Ab}), the category $\mathcal{K}_{\mathcal{M}}$ will no longer have the pushouts available to us in \mathcal{K} . A central theme of [LRV] is that stable independence—or, rather, stably independent squares—provide a workable alternative to the missing pushouts, sufficient for many applications.

Of greater importance, perhaps, is [LRV, 3.1], which asserts, roughly speaking, that given a pair $(\mathcal{K}, \mathcal{M})$, \mathcal{K} cocomplete, $\mathcal{K}_{\mathcal{M}}$ admits a stable independence notion just in case the family of morphisms \mathcal{M} is cofibrantly generated; that is, generated by pushouts, transfinite compositions and retracts from a *set* (as opposed to a proper class) of morphisms. In this case, the stable independence notion must be precisely the one given by \mathcal{M} -effective squares, generalizing the effective unions

Date: August 31, 2022.

2020 *Mathematics Subject Classification.* Primary: 03C45. Secondary: 03C48, 03C52, 13C60, 18C35.

Key words and phrases. Stable independence, accessible categories, classification theory, categories of modules.

The second author is supported by the Grant agency of the Czech Republic under the grant 19-00902S.

of [Bar88]. This provides a useful link between stable independence and, for example, (combinatorial, i.e. cofibrantly generated) cellular categories, which occur naturally in topology and homological algebra. The translation it offers has already proved fruitful: arguing by way of stable independence, one can give a very brief proof of (a special case of) the fact that combinatorial structures are left-induced, [LRV, 3.11]; the original proof of this fact, in [MR14], requires a great deal of heavy category-theoretic machinery, including the *good colimits* of Lurie (see [MRV14]). On the other hand, an analysis of questions involving stable independence by way of cofibrant generation often leads to more efficient proofs, and useful new results, particularly in algebra. For example, [LRV, 4.3] resolves an open question concerning the circumstances under which Ext-orthogonality classes of modules admit a stable independence notion, and, while the proof of the cofibrant generation of pure monomorphisms in locally presentable additive categories in [LPRV20] does not use stable independence, it would not have been evident were it not for this connection.

We pursue further applications here, first deriving stable independence for a host of algebraic categories. For this, we require an essential technical lemma. A crucial part of the definition of a stable independence relation on a category \mathcal{K} is that it be accessible (Definition 2.2(2)), meaning, roughly speaking, that the independent squares form an accessible subcategory of the arrow category, \mathcal{K}^2 : without this property, we say the independence relation is *weakly stable*. Accessibility, which corresponds loosely to the local character of nonforking, is neither particularly natural nor easy to verify in practice. For a *continuous* independence notion, on the other hand, we require only that the appropriate subcategory of \mathcal{K}^2 be closed under directed colimits—this is far more common, as we will see in Section 3. The essential result of Section 2 is that, in many cases, we can infer the existence of a stable independence notion from that of a continuous weakly stable independence notion (Theorem 2.7). In particular, if an accessible category \mathcal{K} has a continuous weakly stable independence notion, and has a stable independence notion on a sufficiently nicely embedded subcategory, then in fact the independence notion on \mathcal{K} must be stable. Thanks to recent work on stability—in the sense of Galois types—in categories of groups and modules (e.g. [KMA20], [MA21a]), we obtain continuous weakly stable independence notions for free in, for example, Abelian p -groups with pure embeddings, or torsion R -modules with pure embeddings. A model-theoretic argument shows that these weakly stable notions restrict to stable independence on subcategories of sufficiently saturated (that is, universal injective) objects, and Theorem 2.7 allows us to lift this stability to the categories themselves. The results are summarised in Theorems 3.8 and 3.9.

Finally, we show that in the setting of [LRV], the existence of a stable independence notion—essentially, a family of nice commutative squares—implies the existence of higher-dimensional stable independence: nice families of cubes, hypercubes, etc. We note that higher-dimensional amalgams of this form have played an important role in the analysis of categoricity in $L_{\omega_1, \omega}$ ([She83a] and [She83b], where they were first introduced) and, more recently, in connection with the categoricity conjecture for abstract elementary classes ([SV]). In the aforementioned cases, it is a significant technical challenge to ensure that the existence of such amalgams

in low dimensions can be pushed to higher dimensions ([SV] needs the weak generalized continuum hypothesis, for example). As our underlying category is locally presentable—hence has arbitrary pushouts—existence is more or less automatic.

We assume a familiarity with accessible categories ([AR94], [MP89]). A passing acquaintance with the category-theoretic formulation of stable independence relations ([LRV19a], [LRV]) would be very useful in providing motivation for the discussion that follows.

We are grateful to Marcos Mazari-Armida for his detailed feedback on an early draft of this paper. We are grateful, too, to the anonymous referee for comments that have led to several important clarifications.

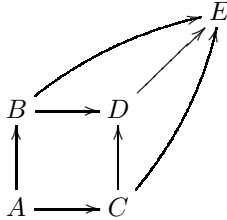
2. LIFTING STABLE INDEPENDENCE

While a much fuller picture can be given of the transfer of stable independence notions along adjunctions, we here concern ourselves with a very limited special case. In particular, we show that if an accessible category \mathcal{K} has a \aleph_0 -continuous weakly stable independence notion, and has a stable independence notion on a sufficiently nice subcategory, then in fact \mathcal{K} has a stable independence notion. This allows us to extend certain stability results concerning classes of modules in [MA21a]: while that paper establishes that a number of such classes are stable in the sense of Galois types, we show that they possess stable independence notions. This is stronger: for example, the category of \aleph_1 -free groups and pure embeddings is Galois-stable (see [MA, 5.9]), but does not have a stable independence notion, albeit for a rather trivial reason—this category does not have the amalgamation property.

