# Redundancy in Logic III: Non-Mononotonic Reasoning 

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#### Abstract

Results about the redundancy of circumscriptive and default theories are presented. In particular, the complexity of establishing whether a given theory is redundant is establihsed.


## 1 Introduction

In this paper, we study the problem of whether a circumscriptive [McC80] or default [Rei80] theory is redundant, that is, it contains unnecessary parts. Formally, a theory is redundant if it is equivalent to one of its proper subsets; a part is redundant in a theory if the theory is not semantically changed by the removal of the part. The redundancy of propositional theories in CNF, 2CNF, and Horn form has already been analyzed in other papers [Lib05b, Lib05c] where motivations are also given. Other problems related to redundancy have been considered by various authors [Gin88, SS97, MS72, Mai80, ADS86, HK93, HW97, LM00, Uma98, GF93, BZ05, PW88, FKS02, Bru03].

Circumscription and default logic are two forms of non-monotonic reasoning, as opposite to classical logic, which is monotonic. A logic is monotonic if the consequences of a set of formulae monotonically non-decrease with the set. In other words, all formulae that are entailed by a set are also entailed by every superset of it. Circumscription and default logic do not have this property, and are therefore non-monotonic.

The difference between monotonic and non-monotonic logic is important in the study of redundancy. In propositional logic, as in all forms of monotonic logic, if a set does not entail a formula, the same is true for all of its subsets. As a result, if $\Pi$ is a set of clauses, $\gamma$ is one of its clauses, and $\Pi \backslash\{\gamma\}$ is not equivalent to $\Pi$, no subset of $\Pi \backslash\{\gamma\}$ is equivalent to $\Pi$. In the other way around, if all clauses of a set of clauses $\Pi$ are irredundant, then $\Pi$ is irredundant. We call this property local redundancy. The converse of this property is obviously true: if a clause is redundant in a formula, the formula is redundant because it is equivalent to the subset composed of all its clauses but the redundant one.

Local redundancy holds for all monotonic logic. In nonmonotonic logics, removing a clause from a formula might result in a decrease of the set of consequences, which can

[^0]however grow to the original one when another clause is further removed. However, some nonmonotonic logics have the local redundancy property. We prove that local redundancy holds for circumscriptive entailment and for the redundancy of the background theory in default logic when all defaults are categorical (prerequisite-free) and normal. In the general case, default logic does not have the local redundancy property.

Since redundancy is defined in terms of equivalence (namely, equivalence of a formula with a proper subset of it), it is affected by the kind of equivalence used. In particular, equivalence can be defined in two ways for default logic: equality of extensions and equality of consequences. This lead to two different definitions of redundancy in default logic.

Regarding the complexity results, we show that checking whether a clause is redundant in a formula according to circumscriptive inference is $\Pi_{2}^{p}$-complete. For default logic, we mainly considered redundancy in the background theory according to Reiter semantic using both kinds of equivalence, but we also considered justified default logic [Luk88], constrained default logic [Sch92, DSJ94], and rational default logic [MT95]. The results are as follows: the redundancy of a clause in the background theory is $\Pi_{2}^{p}$-complete and $\Pi_{3}^{p}$-complete for equivalence based on extensions and consequences, respectively. The problems of redundancy of the background theory are $\Sigma_{3}^{p}$-complete and $\Sigma_{4}^{p}$-complete, respectively. The proofs of the latter two results are of some interest, as they are done by first showing that the problems are $\Pi_{2}^{p}$-complete and $\Pi_{3}^{p}$-complete, respectively, and then showing that such complexity results can be raised of one level in the polynomial hierarchy. This technique allows for a proof of hardness for a class such as $\Sigma_{4}^{p}$ without involving complicated QBFs such as $\exists W \forall X \exists Y \forall Z . F$.

We also considered the redundancy of defaults in a default theory. We show that these problems are at least as hard as the corresponding problems for the redundancy of the background theory for Reiter and justified default logics.

## 2 Preliminaries

If $\Pi$ and $\Gamma$ are sets, $\Pi \backslash \Gamma$ denotes the set of elements that are in $\Pi$ but not in $\Gamma$. This operator is often called set subtraction, because the elements of $\Gamma$ are "subtracted" from $\Pi$. An alternative definition of this operator is: $\Pi \backslash \Gamma=\Pi \cap \bar{\Gamma}$, where $\bar{\Gamma}$ is the complement of $\Gamma$.

All formulae considered in this paper are propositional and finite Boolean formulae over a finite alphabet. We typically use formulae in CNF, that is, sets of clauses. We simply refer to sets of clauses as formulae. We assume that no clause is tautological (e.g., $x \vee \neg x$ ): formulae containing tautological clauses can be simplified in linear time. By $\operatorname{Var}(\Pi)$ we mean the set of variables mentioned in the formula $\Pi$.

In some places, we use the notation $\neg \gamma$, where $\gamma$ is a clause, to denote the formula $\{\neg l \mid l \in \gamma\}$. Note that $\gamma$ is a clause, while both $\{\gamma\}$ and $\neg \gamma$ are formulae (sets of clauses). A clause is positive if and only if it contains only positive literals.

A propositional model is an assignment from a set of propositional variables to the set \{true, false\}. We denote a model by the set of variables it assigns to true. We use the notation $\operatorname{Mod}(\Pi)$ to denote the set of models of a formula $\Pi$. We sometimes use models as formulae, e.g., $\Pi \wedge \omega$ where $\Pi$ is a formula and $\omega$ is a model. In the context where a formula is expected, a model $\omega$ represents the formula $\{x \mid x \in \omega\} \cup\{\neg x \mid x \notin \omega\}$. If $\Pi$ is a formula and $\omega_{X}$
is a model over the variables $X$, we denote by $\left.\Pi\right|_{\omega_{X}}$ the formula obtained by replacing each variable of $X$ with its value assigned by $\omega_{X}$ in $\Pi$.

A quasi-order is a reflexive and transitive relation (formally, a quasi-order is a pair composed of a set and a reflexive and transitive relation on this set, but the set will be implicit in this paper). The set containment relation $\subseteq$ among models is a quasi-order. According to our definition, a model is a set of positive literals; as a result $M \subseteq M^{\prime}$ holds if and only if $M$ assigns to false all variables that $M^{\prime}$ assign to false.

A clause is (classically) redundant in a CNF formula $\Pi$ if $\Pi \backslash\{\gamma\} \models \gamma$. A CNF formula is (classically) redundant if it is equivalent to one of its proper subsets. Propositional logic has the local redundancy property: a formula is redundant if and only if it contains a redundant clause. The local redundancy property is defined as follows.

Definition 1 (Local redundancy) A logic has the local redundancy property if, in this logic, a theory is redundant only if it contains a redundant clause.

Propositional logic has the local redundancy property. This is however not true for all logics.

## 3 Circumscription

Circumscriptive inference is based on the minimal models of a theory, i.e., the models that assign the maximum quantity of literals to false. Formally, we define the set of minimal models as follows.

Definition 2 The set of minimal models of a propositional formula $\Pi$, denoted by $\operatorname{CIRC}(\Pi)$, is defined as follows.

$$
\operatorname{CIRC}(\Pi)=\min _{\subseteq}(\operatorname{Mod}(\Pi))
$$

We define $\operatorname{CIRC}(\Pi)$ to be a set of models instead of a formula, although the latter is more common in the literature. Circumscriptive entailment is defined like classical entailment but only minimal models are taken into account.

Definition 3 The circumscriptive inference $\models_{M}$ is defined by: $\Pi \models_{M} \Gamma$ if and only if $\Gamma$ is satisfied by all minimal models of $\Pi$ :

$$
\Pi \models_{M} \Gamma \quad \text { if and only if } \quad \mathrm{CIRC}(\Pi) \subseteq \operatorname{Mod}(\Gamma)
$$

Equivalence in propositional logic can be defined in two equivalent ways: either by equality of the models or by equality of the sets of entailed formulae. These two definitions of equivalence coincide for circumscriptive inference as well. We define $\equiv_{M}$ as follows: $\Pi \equiv_{M} \Gamma$ if and only if $\operatorname{CIRC}(\Pi)=\operatorname{CIRC}(\Gamma)$. Redundancy of a clause is defined as follows.

Definition 4 A clause $\gamma \in \Pi$ is CIRC-redundant in the CNF formula $\Pi$ if and only if $\Pi \backslash\{\gamma\} \equiv_{M} \Pi$.

A formula is redundant if some of its clauses can be removed without changing its semantics.

Definition 5 A formula is CIRC-redundant if it is $\equiv_{M^{-}}$-equivalent to one of its proper subsets.

A formula is therefore redundant if some clauses can be removed from it while preserving equivalence. In the next section we show that a formula is CIRC-redundant if and only if it contains a CIRC-redundant clause, that is, circumscription has the local redundancy property.

### 3.1 Clause-Redundancy vs. Formula-Redundancy

Propositional logic has the local redundancy property. Showing why is interesting for comparison with logics not allowing the same proof to be used. If $\Pi$ does not contain a redundant clause, then $\Pi \backslash\{\gamma\} \not \equiv \Pi$ for any clause $\gamma \in \Pi$. Therefore, $\operatorname{Mod}(\Pi) \neq \operatorname{Mod}(\Pi \backslash\{\gamma\})$. Since $\Pi \backslash\{\gamma\}$ is a subset of $\Pi$, we have $\operatorname{Mod}(\Pi) \subseteq \operatorname{Mod}(\Pi \backslash\{\gamma\})$ in general and $\operatorname{Mod}(\Pi) \subset$ $\operatorname{Mod}(\Pi \backslash\{\gamma\})$ in this case. If $\Pi^{\prime} \subset \Pi$ then $\Pi^{\prime} \subseteq \Pi \backslash\{\gamma\}$ for a clause $\gamma$. Therefore, $\operatorname{Mod}(\Pi) \subset$ $\operatorname{Mod}(\Pi \backslash\{\gamma\}) \subseteq \operatorname{Mod}\left(\Pi^{\prime}\right)$, which proves that $\Pi$ and $\Pi^{\prime}$ are not equivalent.

This proof does not work for circumscription because the set of minimal models of a formula can grow or shrink in response to a clause deletion. In principle, $\Pi$ and $\Pi \backslash\{\gamma\}$ might have different sets of minimal models and yet $\Pi$ and $\Pi^{\prime} \subset \Pi \backslash\{\gamma\}$ have the same minimal models. We show that this is not possible. The proof is based on the following simple result about quasi-orders (reflexive and transitive relations.)

Lemma 1 If $\leq$ is a quasi-order (a reflexive and transitive relation) and $A$ and $B$ are two finite sets such that $A \subseteq B$ and $\min _{\leq}(A) \neq \min _{\leq}(B)$, then $\min _{\leq}(B) \backslash A$ is not empty.

Proof. Since $\min _{\leq}(A) \neq \min _{\leq}(B)$, then either $\min _{\leq}(A) \backslash \min _{\leq}(B)$ or $\min _{\leq}(B) \backslash \min _{\leq}(A)$ is not empty. We consider these two cases separately.

Let $x \in \min _{\leq}(B) \backslash \min _{\leq}(A)$. We prove that $x \notin A$. Since $x$ is minimal in $B$, there is no element of $y \in B$ such that $y<x$. Since $A \subseteq B$, the same holds for every element of $A$ in particular. As a result, if $x \in A$ then $x \in \min _{\leq}(A)$, contradicting the assumption.

Let us instead assume that $\min _{\leq}(A) \backslash \min _{\leq}(B)$ is not empty. Let $x \in \min _{\leq}(A) \backslash \min _{\leq}(B)$. Since $x \in A$, it holds $x \in B$. Since $x \in B, x \notin \min _{\leq}(B)$, and $B$ is a finite set, there exists $y \in \min _{\leq}(B)$ such that $y<x$. Since $x$ is minimal in $A$, we have that $y \notin A$.

The order $\subseteq$ on propositional models is a quasi-order. As a result, if $A$ and $B$ are two sets of models such that $A \subseteq B$ and the set of minimal elements of $A$ and $B$ are different, then $B$ has a minimal element that is not in $A$. When applied to circumscription, this result tells that a formula can be non-equivalent to a stronger one only because of a minimal model that is not a model of the stronger formula. In the other way around, if a formula is weakened, the set of minimal models either remains the same or acquires a new element.

Theorem 1 If $\operatorname{Mod}(\Pi) \subseteq \operatorname{Mod}\left(\Pi^{\prime}\right) \subseteq \operatorname{Mod}\left(\Pi^{\prime \prime}\right)$ and $\operatorname{CIRC}(\Pi) \neq \operatorname{CIRC}\left(\Pi^{\prime}\right)$ then $\operatorname{CIRC}(\Pi) \neq$ $\operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$.

Proof. Let us assume that $\operatorname{CIRC}(\Pi)=\operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$. Since $\operatorname{CIRC}(\Pi) \neq \operatorname{CIRC}\left(\Pi^{\prime}\right)$, we have $\operatorname{CIRC}\left(\Pi^{\prime}\right) \neq \operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$. Since $\subseteq$ on propositional models is a quasi-order, Lemma 1 applies: there exists $M$ such that $M \in \operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$ and $M \notin \operatorname{Mod}\left(\Pi^{\prime}\right)$. Since $\operatorname{Mod}(\Pi) \subseteq \operatorname{Mod}\left(\Pi^{\prime}\right)$, the latter implies $M \notin \operatorname{Mod}(\Pi)$. As a result, $M \notin \operatorname{CIRC}(\Pi)$. Since $M \in \operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$, we have $\operatorname{CIRC}(\Pi) \neq \operatorname{CIRC}\left(\Pi^{\prime \prime}\right)$.

The redundancy of a formula and the presence of a redundant clause in the formula are related by the following theorem, which is an application of the above to the case in which $\Pi^{\prime}=\Pi \backslash\{\gamma\}$ and $\Pi^{\prime \prime}$ is a subset of $\Pi^{\prime}$.

Theorem $2 A C N F$ formula $\Pi$ is CIRC-redundant if and only if it contains a CIRCredundant clause.

