First publ.in: Artificial intelligence 170 (2006), 4/5, pp. 462-471

Ranking functions and rankings on languages

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Abstract

The Spohnian paradigm of ranking functions is in many respects like an order-of-magnitude reverse of subjective probability theory. Unlike probabilities, however, ranking functions are only indirectly—via a pointwise ranking function on the underlying set of possibilities W—defined on a field of propositions $\mathcal{A}$ over W. This research note shows under which conditions ranking functions on a field of propositions $\mathcal{A}$ over W and rankings on a language $\mathcal{L}$ are induced by pointwise ranking functions on W and the set of models for $\mathcal{L}$, $Mod_{\mathcal{L}}$, respectively.

Keywords: Extension theorem for rankings on languages; Probabilities; Ranking functions; Rankings on languages; Spohn

1. Introduction: Pointwise ranking functions

The Spohnian paradigm of ranking functions [16,17] is in many respects like an order-of-magnitude reverse of subjective probability theory [9]. "Ranks represent degrees"—or rather: *grades*—"of disbelief" ([19]: 6). Whereas a high probability indicates a high degree of belief, a high rank indicates a high grade of disbelief.

There are many parallels between probability theory and ranking theory [16,18], and in Footnote 22 of his [16] Spohn "wonder[s] how far the mathematical analogy [of his ranking functions to probabilities] could be extended".[1] The starting point of this paper is one of the few places where ranking theory differs from subjective probability theory as well as qualitative-logical approaches to the representation of epistemic states such as entrenchment orderings in belief revision theory: the domain on which these models are defined, that is, what they take to be the objects of belief.

Unlike probabilities, ranking functions are only indirectly—via a pointwise ranking function on a non-empty set of possibilities (possible worlds, models) W—defined on some finitary/σ-/complete field $\mathcal{A}$ over W, i.e., a set of subsets of W containing the empty set and closed under complementation and finite/countable/arbitrary intersections. Let us have a closer look.

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[1] Ranking theory is very similar to possibility theory [5], and it would be highly desirable to know to what extent the results below also hold for possibility measures. Unfortunately this goes beyond the scope of this research note.

A function κ from W into the set of natural numbers N is a *pointwise ranking function on* W iff $\kappa(\omega) = 0$ for at least on $\omega \in W$. A pointwise ranking function $\kappa : W \to N$ is extended to a function ϱ_κ on a field $\mathcal{A}$ over W with range $N \cup \{\infty\}$ by defining, for each $A \in \mathcal{A}$,

$$\varrho_\kappa(A) = \begin{cases} \min\{\kappa(\omega) \colon \ \omega \in A\}, & \text{if } A \neq \emptyset, \\ \infty, & \text{if } A = \emptyset. \end{cases}$$

As will be seen below, it is useful to allow that some possibility $\omega \in W$ is sent to ∞, which amounts to ω being a "virtually impossible possibility" (according to κ). In order to distinguish the more restricted notion of a pointwise ranking function as defined above from the more liberal one allowing for virtually impossible possibilities, let us call the former *natural* pointwise ranking functions (because the range of κ is restricted to the set of natural numbers N).

Pointwise ranking functions κ are functions defined on a non-empty set of possibilities W that take natural numbers or ∞ as values. They are extended to functions ϱ_κ on a field $\mathcal{A}$ over W by stipulating that the rank of any non-empty proposition $A \in \mathcal{A}$ equals the minimum rank of the possibilities in A, i.e., $\varrho_\kappa(A) = \min\{\kappa(\omega) \colon \ \omega \in A\}$, and the empty proposition is sent to ∞.

In case W is a finite set of possibilities and $\mathcal{A}$ its powerset, every possibility corresponds to a proposition (viz. the singleton containing it). But already when W is the set of all models $Mod_{\mathcal{L}}$ for a propositional language $\mathcal{L}$ with infinitely many propositional variables and $\mathcal{A}$ is the field $\{Mod(\alpha) \subseteq W \colon \ \alpha \in \mathcal{L}\}$, no possibility corresponds to a proposition. Furthermore, one has to specify a ranking over uncountably many possibilities in order to assign a positive finite rank to a single proposition. But clearly, we often have a definite opinion about a single proposition (represented in terms of a sentence) even if we do not have an idea of what the underlying set of possibilities looks like—let alone what our ranking over these possibilities might be. For instance, I strongly disbelieve that one can buy a bottle of Schilcher for less than 1 Euro, though I lack the relevant enological vocabulary in order to know what all the possibilities are. Indeed, it seems the underlying set of possibilities *should not* matter for my disbelief in this proposition.

More generally, we should be able to theorize about our epistemic states even if all we are given is a ranking over the sentences or propositions of some language or field, and we have no ranking over the underlying set of possibilities. After all, what we as ordinary or scientific believers do have are plenty of beliefs and grades of belief in various propositions—usually if not always via beliefs and grades of belief in sentences or other representations of these propositions. When we want to attach ranks to sentences, pointwise ranking theory first has us specify a set of possible worlds for the language the sentences are taken from; then we have to specify a ranking over these possible worlds, which in turn induces a ranking over sets of possible worlds; and only then can we identify the rank of a sentence with the rank of the proposition containing exactly the possible worlds making our sentence true.

This is a bit awkward. What one would like to do is to start with a ranking of the sentences in $\mathcal{L}$, and then be able to induce a pointwise ranking function on the corresponding set of possible worlds that yields the original ranking. The question is whether this is always possible. In order to answer it, let us first define ranking functions on fields of propositions and rankings on languages. (For a similar generalization of pointwise ranking functions see [21].)

2. Ranking functions and rankings on languages

(*Finitely minimitive*) *ranking functions* are functions ϱ from a field $\mathcal{A}$ over a set of possibilities W into the set of natural numbers extended by ∞[2] such that for all $A, B \in \mathcal{A}$:

(1) $\varrho(\emptyset) = \infty$;
(2) $\varrho(W) = 0$;
(3) $\varrho(A \cup B) = \min\{\varrho(A), \varrho(B)\}$.