For the sake of completeness, we briefly recall the definitions of the essential notions here. More detailed treatments can be found in [LRV19a] and [LRV].

Definition 2.1. Let \mathcal{K} be a category.

- (1) We define an *independence notion* (or *independence relation*) on \mathcal{K} as a class \perp of commutative squares (called \perp -independent, or simply *independent*) such that, for any commutative diagram



the square spanning A , B , C , and D is independent if and only if the square spanning A , B , C , and E is independent.

- (2) We say that an independence notion on \mathcal{K} is *weakly stable* if it satisfies the following conditions:
 - (a) Symmetry: \perp is closed under reflection across the diagonal from bottom left to top right.
 - (b) Transitivity: \perp is closed under vertical and horizontal composition of squares.
 - (c) Existence: Any span $B \leftarrow A \rightarrow C$ can be completed to a \perp -independent square.

- (d) Uniqueness: Any two completions of a span $B \leftarrow A \rightarrow C$ to \perp -independent squares are equivalent up to amalgamation, in the sense of the diagram in (1) above.

In this section, we will concern ourselves largely with the tension between *weakly stable* and *stable* independence, where the latter builds in a crucial accessibility condition.

Definition 2.2. Let \perp be an independence notion on category \mathcal{K} , satisfying the existence and transitivity properties.

- (1) We denote by \mathcal{K}_\perp the subcategory of the arrow category \mathcal{K}^2 whose objects are \mathcal{K} -morphisms, and whose morphisms are \perp -independent squares. (Our assumption on \perp in the preamble of this definition is needed only to ensure that \mathcal{K}_\perp does indeed form a category.)
- (2) We say that \perp is λ -*accessible*, λ an infinite regular cardinal, if the category \mathcal{K}_\perp is λ -accessible. We say that \perp is *accessible* if it is λ -accessible for some λ .
- (3) We say that \perp is a *stable independence notion* if it is weakly stable and accessible.

We note that in the model-theoretic context, weakenings of stability—for example, simplicity—are obtained by weakening existence and/or uniqueness, these being the more difficult property to verify, model-theoretically. Here accessibility (which corresponds roughly to what model theorists might refer to as *local and finite character*) is far thornier: by and large, it is easier to detect when a weakly stable independence notion satisfies the (weaker) condition of λ -continuity:

Definition 2.3. Let \perp be an independence notion on category \mathcal{K} , satisfying the existence and transitivity properties. We say that \perp is λ -continuous if \mathcal{K}_\perp is closed in \mathcal{K}^2 under λ -directed colimits.

That λ -continuity follows from λ -accessibility, incidentally, is [LRV19a, 3.26].

We now turn to the central result of this section, which ensures that if an accessible category \mathcal{K} with all morphisms monomorphisms has a continuous, weakly stable independence notion and a stable independence notion on a sufficiently nice subcategory \mathcal{L} , then in fact there is a stable independence notion on \mathcal{K} itself.

In this context, “sufficiently nice” will mean, precisely, that \mathcal{L} is a *cofinal* subcategory of \mathcal{K} :

Definition 2.4. We say that a functor $F : \mathcal{L} \rightarrow \mathcal{K}$ is *cofinal* if for any $K \in \mathcal{K}$, and any finite sequence $(FL_i \xrightarrow{f_i} K)_{i \in I}$, there exists $L \in \mathcal{L}$, $K \xrightarrow{g} FL$ and $(FL_i \xrightarrow{Fg_i} FL)_{i \in I}$ such that $Fg_i = g \circ f_i$ for all $i \in I$.

We say that a subcategory \mathcal{L} of a category \mathcal{K} is *cofinal* if the inclusion is cofinal.

Example 2.5.

- (1) If \mathcal{K}_* is a full subcategory of \mathcal{K} and for every $M \in \mathcal{K}$ there exists $M \rightarrow N$ with $N \in \mathcal{K}_*$, then \mathcal{K}_* is a cofinal subcategory of \mathcal{K} . Of course, by choosing $I = \emptyset$, every cofinal subcategory has the latter property.
- (2) The category of λ -saturated models of an elementary class is a cofinal subcategory. This generalizes to μ -AECs, [LRV19a, 7.7].

Remark 2.6. We note that this notion of cofinality is somewhat weaker than the one found elsewhere in the category-theoretic literature, e.g. [AR94, 0.11]. One can show, however, that the two definitions coincide if one assumes the amalgamation property.

Theorem 2.7. Let \mathcal{K} be an accessible category and let \mathcal{L} be an accessibly embedded cofinal full subcategory of \mathcal{K} . If:

- (1) All morphisms of \mathcal{K} are monos.
- (2) \mathcal{K} has an \aleph_0 -continuous weakly stable independence notion.
- (3) \mathcal{L} has a stable independence notion.

Then \mathcal{K} has a stable independence notion.

Proof. Let \downarrow be an \aleph_0 -continuous weakly stable independence notion on \mathcal{K} . The \aleph_0 -continuity of \downarrow implies in particular that \mathcal{K} has directed colimits, so \mathcal{L} has directed bounds. It is easy to check that \downarrow restricted to \mathcal{L} is a weakly stable independence notion, hence by the canonicity theorem ([LRV, A.6]) it must be stable. We wish to show that \downarrow is stable, so must show, in particular, that \mathcal{K}_\downarrow is accessible. Since \downarrow restricted to \mathcal{L} is accessible, \mathcal{L} itself must be an accessible category. Taking μ bigger if needed, we may assume without loss of generality that \mathcal{K} , \mathcal{L} , and \mathcal{K}_\downarrow are μ -accessible. Let \mathcal{K}_μ denote the full subcategory of \mathcal{K} consisting of μ -presentable objects. Let \mathcal{K}^* consist of those morphisms in \mathcal{K} which are μ -directed colimits in \mathcal{K}_\downarrow of morphisms in \mathcal{K}_μ^2 . Recalling that the morphisms in \mathcal{K}_μ^2 are precisely the μ -presentable objects of \mathcal{K}^2 , it suffices to see that $\mathcal{K}^* = \mathcal{K}^2$. Clearly, $\mathcal{K}_\mu^2 = \mathcal{K}_\mu^*$ and $\mathcal{L}^2 \subseteq \mathcal{K}^*$.