Proof. The "if" direction is obvious: if $\gamma$ is redundant in $\Pi$, then $\Pi \backslash\{\gamma\} \equiv_{M} \gamma$, and $\Pi \backslash\{\gamma\}$ is therefore a strict subset of $\Pi$ that is equivalent to it. The "only if" direction is a consequence of the above theorem. Assume that $\operatorname{CIRC}(\Pi \backslash\{\gamma\}) \neq \operatorname{CIRC}(\Pi)$ holds for every $\gamma \in \Pi$. Let us consider $\Pi^{\prime \prime} \subset \Pi$ : we prove that $\operatorname{CIRC}\left(\Pi^{\prime \prime}\right) \neq \operatorname{CIRC}(\Pi)$. Since $\Pi^{\prime \prime} \subset \Pi$, there exists $\gamma \in \Pi^{\prime \prime} \backslash \Pi$. Consider one such clause $\gamma$. Since $\Pi \subset \Pi \backslash\{\gamma\} \subseteq \Pi^{\prime \prime}$, we have that $\operatorname{Mod}\left(\Pi^{\prime \prime}\right) \subseteq \operatorname{Mod}(\Pi \backslash\{\gamma\}) \subset \operatorname{Mod}(\Pi)$. We are thus in the conditions to apply the above theorem: since $\operatorname{CIRC}(\Pi \backslash\{\gamma\}) \neq \operatorname{CIRC}(\Pi)$, we have that $\operatorname{CIRC}\left(\Pi^{\prime \prime}\right) \neq \operatorname{CIRC}(\Pi)$. Therefore, $\Pi$ is not equivalent to any of its proper subsets.

This theorem shows that circumscription, although nonmonotonic, has the local redundancy property.

### 3.2 Redundant Clauses

The following lemma characterizes the clauses that are redundant in a formula.
Lemma 2 The following three conditions are equivalent:

1. the clause $\gamma \in \Pi$ is CIRC-redundant in $\Pi$;
2. for each $M \in \operatorname{Mod}(\Pi \backslash\{\gamma\} \cup \neg \gamma)$ there exists $M^{\prime} \in \operatorname{Mod}(\Pi)$ such that $M^{\prime} \subset M$;
3. for each $M \in \operatorname{Mod}(\Pi \backslash\{\gamma\} \cup \neg \gamma)$ there exists $M^{\prime} \in \operatorname{Mod}(\Pi \backslash\{\gamma\})$ such that $M^{\prime} \subset M$.

Proof. The models of $\Pi \backslash\{\gamma\}$ that are not models of $\Pi$ are exactly the models of $\Pi \backslash\{\gamma\} \cup \neg \gamma$. The two formulae $\Pi$ and $\Pi \backslash\{\gamma\}$ are $\models_{M}$-equivalent if none of these models (if any) is minimal, that is, all these models contain other models of $\Pi$. In other words, $\gamma$ is redundant if and only if every model of $\Pi \backslash\{\gamma\} \cup \neg \gamma$ contains a model of $\Pi$.

The fact that we can check $M^{\prime} \in \operatorname{Mod}(\Pi \backslash\{\gamma\})$ instead of $M^{\prime} \in \operatorname{Mod}(\Pi)$ follows from the fact that $\operatorname{Mod}(\Pi \backslash\{\gamma\})$ is composed of all models of $\Pi$ and all models of $\Pi \backslash\{\gamma\} \cup \neg \gamma$. Consider
a model $M$ that is a minimal model of $\Pi \backslash\{\gamma\} \cup \neg \gamma$. The condition $M^{\prime} \subset M$ implies that $M^{\prime}$ is not a model of $\Pi \backslash\{\gamma\} \cup \neg \gamma$, and is therefore a model of $\Pi$. By transitivity, the condition that there exists $M^{\prime} \in \operatorname{Mod}(\Pi)$ such that $M^{\prime} \subset M$ holds for all models of $\Pi \backslash\{\gamma\} \cup \neg \gamma$.

Computationally, checking the second or third condition of this lemma can be done by checking whether for all $M \in \ldots$ there exists $M^{\prime} \in \ldots$ such that a simple condition is met. As a result, the problem is in $\Pi_{2}^{p}$. For positive clauses, checking CIRC-redundancy is easier, as it amounts to checking classical redundancy.

Lemma 3 A positive clause is CIRC-redundant in $\Pi$ if and only if it is classically redundant in $\Pi$.

Proof. If a clause is redundant in $\Pi$ it is also CIRC-redundant in $\Pi$. Let us now prove the converse: assume that $\gamma$ is a positive clause that is CIRC-redundant in $\Pi$. By Lemma 2, every model of $\Pi \backslash\{\gamma\} \cup \neg \gamma$ contains a model of $\Pi$, which is the same as $\Pi \backslash\{\gamma\} \cup\{\gamma\}$. Let $\gamma=x_{i_{1}} \vee \cdots \vee x_{i_{k}}$. Since $\gamma$ is CIRC redundant, each model of $\Pi \backslash\{\gamma\} \cup\left\{\neg x_{i_{i}}, \ldots, \neg x_{i_{k}}\right\}$ contains at least a model of $\Pi \backslash\{\gamma\} \cup\left\{x_{i_{i}} \vee \cdots \vee x_{i_{k}}\right\}$. All models of the latter formula contain at least a variable among $x_{i_{i}}, \ldots, x_{i_{k}}$ while no models of the former contain any of them. Therefore, no model of the first formula contains a model of the second. Therefore, the condition can be true only if $\Pi \backslash\{\gamma\} \cup\left\{\neg x_{i_{1}} \vee \cdots \vee \neg x_{i_{k}}\right\}$ has no models, that is, $\Pi \backslash\{\gamma\} \models \gamma$ : the clause $\gamma$ is classically redundant in $\Pi$.

Intuitively, positive clauses only exclude models with all their literals assigned to false. Therefore, whenever a positive clause is irredundant w.r.t. $\models$, it is because such models were not otherwise excluded; therefore, it is also irredundant w.r.t. minimal models.

According to this argument, it may look like all negative clauses are redundant because they exclude models with positive literals, and these models are not minimal. This is however not the case: a model with some positive literals might be minimal because no other model of the formula has less positive literal. Consider, for example, the following formula:

$$
\Pi=\left\{\neg x_{1} \vee \neg x_{2}, x_{1} \vee x_{3}, x_{2} \vee x_{3}\right\}
$$

The clause $\neg x_{1} \vee \neg x_{2}$, although negative, is irredundant. Indeed, $\Pi \backslash\left\{\neg x_{1} \vee \neg x_{2}\right\}=$ $\left\{x_{1} \vee x_{3}, x_{2} \vee x_{3}\right\}$, and this formula has $\left\{x_{1}, x_{2}\right\}$ and $\left\{x_{3}\right\}$ as its minimal models. The first one is not a model of $\Pi$ because of the clause $\neg x_{1} \vee \neg x_{2}$. Therefore, $\neg x_{1} \vee \neg x_{2}$ is CIRC-irredundant in $\Pi$.

Intuitively, a negative clause excludes the possibility of setting all variables to true, while minimal inference only tries to set variables to false. Therefore, removing the clause may generate a model that have its variables set to true ( $\left\{x_{1}, x_{2}\right\}$ in the example), but is minimal because of the values of the other variables ( $x_{3}$ in the example).

Lemma 3 can be extended to clauses containing negative literals via the addition of new clauses and new variables. To this aim, the following property of quasi-orders is needed.

Lemma 4 If $\leq$ is a quasi-order, $X \in \min _{\leq}(A), X \in B$, and $B \subseteq A$, then $X \in \min _{\leq}(B)$.

Proof. Since $X \in \min _{\leq}(A)$, there is no element $Y \in A$ such that $Y<X$. Since $B \subseteq A$, there is no element of $\bar{B}$ with the same property. Since $X$ is an element of $B$ such that $Y<X$ does not hold for any $Y \in B$, it holds $X \in \min _{\leq}(B)$ by definition.

Applied to formulae: if $M$ is a minimal model of $\Pi$ and satisfies $\Pi^{\prime}$, then $M$ is a minimal model of $\Pi \cup \Pi^{\prime}$.

Lemma 5 A clause $\gamma$ is classically redundant in $\Pi$ if and only if it is CIRC-redundant in $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$.

Proof. If $\gamma$ is redundant in $\Pi$ then $\Pi \backslash\{\gamma\} \models \gamma$ and therefore $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\} \backslash\{\gamma\} \models \gamma$. Since $\gamma$ is redundant in $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$, it is also CIRC-redundant.

Let us now assume that $\gamma$ is irredundant in $\Pi$, that is, $\Pi \backslash\{\gamma\} \cup \neg \gamma$ has some models. Let $M$ be one such model. Since this model satisfies $\neg \gamma$, it assigns false to any variable $x$ such that $x \in \gamma$ and true to any variable $x$ such that $\neg x \in \gamma$. Extending $M$ to assign false to all variables $x^{\prime}$, this model also satisfies $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\} \backslash\{\gamma\} \cup \neg \gamma$.

We show that $M$ cannot contain a model of $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$. This model assigns false to all $x \in \gamma$ and also false to all $x^{\prime}$ such that $\neg x \in \gamma$. On the other hand, $\{\gamma\} \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$ entails the clause $\bigvee\{x \mid x \in \gamma\} \vee \bigvee\left\{x^{\prime} \mid \neg x \in \gamma\right\}$; this can be proved for example by iteratively resolving upon all literals $x$ such that $\neg x \in \gamma$. As a result, no model of $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$ has a model that assign false to all $x \in \gamma$ and all $x^{\prime}$ such that $\neg x \in \gamma$. Since this is instead done by $M$, it follows that no model of $\Pi \cup\left\{x \vee x^{\prime} \mid \neg x \in \gamma\right\}$ is contained in $M$.

Note that the clauses $x \vee x^{\prime}$ are not necessarily CIRC-irredundant in the considered formula. On the other hand, Lemma 3 can be applied to them: they are CIRC-redundant if and only if they are classically redundant.

### 3.3 Complexity of Clause Redundancy

Let us now turn to the hardness of the problem of checking the redundancy of a clause in a formula. We first show a reduction that proves the hardness of the problem of redundancy of a clause and then show how this result can be used to prove that the problem of redundancy of a formula has the same complexity.

Theorem 3 Checking the CIRC-redundancy of a clause in a formula is $\Pi_{2}^{p}$-complete.
Proof. Lemma 2 proves that the redundancy of a clause in a formula can be checked by solving a $\forall \exists \mathrm{QBF}$ (for all $M \ldots$ there exists $M^{\prime} \ldots$ ), and is therefore in $\Pi_{2}^{p}$.

Let us now show hardness. We show that the QBF formula $\forall X \exists Y . \Gamma$, where $\Gamma=$ $\left\{\delta_{1}, \ldots, \delta_{m}\right\}$ and $n=|Y|$, is valid if and only if $\gamma$ is CIRC-redundant in $\Pi$, where:

$$
\begin{aligned}
\Pi & =\left\{x_{i} \vee p_{i}\right\} \cup\left\{\neg a \vee y_{i}\right\} \cup\left\{a \vee \delta_{i} \mid \delta_{i} \in \Gamma\right\} \cup\{\gamma\} \\
\gamma & =\neg a \vee \neg y_{1} \vee \cdots \vee \neg y_{n}
\end{aligned}
$$

The clause $\gamma$ is CIRC-redundant in $\Pi$ if and only if all minimal models of $\Pi \backslash\{\gamma\} \cup \neg \gamma$ contain some models of $\Pi$. The following equivalences holds:

$$
\begin{aligned}
\Pi & \equiv\left\{x_{i} \vee p_{i}\right\} \cup\{\neg a\} \cup \Gamma \\
\Pi \backslash\{\gamma\} \cup \neg \gamma & \equiv\left\{x_{i} \vee p_{i}\right\} \cup\left\{a, y_{1}, \ldots, y_{n}\right\}
\end{aligned}
$$

The first equivalence holds because $\left\{\neg a \vee y_{i}\right\} \cup\left\{\neg a \vee \neg y_{1} \vee \cdots \vee \neg y_{n}\right\}$ is equivalent to $\neg a$, as can be checked by resolving upon each $y_{i}$ in turn. The second equivalence holds because $\neg \gamma=\left\{a, y_{1}, \ldots, y_{n}\right\}$ and this set implies all clauses $\neg a \vee y_{i}$ and $a \vee \delta_{i}$.

The formula $\Pi \backslash\{\gamma\} \cup \neg \gamma \equiv\left\{x_{i} \vee p_{i}\right\} \cup\left\{a, y_{1}, \ldots, y_{n}\right\}$ has a minimal model for each truth evaluation $\omega_{X}$ over the variables $x_{i}$ :

$$
I_{\omega_{X}}=\omega_{X} \cup\left\{p_{i} \mid x_{i} \notin \omega_{X}\right\} \cup\{a\} \cup\left\{y_{i} \mid 1 \leq i \leq n\right\}
$$

We show that the model $I_{\omega_{X}}$ contains a model of $\Pi$ if and only if $\left.\Gamma\right|_{\omega_{X}}$ is satisfiable. By Lemma 2, the redundancy of $\gamma$ corresponds to this condition being true for all possible models of $\Pi^{\prime} \cup \neg \gamma$. This would therefore prove that the QBF is valid if and only if $\gamma$ is redundant in $\Pi$.

Since $\Pi \equiv\left\{x_{i} \vee p_{i}\right\} \cup\{\neg a\} \cup \Gamma$, if $\Gamma$ has a model with a given value of $\omega_{X}$ then $\Pi$ has a model that is strictly contained in $I_{\omega_{X}}$ : add to the satisfying assignment of $\Gamma$ the setting of every $p_{i}$ to the opposite of $x_{i}$ and $a$ to false.

On the converse, if $\Pi$ contains a model that is strictly contained in $I_{\omega_{X}}$, this model must have exactly the same value of $X \cup P$ because $\Pi$ contains $x_{i} \vee p_{i}$ and either $x_{i}$ or $y_{i}$ is false in $I_{\omega_{X}}$. On the other hand, this model of $\Pi$ must also set $a$ to false and satisfy $\Gamma$, thus showing that there exists an assignment extending $\omega_{X}$ and satisfying $\Gamma$.