If $\mathcal{A}$ is a σ-field/complete field, ϱ is a σ*-minimitive/completely minimitive* ranking function iff, in addition to (1)–(3), we have for every countable/possibly uncountable $\mathcal{B} \subseteq A$:

[2] One can also take the set of ordinal numbers smaller than or equal to some limit ordinal β and send $\emptyset$ to β, but we do not need this generality here.

(4) $\varrho(\bigcup \mathcal{B}) = \min\{\varrho(B)\colon B \in \mathcal{B}\}$.

In case $\mathcal{A}$ is finite, i.e., if $\mathcal{A}$ contains only finitely many elements, these distinctions collapse. According to (4), the range of ranking functions has to be well-ordered. Therefore N is a natural choice. A ranking function ϱ on $\mathcal{A}$ is a *pre-ranking* iff ϱ is a finitely minimitive ranking function on $\mathcal{A}$ such that

$$\varrho\left(\bigcup \mathcal{B}\right) = \min\{\varrho(A)\colon A \in \mathcal{B}\}$$

for every countable $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B} \in \mathcal{A}$. A ranking function ϱ is *regular* iff $\varrho(A) < \varrho(\emptyset)$ for every non-empty $A \in \mathcal{A}$. The conditional ranking function $\varrho(\cdot \mid \cdot) : \mathcal{A} \times \mathcal{A} \to N \cup \{\infty\}$ based on the ranking function $\varrho : \mathcal{A} \to N \cup \{\infty\}$ is defined such that for all $A, B \in \mathcal{A}$ with $B \neq \emptyset$,

$$(5)\quad \varrho(B \mid A) = \begin{cases} \varrho(B \cap A) - \varrho(A), & \text{if } \varrho(A) < \infty, \\ 0, & \text{if } \varrho(A) = \infty. \end{cases}$$

The second clause says that, conditional on a (virtually) impossible proposition, no non-tautological proposition is believed in ϱ. Goldszmidt and Pearl ([9]: 63) define $\varrho(B \mid A) = \infty$ for $A = \emptyset$, which means that, conditional on the impossible proposition, every proposition is maximally believed in ϱ. We further stipulate that $\varrho(\emptyset \mid A) = \infty$ for every $A \in \mathcal{A}$, which completes the definition of a conditional ranking function and ensures that $\varrho(\cdot \mid A) : \mathcal{A} \to N \cup \{\infty\}$ is a ranking function.

If the function $\varrho_\kappa : \mathcal{A} \to N \cup \{\infty\}$ is induced by a (natural) pointwise ranking function $\kappa : W \to N$, ϱ_κ is a (regular and) completely minimitive ranking function. The converse is not true. The triple $\mathbf{A} = \langle W, \mathcal{A}, \varrho\rangle$ with W a set of possibilities, $\mathcal{A}$ a finitary/σ-/complete field over W, and $\varrho : \mathcal{A} \to N \cup \{\infty\}$ a ranking function is called a finitary/σ-/complete *ranking space*. $\mathbf{A}$ is called *regular* iff ϱ is regular, and $\mathbf{A}$ is called *natural* iff ϱ is induced by some natural pointwise ranking function κ.

A proposition $A \in \mathcal{A}$ is *believed* in ϱ iff $\varrho(\overline{A}) > 0$. ϱ's belief set $Bel_\varrho = \{A \in \mathcal{A}\colon \varrho(\overline{A}) > 0\}$ is consistent and deductively closed in the finite/countable/complete sense whenever ϱ is finitely/σ-/completely minimitive. Here Bel is consistent in the finite/countable/complete sense iff $\bigcap \mathcal{B} \neq \emptyset$ for every finite/countable/possibly uncountable $\mathcal{B} \subseteq Bel$; and Bel is deductively closed in the finite/countable/complete sense iff for all $A \in \mathcal{A}$: $A \in Bel$ whenever $\bigcap \mathcal{B} \subseteq A$ for some finite/countable/possibly uncountable $\mathcal{B} \subseteq Bel$.[3]

Observation 1. *For any ranking space* $\mathbf{A} = \langle W, \mathcal{A}, \varrho\rangle$ *and all* $A, B \in \mathcal{A}$:

1. $\min\{\varrho(A), \varrho(\overline{A})\} = 0$.
2. $A \subseteq B \Rightarrow \varrho(B) \leqslant \varrho(A)$.

Rankings $\kappa : \mathcal{L} \to N \cup \{\infty\}$ on languages $\mathcal{L}$ *are defined such that for all* $\alpha, \beta \in \mathcal{L}$:

0. $\alpha \dashv\vdash \beta \Rightarrow \varrho(\alpha) = \varrho(\beta)$.
1. $\alpha \vdash \bot \Rightarrow \varrho(\alpha) = \infty$.
2. $\vdash \alpha \Rightarrow \varrho(\alpha) = 0$.
3. $\varrho(\alpha \vee \beta) = \min\{\varrho(\alpha), \varrho(\beta)\}$.
4. $\beta \nvdash \bot \Rightarrow \varrho(\beta \mid \alpha) = \varrho(\alpha \wedge \beta) - \varrho(\alpha)$ $(= 0$ *if* $\varrho(\alpha) = \infty)$.
5. $\beta \vdash \bot \Rightarrow \varrho(\beta \mid \alpha) = \infty$.

To be sure: $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ is the *classical* consequence relation (and singletons on the left hand side are identified with the wff they contain). The corresponding definitions and observations for finitely minimitive ranking functions also apply for rankings on languages. Finally, the minimitivity labels correspond to the additivity labels of probabilities, where it is to be noted that complete additivity does not make sense for probabilities.

[3] If possibility theory is interpreted in terms of uncertainty rather than imprecision, one can define a notion of belief—positive degree of necessity, or equivalently, degree of possibility smaller than 1—that is consistent and deductively closed in the finite, though not in the countable sense.