Concretely, we will prove the following:

- (i) \mathcal{K}^* is closed under composition.
- (ii) \mathcal{K}^* is left cancellable in \mathcal{K}^2 .
- (iii) If $M \in \mathcal{K}$, there exists $M \rightarrow N$ in \mathcal{K}^* with N in \mathcal{L} .

To verify that this is sufficient, take $f : M \rightarrow N$ in \mathcal{K}^2 . We will show that, under assumptions (i) and (ii), if M satisfies the condition of (iii), then f must be in \mathcal{K}^* . To begin, since M satisfies (iii), there is $g : M \rightarrow M'$ in \mathcal{K}^* with M' in \mathcal{L} . Without loss of generality, $f \in \mathcal{K}_\lambda^2$ and \mathcal{K}^2 is λ -accessible for some $\lambda > \mu$. Following [Ros97], there is a morphism $h : M' \rightarrow M^*$ where M^* is λ -saturated in \mathcal{K} . Thus there is $t : N \rightarrow M^*$ such that $tf = hg$. Following 2.5(1), there is $p : M^* \rightarrow L$ with L in \mathcal{L} . Since $ph \in \mathcal{L}^2 \subseteq \mathcal{K}^*$, (i) implies that $phg \in \mathcal{K}^*$. Hence, because $ptf = phg$, (ii) implies that $f \in \mathcal{K}^*$, as desired. We now prove statements (i)-(iii):

(i) Consider $f : K \rightarrow L$ and $g : L \rightarrow M$ in \mathcal{K}^* . Let $f = \text{colim}_i f_i$ and $g = \text{colim}_j g_j$ be μ -directed colimits in \mathcal{K}_\downarrow of morphisms in \mathcal{K}_μ^2 . We may assume that $(k_{ii'}, l_{ii'}^1) : f_i \rightarrow f_{i'}$ and $(l_{jj'}^2, m_{jj'}) : g_j \rightarrow g_{j'}$ are canonical diagrams of f and g with respect to \mathcal{K}_λ^2 in \mathcal{K}_\downarrow . Given $i_0 \in I$ and $j_0 \in J$, there is $j_1 \geq j_0$ in J such that $l_{i_0}^1 : L_{i_0}^1 \rightarrow L$ factorizes through $l_{j_1}^2 : L_{j_1}^2 \rightarrow L$. Similarly, there is $i_1 \geq i_0$ in I such that $l_{j_1}^2$ factorizes through $l_{i_1}^1$. Continuing this procedure and taking a colimit, we get $L_{i_\omega}^1 \cong L_{j_\omega}^2$. Then f_{i_ω} and g_{j_ω} are composable and $g_{i_\omega} f_{j_\omega}$ is in \mathcal{K}^* . Continuing in this way, we obtain that $gf \in \mathcal{K}^*$.

(ii) Consider $f : K \rightarrow L$ and $g : L \rightarrow M$ such that $gf \in \mathcal{K}^*$. Let $\mathcal{K}^{\rightarrow\rightarrow}$ be the category of composable pairs of morphisms in \mathcal{K} and $(f, g) = \text{colim}_i (f_i, g_i)$ be a

μ -directed colimit in $\mathcal{K}^{\rightarrow\rightarrow}$ of morphisms in $\mathcal{K}_\mu^{\rightarrow\rightarrow}$. Let $gf = \text{colim}_j h_j$ be a μ -directed colimit in \mathcal{K}_\downarrow of morphisms in \mathcal{K}_μ^2 . Here, $f_i : K_i^1 \rightarrow L_i$, $g_i : L_i \rightarrow M_i^1$ and $h_j : K_j^2 \rightarrow M_j^2$. As in the preceding argument, for every $i_0 \in I$ and $j_0 \in J$ there are $i_\omega \geq i_0$ in I and $j_\omega \geq j_0$ in J such that $K_{i_\omega}^1 \cong K_{j_\omega}^2$. In the same way, for every $i_0 \in I$ and $j_0 \in J$ there are $i_\omega \geq i_0$ in I and $j_\omega \geq j_0$ in J such that $M_{i_\omega}^1 \cong M_{j_\omega}^2$. By iterating both procedures, we show that for every $i_0 \in I$ and $j_0 \in J$ there are $i_\omega \geq i_0$ in I and $j_\omega \geq j_0$ in J such that $M_{i_\omega}^1 \cong M_{j_\omega}^2$ and $K_{i_\omega}^1 \cong K_{j_\omega}^2$. Then $h_{j_\omega} = g_{i_\omega} f_{i_\omega}$, which implies that $f_{i_\omega} \rightarrow f$ is a morphism in \mathcal{K}_\downarrow . Hence $f \in \mathcal{K}^*$.