### 3.4 Complexity of Formula Redundancy

In order to characterize the complexity of the problem of checking the CIRC-redundancy of a formula, we use the fact that a formula is CIRC-redundant if and only if it contains a CIRC-redundant clause by Theorem 2. In particular, Lemma 3 shows that the problem of checking the CIRC-redundancy of a clause $\gamma$ in $\Pi$ is $\Pi_{2}^{p}$-hard. In order for this result to be used as a proof of hardness for the problem of CIRC-redundancy of formulae, we need to modify the formula $\Pi$ in such a way all its clauses but $\gamma$ are made CIRC-irredundant. This is the corresponding of Lemma 4 of the paper of redundancy of propositional CNF formulae [Lib05b], which has been useful because it allows to "localize" problems about redundancy.

Lemma 6 For every consistent formula $\Pi$ and $\Pi^{\prime} \subseteq \Pi$, the only CIRC-redundant clauses of $I\left(\Pi, \Pi^{\prime}\right)$ are the clauses $\neg s \vee \neg t \vee \gamma_{i}$ such that $\gamma_{i} \in \Pi^{\prime}$ and $\gamma_{i}$ is CIRC-redundant in $\Pi$.

$$
\begin{aligned}
I\left(\Pi, \Pi^{\prime}\right)= & \{s \vee t\} \cup\{s \vee a, t \vee b\} \cup \\
& \left\{\neg s \vee t \vee c_{i} \vee d_{i}\right\} \cup\left\{\neg s \vee \neg c_{i}\right\} \cup \\
& \left\{\neg t \vee c_{i} \vee \gamma_{i} \mid \gamma_{i} \in \Pi \backslash \Pi^{\prime}\right\} \cup\left\{s \vee \neg t \vee x \vee x^{\prime} \mid x \in \operatorname{Var}(\Pi)\right\} \cup \\
& \left\{\neg s \vee \neg t \vee \gamma_{i} \mid \gamma_{i} \in \Pi^{\prime}\right\}
\end{aligned}
$$

Proof. There are four possible assignment to the variables $s$ and $t$. Since the models of $I\left(\Pi, \Pi^{\prime}\right)$ can be partitioned into the models of $I\left(\Pi, \Pi^{\prime}\right) \cup\{\neg s, \neg t\}, I\left(\Pi, \Pi^{\prime}\right) \cup\{s, \neg t\}$, $I\left(\Pi, \Pi^{\prime}\right) \cup\{\neg s, t\}$, and $I\left(\Pi, \Pi^{\prime}\right) \cup\{s, t\}$, the minimal models of $I\left(\Pi, \Pi^{\prime}\right)$ are necessarily some of the minimal models of these formulae.

In the table below we show what remains of $I\left(\Pi, \Pi^{\prime}\right) \backslash\{s \vee t\}$ in each of the four possible assignment to $s$ and $t$ after removing entailed clauses and false literals. We also show the minimal models of the resulting formulae.

| assignment | subformula | minimal models |
| :--- | :--- | :--- |
| $\{\neg s, \neg t\}$ | $\{a, b\}$ | $\{a, b\}$ |
| $\{s, \neg t\}$ | $\{b\} \cup\left\{c_{i} \vee d_{i}\right\} \cup\left\{\neg c_{i}\right\}$ | $\{s, b\} \cup\left\{d_{i}\right\}$ |
| $\{\neg s, t\}$ | $\{a\} \cup\left\{c_{i} \vee \gamma_{i} \mid \gamma_{i} \in \Pi \backslash \Pi^{\prime}\right\} \cup$ | $\{t, a\}+$ some subsets of $\left(C \cup X \cup X^{\prime}\right)$ |
|  | $\left\{x \vee x^{\prime} \mid x \in \operatorname{Var}(\Pi)\right\}$ |  |
| $\{s, t\}$ | $\left\{\neg c_{i}\right\} \cup\left\{c_{i} \vee \gamma_{i} \mid \gamma_{i} \in \Pi \backslash \Pi^{\prime}\right\} \cup \Pi^{\prime}$ | $\{s, t\}+$ a minimal model of $\Pi$ |

The four subformulae are all satisfiable. Moreover, no minimal model of one is contained in the minimal models of the other ones because of either the values of $\{s, t\}$ and $\{a, b\}$. As a result, the minimal models of $I\left(\Pi, \Pi^{\prime}\right) \backslash\{s \vee t\}$ are exactly the minimal models of the four subformulae. The clause $s \vee t$ is irredundant because its addition deletes the minimal model $\{a, b\}$. The minimal models of $I\left(\Pi, \Pi^{\prime}\right)$ are therefore exactly the minimal models of the remaining three subformulae.

We show that the remaining clauses but the ones derived from $\Pi^{\prime}$ are irredundant. This is shown by removing a clause from the set and showing that some of the minimal models of a subformula can be removed some elements. Since the minimal models of these three subformulae are exactly the minimal models of $\Pi$, this is a proof that the clause is irredundant.

1. The clauses $s \vee a$ and $t \vee b$ are irredundant because their removal would allow $a$ and $b$ to be set to false in the minimal models of the third and second subformula, respectively.
2. The clauses $\neg s \vee t \vee c_{i} \vee d_{i}$ and $\neg s \vee \neg c_{i}$ are irredundant because their removal would allow $d_{i}$ to be set to false in the minimal model of the second subformula.
3. The clauses $\neg t \vee c_{i} \vee \gamma_{i}$ and $s \vee \neg t \vee x \vee x^{\prime}$ require a longer analysis. In the third assignment, $I\left(\Pi, \Pi^{\prime}\right)$ becomes:

$$
C=\{a\} \cup\left\{c_{i} \vee \gamma_{i}\right\} \cup\left\{x \vee x^{\prime} \mid x \in \operatorname{Var}(\Pi)\right\}
$$

The clauses $x \vee x^{\prime}$ are positive. By Lemma 3, they are CIRC-redundant if and only if they are redundant. In turn, they are not redundant because $\{a\} \cup\left\{c_{i}\right\} \cup\{y \mid y \neq x\}$ is a model of all clauses but $x \vee x^{\prime}$.
Since $c_{i}$ occurs positive in $c_{i} \vee \gamma_{i}$, Lemma 5 ensures that this clause is CIRC-redundant in $C$ if and only if it is redundant in $C \backslash\left\{x \vee x^{\prime} \mid \neg x \in \gamma_{i}\right\}$. This is false because the removal of $c_{i} \vee \gamma_{i}$ creates the following new model:

$$
M=\left\{c_{j} \mid j \neq i\right\} \cup\left\{x^{\prime}\right\} \cup\left\{x \mid \neg x \in \gamma_{i}\right\}
$$

This model $M$ satisfies $C \backslash\left\{x \vee x^{\prime} \mid \neg x \in \gamma_{i}\right\} \backslash\left\{c_{i} \vee \gamma_{i}\right\}$ : all clauses $c_{j} \vee \gamma_{j}$ are satisfied because $c_{j} \in M$ and all clauses $x \vee x^{\prime}$ are satisfied because $x^{\prime} \in M$. On the other hand, $M$ does not satisfy $c_{i} \vee \gamma_{i}$ because it assigns all its literals to false.

The only clauses that can therefore be redundant are those corresponding to the clauses of $\Pi^{\prime}$. In particular, these clauses only occur in the fourth subformula, which is equivalent to $\{c\} \cup\left\{\neg c_{i}\right\} \cup \Pi$. A clause $\neg s \vee \neg t \vee \gamma_{i}$ with $\gamma_{i} \in \Pi^{\prime}$ is therefore CIRC-redundant in $I\left(\Pi, \Pi^{\prime}\right)$ if and only if $\gamma_{i}$ is CIRC-redundant in $\Pi$.

More precisely, this theorem shows a way to make the clauses of $\Pi^{\prime}$ necessary, that is, contained in all equivalent subsets of $\Pi$. The theorem allows to characterize the complexity of formula CIRC-redundancy.

Theorem 4 The problem of CIRC-redundancy is $\Pi_{2}^{p}$-complete.
Proof. By Theorem 2, $\Pi$ is redundant if and only if it contains a redundant clause. Therefore, we have to solve a linear number of problems in $\Pi_{2}^{p}$. Since these problems can be solved in parallel, the whole problem is in $\Pi_{2}^{p}$.

Hardness is proved by reduction from the problem of CIRC-redundancy of a single clause. By Lemma 6, a clause $\gamma$ is CIRC-redundant in $\Pi$ if and only if $\neg s \vee \neg t \vee \gamma$ is CIRC-redundant in $I(\Pi,\{\gamma\})$ and all other clauses of $I(\Pi,\{\gamma\})$ are irredundant.

## 4 Default Logic

A default theory is a pair $\langle D, W\rangle$, where $W$ is formula and $D$ is a set of default rules, each rule being in the form:

$$
\frac{\alpha: \beta}{\gamma}
$$

The formulae $\alpha, \beta$, and $\gamma$ are called the precondition, the justification, and the consequence of the default, respectively. In this paper, we assume that $W$ is a CNF finite formula (a finite set of clauses) and that the set of variables and defaults are finite. We also assume that each default has a single justification, rather than a set of justifications. Given a default $d=\frac{\alpha: \beta}{\gamma}$, its parts are denoted by $\operatorname{prec}(d)=\alpha, \operatorname{just}(d)=\beta$, and $\operatorname{cons}(d)=\gamma$.

We use the operational semantics of default logics [AS94, Ant99, FM92, FM94], which is based on sequences of defaults with no duplicates. If $\Pi$ is such a sequence, we denote by $\Pi[d]$ the sequence of defaults preceeding $d$ in $\Pi$, and by $\Pi \cdot[d]$ the sequence obtained by adding $d$ at the end of $\Pi$. We extend the notation from defaults to sequences, so that $\operatorname{prec}(\Pi)$ is the conjunction of all preconditions of the defaults in $\Pi$, just $(\Pi)$ is the conjunction of all justifications, and cons $(\Pi)$ is the conjunction of all consequences.

Implication is denoted by $\models$, $\top$ indicates (combined) consistency, and $\perp$ indicates inconsistency. For example, $A \top B$ means that $A \wedge B$ is consistent, while $A \perp B$ means that $A \wedge B$ is inconsistent.

Default logic can be defined in terms of the selected processes, that are the sequences of defaults that are considered applicable by the semantics [Ant99]. A sequence of defaults $\Pi$ is a process if $W \cup \operatorname{cons}(\Pi[d]) \models \operatorname{prec}(d)$ holds for any $d \in \Pi$. A default $d$ is locally applicable in a sequence $\Pi$ if $\operatorname{cons}(\Pi) \cup W \models \operatorname{prec}(d)$ and $\operatorname{cons}(\Pi) \cup W$ Tjust $(d)$. Global applicability also requires cons $(\Pi) \cup W T j u s t(\Pi \cdot[d])$. Each semantics defines the sequences of defaults that are applied in a particular theory. Formally, the definitions are as follows:

Reiter: a process $\Pi$ is selected if cons $(\Pi) \cup W \top$ just $(d)$ for each $d \in \Pi$ and no default $d^{\prime} \notin \Pi$ is locally applicable in $\Pi$;

Justified: a process is selected if it is a maximal process such that cons $(\Pi) \cup W T j u s t(d)$ for each $d \in \Pi$;

Constrained: a process is selected if it is a maximal process such that cons $(\Pi) \cup W \top$ just $(\Pi)$;
Rational: a process is selected if $\operatorname{cons}(\Pi) \cup W \models \operatorname{prec}(d)$ and no default $d^{\prime} \notin \Pi$ is globally applicable in $\Pi$.

The conditions on selected processes can be all broken in two parts: success (the consistency condition) and closure (the non-extendibility of the process). For example, for constrained default logic the condition of success is cons $(\Pi) \cup W T j u s t(d)$ and the condition of closure is that $\Pi \cdot[d]$ is not successful for any $d^{\prime} \notin \Pi$.

Remarkably, the conditions above only mention the background theory $W$ in conjunction with cons $(\Pi)$, that is, $W$ only occurs in subformulae of the form $W \cup \operatorname{cons}(\Pi)$. The only conditions for which this is not true is that of $\Pi$ being a process.

If $\Pi$ is a selected process of $\langle D, W\rangle$, the formula $C n(W \cup \operatorname{cons}(\Pi))$ is an extension of $\langle D, W\rangle$. We denote by $\operatorname{Ext}(\langle D, W\rangle)$ or $\operatorname{Ext}_{D}(W)$ the set of all formulae that are equivalent to an extension of $\langle D, W\rangle$. Including formulae that are equivalent to the extensions in this set allows to write $E \in \operatorname{Ext}_{D}(W)$ to denote the equivalence of $E$ with an extension of $\langle D, W\rangle$.

A default theory $\langle D, W\rangle$ entails a formula $W^{\prime}$ if and only if $E \models W^{\prime}$ for every $E \in$ $\operatorname{Ext}_{D}(W)$. This condition is equivalent to $\vee \operatorname{Ext}_{D}(W) \models W^{\prime}$; as a result, the set of all consequences of a default theory is equivalent to $\vee \operatorname{Ext}_{D}(W)$. The condition that $\langle D, W\rangle$ entails $W^{\prime}$ is denoted by $\langle D, W\rangle \models W^{\prime}$ or $W \models_{D} W^{\prime}$. The latter notation emphasizes that every fixed set of defaults $D$ induces a nonmonotonic inference operator $\models_{D}$.

Some semantics of default logic do not assign any extension to some theories. In this paper, we try to derive the hardness results using only theories having extensions.

### 4.1 Equivalence in Default Logics

The monotonic inference operator $\models_{D}$ induced by a set of default $D$ is a consequence relation. Therefore, the definitions of redundancy of a clause and of a formula for $\models$ and $\models_{M}$ can be given for $\models_{D}$ as well: a clause $\gamma$ of a formula $W$ is redundant w.r.t. default $D$ if and only
if $W$ and $W \backslash\{\gamma\}$ are equivalent; a formula $W$ is redundant if there exists $W^{\prime} \subset W$ that is equivalent to it.