3. Extending rankings on languages

In probability theory we can start with a probability Pr on a language $\mathcal{L}$, i.e., a function assigning non-negative real numbers to all sentences in $\mathcal{L}$ such that logically equivalent sentences are assigned the same number, tautologies are assigned probability 1, and a disjunction of two logically incompatible sentences is assigned the sum of the probabilities of its two disjuncts. This probability Pr on $\mathcal{L}$ induces a finitely additive probability measure, in fact, a pre-probability Pr_0^* on the field $\mathcal{A} = \{Mod(\alpha)\colon \alpha \in \mathcal{L}\}$ by defining $\mathrm{Pr}_0^*(Mod(\alpha)) = \mathrm{Pr}(\alpha)$. By Carathéodory's theorem, Pr_0^* is then uniquely extended to a σ-additive probability measure Pr^* on the smallest σ-field $\sigma(\mathcal{A})$ containing $\mathcal{A}$.

More precisely, Carathéodory's theorem says that whenever we have a pre-probability, i.e., a finitely additive probability measure Pr_0^* on a finitary field $\mathcal{A}$ such that

$$\mathrm{Pr}_0^*\Big(\bigcup \mathcal{B}\Big) = \sum_{A \in \mathcal{B}} \mathrm{Pr}_0^*(A)$$

whenever $\mathcal{A}$ contains the union $\bigcup \mathcal{B}$ of a countable set $\mathcal{B}$ of disjoint elements $A \in \mathcal{A}$, then we are guaranteed the existence and uniqueness of a σ-additive Pr^* on $\sigma(\mathcal{A})$ that coincides with Pr_0^* on $\mathcal{A}$.

This is different in ranking theory. If we start with a ranking ϱ on a language $\mathcal{L}$, i.e., a function that assigns the same rank to logically equivalent sentences, that sends contradictions to ∞ and tautologies to 0, and that assigns to a disjunction as its rank the minimum of the ranks of the two disjuncts, then we also get a finitely minimitive ranking function, in fact, a pre-ranking ϱ_0^* on $\mathcal{A}$ by setting $\varrho_0^*(Mod(\alpha)) = \varrho(\alpha)$. However, there may be uncountably many pairs of σ-minimitive (and also completely minimitive) ranking functions ϱ_1^*, ϱ_2^* on $\sigma(\mathcal{A})$ that extend ϱ_0^*, i.e., $\varrho_0^*(A) = \varrho_1^*(A) = \varrho_2^*(A)$ for every $A \in \mathcal{A}$, but that are not even ordinally equivalent in the sense that there are $B, C \in \sigma(\mathcal{A})$ such that $\varrho_1^*(B) \leqslant \varrho_1^*(C)$ and $\varrho_2^*(B) > \varrho_2^*(C)$. This is shown by the following example.

Example 1 *(No unique extension).* The first example shows that a regular pre-ranking cannot always be uniquely extended to a σ-minimitive ranking function. This means in particular that there need not be a unique pointwise ranking function inducing a given pre-ranking.

Consider the smallest set of wffs closed under the propositional connectives $\neg$ and $\wedge$ (with $\vee$, $\rightarrow$, and $\leftrightarrow$ defined in the usual way) and containing the set of propositional variables $PV = \{p_i\colon i \in N\}$. ϱ on $\mathcal{L}$ is defined by assigning each consistent sentence rank 0, and contradictions are sent to ∞. As mentioned, ϱ induces a finitely minimitive ranking function ϱ_0^* on $\mathcal{A} = \{Mod(\alpha)\colon \alpha \in \mathcal{L}\}$ by defining $\varrho_0^*(Mod(\alpha)) = \varrho(\alpha)$. Indeed, ϱ_0^* is a regular pre-ranking. Note that for every $\alpha \in \mathcal{L}$, $Mod(\alpha) \in \mathcal{A}$ is either empty or uncountable.

The smallest σ-field $\sigma(\mathcal{A})$ containing $\mathcal{A}$ has as elements, among others, the singletons containing ω, for every $\omega \in Mod_{\mathcal{L}}$, because

$$\{\omega\} = \bigcap \big\{Mod(\alpha) \in \mathcal{A}\colon \omega \models \alpha, \alpha \in \mathcal{L}\big\} \in \sigma(\mathcal{A})$$

(there are but countably many wffs $\alpha \in \mathcal{L}$, so this is an intersection of countably many elements of $\mathcal{A}$). Now consider any of the uncountably many countable subset S of $Mod_{\mathcal{L}}$, and let κ be any pointwise ranking function on $Mod_{\mathcal{L}}$ such that $\kappa(\omega) > 0$ for $\omega \in S$, and $\kappa(\omega) = 0$ for $\omega \in Mod_{\mathcal{L}} \setminus S$. $\varrho_\kappa(Mod(\alpha)) = 0 = \varrho_0^*(Mod(\alpha))$ for every non-empty $Mod(\alpha) \in \mathcal{A}$, and $\varrho_\kappa(\emptyset) = \infty = \varrho_0^*(\emptyset)$.

Still, one might argue, the interesting question is not uniqueness, but whether there *exists* a pointwise ranking function that induces the pre-ranking ϱ_0^* one started with. In case of existence, one can further ask whether there is a unique *minimal* pointwise ranking function κ^* that induces the pre-ranking ϱ_0^*, i.e., a pointwise ranking function κ^* inducing ϱ_0^* and such that no pointwise ranking function κ with $\kappa(\omega) < \kappa^*(\omega)$, for some $\omega \in W$, also induces κ_0^*. As shown by the following example, one cannot expect there to be a *natural* pointwise ranking function inducing the pre-ranking ϱ_0^*, even if ϱ_0^* is regular.

Example 2 *(No regular σ-minimitive and no natural pointwise extension).* The second example shows that a regular pre-ranking cannot always be extended to a regular and σ-minimitive ranking function. This means in particular that a regular pre-ranking need not be induced by a natural pointwise ranking function.