(iii) Assume that the claim does not hold and let M have the smallest presentation rank r among objects violating (iii). Following [BR12, 4.2], $r = \lambda^+ \geq \mu^+$. Under the hypothesis of the theorem, \mathcal{K} is well λ^+ -filtrable (see [LRV20, 8.8(2)]), noting that, since the morphisms in \mathcal{K} are monos, filtrability and well-filtrability coincide). In particular, $M = \text{colim } M_i$ can be expressed as the colimit of a smooth chain of λ -presentable objects where $i \leq \lambda^+$. There is $h_0 : M_0 \rightarrow N_0$ in \mathcal{K}^* with $N_0 \in \mathcal{L}$. There is a \downarrow -independent square

$$\begin{array}{ccc} M_1 & \xrightarrow{h_1} & N_1 \\ m_{01} \uparrow & & \uparrow n_{01} \\ M_0 & \xrightarrow{h_0} & N_0 \end{array}$$

in \mathcal{K} and, since \mathcal{L} is cofinal in \mathcal{K} , we may assume that N_1 is in \mathcal{L} . Since M_1 satisfies (iii), h_1 is in \mathcal{K}^* . We iterate this procedure, proceeding as above at successor stages. At limit stages, we take colimits, although this requires somewhat more care: for short chains, we must make use of the cofinality of \mathcal{L} to ensure that the object in the upper right corner is still in \mathcal{L} . Taking the colimit of the resulting λ^+ -chain, we have $h : M \rightarrow N$ in \mathcal{K}^* with N in \mathcal{L} (thanks to the λ^+ -accessibility of \mathcal{L}), which contradicts our initial assumption. \square

3. STABLE INDEPENDENCE IN CATEGORIES OF GROUPS AND MODULES

That we are able to lift stable independence in the sense of Theorem 2.7 yields immediate benefits, namely the proof of stable independence in a host of categories that arise naturally in algebra. This is a consequence not only of the theorem, but the following recent developments:

- (1) One of the essential ideas of [LRV19b] is that continuous weakly stable independent relations are abundant and easily detectable in the algebraic context, typically taking the form of *effective squares*. We briefly recall some of the necessary terminology, as it will also be required in Section 4.
- (2) In [KMA20] and [MA21a], a large number of algebraic categories are shown to have precisely the model-theoretic properties required to ensure the existence of a cofinal full subcategory equipped with a stable independence relation.

Taken together, this yields a host of algebraic categories with stable independence.

We begin by recalling a few pieces of necessary terminology from [LRV], which will allow us to give sufficient conditions for the existence of a \aleph_0 -continuous weakly stable independence relation.

Definition 3.1. Let \mathcal{K} be a category, and let \mathcal{M} be a class of morphisms in \mathcal{K} .

- (1) We say that \mathcal{M} is *almost nice* if it satisfies the following conditions:
 - (a) \mathcal{M} is *normal*: it contains all isomorphisms in \mathcal{K} and is closed under composition.
 - (b) \mathcal{M} is *coherent*: whenever f and g are composable morphisms with $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, then $f \in \mathcal{M}$ as well.
 - (c) \mathcal{M} is a *coclan*: the pushout of any two morphisms, at least one of which is in \mathcal{M} , exists, and \mathcal{M} is closed under pushouts.

Incidentally, we say that \mathcal{M} is *nice* if it is also closed under retracts in \mathcal{K}^2 . This means that if $(u, v) : g \rightarrow f$ and $(r, s) : f \rightarrow g$ are morphisms in \mathcal{K}^2 such that $(r, s)(u, v) = \text{id}_g$ then $f \in \mathcal{M}$ implies that $g \in \mathcal{M}$.

- (2) We say that \mathcal{M} is λ -*continuous*, λ an infinite regular cardinal, if \mathcal{K} has λ -directed colimits, and \mathcal{M} is closed under λ -directed colimits in \mathcal{K} .
- (3) We say that \mathcal{M} is λ -*accessible* if it is λ -continuous and both \mathcal{K} and $\mathcal{K}_{\mathcal{M}}$ are λ -accessible. \mathcal{M} is *accessible* if it is λ -accessible for some λ .

Notation 3.2. Note that if \mathcal{M} is normal, we can form a subcategory $\mathcal{K}_{\mathcal{M}}$ of \mathcal{K} whose objects are those of \mathcal{K} and whose morphisms are precisely those in \mathcal{M} .

We define a natural candidate for an independence relation on $\mathcal{K}_{\mathcal{M}}$ in the form of \mathcal{M} -*effective squares*, following [LRV, 2.2, 2.3].

Definition 3.3. Let \mathcal{K} be a category, and let \mathcal{M} be a class of morphisms in \mathcal{K} . An \mathcal{M} -*effective square* is a commutative square of \mathcal{K} -morphisms

$$\begin{array}{ccc} B & \xrightarrow{h} & D \\ f \uparrow & & \uparrow k \\ A & \xrightarrow{g} & C \end{array}$$

such that the pushout P of f and g exists, and the induced morphism $P \rightarrow D$ is in \mathcal{M} .

We note that, in case \mathcal{M} consists of the regular monomorphisms in \mathcal{K} , \mathcal{M} -effective squares are precisely the *effective unions* of [Bar88].

Fact 3.4. If \mathcal{K} has pushouts and \mathcal{M} is almost nice, \mathcal{M} -effective squares form a weakly stable independence relation on $\mathcal{K}_{\mathcal{M}}$ [LRV, 2.7]. If, moreover, \mathcal{M} is λ -continuous, this independence relation is λ -continuous [LRV, 2.11].

This will guarantee the existence of \aleph_0 -continuous, weakly stable independence relations in a number of familiar algebraic categories that have recently been the subject of analyses using the tools of abstract model theory. The essential fact we require, which allows us to isolate a cofinal, full subcategory equipped with a stable independence relation—and thus apply Theorem 2.7—is fundamentally model-theoretic.