Both definitions are based on equivalence of two formulae, and in particular the equivalence of a formula with one of its proper subsets. In this section, we show that three different form of equivalence can be defined; we compare them in general and in the particular case of equivalence of a formula with one of its proper subsets. The first form of equivalence is based on entailment.

Definition 6 (Entailment and Mutual Equivalence) For a given set of defaults D, formula $W$ entails $W^{\prime}$, denoted by $W \models_{D} W^{\prime}$, if $\vee \operatorname{Ext}_{D}(W) \models W^{\prime}$. These two formulae are mutual equivalent, denoted by $W \equiv_{D}^{m} W^{\prime}$, if $W \models_{D} W^{\prime}$ and $W^{\prime} \models_{D} W$.

In classical logic, this definition of equivalence is the same as $W$ and $W^{\prime}$ having the same set of consequences and the same set of models. In default logic, this is not the case. We define the equivalence based on the set of consequences as follows.

Definition 7 (Consequence-Entailment and Consequence-Equivalence) For a given set of defaults $D$, formula $W$ consequence-entails $W^{\prime}$, denoted $W \models_{D}^{c} W^{\prime}$, if $\vee \operatorname{Ext}_{D}(W) \models$ $\vee \operatorname{Ext}_{D}\left(W^{\prime}\right)$. These two two formulae are consequence-equivalent, denoted $W \equiv_{D}^{c} W^{\prime}$, if $\vee \operatorname{Ext}_{D}(W) \equiv \vee \operatorname{Ext}_{D}\left(W^{\prime}\right)$.

Note that the comparison of the two formulae is based on a fixed set of defaults $D$. A more stringent condition of equivalence of two defaults theories is that of having the same extensions.

Definition 8 (Faithful Entailment and Faithful Equivalence) For a given set of defaults $D$, a formula $W$ faithfully entails $W^{\prime}$, denoted $W \models_{D}^{e} W^{\prime}$, if $\operatorname{Ext}_{D}(W) \subseteq \operatorname{Ext}_{D}\left(W^{\prime}\right)$. These two formulae are faithfully equivalent, denoted $W \equiv \equiv_{D}^{e} W^{\prime}$, if $\operatorname{Ext}_{D}(W)=\operatorname{Ext}_{D}\left(W^{\prime}\right)$.

For all three definition, equivalence is the same as each formula implying the other one. We are especially interested into $\equiv_{D}^{c}$ and $\equiv_{D}^{e}$, that is, equality of consequences and equality of extensions. Mutual equivalence has been defined for technical reasons. Redundancy in default logic is defined as follows.

Definition 9 (Redundancy of a Clause) For a given set of defaults D, a clause $\gamma$ is redundant in a formula $W$ according to equivalence $\equiv_{D}^{x}$ if $W \equiv_{D}^{x} W \backslash\{\gamma\}$.

Definition 10 (Redundancy of a Formula) For a given set of defaults $D$, a formula $W$ is redundant according to equivalence $\equiv_{D}^{x}$ if there exists $W^{\prime} \subset W$ such that $W \equiv_{D}^{x} W^{\prime}$.

In both cases, we are comparing for equivalence a formula and one of its subsets. In the following section we study the equivalence of $W^{\prime}$ and $W$ when $W^{\prime} \subseteq W$.

### 4.1.1 Correspondence, General

We now compare the three forms of equivalence defined above. The following chain of implications is easy to prove:

$$
W^{\prime} \models_{D}^{e} W \Rightarrow W^{\prime} \models_{D}^{c} W \Rightarrow W^{\prime} \models_{D} W
$$

The latter implication is proved by the following lemma.
Lemma 7 If $W^{\prime} \models_{D}^{c} W$ then $W^{\prime} \models_{D} W$.
Proof. By assumption, $\vee \operatorname{Ext}_{D}\left(W^{\prime}\right) \models \vee \operatorname{Ext}_{D}(W)$. Since every extension of $\langle D, W\rangle$ entails $W$, we have $\vee \operatorname{Ext}_{D}(W) \models W$. As a result, $\vee \operatorname{Ext}_{D}\left(W^{\prime}\right) \models W$, which is by definition $W^{\prime} \models_{D}$ $W$.

Redundancy is defined in terms of equivalence of two formulae, one contained in the other. As a result, it makes sense to study the conditions of equivalence in the particular case in which $W^{\prime} \subseteq W$. We prove that the above chain of implication can be wrapped around in this case, thus proving that the three conditions are equivalent.

Lemma 8 If $W^{\prime} \subseteq W$ and $W^{\prime} \models_{D} W$, then $W^{\prime} \models_{D}^{e} W$.
Proof. Let $\Pi$ be a selected process of $\left\langle D, W^{\prime}\right\rangle$. We prove that it is also a selected process of $\langle D, W\rangle$. Since $W^{\prime} \models_{D} W$, the formula $W$ is entailed by every extension in $\operatorname{Ext}_{D}\left(W^{\prime}\right)$. In particular, $W^{\prime} \cup \operatorname{cons}(\Pi) \models W$. Therefore, $W^{\prime} \cup \operatorname{cons}(\Pi) \equiv W \cup \operatorname{cons}(\Pi)$. As a result, all conditions (such as success and closure) where $W^{\prime}$ only occurs in the subformula $W^{\prime} \cup \operatorname{cons}(\Pi)$ are not changed by replacing $W^{\prime}$ with $W$. This is in particular true for all considered conditions of success and closure.

The only condition that mentions the background theory not in conjunction with cons $(\Pi)$ is the condition of a sequence being a process: $\Pi$ is a process of $\left\langle D, W^{\prime}\right\rangle$ if and only if $W^{\prime} \cup \operatorname{cons}(\Pi[d]) \models \operatorname{prec}(d)$ for any $d \in \Pi$. The same condition is however true for $W$ because $W^{\prime} \subseteq W$ implies $W \models W^{\prime}$.

The following is a consequence of the above.
Corollary 1 If $W^{\prime} \subseteq W$, then:

$$
W^{\prime} \models_{D} W \quad \Leftrightarrow \quad W^{\prime} \models_{D}^{c} W \quad \Leftrightarrow \quad W^{\prime} \models_{D}^{e} W
$$

### 4.1.2 Non-Correspondence, General

The last corollary proves that the three definitions of entailment from $W^{\prime}$ to $W$ are equivalent if $W^{\prime} \subseteq W$. The same does not hold for equivalence, and therefore does not hold for entailment from $W$ to $W^{\prime}$.

Theorem 5 There exists $D, W$, and $W^{\prime} \subset W$ such that $W^{\prime} \equiv_{D}^{m} W$ and $W^{\prime} \not \equiv_{D}^{c} W$

Proof. Since $W^{\prime} \models_{D} W$, every extension of $\left\langle D, W^{\prime}\right\rangle$ implies $W$. A wrong proof of $W^{\prime} \equiv_{D}^{e} W$ could then be based on the fact that, once $W$ is derived from $\left\langle D, W^{\prime}\right\rangle$ applying some defaults, we can proceed by applying the defaults of an arbitrary process of $\langle D, W\rangle$.


This figure shows why a process $\Pi^{\prime}$ of $\left\langle D, W^{\prime}\right\rangle$ and a process $\Pi$ of $\langle D, W\rangle$ cannot always be concatenated: while $\Pi^{\prime}$ allows the derivation of $W$, this process might also derive another formula $W^{\prime \prime}$ that makes the process $\Pi$ inapplicable. An example in which this situation arises is the following one:

$$
\begin{gathered}
D=\left\{d_{1}, d_{2}\right\} \\
\text { where: } \\
\\
\quad d_{1}=\frac{: a \wedge \neg b}{a \wedge \neg b} \\
\\
\quad d_{2}=\frac{a: b}{b} \\
W= \\
W a\} \\
W^{\prime}=
\end{gathered}
$$

The only process of $\left\langle D, W^{\prime}\right\rangle$ is $\left[d_{1}\right]$, which generates the extension $C n(a \wedge \neg b)$. This extension entails $W$, but it also entails $\neg b$. The theory $\langle D, W\rangle$ has also the process $\left[d_{2}\right.$ ], generating the extension $C n(a \wedge b)$. These two processes cannot however be concatenated, as the consequence $\neg b$ of $d_{1}$ is inconsistent with the justification of $d_{2}$.

Since $W^{\prime} \subset W$, we have that $W \models_{D} W^{\prime}$. In this example, we also have $W^{\prime} \models_{D} W$ because the single extension of $W^{\prime}$ entails $W=\{a\}$. However, $W^{\prime}$ and $W$ have different set


A similar result can be proved about $\equiv_{D}^{c}$ and $\equiv{ }_{D}^{e}$.
Theorem 6 There exists $D, W$, and $W^{\prime} \subset W$ such that $W^{\prime} \equiv_{D}^{c} W$ but $W^{\prime} \not \equiv_{D}^{e} W$ in Reiter and justified default logic.

Proof. Rather than the counterexample itself, it is interesting to show how it has been derived. The idea is the same as that of Theorem 5: a process of $\left\langle D, W^{\prime}\right\rangle$ that cannot be concatenated with a process of $\langle D, W\rangle$. The proof for this case, however, is complicated by the fact that we assume $W^{\prime} \equiv_{D}^{c} W$, that is, $\vee \operatorname{Ext}_{D}\left(W^{\prime}\right)$ and $\vee \operatorname{Ext}_{D}(W)$ are equivalent. The theories used in Theorem 5 do not work, as any other pair of theories having one extension each: in this cases, indeed, $\vee \operatorname{Ext}_{D}(W)$ is equivalent to $\operatorname{Ext}_{D}(W)$. In order for the counterexample to work, $\left\langle D, W^{\prime}\right\rangle$ must have multiple processes, each entailing $W$ and some other formula.


In order for the counterexample to work, some defaults of $\Pi$ cannot be applied after $\Pi_{1}$ or $\Pi_{2}$ because their justifications are inconsistent with $W_{1}$ or $W_{2}$. In order for $W \equiv_{D}^{c} W^{\prime}$ to hold, however, $W \cup \operatorname{cons}(\Pi)$ must be equivalent to $W \cup\left(W_{1} \vee W_{2}\right)$. As a result, every model of $W \cup \operatorname{cons}(\Pi)$ is a model of $W \cup W_{1}$ or $W \cup W_{2}$.

The precondition of the first default of $\Pi$ is entailed by $W \cup W_{1}$ and $W \cup W_{2}$. Since the justifications of the defaults in $\Pi$ are consistent with the consequences of $\Pi$, there is a model that is both a model of $\operatorname{cons}(\Pi) \cup W$ and a model of just $(d)$ for any $d \in \Pi$. But this is also a model of $W \cup W_{1}$ or $W \cup W_{2}$. As a result, the default $d$ is applicable in $\Pi_{i}$.

This arguments cannot be extended further, however. Indeed, $\Pi$ may be composed of two defaults, one applicable to $W \cup W_{1}$ and one applicable to $W \cup W_{2}$. This is possible in a selected process because Reiter and justified semantics does not enforce joint consistency of justifications.

A minimal counterexample requires two defaults that can be applied in $W^{\prime}$ leading to two disjoint extensions $W \cup W_{1}$ and $W \cup W_{2}$, and two other defaults that can be applied in sequence from $W$, but not from $W \cup W_{1}$ or $W \cup W_{2}$.

The background theory $W$ of this counterexample is composed of the four possible models $A, B, C$, and $D$. We define $D$ so that $W$ has a process that generates an extension $E$ having $A$ and $B$ as its models. In order for the consequences to be the same of those of $W^{\prime}$, both $A$ and $B$ have to be part of some extensions of $W^{\prime}$. We define the defaults so that $W^{\prime}$ has two processes generating $A$ and $B$, respectively. Namely, the first process generates $A$, but its justifications are satisfiable because the extension contains $C$; the other one contains $B$, but consistency with justifications are ensured by the model $D$. This trick is necessary to avoid these processes to be extended with the defaults that generate $W^{\prime}$.

In order to make the discussion more intuitive, we identify models with terms, and define formulae and defaults based on terms. We then convert terms into real formulae. The defaults that are applicable from $W^{\prime}$ are defined as follows.

$$
\begin{aligned}
d_{1} & =\frac{: C}{A \vee C} \\
d_{2} & =\frac{: D}{B \vee D}
\end{aligned}
$$

Both $d_{1}$ and $d_{2}$ can be applied from $W^{\prime}$, leading to a consequence that is inconsistent with the justification of the other default. Moreover, each extension contains a model of $\{A, B\}$,
as required. For example, $d_{1}$ produces $A$. However, the consistency of the extension with the justification is ensured by the other model $C$. This is necessary to avoid these defaults to be applicable in the new extension $E$ of $W$. Let us now define the two defaults that are applicable from $W$ only and generate this extension $E$.

$$
\begin{aligned}
d_{3} & =\frac{W: A}{A \vee B \vee C} \\
d_{4} & =\frac{A \vee B \vee C: B}{A \vee B}
\end{aligned}
$$

The processes of $W^{\prime}$ are $\left[d_{1} d_{3}\right]$ and $\left[d_{2}\right]$. There is no way to avoid the first default $d_{3}$ of the new extension to be part of some process of $W^{\prime}$ as well. However, as in this case, it may have no effects. Indeed, the extensions are $A \vee C$ and $B \vee D$. Let us now consider which defaults can be applied to $W$. Both processes of $W^{\prime}$ are still processes of $W$. However, $d_{3}$ is applicable to $W$, leading to a state in which both $d_{1}$ and $d_{2}$ are applicable. However, both processes only contain models that are among the previous ones.