For PV, $\mathcal{L}$, and $\mathcal{A}$ as in Example 1, let ϱ be defined as follows:

$$\varrho(p_i) = i+1,$$

$$\varrho(\neg p_i) = 0,$$

$$\varrho(\pm p_{i_1} \wedge \cdots \wedge \pm p_{i_n}) = \begin{cases} \max\{\varrho(p_{i_j})\colon \pm p_{i_j} = p_{i_j}, 1 \leqslant j \leqslant n\}, & \text{if } \pm p_{i_1} \wedge \cdots \wedge \pm p_{i_n} \nvdash \bot, \\ \infty, & \text{if } \pm p_{i_1} \wedge \cdots \wedge \pm p_{i_n} \vdash \bot, \end{cases}$$

$$\varrho(\alpha_1 \vee \cdots \vee \alpha_n) = \min\{\varrho(\alpha_i)\colon 1 \leqslant i \leqslant n\},$$

where $\max \emptyset = 0$. By putting every wff $\alpha \in \mathcal{L}$ into disjunctive normal form we get a regular ranking on $\mathcal{L}$, and hence a regular pre-ranking ϱ_0^* on $\mathcal{A}$. However, in order to extend ϱ_0^* to a σ-minimitive ranking function on $\sigma(\mathcal{A})$—and hence also in order for ϱ_0^* to be induced by a pointwise ranking function on $Mod_{\mathcal{L}}$—all but countably many of the (singletons $\{\omega\}$ containing the) possibilities $\omega \in Mod_{\mathcal{L}}$ must be sent to ∞.

This is seen as follows: Every $\omega \in Mod_{\mathcal{L}}$ can be represented by an infinite sequence $\omega = \langle \pm p_1, \ldots, \pm p_n, \ldots \rangle$, where $+p_n$ means $\omega(p_n) = 1$, and $-p_n$ means $\omega(p_n) = 0$. If there are infinitely many $i \in N$ such that $\omega(p_i) = 1$, then ω must get rank ∞. (Suppose the rank of ω is $n < \infty$. Then there is $m \geqslant n$ such that $\omega(p_m) = 1$. $\varrho_0^*(Mod(p_m)) = m+1 > n$, although $\omega \models p_m$—a contradiction.) So ω has a finite rank only if $\omega(p_i) = 0$ for all but finitely many $i \in N$. For each $n \in N$ there are but countably many ωs such that $\omega(p_i) = 1$ for exactly n natural numbers $i \in N$. So there are only countably many ωs with $\omega(p_i) = 1$ for all but finitely many $i \in N$, and hence only countably many ωs with a finite rank.

Still, one might continue to argue, the naturalness of pointwise ranking functions—in contrast to the regularity of rankings—is too restrictive anyway, and the above example is not sufficient to rule out the existence of an "unnatural" pointwise ranking function that induces ϱ_0^*. After all, the important thing is that we do not send any consistent sentence from $\mathcal{L}$ or any non-empty proposition from $\mathcal{A}$ to ∞, even though we may have to consider some possibilities as virtually impossible. This is a familiar phenomenon from probability theory, where the Lebesgue measure on the σ-field of Borel sets over the reals assigns any singleton containing a real number—indeed, any countable set of real numbers—measure 0, though no non-trivial interval gets Lebesgue measure 0.

So, when we start with a ranking ϱ on $\mathcal{L}$, and thus get a pre-ranking ϱ_0^* on $\mathcal{A}$, is it the case that we always get a unique minimal pointwise ranking function κ^* on $Mod_{\mathcal{L}}$ that induces ϱ_0^* on $\mathcal{A}$, and hence ϱ on $\mathcal{L}$, even though one is sometimes forced to send some possibilities $\omega \in Mod_{\mathcal{L}}$ to ∞? The answer is given by

Theorem 1 *(Extension theorem for rankings on languages). Let $\mathcal{L}$ be a language, i.e., a countable set of wffs closed under negation and conjunction, and let ϱ be a ranking on $\mathcal{L}$ so that ϱ_0^* is a pre-ranking on the field $\mathcal{A} = \{Mod(\alpha)\colon \alpha \in \mathcal{L}\}$, where $\varrho_0^*(Mod(\alpha)) = \varrho(\alpha)$.*

Then there is a unique minimal pointwise ranking function κ^* on $Mod_{\mathcal{L}}$ that induces ϱ_0^*. That is, $\varrho_0^*(A) = \min\{\kappa^*(\omega)\colon \omega \in A\}$ for every non-empty $A \in \mathcal{A}$; and for every pointwise ranking function κ on $Mod_{\mathcal{L}}$ such that $\kappa(\omega) < \kappa^*(\omega)$ for at least one $\omega \in Mod_{\mathcal{L}}$, $\varrho_0^*(A) \neq \min\{\kappa(\omega)\colon \omega \in A\}$ for some $A \in \mathcal{A}$.

Proof. Let $A_1 = Mod(\alpha_1), \ldots, A_n = Mod(\alpha_n), \ldots$ be an enumeration of all the countably many elements of $\mathcal{A}$, and define κ_n^* as follows:

$$\kappa_n^*(\omega) = \varrho_0^*\big((\pm A_1 \cap \cdots \cap \pm A_n)_\omega\big),$$

where $(\pm A_1 \cap \cdots \cap \pm A_n)_\omega$ is the unique element of the finite partition

$$P_n = \{\pm A_1 \cap \cdots \cap \pm A_n\} \subseteq \mathcal{A}$$

of $W = Mod_{\mathcal{L}}$ such that $\omega \in (\pm A_1 \cap \cdots \cap \pm A_n)_\omega$. For each $\omega \in W$, $\kappa_1^*(\omega), \ldots, \kappa_n^*(\omega), \ldots$ is a non-decreasing sequence of natural numbers, i.e., $\kappa_m^*(\omega) \leqslant \kappa_n^*(\omega)$ for $m \leqslant n$. $\kappa^*(\omega)$ is defined as the limit of this sequence, if this limit exists, and as ∞ otherwise, i.e., $\kappa^*(\omega) = \lim_{n\to\infty} \kappa_n^*(\omega)$.