We recall that in an abstract elementary class \mathcal{K} , the syntactic types familiar from classical model theory are replaced by *Galois* (or *orbital*) *types*: given a model $M \in \mathcal{K}$, Galois types over M are typically identified with orbits of tuples in a large,

strongly homogeneous *monster model* under automorphisms fixing M . A class is said to be λ -Galois stable if there are at most λ Galois (1-)types over any $M \in \mathcal{K}$ of cardinality λ . Moreover, Galois types in a class \mathcal{K} are said to be $< \aleph_0$ -short if, roughly speaking, the type of any tuple is completely determined by the types of its finite subtuples. Readers unfamiliar with these properties may wish to treat the following as a black box:

Lemma 3.5. Let \mathcal{K} be an *abstract elementary class*. If

- (1) \mathcal{K} has the amalgamation property,
- (2) \mathcal{K} is Galois-stable, and
- (3) types in \mathcal{K} are $< \aleph_0$ -short over models,

then there is a full, cofinal subcategory of \mathcal{K} —consisting of sufficiently saturated models—on which there is a stable independence notion.

Proof sketch. In essence, finite shortness puts us in the realm of homogeneous model theory, where the desired result is already known. While we omit the full argument, we hope that the following outline will be sufficient for the interested reader. In the process, we will make free use of the technique of *Galois Morleyization* introduced in [Vas1]: in an $< \aleph_0$ -short AEC one can identify types of finite sequences over the empty set with finitary quantifier-free formulas (formally, by expanding the language).

Following [HS00, §3], the assumptions of the lemma yield a relation p is free over M , for p a Galois type over a model $N \leq M$, that satisfies all the properties of stable independence provided M and N are sufficiently saturated (in particular, in the terminology of that paper, M and N must be a -saturated, a consequence of λ -saturation in some sufficiently big λ , see [HS00, 1.9.4]). Note that the properties of independence verified in that paper are the model-theoretic analogues of the category-theoretic definition we discuss here, but the two definitions are equivalent, [LRV, 8.14]. There is the issue, too, that in [HS00], p is assumed to be the type of a finite sequence—as we are concerned with types of infinite sequences, we must show that the existence/extension property of [HS00] can be transferred to this context. This can be done relatively easily, making use of the compactness theorem for homogeneous model theory: the complete type p of a sequence of arbitrary length is satisfiable just in case its restrictions to finite subsequences are satisfiable (see [HS00, 1.1], or, more explicitly, [ABV19, 3.8, 3.9]). Given the type p of a sequence of arbitrary length over M , and \bar{x} a finite subsequence, the restriction of p to \bar{x} has a free extension over N . Consider the set of all such free extensions, regarded as quantifier-free formulas (via Galois Morleyization, if necessary). The resulting set is complete and, by the extension property of freeness for types of *finite* sequences, all of its restrictions to finite sets of variables are consistent. By construction, the resulting type is the free extension of p over N . \square

We now obtain stable independence relations on a wide array of algebraic categories using Theorem 2.7, Fact 3.4, and Lemma 3.5—we note that [MA] constructs stable independence relations in many of the same cases, by more concrete means.

As a template for our approach, consider:

Theorem 3.6. For any ring with unit R , the category of (left) R -modules and pure monomorphisms, $\mathbf{RMod}_{\text{pure}}$, has a stable independence notion.

Proof. By [KMA20], \mathbf{RMod}_{pure} forms an AEC, has amalgamation, is stable, and types are $(< \aleph_0)$ -short over models; that is, it satisfies all the hypotheses of Lemma 3.5. Thus \mathcal{K} must have a stable independence relation on its sufficiently saturated models, which form a cofinal, full subcategory. By Theorem 2.7 and Fact 3.4, then, \mathbf{RMod}_{pure} has a stable independence relation. \square

As an aside, in light of Fact 4.3 below (originally appearing as [LRV, 3.1]), it follows that:

Corollary 3.7. Pure monomorphisms are cofibrantly generated (generated from a set of morphisms by pushouts, transfinite composition, and retracts) in \mathbf{RMod} .

Note that this is a special case of [LPRV20, 3.13], which holds not just for R -modules but arbitrary locally finitely presentable additive categories. Of greater interest are the other applications of this style of argument. In particular,

Theorem 3.8. Let R be an integral domain. The following categories of modules have a stable independence relation:

- (1) Torsion R -modules with pure monomorphisms.
- (2) R -divisible modules with pure monomorphisms (recall that a module M is R -divisible if for any nonzero $m \in M$ and nonzero $r \in R$, there is $n \in M$ with $rn = m$).

Proof. (1) The category of torsion R -modules and pure monomorphisms satisfies the condition of Lemma 3.5, by [MA21a, 4.2(4)] and [MA21a, 4.8(2)]
 (2) Similarly to (1), using [MA21a, 4.2(5)] in place of [MA21a, 4.2(4)]. \square

We obtain stable independence relations on an assortment of familiar categories of groups, as well, again taking advantage of recent model-theoretic results—again, [MA] actually obtains similar results, by different means.

Theorem 3.9. The following categories of groups all have a stable independence relation.

- (1) Abelian groups with monomorphisms (respectively, pure monomorphisms).
- (2) Torsion-free abelian groups with pure monomorphisms. Similarly, reduced torsion-free abelian groups with pure monomorphisms.
- (3) Abelian p -groups with monomorphisms (respectively, pure monomorphisms), p any prime.
- (4) Torsion abelian groups with monomorphisms (respectively, pure monomorphisms).
- (5) Divisible abelian groups with monomorphisms.

Proof. As in the proof of Theorem 3.8, we satisfy ourselves with indicating the model-theoretic sources that ensure the category satisfies the hypotheses of Lemma 3.5.

- (1) [BCG⁺] and [MA21b, 3.12], and [KMA20, 3.16] (respectively).
- (2) The reduced case is [She17, 1.2(3)]; general torsion free groups are addressed in, e.g. [KMA20, 3.14].
- (3) By [MA21a, 4.8(3)] and [MA21a, 3.5].
- (4) By [MA21a, 4.8(3)] and [MA21a, 4.8(1)], respectively.
- (5) By [MA21a, 4.8(3)]. \square

We expect that considerably more applications of Theorem 2.7 of this form are within easy reach: the forthcoming paper [MA], for example, provides clear avenues for future work along these lines.