The above terms can be translated into the following formuale:

$$
\begin{aligned}
& A=a b c \\
& B=a b \neg c \\
& C=a \neg b c \\
& D=a \neg b \neg c
\end{aligned}
$$

The defaults will be then defined as follows.

$$
\begin{aligned}
d_{1} & =\frac{: \neg b \wedge c}{a \wedge c} \\
d_{2} & =\frac{: \neg b \wedge \neg c}{a \wedge \neg c} \\
d_{3} & =\frac{a: b \wedge c}{b \vee c} \\
d_{4} & =\frac{a \wedge(b \vee c): b \wedge \neg c}{b}
\end{aligned}
$$

In $W^{\prime}$, only $d_{1}$ and $d_{2}$ are applicable. The first one leads to $a \wedge c$, which is consistent with the justification of $d_{3}$. The first selected process of $W^{\prime}$ is therefore $\left[d_{1} d_{3}\right]$, leading to the extension $a \wedge c$.

The second process from $W^{\prime}$ starts with $d_{2}$, which generates $a \wedge \neg c$, which is not consistent with $d_{1}$ and $d_{3}$, and does not imply the precondition of $d_{4}$. As a result, $\left[d_{2}\right]$ is the second
selected process of $W^{\prime}$, leading to the extension $a \wedge \neg c$. We therefore have $\vee \operatorname{Ext}_{D}\left(W^{\prime}\right) \equiv$ $(a \vee c) \wedge(a \vee \neg c) \equiv a$.

Let us now consider the extensions from $W$. All selected processes of $W^{\prime}$ are also selected processes of $W$. However, we can now apply $d_{3}$, as $a$ is true in the background theory. We therefore obtain $b \vee c$. This conclusion is inconsistent with the justification of $d_{2}$, but $d_{1}$ and $d_{4}$ can be applied. The first one leads to the extension $a \wedge c$, which is also an extension of $\left\langle D, W^{\prime}\right\rangle$. On the other hand, $\left[d_{3} d_{4}\right]$ leads to $a \wedge b$, which is a new extension. Nevertheless, $\vee \operatorname{Ext}_{D}(W) \equiv(a \vee c) \wedge(a \vee \neg c) \wedge(a \vee b) \equiv a$ : the theory $\langle D, W\rangle$ has some extensions that $\left\langle D, W^{\prime}\right\rangle$ does not have, but the skeptical consequences are the same.

In the proof, we used two defaults that are applicable in $W$ but not in the processes of $\left\langle D, W^{\prime}\right\rangle$. These two defaults cannot have mutually consistent justifications; otherwise, they would be both applicable in some process of $\left\langle D, W^{\prime}\right\rangle$ thanks to the fact that any extension of $\langle D, W\rangle$ contains only models of some extensions of $\left\langle D, W^{\prime}\right\rangle$. This proof does not work for constrained and rational default logic; however, the same claim can be proved in a different way.

Theorem 7 There exists $D, W$, and $W^{\prime} \subset W$ such that $W^{\prime} \equiv_{D}^{c} W$ but $W^{\prime} \not \equiv_{D}^{e} W$, for constrained and rational default logic.

Proof. The idea is as follows: the constrained extensions of a default theory are each characterized by a model that is consistent with all justifications and consequences of the defaults used to generate the extensions [Lib05d]. Therefore, we might have a situation like the one depicted below:


The arrows indicate the model associated to each extension: $C$ is the model associated with $E_{1}$, etc. Note that $E_{1} \vee E_{2} \vee E_{3} \equiv E_{1} \vee E_{2}$. As a result, a default theory having only the extensions $E_{1}$ and $E_{2}$ is equivalent to a theory having all three extensions, but yet these two theories have the same consequences.

We define $\langle D, W\rangle$ in such a way it has all three extensions, but $\langle D, \emptyset\rangle$ does not have the extension $E_{3}$ because the model $B$ is excluded by a default that generates $W$. The above condition can be realized using two variables $a$ and $b$ to distinguish the four models $A, B$, $C$, and $D$, and a variable $x$ to distinguish $W$ from $W^{\prime}$.

$$
\begin{array}{rlrl}
D & = & \left\{d_{1}, d_{2}, d_{3}, d_{4}\right\} & \\
W & =\{x\} & \\
W^{\prime} & =\emptyset & & \\
& \text { the defaults are: } & & \\
& d_{1}=\frac{: x \wedge a}{x} & & \text { generates } W \text { from } W^{\prime} \\
& d_{2}=\frac{x: a \wedge b}{b} & & \text { generates the extension } E_{1}=C n(x \wedge b) \\
& d_{3}=\frac{x: a \wedge \neg b}{\neg b} & & \text { generates the extension } E_{2}=C n(x \wedge \neg b) \\
& d_{4}=\frac{x: \neg a \wedge \neg b}{x} & & \text { generates the extension } E_{3}=C n(x)
\end{array}
$$

The justification of $d_{2}, d_{3}$, and $d_{4}$ are mutually inconsistent. In $\langle D, W\rangle$, the three extensions are generated by the processes $\left[d_{2}, d_{1}\right],\left[d_{3}, d_{1}\right]$, and $\left[d_{4}\right]$. The presence of $d_{1}$ in these processes do not change their generated extensions, as cons $\left(d_{1}\right)=x$, which is already in $W$. We have $E_{1} \vee E_{2} \vee E_{3} \equiv x$.

Let us now consider $\left\langle D, W^{\prime}\right\rangle$. The only default that is applicable in $W^{\prime}=\emptyset$ is $d_{1}$, which generates $x$ but also have $a$ as a justification. As a result, the defaults $d_{2}$ and $d_{3}$ are still applicable, but $d_{4}$ is not. As a result, the only extensions of $\left\langle D, W^{\prime}\right\rangle$ are $E_{1}$ and $E_{2}$. We therefore have $W^{\prime} \not \equiv_{D}^{e} W$. On the other hand, $E_{1} \vee E_{2} \equiv x$, which is equivalent to $E_{1} \vee E_{2} \vee E_{3}$. As a result, $W^{\prime} \equiv_{D}^{c} W$.

### 4.1.3 Correspondence, Particular Cases

While $\equiv_{D}^{c}$ and $\equiv_{D}^{e}$ are not the same in general, they coincide when all defaults are normal and one formula is contained in the other one.

Theorem 8 If $W^{\prime} \subseteq W$ and $D$ is a set of normal defaults, then $W^{\prime} \equiv_{D}^{c} W$ implies $W^{\prime} \equiv_{D}^{e} W$ in constrained default logic.

Proof. Given the previous result, we only have to prove that $W \equiv_{D}^{c} W^{\prime}$ implies that $\operatorname{Ext}_{D}(W) \subseteq \operatorname{Ext}_{D}\left(W^{\prime}\right)$, that is, $\langle D, W\rangle$ does not have any extension that is not an extension of $\left\langle D, W^{\prime}\right\rangle$.

To the contrary, assume that such extension exists. Let $\Pi$ be the process that generates the extension of $\langle D, W\rangle$ that is not an extension of $\left\langle D, W^{\prime}\right\rangle$. By definition of process, $\operatorname{cons}(\Pi) \cup W \cup j u s t(\Pi)$ is consistent. Therefore, it has a model $M$. Since this model satisfies both $W$ and $\operatorname{cons}(\Pi)$, it is a model of the extension generated by $\Pi$.

Since the conclusions of the two theories are the same, every model of the extension generated by $\Pi$ is a model of some extensions of $\left\langle D, W^{\prime}\right\rangle$. Let $\Pi^{\prime}$ be the a process of $\left\langle D, W^{\prime}\right\rangle$ that generates an extension that contains the model $M$. We prove that all defaults of $\Pi$ are in $\Pi^{\prime}$.

Since $M$ is a model of the extension generated by $\Pi^{\prime}$, it is a model of $\operatorname{cons}\left(\Pi^{\prime}\right) \cup W^{\prime}$. Therefore, it is a model of cons $\left(\Pi^{\prime}\right)$, and a model of just $\left(\Pi^{\prime}\right)$ because defaults are normal. We have already proved that $M$ is a model of cons $(\Pi)$ and just $(\Pi)$ and of $W$. As a result, the set cons $(\Pi) \cup \operatorname{cons}\left(\Pi^{\prime}\right) \cup W \cup j u s t(\Pi) \cup j u s t\left(\Pi^{\prime}\right)$ is consistent. As a result, we can add all defaults of $\Pi$ to $\Pi^{\prime}$ without contradicting the justifications.

As a result, the defaults of $\Pi$ are not in $\Pi^{\prime}$ only if their preconditions are not entailed from the consequences of $\Pi$. This is impossible: since $\Pi$ is a process of $\langle D, W\rangle$, we have $W \models \operatorname{prec}(d)$, where $d$ is the first default of $\Pi$. As a result, $d$ must be part of $\Pi^{\prime}$, otherwise $\Pi^{\prime}$ would not be a maximal process. The consequences of $d$ are therefore part of cons $\left(\Pi^{\prime}\right) \cup W^{\prime}$. Repeating the argument with the second default of $\Pi$ we get the same result. We can therefore conclude that all defaults of $\Pi$ are in $\Pi^{\prime}$.

Since Reiter, justified, constrained, and rational default logics coincide on normal default theories, the equality of the definitions of equivalence holds when defaults are normal.

Theorem 9 If $D$ is a set of normal defaults and $W^{\prime} \subseteq W$, then $W^{\prime} \equiv_{D}^{c} W$ if and only if $W^{\prime} \equiv_{D}^{e} W$.

When all defaults are categorical (prerequisite-free), the following lemma allows proving that the three considered forms of equivalence coincide.

Lemma 9 If $D$ is a set of categorical defaults, $W^{\prime} \subseteq W$, and $W^{\prime} \models_{D} W$, then $W \models_{D}^{e} W^{\prime}$ in constrained default logic.

Proof. Let $\Pi$ be a selected process of $\langle D, W\rangle$. We prove that $\Pi$ is a selected process of $\left\langle D, W^{\prime}\right\rangle$ generating the same extension.

Since $\Pi$ is a selected process of $\langle D, W\rangle$, it holds that $W \cup \operatorname{cons}(\Pi) \cup$ just $(\Pi)$ is consistent. Since $W^{\prime} \subseteq W$, it also holds that $W^{\prime} \cup \operatorname{cons}(\Pi) \cup$ just $(\Pi)$ is consistent. Since no default has preconditions, $\Pi$ is a successful process of $\left\langle D, W^{\prime}\right\rangle$. Since constrained default logic is a failsafe semantics [Lib05e], there exists $\Pi^{\prime}$ such that $\Pi \cdot \Pi^{\prime}$ is a selected process of $\langle D, W\rangle$.

Since every extension of $W^{\prime}$ entails $W$, this is in particular true for the extension generated by $\Pi \cdot \Pi^{\prime}$. In other words, $W^{\prime} \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \models W$. As a result, $W^{\prime} \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \equiv$ $W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right)$. Since $\Pi \cdot \Pi^{\prime}$ is a process of $\left\langle D, W^{\prime}\right\rangle$, we have that $W^{\prime} \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \cup j u s t\left(\Pi \cdot \Pi^{\prime}\right)$ is consistent, which is therefore equivalent to the consistency of $W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \cup j u s t\left(\Pi \cdot \Pi^{\prime}\right)$. Therefore, $\Pi \cdot \Pi^{\prime}$ is a successful process of $\langle D, W\rangle$. Since $\Pi$ is by assumption a maximal successful process of $\langle D, W\rangle$, it must be $\Pi^{\prime}=[]$, that is, $\Pi \cdot \Pi^{\prime}=\Pi$. We have already proved that $W^{\prime} \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \equiv W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right)$, that is, $\Pi$ generates the same extension in $W$ and in $W^{\prime}$.

Since constrained and Reiter default logics coincide on normal default theories, we have the following consequence.

Corollary 2 If $D$ is a set of normal and categorical defaults and $W^{\prime} \subseteq W$, the conditions $W^{\prime} \equiv{ }_{D}^{m} W, W^{\prime} \equiv_{D}^{c} W$, and $W^{\prime} \equiv_{D}^{e} W$ are equivalent.

### 4.2 Redundancy of Clauses vs. Theories

The redundancy of a clause $\gamma$ in a formula $W$ is defined as the equivalence of $W$ and $W \backslash\{\gamma\}$. The redundancy of a formula $W$ is defined as its equivalence to one of its proper subsets. A formula containing a redundant clause is redundant, but the converse is not always true: a formula might contain no redundant clause but yet it is equivalent to one of its proper subsets.

In this section, we compare the redundancy of a set of clauses with the redundancy of a single clause in default logic. In propositional logics, these two concepts are the same: $\Pi$ is equivalent to one of its proper subsets if and only if it contains a redundant clause. In default logic, it may be that $\gamma_{1}$ and $\gamma_{2}$ are both irredundant in $\left\{\gamma_{1}, \gamma_{2}\right\}$ while $\left\{\gamma_{1}, \gamma_{2}\right\}$ is redundant, as shown by the theory $\langle D, W\rangle$ defined below.

$$
\begin{array}{r}
W=\{a, b\} \\
D= \\
\text { where } \\
\left.\quad d_{1}=\frac{a: \neg b}{\neg b}, d_{1}, d_{3}\right\} \\
\quad d_{2}=\frac{b: \neg a}{\neg a} \\
\quad d_{3}=\frac{: a \wedge b}{a \wedge b}
\end{array}
$$

The theory $\langle D, W\rangle$ has the single extension $C n(\{a, b\})$. Indeed, $d_{1}$ and $d_{2}$ are not applicable because their justifications are inconsistent with $W$. The third default is applicable, but its consequence is $a \wedge b$, which is already in the theory.

The theory $\langle D, W \backslash\{b\}\rangle$ still has the extension $\{a, b\}$, which results from the application of $d_{3}$, which then blocks the application of $d_{1}$ and $d_{2}$. However, it also has a new extension: since $d_{1}$ is applicable, it generates $\neg b$, which blocks the application of $d_{3}$. This produces the extension $\{a, \neg b\}$. In the same way, $\langle D, W \backslash\{a\}\rangle$ has the two extensions $\{a, b\}$ and $\{\neg a, b\}$

The theory $\langle D, W \backslash\{a, b\}\rangle$ has again a single extension: $d_{3}$ is the only applicable default, leading to the addition of $a \wedge b$. Neither $d_{1}$ nor $d_{2}$ are applicable. Therefore, $\{a, b\}$ is the only extension of this theory.