We first show that κ^* is a pointwise ranking function on W, i.e., that at least one $\omega \in W$ is assigned κ^*-rank 0. Either $\varrho_0^*(A_1) = 0$ or $\varrho_0^*(\overline{A_1}) = 0$. Let $B_1 = A_1$, if $\varrho_0^*(A_1) = 0$, and $B_1 = \overline{A_1}$ otherwise. Hence

$$\varrho_0^*(B_1) = 0 = \min\{\varrho_0^*(B_1 \cap A_2), \varrho_0^*(B_1 \cap \overline{A_2})\}.$$

Let $B_2 = A_2$, if $\varrho_0^*(B_1 \cap A_2) = 0$, and $B_2 = \overline{A_2}$ otherwise. In general, let $B_n = A_n$, if $\varrho_0^*(B_1 \cap \dots \cap B_{n-1}) = 0 = \varrho_0^*(B_1 \cap \dots \cap B_{n-1} \cap A_n)$, and $B_n = \overline{A_n}$ otherwise. So for each n,

$$\varrho_0^*(B_1 \cap \dots \cap B_n) = 0 = \kappa_n^*(\omega) \quad \text{for all } \omega \in B_1 \cap \dots \cap B_n.$$

As $\kappa_n^{*-1}(0) \supseteq B_1 \cap \dots \cap B_n$, for each n, we have $\kappa^{*-1}(0) = \bigcap_{n=1}^{\infty} \kappa_n^{*-1}(0) \supseteq \bigcap_{n=1}^{\infty} B_n$. It remains to be shown that $\bigcap_{n=1}^{\infty} B_n \neq \emptyset$. Suppose for reductio that $\bigcap_{n=1}^{\infty} B_n = \emptyset$. This means that the set of wffs $B = \{\beta_i \in \mathcal{L}\colon Mod(\beta_i) = B_i\}$ is inconsistent. By the compactness of classical logic, there is a finite subset $B_{\text{fin}} = \{\beta_{i_1}, \dots, \beta_{i_n}\} \subseteq B$ that is inconsistent, i.e.,

$$\bigcap_{j=1}^{n} \{Mod(\beta_{i_j}) \in \mathcal{A}\colon 1 \leqslant j \leqslant n\} = \emptyset.$$

Let $m = \max\{i_j\colon 1 \leqslant j \leqslant n\}$. Then $B_1 \cap \dots \cap B_m = \emptyset$, and, by construction of the B_i, $\varrho_0^*(B_1 \cap \dots \cap B_m) = 0$—a contradiction.

So κ^* is a pointwise ranking function on W: κ^* sends at least one ω to 0, but it may send uncountably many ωs to ∞. (For each $n \in N$, κ_n^* is a natural pointwise ranking function on W that sends uncountably many ωs to 0.) Let us show next that κ^* induces ϱ_0^*, i.e., for every non-empty $A \in \mathcal{A}$:

$$\varrho_0^*(A) = \min\{\kappa^*(\omega)\colon \omega \in A\}.$$

For every $A \in \mathcal{A}$ there is an m_A such that for all $n \geqslant m_A$, A is equal to the finite union of all (at most 2^n) elements of P_n that are subsets of A. Let $\varrho_0^*(A) = r \in N \cup \{\infty\}$. By finite minimitivity,

$$\begin{aligned}\varrho_0^*(A) &= \varrho_0^*\Big(\bigcup\{\pm A_1 \cap \dots \cap \pm A_{m_A} \in P_{m_A}\colon \pm A_1 \cap \dots \cap \pm A_{m_A} \subseteq A\}\Big)\\ &= \min\{\varrho_0^*(\pm A_1 \cap \dots \cap \pm A_{m_A})\colon P_{m_A} \ni \pm A_1 \cap \dots \cap \pm A_{m_A} \subseteq A\}.\end{aligned}$$

Let $D_1, \dots, D_l$ be the $l \leqslant 2^{m_A}$ disjoint "disjuncts" $\pm A_1 \cap \dots \cap \pm A_{m_A} \subseteq A$ in this union, and pick any $A' := \pm A_1 \cap \dots \cap A_{m_A}$ such that $\varrho_0^*(A) = \varrho_0^*(A')$. For each n, each of the $l \cdot 2^n$ elements of P_{m_A+n} whose union is equal to A, and each i, $1 \leqslant i \leqslant l$:

$$\begin{aligned}\varrho_0^*(A) = \varrho_0^*(A') &\leqslant \varrho_0^*(D_i \cap \pm A_{m_A+1} \cap \dots \cap \pm A_{m_A+n})\\ &= \kappa_{m_A+n}(\omega) \quad \text{for all } \omega \in D_i \cap \pm A_{m_A+1} \cap \dots \cap \pm A_{m_A+n}.\end{aligned}$$

As each $\omega \in A$ is in exactly one $D_i \cap \pm A_{m_A+1} \cap \dots \cap \pm A_{m_A+n}$ we have for every n and every $\omega \in A$:

$$\varrho_0^*(A) \leqslant \kappa_{m_A+n}^*(\omega) \leqslant \lim_{n\to\infty} \kappa_n^*(\omega).$$

If $\varrho_0^*(A) = \infty$, we are already done. So suppose $\varrho_0^*(A) = r < \infty$, whence A is non-empty. As before,

$$\varrho_0^*(A) = \varrho_0^*(A') = \min\{\varrho_0^*(A' \cap A_{m_A+1}), \varrho_0^*(A' \cap \overline{A_{m_A+1}})\}.$$