4. HIGHER-DIMENSIONAL INDEPENDENCE

We turn now to a different, and perhaps more natural question: we know that if we have a stable independence relation on a category \mathcal{K} , we obtain a well behaved subcategory \mathcal{K}_\downarrow of the category of morphisms \mathcal{K}^2 consisting of the independent squares. Is it the case, too, that there is a stable independence notion on \mathcal{K}_\downarrow —consisting now of commutative *cubes* in \mathcal{K} —and under what conditions? Is there, in turn, a stable independence relation on these cubes?

We wish to examine, in short, the existence and behavior of *higher-dimensional* stable independence relations.

It should be noted that this is not an exercise in abstraction: higher-dimensional independence relations have played a significant role in recent advances in model theory. The idea, introduced by Shelah, is vital in his analysis of the classification theory of $L_{\omega_1, \omega}$ in [She83a] and [She83b], and in his proof of the first-order Main Gap, cf. [She90, Ch. XII]. More recently, these notions have made crucial appearances in a number of categoricity transfer arguments, most notably in [Zil05], for quasiminimal pregeometry classes, and [SV], for abstract elementary classes with amalgamation, assuming weak GCH. Of particular interest are *excellent* classes, which possess independence notions in all finite dimensions.

To be precise, we propose the following notions of n -dimensional stable independence and excellence, adapted to our context—these should specialize to the standard ones in the model-theoretic examples mentioned above.

Definition 4.1. Let \mathcal{K} be a category. For $n \geq 1$, we define an *n -dimensional stable independence notion* on \mathcal{K} , Γ , and its induced category, \mathcal{K}^Γ , proceeding by induction on n :

- We say that Γ is a 1-dimensional stable independence notion on \mathcal{K} just in case it is the collection of all morphisms in \mathcal{K} . In this case, we define $\mathcal{K}^\Gamma = \mathcal{K}$.
- An $(n+1)$ -dimensional stable independence notion on \mathcal{K} consists of a pair (Γ_n, Γ) , where
 - Γ_n is an n -dimensional stable independence notion on \mathcal{K} , and
 - Γ is a stable independence notion on \mathcal{K}^{Γ_n} , in the sense of Definition 2.2(3).
- Given an $(n+1)$ -dimensional stable independence notion $\Gamma_{n+1} = (\Gamma_n, \Gamma)$ on \mathcal{K} , we define $\mathcal{K}^{\Gamma_{n+1}}$ to be the category $(\mathcal{K}^{\Gamma_n})^\Gamma$, whose objects are morphisms of \mathcal{K}^{Γ_n} and whose morphisms are the Γ -independent squares. Note that this is precisely $(\mathcal{K}^{\Gamma_n})_\downarrow$ with $\downarrow = \Gamma$.

Note that the stable independence notions considered in Sections 2 and 3 are precisely 2-dimensional independence notions—in the sense above—on the appropriate categories. The best case scenario is the following:

Definition 4.2. We say that a category \mathcal{K} is *excellent* if for all $n \geq 1$, \mathcal{K} has an n -dimensional stable independence notion Γ_n so that \mathcal{K}^{Γ_n} has directed colimits.

As noted in the introduction, excellence is far from the norm in the model-theoretic context, as the existence property will typically fail for sufficiently high-dimensional relations: [GKK15a] and [GKK15b] develop a comprehensive theory of such obstructions. Here we restrict ourselves to the setting of [LRV]; that is, with locally presentable ambient category \mathcal{K} and well behaved class of morphisms \mathcal{M} . With this added structure, the obstructions disappear: if there is a stable independence relation on $\mathcal{K}_{\mathcal{M}}$, it is excellent.

All of the difficulty lies in the inductive step: given an n -dimensional stable independence notion Γ on \mathcal{K} , how do we construct a stable independence notion Γ' so that $(\mathcal{K}^{\Gamma})^{\Gamma'}$ has directed colimits?

In fact, we consider a simpler—but entirely equivalent—problem, whose solution, Proposition 4.4 below, should be of independent interest. In particular, we take advantage of the fact that, in this framework, existence of stable independence notions of a particular dimension is equivalent to cofibrant generation of a suitable family of morphisms, via the central result of [LRV]. For the sake of completeness, we include that result here, phrased in terms better suited to the current context:

Fact 4.3. ([LRV, 3.1]) Let \mathcal{K} be a locally presentable category, and let \mathcal{M} be a nice, accessible, and \aleph_0 -continuous class of morphisms in \mathcal{K} . The following are equivalent:

- (1) $\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.
- (2) \mathcal{M} -effective squares form a stable independence notion in $\mathcal{K}_{\mathcal{M}}$.
- (3) \mathcal{M} is cofibrantly generated in \mathcal{K} .

Note that we have unpacked much of the terminology used in [LRV, 3.1], for the benefit of the reader. Recall that a class of morphisms \mathcal{M} is *cofibrantly generated* if it can be generated from a set—as opposed to a proper class—of morphisms by pushouts, transfinite compositions, and retracts.

Proposition 4.4. Let \mathcal{K} be locally presentable and \mathcal{M} be nice and \aleph_0 -continuous in \mathcal{K} . Let $\mathcal{M}!$ consist of \mathcal{M} -effective morphisms in \mathcal{K}^2 . Then $\mathcal{M}!$ is nice and \aleph_0 -continuous in \mathcal{K}^2 .