The set of extensions of the theory is changed by removing any single clause, but is not changed by the removal of both clauses. In other words, both $a$ and $b$ are irredundant in $\{a, b\}$, but $\{a, b\}$ is redundant. Since $D$ is a set of normal default, this counterexample holds even for normal default theories.

The two theories obtained by removing a single clause of $W$ differ from $W$ because of a new extension. This can be proved to be always the case if the removal of both clauses leads to the original set of extensions. This is proved by first showing a sort of "continuity" of extensions.

Lemma 10 If $E$ is an extension of $\left\langle D, W^{\prime}\right\rangle$ and of $\langle D, W\rangle$ with $W^{\prime} \subseteq W$, then every selected process of $\left\langle D, W^{\prime}\right\rangle$ generating $E$ is a selected process of $\langle D, W\rangle$ generating $E$.

Proof. Let $\Pi$ be a selected process of $\left\langle D, W^{\prime}\right\rangle$ that generates $E$. Since $W^{\prime} \subseteq W$ we have $W \models W^{\prime}$; therefore, $\Pi$ is a process of $\langle D, W\rangle$. Remains to prove that it is also selected. However, all conditions for a process to be selected in $\left\langle D, W^{\prime}\right\rangle$ contains $W^{\prime}$ only in the subformula $W^{\prime} \cup \operatorname{cons}(\Pi)$. Since $E=W^{\prime} \cup \operatorname{cons}(\Pi)$, and $E$ is an extension of $\langle D, W\rangle$, we have that $E \models W$. As a result, $W^{\prime} \cup \operatorname{cons}(\Pi) \equiv W \cup \operatorname{cons}(\Pi)$. Therefore, every condition for $\Pi$ in $\left\langle D, W^{\prime}\right\rangle$ is equivalent to the same condition for $\langle D, W\rangle$.

The following lemma relates the selected processes of three formulae.
Lemma 11 If $\Pi$ is a selected process of both $\left\langle D, W^{\prime}\right\rangle$ and $\langle D, W\rangle$ and generates the same extension in both theories, it is also a selected process of every $\left\langle D, W^{\prime \prime}\right\rangle$ with $W^{\prime} \subseteq W^{\prime \prime} \subseteq W$ and generates the same extension in $\left\langle D, W^{\prime \prime}\right\rangle$.

Proof. If $W^{\prime} \subseteq W$ does not hold, the claim is trivially true because there is no $W^{\prime \prime}$ such that $W^{\prime} \subseteq W^{\prime \prime} \subseteq W$.

Since $\Pi$ is a process of $\left\langle D, W^{\prime}\right\rangle$, it is also a process of $\left\langle D, W^{\prime \prime}\right\rangle$ because $W^{\prime \prime} \models W^{\prime}$. Since $\Pi$ generates the same extensions in $W^{\prime}$ and $W$, we have that $W^{\prime} \cup \operatorname{cons}(\Pi) \equiv W \cup \operatorname{cons}(\Pi)$. Since $W \models W^{\prime \prime} \models W^{\prime}$, we also have $W^{\prime} \cup \operatorname{cons}(\Pi) \equiv W^{\prime \prime} \cup \operatorname{cons}(\Pi)$. Therefore, every condition that is true for $W^{\prime} \cup \operatorname{cons}(\Pi)$ is also true for $W^{\prime \prime} \cup \operatorname{cons}(\Pi)$.

The following lemma proves that extensions of both a theory and a subset of it are also extensions of any theory "between them".

Lemma 12 If $E$ is an extension of both $\left\langle D, W^{\prime}\right\rangle$ and $\langle D, W\rangle$, with $W^{\prime} \subseteq W$, then it is an extension of any $\left\langle D, W^{\prime \prime}\right\rangle$ with $W^{\prime} \subseteq W^{\prime \prime} \subseteq W$

Proof. By Lemma 10, every selected process $\Pi$ of $\left\langle D, W^{\prime}\right\rangle$ that generates $E$ is also a selected process of $\langle D, W\rangle$ and generates the same extension $E$ in this theory. As a result, Lemma 11 applies, and $\Pi$ is a selected process of $\left\langle D, W^{\prime \prime}\right\rangle$ and generates the same extension.

This theorem shows that extensions have a form of "partial monotonicity": an extension of both a subset and a superset of a formula is also an extension of the formula. This is important to our aims, as it shows that the equivalence $W^{\prime} \equiv_{D}^{e} W$ implies that all default theories $\left\langle D, W^{\prime \prime}\right\rangle$ with $W^{\prime} \subseteq W^{\prime \prime} \subseteq W$ have the same extensions of $\langle D, W\rangle$. Therefore, $\left\langle D, W^{\prime \prime}\right\rangle$ can differ from $\langle D, W\rangle$ only because of new extensions.

Corollary 3 If $W^{\prime} \equiv_{D}^{e} W, W^{\prime} \subseteq W^{\prime \prime} \subseteq W$, and $W^{\prime \prime} \not \equiv_{D}^{e} W$, then $\operatorname{Ext}_{D}(W) \subset \operatorname{Ext}_{D}\left(W^{\prime \prime}\right)$.
The existence of extensions of $W^{\prime \prime}$ that are not extensions of $W$ does not imply that $W^{\prime \prime}$ theory is not consequence-equivalent to $W$ and $W^{\prime \prime}$. On the other hand, $W^{\prime \prime} \not \equiv_{D}^{c} W$ implies $W^{\prime \prime} \not \equiv_{D}^{e} W$, which leads to the following consequence.

Corollary 4 If $W^{\prime} \equiv_{D}^{e} W, W^{\prime} \subseteq W^{\prime \prime} \subseteq W$, and $W^{\prime \prime} \not \equiv_{D}^{c} W$, then $\operatorname{Ext}_{D}(W) \subset \operatorname{Ext}_{D}\left(W^{\prime \prime}\right)$.
While it is not true that the irredundancy of two clauses proves the irredundancy of the set composed of them, it is however true that this can only happen because of new extensions that are created by removing a single clause. For some special cases of default logics, such creation is not possible.

Theorem 10 If $D$ is a set of normal and categorical defaults, then $W^{\prime} \equiv_{D}^{e} W$ implies that $W \equiv_{D}^{e} W^{\prime \prime}$ for any $W^{\prime \prime}$ such that $W^{\prime} \subseteq W^{\prime \prime} \subseteq W$.

Proof. By Lemma 12, all extensions of $\langle D, W\rangle$ are also extensions of $\left\langle D, W^{\prime \prime}\right\rangle$. We therefore only have to prove the converse. Let $\Pi$ be a process of $\left\langle D, W^{\prime \prime}\right\rangle$. Since the theory has no preconditions, all defaults $d \in \Pi$ satisfy $W^{\prime} \cup \operatorname{cons}(\Pi[d]) \models d$. In other words, $\Pi$ is a process of $\left\langle D, W^{\prime}\right\rangle$. Since $W^{\prime \prime} \cup \operatorname{cons}(\Pi) \cup$ just $(d)$ is consistent for every $d \in \Pi$, and $W^{\prime}$ is logically weaker than $W^{\prime \prime}$, the same condition is true for $W^{\prime}$. Since normal default logic is fail-safe [Lib05e], there exists $\Pi^{\prime}$ such that $\Pi \cdot \Pi^{\prime}$ is a selected process of $\left\langle D, W^{\prime}\right\rangle$. By Lemma 10 and Lemma 12, $W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right)$ is an extension of $\left\langle D, W^{\prime \prime}\right\rangle$.

Let us assume that $W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right)$ and $W^{\prime \prime} \cup \operatorname{cons}(\Pi)$ are not equivalent. This is only possible if $W \cup \operatorname{cons}\left(\Pi \cdot \Pi^{\prime}\right) \models W^{\prime \prime} \cup \operatorname{cons}(\Pi)$ but not vice versa. This is however impossible, because in normal default logic all extensions are mutually inconsistent [Rei80].

Since the two definitions of equivalence are the same on normal default theories, as proved in Theorem 9, this result extends to the definition of redundancy based on consequences.

Corollary 5 If $D$ is a set of normal and categorical defaults, then $W \equiv_{D}^{c} W^{\prime \prime}$ implies that $W \equiv_{D}^{c} W^{\prime}$ for any $W^{\prime}$ such that $W^{\prime \prime} \subseteq W^{\prime} \subseteq W$.

This result does not hold for normal default theories with preconditions, as the counterexample at the beginning of the section is only composed of normal defaults with preconditions.

Corollary 6 If $D$ is a set of normal and categorical defaults, then a formula is redundant if and only if it contains a redundant clause.

In other words, default logic restricted to the case of normal and categorical defaults has the local redundancy property.

### 4.3 Making Clauses Irredundant

Modifying a theory in order to make some parts irredundant proved useful for classical and circumscriptive logics. We show a similar result for default logic.

Definition 11 The $M$-irredundant version of a default theory $\langle D, W\rangle$, where $M \subseteq W=$ $\left\{\gamma_{1}, \ldots, \gamma_{m}\right\}$, is the following theory, where $\left\{c_{1}, \ldots, c_{m}\right\}$ are new variables.

$$
\begin{aligned}
& I(\langle D, W\rangle, M)=\left\langle D^{\prime}, W^{\prime}\right\rangle \\
& \text { where: } \\
& W^{\prime}=\left\{c_{i} \vee \gamma_{i} \mid \gamma_{i} \in W\right\} \\
& D^{\prime}=D_{1} \cup D_{2} \cup D_{3} \\
& D_{1}=\left\{\frac{: \neg c_{1} \wedge \cdots \wedge \neg c_{m}}{\neg c_{1} \wedge \cdots \wedge \neg c_{m}}\right\} \cup
\end{aligned}
$$

$$
\begin{aligned}
D_{2} & =\left\{\left.\frac{c_{i} \vee \gamma_{i}: c_{i} \wedge\left\{\neg c_{i} \mid 1 \leq j \leq k, i \neq i\right\}}{c_{i} \wedge\left\{\neg c_{i} \mid 1 \leq j \leq k, i \neq i\right\}} \right\rvert\, \gamma_{i} \in M\right\} \cup \\
D_{3} & =\left\{\left.\frac{\neg c_{1} \wedge \cdots \wedge \neg c_{m} \wedge \alpha: \beta}{\gamma} \right\rvert\, \frac{\alpha: \beta}{\gamma} \in D\right\}
\end{aligned}
$$

The clauses of $M$ are made irredundant by this transformation, while the redundancy of the other clauses does not change.

Theorem 11 If $M \subseteq W,\left\langle D^{\prime}, W^{\prime}\right\rangle=I(\langle D, W\rangle, M)$, and $W^{\prime \prime} \subset W$, then $W^{\prime} \equiv_{D^{\prime}}^{e} W^{\prime \prime}$ holds if and only if $\left\{\gamma_{i} \mid c_{i} \vee \gamma_{i} \in W^{\prime \prime}\right\} \equiv_{D}^{e} W$ and $W^{\prime \prime}$ contains all clauses $c_{i} \vee \gamma_{i}$ such that $\gamma_{i} \in M$. The same holds for consequence-equivalence.

Proof. The default of $D_{1}$ can be applied provided that $W$ is consistent. Its application makes all defaults of $D_{2}$ inapplicable and makes the background theory and the defaults of $D_{3}$ equivalent to $W$ and to $D$, respectively. As a result, $I(\langle D, W\rangle, M)$ has an extension $C n\left(E \wedge \neg c_{1} \wedge \cdots \wedge \neg c_{m}\right)$ for any extension $E$ of $\langle D, W\rangle$. As a result, a subset of $W^{\prime}$ has this extension if and only if the corresponding subset of $W$ has the same extension.

The $i$-th default of $D_{2}$ is applicable to $W$ because $c_{i} \vee \gamma_{i}$ is in the background theory. Since all clauses of $W$ contain the literals $c_{j}$ only positively, these literals cannot be removed by resolution. As a result, every non-tautological consequence of $W \backslash\left\{c_{i} \vee \gamma_{i}\right\}$ is disjoined with at least a variable $c_{j}$ with $j \neq i$. As a result, no subset of $W^{\prime}$ allows for the application of this default unless it contains the clause $c_{i} \vee \gamma_{i}$.

The application of this default makes all other defaults of $D^{\prime}$ inapplicable. The generated extension is moreover inconsistent with all other extensions of the theory. As a result, any subset of $W^{\prime}$ not containing $c_{i} \vee \gamma_{i}$ necessarily has a different set of extensions and consequences than $W^{\prime}$.

### 4.4 Complexity of Clause Redundancy

In this section, we analyze the complexity of checking the redundancy of a clause in a formula. Formally, this is the problem of whether $W \backslash\{\gamma\}$ is equivalent to $W$ according to $\equiv_{D}^{c}$ or $\equiv_{D}^{e}$. By Corollary 1, these two forms of equivalence are related, as $W^{\prime} \models_{D}^{c} W$ is equivalent to $W^{\prime} \models_{D}^{e} W$ and also to $W^{\prime} \models_{D} W$, if $W^{\prime} \subseteq W$. As a result, checking whether $W^{\prime} \models_{D} W$ allows for telling whether the "first part of equivalence" between $W^{\prime}$ and $W$ holds, for both kinds of equivalence. In other words, in order to check whether $W^{\prime}$ and $W$ are equivalent with $W^{\prime} \subseteq W$, we can first check whether $W^{\prime} \models_{D} W$; if this condition is true, we then proceed checking whether $W \models_{D}^{c} W^{\prime}$ or $W \models_{D}^{e} W^{\prime}$ depending on which equivalence is considered.