Let $C_1 = A_{m_A+1}$, if $\varrho_0^*(A') = \varrho_0^*(A' \cap A_{m_A+1})$, and let $C_1 = \overline{A_{m_A+1}}$ otherwise. In general, let $C_{n+1} = A_{m_A+n+1}$, if

$$\varrho_0^*(A' \cap C_{m_A+1} \cap \dots \cap C_{m_A+n}) = \varrho_0^*(A' \cap C_{m_A+1} \cap \dots \cap C_{m_A+n} \cap A_{m_A+n+1}),$$

and $C_{n+1} = \overline{A_{m_A+n+1}}$ otherwise. Then we have for each n:

$$\varrho_0^*(A) = \varrho_0^*(A' \cap C_1 \cap \dots \cap C_n) = \kappa_{m_A+n}^*(\omega) = r \quad \text{for all } \omega \in A' \cap C_1 \cap \dots \cap C_n.$$

As $\kappa_{m_A+n}^{*-1}(r) \supseteq A' \cap \bigcap_{i=1}^{n} C_i$, for each n, we have $\kappa^{*-1}(r) = \bigcap_{n=1}^{\infty} \kappa_n^{*-1}(r) \supseteq A' \cap \bigcap_{n=1}^{\infty} C_n$. We only have to show that $A' \cap \bigcap_{n=1}^{\infty} C_n \neq \emptyset$; for then $\kappa^*(\omega) = r = \varrho_0^*(A)$ for at least one $\omega \in A$. As before, suppose for reductio that $A' \cap \bigcap_{n=1}^{\infty} C_n = \emptyset$. Then the set of wffs

$$C = \{\alpha' \in \mathcal{L}\colon A' = Mod(\alpha')\} \cup \{\gamma_n \in \mathcal{L}\colon C_n = Mod(\gamma_n), n \in N\}$$

is inconsistent. By the compactness of classical logic, there is a finite subset $C_{fin} = \{\alpha', \gamma_{i_1}, \dots, \gamma_{i_n}\} \subseteq C$ that is inconsistent, which implies that $A' \cap C_1 \cap \dots \cap C_m = \emptyset$, where $m = \max\{i_j\colon 1 \leqslant j \leqslant n\}$. But by construction of the C_n, $\varrho_0^*(A' \cap C_1 \cap \dots \cap C_m) = r < \infty$—a contradiction.

It remains to be shown that κ^* is minimal. Suppose there is a pointwise ranking function κ on W such that $\kappa(\omega) < \kappa^*(\omega)$ for some $\omega \in W$. This means $\kappa(\omega) < \lim_{n\to\infty}\kappa_n^*(\omega)$, where $\kappa^*(\omega) = \infty$ if this limit does not exist. If this limit exists, there is n such that for all $m \geqslant n$, $\kappa(\omega) < \kappa_n^*(\omega) = \kappa_m^*(\omega) < \infty$. If this limit does not exist, then for each n there is $m > n$ such that $\kappa_n^*(\omega) < \kappa_m^*(\omega) < \infty$ (remember: κ_m^* is a natural pointwise ranking function, for each $m \in N$). So in both cases there is n such that $\kappa(\omega) < \kappa_n^*(\omega) < \infty$. As $\kappa_n^*(\omega) = \varrho_0^*(A')$ for that element $A' := \pm A_1 \cap \cdots \cap \pm A_n$ of P_n such that $\omega \in A'$, we have $\kappa(\omega) < \varrho_0^*(A')$ for some $\omega \in A' \in \mathcal{A}$. Hence κ does not induce ϱ_0^*. □

Theorem 1 is encouraging, but does not extend to pre-rankings on arbitrary fields.

Example 3 *(No pointwise extension on arbitrary fields).* The third example shows that a regular and σ-minimitive ranking function on a σ-field cannot always be induced by a pointwise ranking function. This means in particular that a regular pre-ranking on a field need not be induced by a pointwise ranking function.

Let the σ-field over $\mathfrak{R}$ be

$$\mathcal{R} = \{A \subseteq \mathfrak{R}\colon\ A \text{ is countable or } \overline{A} \text{ is countable}\},$$

and let $\varrho(A) = \infty$, if A is empty, $\varrho(A) = 1$ if A is non-empty and countable, and $\varrho(A) = 0$ if A is uncountable. ϱ is a regular and σ-minimitive ranking function: $\varrho(\emptyset) = \infty$, $\varrho(\mathfrak{R}) = 0$, and for every countable $\mathcal{B} \subseteq R$, $\varrho(\bigcup\mathcal{B}) = \min\{\varrho(A)\colon\ A \in \mathcal{B}\}$. This is seen as follows: If $\bigcup\mathcal{B}$ is empty, then so is every $A \in \mathcal{B}$; and if $\bigcup\mathcal{B}$ is non-empty and countable, then every $A \in \mathcal{B}$ is countable, and at least one $A \in \mathcal{B}$ is non-empty. Finally, if $\bigcup\mathcal{B}$ is uncountable, then at least one $A \in \mathcal{B}$ must be uncountable, too.

Clearly ϱ cannot be induced by a pointwise ranking function κ. $\varrho(\{r\}) = 1$, and so $\kappa(r) = 1$ for every $r \in \mathfrak{R}$. But then $\min\{\kappa(r)\colon\ r \in \mathfrak{R}\} = 1 > \varrho(\mathfrak{R})$.

Note, though, that Example 3 leaves open the question whether a pre-ranking on a field $\mathcal{A}$ can be extended to a σ-minimitive ranking function on $\sigma(\mathcal{A})$.

Given that logically equivalent sentences are assigned the same rank, it might seem it should not matter whether one works with rankings on languages or ranking functions on fields. However, the above shows that this is not quite correct. The propositions on a set of models $Mod_{\mathcal{L}}$ induced by the sentences of a language $\mathcal{L}$ are not just any subsets of an arbitrary set of possibilities W—as they often are when one considers measure-theoretic fields in general. Rather, they come with their own structure—most notably, closure under finite intersections only and compactness—that is inherited from the structure of $\mathcal{L}$. Ranking functions behave nicely on this structure, but they do not do so in general. Assuming that we believe in representations of propositions, say sentences, and not propositions themselves—that is, assuming that belief is a sentential or representational, and not a propositional attitude—and assuming that the structure of its objects is of importance for the representation of belief, this might be taken to be another reason for modeling epistemic states by ranking functions.