Proof. (a) $\mathcal{M}!$ is normal: Isomorphisms in \mathcal{K}^2 are commutative squares whose horizontal arrows are isomorphisms. Such squares are pushouts, hence \mathcal{M} -effective. The composition of two \mathcal{M} -effective morphisms is \mathcal{M} -effective (see [LRV, 2.7]).

(b) $\mathcal{M}!$ is coherent: see [LRV, 2.10].

(c) $\mathcal{M}!$ is a coclan: We have to show that pushouts of \mathcal{M} -effective squares are \mathcal{M} -effective. Let

$$\begin{array}{ccc} C & \xrightarrow{m_2} & D \\ m_1 \uparrow & & \uparrow m_3 \\ A & \xrightarrow{m_0} & B \end{array}$$

be an \mathcal{M} -effective square considered as a morphism $(m_0, m_2) : m_1 \rightarrow m_3$ in \mathcal{K}^2 . Let

$$\begin{array}{ccc} C & \xrightarrow{c} & C' \\ m_1 \uparrow & & \uparrow m'_1 \\ A & \xrightarrow{a} & A' \end{array}$$

be a commutative square considered as a morphism $(a, c) : m_1 \rightarrow m'_1$. Consider the pushout

$$\begin{array}{ccc} m_3 & \xrightarrow{(b,d)} & m'_3 \\ (m_0, m_2) \uparrow & & \uparrow (m'_0, m'_2) \\ m_1 & \xrightarrow{(a,c)} & m'_1 \end{array}$$

in \mathcal{K}^2 . This means that

$$\begin{array}{ccc} A' & \xrightarrow{m'_0} & B' \\ a \uparrow & & \uparrow b \\ A & \xrightarrow{m_0} & B \end{array}$$

and

$$\begin{array}{ccc} C' & \xrightarrow{m'_2} & D' \\ c \uparrow & & \uparrow d \\ C & \xrightarrow{m_2} & D \end{array}$$

are pushouts and m'_3 is the induced morphism, i.e., $m'_3 m'_0 = m'_2 m'_1$ and $m'_3 b = d m_3$. It suffices to show that the square

$$\begin{array}{ccc} C' & \xrightarrow{m'_2} & D' \\ m'_1 \uparrow & & \uparrow m'_3 \\ A' & \xrightarrow{m'_0} & B' \end{array}$$

is \mathcal{M} -effective.

Consider a pushout

$$\begin{array}{ccc} C' & \xrightarrow{p'_0} & P' \\ m'_1 \uparrow & & \uparrow p'_1 \\ A' & \xrightarrow{m'_0} & B' \end{array}$$

We must show that the induced morphism $t' : P' \rightarrow D'$ is in \mathcal{M} . Returning to the original square, we know that the induced morphism $t : P \rightarrow D$ is in \mathcal{M} where

$$\begin{array}{ccc} C & \xrightarrow{p_0} & P \\ m_1 \uparrow & & \uparrow p_1 \\ A & \xrightarrow{m_0} & B \end{array}$$

is a pushout. For this, it suffices to show that

$$\begin{array}{ccc} P' & \xrightarrow{t'} & D' \\ p \uparrow & & \uparrow d \\ P & \xrightarrow{t} & D \end{array}$$

is a pushout where $p : P \rightarrow P'$ is the induced morphism; that is, $pp_0 = p'_0c$ and $pp_1 = p'_1b$.

Consider morphisms $u : P' \rightarrow X$ and $v : D \rightarrow X$ such that $up = vt$. Then

$$up'_0c = upp_0 = vtp_0 = vm_2.$$

Thus there exists a unique $q : D' \rightarrow X$ such that $qm'_2 = up'_0$ and $qd = v$. It remains to show that $qt' = u$. We have

$$qt'p'_0 = qm'_2 = up'_0$$

and $qt'p'_1 = qm'_3$. To finish the proof, we need that $up'_1 = qm'_3$ because then $qt' = u$. We have

$$qm'_3m'_0 = qm'_2m'_1 = up'_0m'_1 = up'_1m'_0$$

and

$$qm'_3b = qdm_3 = vm_3 = vtp_1 = upp_1 = up'_1b.$$

Hence $qm'_3 = up'_1$.

(d) $\mathcal{M}!$ is nice: Since \mathcal{M} is closed under retracts, $\mathcal{M}!$ is closed under retracts.

(e) $\mathcal{M}!$ is \aleph_0 -continuous: see [LRV, 2.11]. \square

Theorem 4.5. Let \mathcal{K} be a locally presentable category, and let \mathcal{M} be a nice, accessible, and \aleph_0 -continuous class of morphisms in \mathcal{K} . If $\mathcal{K}_{\mathcal{M}}$ has a stable independence notion, it is excellent.

Proof. We have the obvious one-dimensional stable independence notion on $\mathcal{K}_{\mathcal{M}}$, with $(\mathcal{K}_{\mathcal{M}})^{\Gamma_1} = \mathcal{K}_{\mathcal{M}}$ and Γ_1 consisting precisely of \mathcal{M} , the class of morphisms in $\mathcal{K}_{\mathcal{M}}$. Following Fact 4.3, \mathcal{M} -effective squares form a stable, \aleph_0 -continuous independence notion Γ on $\mathcal{K}_{\mathcal{M}}$. Take $\Gamma_2 = (\Gamma_1, \Gamma)$. Then

$$(\mathcal{K}_{\mathcal{M}})^{\Gamma_2} = (\mathcal{K}_{\mathcal{M}})_{\downarrow} \subseteq (\mathcal{K}_{\mathcal{M}})^2 \subseteq \mathcal{K}^2$$

is an accessible category closed under directed colimits in \mathcal{K}^2 .