Lemma 8 tells that $W^{\prime} \models_{D} W$ implies that all processes of $\left\langle D, W^{\prime}\right\rangle$ are also processes of $\langle D, W\rangle$. This condition does not imply equivalence because $\langle D, W\rangle$ may contain some other processes, as in the default theory $\langle D, W\rangle$ below.

$$
W=\{a\}
$$

$$
\begin{gathered}
D=\left\{d_{1}, d_{2}\right\} \\
\text { where: } \\
d_{1}=\frac{a: b}{b} \\
d_{2}=\frac{: a \wedge \neg b}{a \wedge \neg b}
\end{gathered}
$$

The theory $\langle D, W\rangle$ has two extensions: applying either $d_{1}$ or $d_{2}$, the other is not applicable. The resulting extensions are $C n(a \wedge b)$ and $C n(a \wedge \neg b)$. Let $W^{\prime}=\emptyset$. The only default that is applicable in $W^{\prime}$ is $d_{2}$, leading to the only extension $C n(a \wedge \neg b)$. This extension implies $W$. As a result, we have that $W^{\prime} \models_{D} W$ but $W^{\prime}$ and $W$ do not have the same extensions and the same consequences. In particular, $W$ has some extensions that $W^{\prime}$ does not have. This is always the case if $W^{\prime} \models_{D} W$ but $W$ and $W^{\prime}$ are not equivalent.

In order to check equivalence of $W^{\prime}$ and $W$ with $W^{\prime} \subseteq W$, two conditions have to be checked:

1. $W^{\prime} \models_{D} W$; and
2. $W \models_{D}^{c} W^{\prime}$ or $W \models_{D}^{e} W^{\prime}$.

An upper bound on the complexity of checking the redundancy of a clause is given by the following theorem.

Theorem 12 Checking whether $W^{\prime} \equiv_{D}^{e} W$ in Reiter and justified default logic is in $\Pi_{2}^{p}$ if $W^{\prime} \subseteq W$.

Proof. Checking whether $W^{\prime} \models_{D} W$ is in $\Pi_{2}^{p}$. The other condition to be checked is $W \models_{D}^{e} W^{\prime}$. The converse of this condition is that there exists a formula $E \subseteq W \cup \operatorname{cons}(D)$ such that $E$ is an extension of $\langle D, W\rangle$ but is not an extension of $\left\langle D, W^{\prime}\right\rangle$. Since checking whether a formula is a Reiter or justified default logic is in $\Delta_{2}^{p}[\log n]$ [Ros99], the whole problem is in $\sum_{2}^{p}$. Its converse is therefore in $\Pi_{2}^{p}$. The problem of redundancy of a clause can be solved by solving two problems in $\Pi_{2}^{p}$ in parallel.

The hardness of the problem for the same class is proved by the following theorem.
Theorem 13 Checking whether $W^{\prime} \equiv{ }_{D}^{e} W$ is $\Pi_{2}^{p}$-hard even if $W=W^{\prime} \cup\{a\}$ and all defaults are categorical and normal.

Proof. The claim could be proved from the fact that entailment in default logic is $\Pi_{2}^{p}$-hard even if the formula to entail is a single positive literal, and all defaults are categorical and normal [Got92, Sti92]. If all defaults are categorical and normal, Corollary 2 proves that $W^{\prime} \equiv_{D}^{m} W$ is equivalent to the two other forms of equivalence.

We however use a new reduction from $\forall \exists \mathrm{QBF}$ because this is required by the proof of $\Sigma_{3}^{p}$-hardness of formula redundancy. The formula $\forall X \exists Y . F$ is valid if and only if $a$ is redundant in the theory below:

$$
\left\langle\left\{\frac{: x_{i}}{x_{i}}, \frac{: \neg x_{i}}{\neg x_{i}}\right\} \cup\left\{\frac{: F \wedge a}{a}\right\},\{a\}\right\rangle
$$

This theory has an extension for every possible truth evaluation over the variables $X$. For each such extension, the last default can be applied only if $F$ is consistent with the given evaluation of $X$. As a result, if $F$ is consistent with every truth evaluation over the variables $X$, then $a$ can be removed from the background theory without changing the consequences of these extensions. Otherwise, the removal of $a$ would cause some of these extensions not to entail $a$ any more.

We now consider the problem of redundancy of clauses when consequence-equivalence is used. The difference between the two kinds of equivalence is that two sets of extensions may be different but yet their disjunctions are the same. The necessity of calculating the disjunction of all extensions intuitively explains why checking redundancy for consequenceequivalence is harder than for faithful equivalence.

Theorem 14 The problem of checking whether $W^{\prime} \equiv_{D}^{c} W$ is in $\Pi_{3}^{p}$ if $W^{\prime} \subseteq W$.
Proof. $W^{\prime}$ and $W$ are consequence-equivalent if $W^{\prime} \models_{D} W$ and $W \models_{D}^{c} W^{\prime}$. The first problem is in $\Pi_{2}^{p}$. We prove that the converse of the second condition is in $\Sigma_{3}^{p}$. By definition, $W \not \mathcal{F}_{D}^{c} W^{\prime}$ holds if and only if $\vee \operatorname{Ext}_{D}(W) \not \models \vee \operatorname{Ext}_{D}\left(W^{\prime}\right)$. In terms of models, we have $\cup\left\{\operatorname{Mod}(E) \mid E \in \operatorname{Ext}_{D}(W)\right\} \nsubseteq \cup\left\{\operatorname{Mod}(E) \mid E \in \operatorname{Ext}_{D}\left(W^{\prime}\right)\right\}$, that is, there exists $M$ and $E$ such that $M \in \operatorname{Mod}(E), E \in \operatorname{Ext}_{D}(W)$, but $M$ is not a model of any extension of $W^{\prime}$. The whole condition can therefore be expressed by the following formula.

$$
\exists M \exists E . M \in \operatorname{Mod}(E) \wedge E \in \operatorname{Ext}_{D}(W) \wedge\left(\forall E^{\prime} . E^{\prime} \notin \operatorname{Ext}_{D}\left(W^{\prime}\right) \vee M \notin \operatorname{Mod}\left(E^{\prime}\right)\right)
$$

Since $E^{\prime} \notin \operatorname{Ext}_{D}\left(W^{\prime}\right)$ is in $\Delta_{2}^{p}[\log n]$ for Reiter [Ros99] and justified default logic and in $\Pi_{2}^{p}$ for constrained and rational [Lib05a], the problem of checking $W \models_{D}^{c} W^{\prime}$ is in $\Pi_{3}^{p}$. Therefore, the problem of consequence-equivalence is in $\Pi_{3}^{p}$ as well for all four considered semantics.

We show that the problem is hard for the same class.
Theorem 15 The problem of checking whether $W^{\prime} \equiv_{D}^{c} W$ for Reiter and justified default logics is $\Pi_{3}^{p}$-hard even if $W=W^{\prime} \cup\{a\}$.

Proof. Since checking whether $W^{\prime} \equiv_{D}^{e} W$ is in $\Pi_{2}^{p}$, a proof of $\Pi_{3}^{p}$-hardness necessarily requires the use of theories having different extensions but might have the same consequences.

We prove that the problem of non-equivalence of default theories is $\Sigma_{3}^{p}$-hard by reduction from QBF. We reduce a formula $\exists X \forall Y \exists Z . F$ into the problem of checking whether $W \equiv_{D}^{c} W^{\prime}$, where $W^{\prime}=\emptyset, W=\{a\}$, and $D=D_{1} \cup D_{2} \cup D_{3} \cup D_{4} \cup D_{5} \cup D_{6}$. We show each $D_{i}$ at time. First, we generate a complete evaluation over the variables $X$ using the following defaults.

$$
D_{1}=\left\{\frac{: x_{i}}{x_{i} \wedge h_{i}}, \frac{: \neg x_{i}}{\neg x_{i} \wedge h_{i}}\right\}
$$

Since these defaults have no preconditions, they can be applied regardless of whether $W$ or $W^{\prime}$ is the background theory. They generate a process for any truth evaluation $\omega_{X}$ over the variables in $X$. The variables $h_{i}$ are all true only when all variables $x_{i}$ have been set to a value.


The processes of $W$ and $W^{\prime}$ are so far the same. Once all $h_{i}$ are true, we can apply the defaults of $D_{2}=\left\{d_{1}, d_{2}, d_{3}, d_{4}\right\}$, which are the ones used in Theorem 6 to show two theories that have the same consequences but different extensions:

$$
\begin{aligned}
D_{2} & =\left\{d_{1}, d_{2}, d_{3}, d_{4}\right\} \\
d_{1} & =\frac{h_{1} \wedge \cdots \wedge h_{n}: \neg b \wedge c}{a \wedge c} \\
d_{2} & =\frac{h_{1} \wedge \cdots \wedge h_{n}: \neg b \wedge \neg c}{a \wedge \neg c} \\
d_{3} & =\frac{h_{1} \wedge \cdots \wedge h_{n} \wedge a: b \wedge c}{b \vee c} \\
d_{4} & =\frac{h_{1} \wedge \cdots \wedge h_{n} \wedge a \wedge(b \vee c): b \wedge \neg c}{b}
\end{aligned}
$$

Since these default have $h_{1} \wedge \cdots \wedge h_{n}$ as a precondition, they can only be applied once a truth assignment over $X$ has been generated by the previous defaults. They act as in the proof of Theorem 6. Only $\left[d_{1} d_{3}\right]$ and [ $d_{2}$ ] are processes of $W^{\prime}$; their consequences are $a \wedge c$ and $a \wedge \neg c$. The theory $W$ has the same processes, but also $\left[d_{3} d_{1}\right]$ and $\left[d_{3} d_{4}\right]$, which generate the extensions $a \wedge c$ and $a \wedge b$, respectively. While the first is also an extension of $W^{\prime}$, the second is not. The disjunction of all extensions is equivalent to $a$ for both $W$ and $W^{\prime}$.


The idea is as follows: from $a b \omega_{X}$, which is obtained from $W$ but not from $W^{\prime}$, we always generate the extension $a b \omega_{X} d \epsilon_{Y}$, where $\epsilon_{Y}$ is the assignment of false to all variables $Y$; from the two other points $a \neg c \omega_{X}$ and $a c \omega_{X}$ we instead generate an arbitrary assignment $\omega_{Y}$, which then has $a b \omega_{X} d \epsilon_{Y}$ as a model only if $F$ is satisfiable.

This way, if there exists a value $\omega_{X}$ such that for all $\omega_{Y}$ the formula $F$ is satisfiable, then there is no extension of $W^{\prime}$ having the model $a b d \omega_{X} \epsilon_{Y}$. Vice versa, if there exists even a single $\omega_{Y}$ such that $F$ is unsatisfiable, an extension $a c \omega_{X} \ldots$ for $W^{\prime}$ will be generated, and this extension has the model $a b \omega_{X} d \epsilon_{Y}$.


The required defaults are the following ones. First, we generate the considered model from the process that has generated $a b \omega_{X}$ :

$$
D_{3}=\left\{\frac{b: \top}{d \wedge \neg y_{1} \wedge \cdots \wedge \neg y_{n}}\right\}
$$

From $a \neg c \omega_{X}$ and $a c \omega_{X}$ we generate an arbitrary truth evaluation over $Y$. Since the model $a b d \omega_{X} \epsilon_{Y}$ assigns false to all variables $y_{i}$, we cannot simply add $y_{i}$ as a conclusion. A similar effect can be achieved by the following defaults.

$$
\begin{aligned}
& D_{4}=\left\{\frac{\neg c: \neg d \wedge y_{i}}{d \vee\left(y_{i} \wedge l_{i}\right)}, \left.\frac{\neg c: \neg d \wedge \neg y_{i}}{d \vee\left(\neg y_{i} \wedge l_{i}\right)} \right\rvert\, 1 \leq i \leq n\right\} \\
& D_{5}=\left\{\frac{c: \neg d \wedge y_{i}}{d \vee\left(y_{i} \wedge l_{i}\right)}, \left.\frac{c: \neg d \wedge \neg y_{i}}{d \vee\left(\neg y_{i} \wedge l_{i}\right)} \right\rvert\, 1 \leq i \leq n\right\}
\end{aligned}
$$

The two defaults associated with $y_{i}$ and $\neg y_{i}$ cannot be applied both at the same time, as the consequence of one contains the negation of the justification of the other one. Since the following defaults can only be applied when $d \vee\left(l_{1} \wedge \cdots \wedge l_{n}\right)$ has been derived, the current extensions before their application are $a \neg c \omega_{X}\left(d \vee\left(\omega_{Y} \wedge L\right)\right)$ and $a c \omega_{X}\left(d \vee\left(\omega_{Y} \wedge L\right)\right)$, where $\omega_{Y}$ is an arbitrary truth assignment over $Y$.

These extensions have all models of $a b \omega_{X} d \epsilon_{Y}$. The following default removes these models from the extensions if and only if $F$ is satisfiable for these given assignments over $X$ and $Y$.

$$
D_{6}=\left\{\frac{d \vee\left(l_{1} \wedge \cdots \wedge l_{n}\right): \neg d \wedge F}{\neg d}\right\}
$$

This default is not applicable from $a b \omega_{X} d \epsilon_{Y}$ because its justification contains $\neg d$. It is applicable from the other processes but only after the $i$-th default of $D_{4}$ or $D_{5}$ has been applied for each $i$ and only if the consequences of the applied defaults of $D_{4}$ or $D_{5}$ are consistent with $\neg d \wedge F$. In other words, $\left(d \vee\left(\omega_{Y} \wedge L\right)\right) \wedge \neg d \wedge F$ must be consistent, which is equivalent to the consistency of $\omega_{Y} \wedge F$ because $d$ and $L$ are not mentioned in $\omega_{Y}$ and $F$.

We can therefore conclude that:

1. for each truth assignment $\omega_{X}$, three "partial extensions" are generated from $W: a \neg c \omega_{X}$, $a c \omega_{X}$, and $a b \omega_{X}$; the first two ones are also generated by $W^{\prime}$;
2. from $a b \omega_{X}$, the extension $a b \omega_{X} d \epsilon_{Y}$ is generated; if the models of this extension are not models of an extension of $W^{\prime}$, equivalence between $W$ and $W^{\prime}$ does not hold;
3. from $a[\neg] c \omega_{X}$ we generate $a[\neg] c \omega_{X}\left(d \vee\left(\omega_{Y} \wedge L\right)\right)$ for each truth evaluation $\omega_{Y}$ on the variables $Y$; to this formula, $\neg d$ is added if and only if $F$ is consistent with $\omega_{X}$ and $\omega_{Y}$.