There are several other areas where one needs finitely minimitive ranking functions. They are a *sine qua non* when one wants to have the reals as range (or some other set of numbers that is not well-ordered by the smaller-than relation $<$). The reason is that in this case the minimum of a sequence of real-valued ranks need not exist.

As is well known, the lottery-paradox [11] does not arise for ranking functions ϱ_κ induced by pointwise ranking functions κ. Considering a lottery with n tickets where exactly one ticket wins, we have as set of possibilities the set $W_n = \{\omega_i\colon\ i \leqslant n, i \in N\}$, where ω_i is the possibility that ticket i will win (the field is the powerset of W). By definition, a pointwise ranking function assigns rank 0 to at least one possibility $\omega_i \in W_n$. Hence one cannot model the situation that somebody believes of every ticket that it will not win, i.e., $\varrho_\kappa(\{\omega_i\}) > 0$ for every $\omega_i \in W_n$. If, on the other hand, one allows sending all possibilities to a rank greater than 0, then one cannot model the situation that one believes that some ticket will win, i.e., $\varrho_\kappa(\emptyset) > 0$ and $\varrho_\kappa(W_n) = 0$.

In the finite case this is true for arbitrary ranking functions. However, if we turn to an infinite lottery with countably many tickets, the set of possibilities is $W_\infty = \{\omega_i\colon\ i \in N\}$ (we take as field the powerset of W_∞). Now we can send every singleton $\{\omega_i\}$ to a rank greater than 0 and still get a finitely minimitive ranking function that assigns rank 0 to W_∞. For instance, we can assign rank 0 to A whenever A is not finite—say because we go by the slogan: plausibility is cardinality of the set of possibilities; and whenever A is finite, we assign it the minimum of the ranks $\varrho(\{\omega_i\})$, for all possibilities ω_i in A (whatever these singleton ranks are). Then we have a finitely minimitive ranking function that is compatible with any ranking of the singletons $\{\omega_i\}$. In particular, if we believe, for every ticket in this infinite lottery, that it will not win, i.e., $\varrho(\{\omega_i\}) > 0$ for every $\omega_i \in W$, we can nevertheless be maximally convinced that some ticket will win: $\varrho(\emptyset) = \infty$ and $\varrho(W_\infty) = 0$. This is not possible for a ranking function ϱ_κ induced by a pointwise ranking function κ. We can have the above ranking with 0 for every infinite A only if we send at most finitely many ω_is to a rank greater than 0. Similarly for pre-rankings.

4. Probabilities, entrenchments, rankings

Specifying a pointwise ranking function over uncountably many possible worlds is not feasible. In view of this fact it might be surprising that there are applications in artificial intelligence (e.g. [2,9]) that apparently do work with pointwise ranking functions. However, these applications actually work with ranking functions on fields, which are trivially induced by pointwise ranking functions as long as the set of possibilities is finite—and the languages and sets of possible worlds considered in the above mentioned literature are finite so that each possible world corresponds to a sentence.

Ranking theory is a middle course between probabilistic and logical approaches to the representation of partial belief and belief revision—in the sense that ranking functions are measured on a proportional scale, whereas probabilities are measured on an absolute scale, and entrenchments on an ordinal scale.[4] In the literature on AGM belief revision theory [1,6] the objects of belief are sentences—or, because of extensionality, the propositions expressed by these sentences (though not any sets of possibilities). These logical accounts enable one to express that A is more entrenched or believed than B, and that B is more believed than C. But in this framework an epistemic agent is not allowed to quantify the strength of her beliefs. Indeed, she cannot even say that the difference between the strengths of her beliefs in A and B is greater than the difference between the strengths of her beliefs in B and C. Probabilistic accounts more or less share the objects of belief (though the focus is more on the semantic side, and any set of possibilities can be a proposition), but require the epistemic agent to have precise numerical degrees of belief. Ranking theory is a moderate middle course: The epistemic agent can say whether A is more believed than B and that B is in turn more believed than C. In addition, the epistemic agent can express that the difference between her grades of belief in A and B is greater than the difference between her grades of belief in B and C without having to specify with complete accuracy a numerical degree of belief for each of A, B, C. More precisely, the agent can express her grades of belief as multiples of some minimally positive grade of belief.[5]

Given this ranking theory should be welcomed by both subjective probabilists and epistemic logicians. As a matter of fact, however, neither is the case. Logicians object that it is a mystery where the numbers (ranks) come from (see, however, [19]), and probabilists complain about the ordinal nature of the ranking apparatus. Yet there is one feature that is shared by both probabilistic and logical accounts of partial belief and belief revision, but that is not present in pointwise ranking theory: In both approaches the objects of belief are sentences or propositions, whereas in Spohnian pointwise ranking theory the objects of belief are the possible worlds one level below. So by formulating ranking theory in terms of ranking functions on a field and rankings on languages we simultaneously approach probabilistic as well as logical accounts; and we also get rid of the ideal of specifying a ranking over all possible worlds, a requirement no real-world epistemic agent could ever meet.[6]

Continuing this comparison we note that probabilists have the notions of positive and negative relevance and of independence between propositions, which seem to be of utmost importance.[7] Furthermore, they have a way of revising one's epistemic state represented by a probability measure over a field $\mathcal{A}$, viz. Jeffrey conditionalisation,

[4] I am grateful to an anonymous referee for pressing me further on this point.

[5] The epistemic logician will note that the ordering $\alpha < \beta \Leftrightarrow \varrho(\neg\alpha) < \varrho(\neg\beta)$ satisfies all conditions for entrenchment orderings mentioned in Section 4.2 of [7], with $K = \{\alpha \in \mathcal{L}\colon \varrho(\neg\alpha) > 0\}$.