Assume that we have an n -dimensional stable independence notion Γ_n , $n > 1$, on $\mathcal{K}_{\mathcal{M}}$ where

$$(\mathcal{K}_{\mathcal{M}})^{\Gamma_n} \subseteq (\mathcal{K}_{\mathcal{M}})^{2^{n-1}} \subseteq \mathcal{K}^{2^{n-1}}$$

is an accessible category closed under directed colimits in $\mathcal{K}^{2^{n-1}}$ and a nice and \aleph_0 -continuous class \mathcal{M}_{n-1} of morphisms in $\mathcal{K}^{2^{n-1}}$. Following Proposition 4.4,

$\mathcal{M}_n = (\mathcal{M}_{n-1})!$ is nice and \aleph_0 -continuous class of morphisms in \mathcal{K}^{2^n} . We have to show that \mathcal{M}_n -effective squares yield a stable independence notion on $(\mathcal{K}_{\mathcal{M}})^{\Gamma_n}$. Following Fact 3.4, \mathcal{M}_n -effective squares form a weakly stable independence notion on $(\mathcal{K}^{2^n})_{\mathcal{M}_n}$ which is, moreover, \aleph_0 -continuous. $(\mathcal{K}_{\mathcal{M}})^{\Gamma_n}$ is a full subcategory of $(\mathcal{K}^{2^n})_{\mathcal{M}_n}$ consisting of those \mathcal{M}_n -squares which are \mathcal{M}_n -effective. Since \mathcal{M}_n is nice, the proof of [LRV, 2.7] yields that \mathcal{M}_n -effective squares yield a weakly stable independence notion on $(\mathcal{K}_{\mathcal{M}})^{\Gamma_n}$. Similarly, the proof of [LRV, 2.11] yields that this weakly stable independence notion is \aleph_0 -continuous.

It remains to show that the category $(\mathcal{K}_{\mathcal{M}})^{\Gamma_{n+1}} = ((\mathcal{K}_{\mathcal{M}})^{\Gamma_n})_{\downarrow}$ is accessible.

We note that an \mathcal{M}_n -square

$$\begin{array}{ccc} c & \xrightarrow{(f'_0, f'_1)} & d \\ (g_0, g_1) \uparrow & & \uparrow (h_0, h_1) \\ a & \xrightarrow{(f_0, f_1)} & b \end{array}$$

is in fact a cube of the following form:

$$\begin{array}{ccccc} & & D_1 & \xrightarrow{d} & D_2 \\ & \nearrow f'_0 & \uparrow h_0 & & \nearrow f'_1 \\ C_1 & \xrightarrow{c} & C_2 & & \\ \uparrow g_0 & & \uparrow g_1 & & \uparrow h_1 \\ & \nearrow f_0 & B_1 & \xrightarrow{b} & B_2 \\ A_1 & \xrightarrow{a} & A_2 & & \\ & \nearrow f_1 & & & \end{array}$$

where the top, bottom, front and rear squares are all \mathcal{M}_{n-1} -effective.

Such an \mathcal{M}_n -square is \mathcal{M}_n -effective if and only if the derived square induced by pushing out on the right and left hand sides of the cube is \mathcal{M}_{n-1} -effective. That is, if the following squares are pushouts

$$\begin{array}{ccc} C_1 & \xrightarrow{p_0} & P \\ g_0 \uparrow & & \uparrow p_1 \\ A_1 & \xrightarrow{f_0} & B_1 \end{array} \qquad \begin{array}{ccc} C_2 & \xrightarrow{p'_0} & P' \\ g_1 \uparrow & & \uparrow p'_1 \\ A_2 & \xrightarrow{f_1} & B_2 \end{array}$$

and $q : P \rightarrow D_1$, $q' : P' \rightarrow D_2$ and $p : P \rightarrow P'$ are the obvious induced morphisms, the derived square

$$\begin{array}{ccc} P' & \xrightarrow{q'} & D_2 \\ p \uparrow & & \uparrow d \\ P & \xrightarrow{q} & D_1 \end{array}$$

must be \mathcal{M}_{n-1} -effective.

The category $(\mathcal{K}_{\mathcal{M}})^{\Gamma_{n+1}}$ thus consists of the full subcategory of $((\mathcal{K}_{\mathcal{M}})^{\Gamma_n})^2$ on the \mathcal{M}_n -effective squares whose derived squares are \mathcal{M}_{n-1} -effective. In particular, the following is a pullback of categories:

$$\begin{array}{ccc} ((\mathcal{K}_{\mathcal{M}})^{\Gamma_n})^2 & \xrightarrow{F} & (\mathcal{K}_{\mathcal{M}})^{2^{n-1}} \\ \bar{G} \uparrow & & \uparrow G \\ \mathcal{K}^{\Gamma_{n+1}} & \xrightarrow{\bar{F}} & (\mathcal{K}_{\mathcal{M}})^{\Gamma_n} \end{array}$$

where F sends an \mathcal{M}_n -square to its derived square and G is the inclusion. The functor G is transportable because \mathcal{M}_{n-1} -effective squares are closed under isomorphisms of \mathcal{M}_{n-1} -squares. Following [MP89, 5.1.1], the pullback above is in fact a *Pullback* (or *bipullback*) of the corresponding categories. Hence $(\mathcal{K}_{\mathcal{M}})^{\Gamma_{n+1}}$ is an accessible category (see [MP89, 5.1.6]). \square

Remark 4.6. The category $\mathbf{RMod}_{\text{pure}}$ from Theorem 3.6 is excellent. Similarly, the category $\mathbf{RMod}_{\text{emb}}$ of R -modules and regular monomorphisms. The latter follows from the fact that \mathbf{RMod} has effective unions, hence regular monomorphisms are cofibrantly generated. In fact, the same is true in any Grothendieck abelian category or Grothendieck topos ([Bar88]).

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