[6] In his [19] Spohn presents the theory of measurement for his ranking theory, but does so only for the finite case. It should be clear that a theory of measurement for σ-minimitive, let alone completely minimitive or pointwise ranking functions also covering the infinite case is inapplicable. One necessary condition for an ordering of disbelief to be represented by a σ-minimitive (or completely minimitive or pointwise) ranking function is that whenever A is not more disbelieved than any of infinitely many propositions B_i, then A is not less disbelieved than the union $\bigcup_{i\in N} B_i$ of all these propositions B_i. For finitely minimitive ranking functions and rankings on languages this condition reduces to the following finite version: Whenever A is not less disbelieved than either one of B and C, then A is not less disbelieved than $B \cup C$.

[7] Conditional probabilistic independence and its (incomplete) axiomatization, the (semi-)graphoid axioms, started to become of interest with [3,14,15]. Judea Pearl and his group at UCLA started to work with independence in the eighties (e.g. [8,12,13]); for a survey see [18] or [4]. A lot of work on axiomatizing independence has been done by Milan Studený (e.g. [20]).

when the incoming evidence is represented by a probability measure over a subfield of $\mathcal{A}$. Logicians neither have the notions of positive and negative relevance and independence nor do they have an appropriate way of updating their epistemic state represented by a selection function or an entrenchment ordering. Pointwise ranking theory has both of these desirable features [16], and the question is whether they are preserved when we generalize these to ranking functions on fields. The answer is that they are. Copying from Spohn [19], A is *positively relevant for/independent of/negatively relevant for* B *given* C in the sense of the ranking function ϱ iff

$$\varrho(A \cap B \mid C) + \varrho(\overline{A} \cap \overline{B} \mid C) \stackrel{\leq}{\underset{>}{=}} \varrho(A \cap \overline{B} \mid C) + \varrho(\overline{A} \cap B \mid C).$$

If $\varrho : \mathcal{A} \to N \cup \{\infty\}$ is the agent's ranking function on the field $\mathcal{A}$ over W at time t, and between t and t' the agent's ranking function on the field $\mathcal{E} \subseteq A$ changes to $\varrho' : \mathcal{E} \to N \cup \{\infty\}$, and the agent's ranking function does not change on any field $\mathcal{B}$ such that $\mathcal{E} \subset B \subseteq A$, then the agent's ranking function on $\mathcal{A}$ at time t' should be $\varrho_{\varrho \to \varrho'} : \mathcal{A} \to N \cup \{\infty\}$,

$$\varrho_{\varrho \to \varrho'}(\cdot) = \min\{\kappa(\cdot \mid E_i) + \varrho'(E_i) \colon i \in I\},$$

where $\{E_i \in \mathcal{E} \colon i \in I\}$ is a partition of W for which there is no finer partition $\{E_j \in \mathcal{E} \colon j \in J\}$, and I, J are any index sets.

On the other hand, epistemic logicians have the notion of a belief set that is consistent and deductively closed [10]. As shown by the lottery paradox, there is no $\varepsilon > 0$ such that the set of all propositions A with $\Pr(A) \geqslant 1 - \varepsilon$ is deductively closed and consistent. So probabilists lack the notion of a belief set (as long as belief is sufficiently high degree of belief). Any pointwise ranking function κ gives rise to a belief set $Bel = \{A \in \mathcal{A} \colon \varrho_\kappa(\overline{A}) > 0\}$ which is consistent and deductively closed in the following complete sense (even if Bel is uncountable): $\bigcap Bel \neq \emptyset$, and for every $A \in \mathcal{A}$: $A \in Bel$ whenever $\bigcap Bel \subseteq A$.

We have already noted in Section 2 that the same holds true for ranking functions on fields, and conclude by working out this observation for rankings on languages. The belief set $Bel = \{\alpha \in \mathcal{L} \colon \varrho(\neg\alpha) > 0\}$ induced by a ranking ϱ on $\mathcal{L}$ is consistent and deductively closed in the classical finite sense. If $Bel \vdash \beta$, for some $\beta \in \mathcal{L}$, then, by the compactness of classical logic, there is a finite $Bel_{fin} \subseteq Bel$ such that $Bel_{fin} \vdash \beta$. Let $Bel_{fin} = \{\alpha_1, \ldots, \alpha_n\}$. Then $\neg\beta \vdash \neg\alpha_1 \vee \cdots \vee \neg\alpha_n$. $\varrho(\neg\beta) \geqslant \varrho(\neg\alpha_1 \vee \cdots \vee \neg\alpha_n)$ by Observation 1 for rankings on languages, and $\varrho(\neg\alpha_1 \vee \cdots \vee \neg\alpha_n) = \min\{\varrho(\neg\alpha_i) \colon 1 \leqslant i \leqslant n, i \in N\}$ by clause 3 in the definition of rankings on languages. Hence $\varrho(\neg\beta) > 0$, i.e., $\beta \in Bel$. As to consistency, suppose for reductio that Bel is inconsistent. Then $Bel \vdash \bot$, which means $\varrho(\top) > 0$—in contradiction to clause 2 in the definition of rankings on languages.

5. Conclusion

In this paper we have generalized pointwise ranking functions on sets of possibilities to ranking functions on fields of propositions and rankings on languages. In doing so we have kept the important notions of positive and negative relevance as well as independence. Through the belief set induced by a ranking function, we also save the link between belief and degrees of belief—the very feature distinguishing ranking theory from other theories of degrees of belief.[8] Finally, Theorem 1 and Examples 1–3 from Section 3 clarify the conditions under which ranking functions and rankings on languages are induced by pointwise ranking functions.

Acknowledgements

I am grateful to Wolfgang Spohn for comments on an earlier version of this paper, as well as to three anonymous referees for their constructive criticism.

My research was in part supported by the Alexander von Humboldt Foundation, the Federal Ministry of Education and Research, and the Program for the Investment in the Future (ZIP) of the German Government through a Sofja Kovalevskaja Award to Luc Bovens, while I was a member of the *Philosophy, Probability, and Modeling* group at the Center for Junior Research Fellows at the University of Konstanz.

[8] Cf., however, Footnote 3 in Section 2.

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