As a result, the models of $a b \omega_{X} d \epsilon_{Y}$ are not models of an extension of $W^{\prime}$ if and only if $F \wedge \omega_{X}$ is satisfiable for every truth evaluation of $Y$. Since non-equivalence has to be checked for every $\omega_{X}$, we have that non-equivalence holds if and only if $\exists X \forall Y \exists Z . F$.

A similar proof holds for constrained or rational default logics by replacing the default theory of Theorem 6 with that of Theorem 7 . The proof can also slightly simplified in this case, as the defaults of $D_{4}$ and $D_{5}$ can be modified with justifications $y_{i}$ or $\neg y_{i}$ and consequence $d \vee l_{i}$.

Since we have proved that the problem of clause redundancy w.r.t. consequence-equivalence is both in $\Pi_{3}^{p}$ and hard for the same class, we have the following theorem.

Theorem 16 The problem of checking whether $W^{\prime} \equiv_{D}^{c} W$ is $\Pi_{3}^{p}$-complete if $W^{\prime} \subseteq W$; hardness holds even if $W=W^{\prime} \cup\{a\}$.

### 4.5 Complexity of Formula Redundancy

The next problem to analyze is whether a formula (a set of clauses) is redundant, for a fixed set of defaults. The complexity of formula redundancy w.r.t. faithful and consequenceequivalence is in $\Sigma_{3}^{p}$ and $\Sigma_{4}^{p}$, respectively.

Theorem 17 The problem of formula redundancy for faithful and consequence-equivalence is in $\Sigma_{3}^{p}$ and $\Sigma_{4}^{p}$, respectively.

Proof. Both problems can be expressed as the existence of a subset $W^{\prime} \subset W$ such that $W^{\prime}$ is equivalent to $W$. Since equivalence is in $\Pi_{2}^{p}$ and $\Pi_{3}^{p}$, respectively, for faithful and consequence-equivalence, the claim follows.

Regarding hardness, we first show a theorem characterizing the complexity of the problem for the case of faithful equivalence. We then show a more general technique allowing an hardness result to be raised one level in the polynomial hierarchy.

Theorem 18 The problem of formula redundancy based on faithful equivalence is $\Sigma_{3}^{p}$-hard.
Proof. We reduce the problem of validity of $\exists X \forall Y \exists Z . F$ to the problem of redundancy of a formula. Let $n=|X|$. The default theory corresponding to the formula $\exists X \forall Y \exists Z . F$ is the theory $\langle D, W\rangle$ defined as follows.

$$
\begin{aligned}
W & =\left\{s_{i} \mid 1 \leq i \leq n\right\} \cup\left\{r_{i} \mid 1 \leq i \leq n\right\} \\
D & =D_{1} \cup D_{2} \cup D_{3} \cup D_{4} \cup D_{5} \cup D_{6} \\
D_{1} & =\left\{\frac{s_{i} \wedge r_{i}: \neg s_{j}}{a}, \frac{s_{i} \wedge r_{i}: \neg r_{j}}{a} \left\lvert\, \begin{array}{l}
1 \leq i \leq n \\
1 \leq j \leq n
\end{array}\right.\right\} \\
D_{2} & =\left\{\frac{: \neg s_{i} \wedge \neg r_{i}}{a} \left\lvert\, \begin{array}{l}
1 \leq i \leq n \\
1 \leq j \leq n
\end{array}\right.\right\} \\
D_{3} & =\left\{\frac{: y_{i}}{y_{i} \wedge h_{i}}, \left.\frac{: \neg y_{i}}{\neg y_{i} \wedge h_{i}} \right\rvert\, 1 \leq i \leq n\right\} \\
D_{4} & =\left\{\frac{: x_{i}}{p_{i} \wedge x_{i}}, \left.\frac{: \neg x_{i}}{p_{i} \wedge \neg x_{i}} \right\rvert\, 1 \leq i \leq n\right\} \\
D_{5} & =\left\{\frac{x_{i} \wedge r_{i}: \top}{\wedge W}, \left.\frac{\neg x_{i} \wedge s_{i}: \top}{\wedge W} \right\rvert\, 1 \leq i \leq n\right\} \\
D_{6} & =\left\{\frac{p_{1} \ldots p_{n} h_{1} \ldots h_{n}: F}{\wedge W}\right\}
\end{aligned}
$$

The defaults of $D_{1}$ and $D_{2}$ cannot be applied from $W$. The defaults of $D_{3}$ and $D_{4}$ generates an extension for every possible truth evaluation over $X \cup Y$; this extension also contains all variables $h_{i}$ and $p_{i}$. Whether or not the last default is applicable, its consequence is equivalent to the background theory.

Let $W^{\prime} \subset W$. If there is an index $i$ such that both $s_{i}$ and $r_{i}$ are in $W^{\prime}$, one of the defaults of $D_{1}$ is applicable, generating $a$. Therefore, $W^{\prime}$ is not equivalent to $W$. If there exists an index $i$ such that neither $s_{i}$ nor $r_{i}$ is in $W^{\prime}$, the $i$-th default of $D_{2}$ is applicable, still generating $a$.

In order to check for redundancy, we therefore only have to consider subsets $W^{\prime} \subset W$ for which either $s_{i} \in W$ or $r_{i} \in W$ but not both. Let $\omega_{X}$ be the assignment on the variables $X$ such that $x_{i}$ is assigned to true if $s_{i} \in W^{\prime}$ and to false if $r_{i} \in W^{\prime}$. The defaults of $D_{3}$ and
$D_{4}$ generate an arbitrary truth evaluation of the variables $X \cup Y$. If the assignment on $X$ is not equal to $\omega_{X}$, the formula $\wedge W$ is generated, thus leading to an extension that is also an extension of $W$. As a result, all extensions of $W^{\prime}$ that do not match the value $\omega_{X}$ are also extensions of $W$. If the same holds also for the extensions for which the values of $X$ match $\omega_{X}$, then $W^{\prime}$ is equivalent to $W$.

For a given $W^{\prime}$ we consider the extensions consistent with $\omega_{X}$. There is exactly one such extension for each possible truth evaluation over $Y$. If the default of $D_{6}$ can be applied, it generates $\wedge W$, thus making $W^{\prime}$ equivalent to $W$. In turn, the default of $D_{6}$ can be applied for all truth evaluation over $Y$ if and only if for all such truth evaluation, $F$ is satisfiable. As a result, $W^{\prime}$ is equivalent to $W$ if and only if, for all possible truth evaluations over $Y$, the formula $F$ is satisfiable. Since there exists a relevant $W^{\prime}$ for each truth evaluation over $X$, the formula $W$ is redundant if and only if there exists a truth evaluation over $X$ such that, for all possible truth evaluations over $Y$, the formula $F$ is satisfiable.

In order to prove the $\Sigma_{4}^{p}$-hardness of the problem of formula redundancy under consequenceequivalence, we should provide a reduction from $\exists \forall \exists \forall \mathrm{QBF}$ validity into this problem. A simpler proof can however be given, based on the following consideration: checking clause redundancy has been proved $\Pi_{2}^{p}$-hard or $\Pi_{3}^{p}$-hard using reductions from QBFs that results in default theories having $W=\{a\}$ as the background theory. As a result, these reductions also prove that formula redundancy is $\Pi_{2}^{p}$-hard or $\Pi_{3}^{p}$-hard. In other words, we can reduce the validity of a $\forall \exists \mathrm{QBF}$ or a $\forall \exists \forall \mathrm{QBF}$ into the problems of formula redundancy. What we show is that, if such reductions satisfy some assumptions, we can obtain new reductions from QBFs having an additional existential quantifier in the front. The assumptions are that the default theory resulting from the reduction is such that:

1. the background theory that results from the reduction is classically irredundant;
2. the matrix of the QBF is only used in the justification of a single default.

The reductions used for proving the hardness of clause redundancy satisfies both assumptions. In particular, $\forall X \exists Y \forall Z . F$ is valid if and only if the background theory of the following theory is consequence-redundant, where $D, \alpha, \beta, \gamma$, do not depend on $F$ but only on the quantifiers of the QBF and $W$ is classically irredundant.

$$
\left\langle D \cup\left\{\frac{\alpha: \beta \wedge F}{\gamma}\right\}, W\right\rangle
$$

The fact that the matrix of the QBF is copied "verbatim" in the default theory is exploited as follows: if $\omega_{w}$ is a truth evaluation over the variable $w$, then $\forall X \exists Y \forall Z .\left.F\right|_{\omega_{w}}$ is valid if and only if the background theory of $\left\langle D \cup\left\{\frac{\alpha: \beta \wedge F \wedge \omega_{w}}{\gamma}\right\}, W\right\rangle$ is redundant. This default theory can be modified in such a way the subsets of the background theory are in correspondence with the truth evaluations over $\omega_{w}$. This way, the resulting theory is redundant if and only if $\exists w \forall X \exists Y \forall Z . F$. The resulting default theory still satisfies the two assumptions above on the background theory and on the use of the matrix of the QBF; therefore, this procedure can be iterated to obtain a reduction from $\exists \forall \exists \forall \mathrm{QBF}$ validity into the problem of formula
redundancy under consequence-equivalence. A similar technique can be used for faithful equivalence.

The details of this technique are in the following three lemmas. The first one shows that a literal can be moved from the justification of a default to the background theory and vice versa, under certain conditions.

Lemma 13 If the variable of the literal $l$ is not mentioned in $W, D, \operatorname{prec}(d)$, and $\operatorname{cons}(d)$, the processes of the following two theories are the same modulo the replacement of $d$ with $d^{\prime}$ and vice versa.

$$
\begin{aligned}
& \langle D \cup\{d\}, W \cup\{l\}\rangle \\
& \left\langle D \cup\left\{d^{\prime}\right\}, W\right\rangle \\
& \text { where } \\
& d^{\prime}=\frac{\operatorname{prec}(d): \operatorname{just}(d) \wedge l}{\operatorname{cons}(d)}
\end{aligned}
$$

Proof. The literal $l$ and its negation only occur in the background theory $W \cup\{l\}$ and in the justification of $d$ and $d^{\prime}$. The conditions on a process of the first theory being selected either involve $(W \cup\{l\}) \cup \operatorname{just}(d)$ or $W \cup\{l\}$ with other formulae not containing $l$. As a result, moving $l$ from the background theory to the justification of $d$ or vice versa does not change these conditions.

Note that the processes are the same, but the extensions are different in that $l$ is in all extensions of the first theory but not in the extensions of the second.

The second lemma is an obvious consequence of the above: under the same conditions, moving a literal from the justification of a default to the background theory or vice versa does not change the redundancy of a theory.

Lemma 14 If $W^{\prime} \subseteq W$, it holds $W^{\prime} \equiv_{D^{\prime}}^{e} W$ if and only if $W^{\prime} \cup\{l\} \equiv_{D^{\prime \prime}}^{e} W \cup\{l\}$, where $l$ is a literal that is not mentioned in $W, D$, prec $(d)$, and $\operatorname{cons}(d)$, where $D^{\prime}$ and $D^{\prime \prime}$ are as follows.

$$
\begin{aligned}
D^{\prime} & =D \cup\left\{\frac{\operatorname{prec}(d): \operatorname{just}(d) \wedge l}{\operatorname{cons}(d)}\right\} \\
D^{\prime \prime} & =D \cup\{d\}
\end{aligned}
$$

Proof. Obvious consequence of the lemma above: $\left\langle D^{\prime}, W^{\prime}\right\rangle$ and $\left\langle D^{\prime}, W\right\rangle$ have the same processes of $\left\langle D^{\prime \prime}, W^{\prime} \cup\{l\}\right\rangle$ and $\left\langle D^{\prime \prime}, W \cup\{l\}\right\rangle$, respectively.

A consequence of this lemma is that $W$ is redundant according to $D^{\prime}$ if and only if $W \cup\{l\}$ is redundant according to $D^{\prime \prime}$. Indeed, $l$ is not mentioned in the consequences of the defaults; therefore, a subset of $W \cup\{l\}$ can only be equivalent to $W \cup\{l\}$ if it contains $l$. The lemma is formulated in the more complicated way because it is necessary for proving the following
lemma. The same property can be proved using consequence-equivalence because moving $l$ from the justification of the default to the background theory has the only effect of adding $l$ to all extensions.

Lemma 15 If $W$ is classically irredundant, then there exists $W^{\prime} \subset W$ such that $W^{\prime} \cup\{w\} \equiv_{D}^{e}$ $W \cup\{w\}$ or $W^{\prime} \cup\{\neg w\} \equiv_{D}^{e} W \cup\{\neg w\}$ if and only if the following theory is redundant:

$$
\left\langle D_{w} \cup\left\{\left.\frac{p \wedge \alpha: \beta}{\gamma} \right\rvert\, \frac{\alpha: \beta}{\gamma} \in D\right\}, W \cup\left\{w^{+}, w^{-}\right\}\right\rangle
$$

where:

$$
D_{w}=\left\{\frac{w^{+} \wedge w^{-}: \neg W}{\neg p}, \frac{: \neg w^{+} \wedge \neg w^{-}}{\neg p}, \frac{w^{+}: w \wedge p}{w \wedge p}, \frac{w^{-}: \neg w \wedge p}{\neg w \wedge p}\right\}
$$

and $w^{+}, w^{-}$, and $p$ are new variables.
Proof. Since $w^{+}$and $w^{-}$are new variables not contained in $W$ and $W$ is classically irredundant, $W \cup\left\{w^{+}, w^{-}\right\}$is classically irredundant as well.

We now consider the processes that can be generated from $W \cup\left\{w^{+}, w^{-}\right\}$and from its subsets. From $W \cup\left\{w^{+}, w^{-}\right\}$we can apply only one of the last two defaults of $D_{w}$, generating either $w \wedge p$ or $\neg w \wedge p$. From this point on, we have exactly the same processes of $\langle D, W \cup\{w\}\rangle$ and $\langle D, W \cup\{\neg w\}\rangle$, the generated extensions only differing because of the addition of $p$ and $w^{+}$or $w^{-}$.

The proper subsets of $W \cup\left\{w^{+}, w^{-}\right\}$are $W^{\prime} \cup\left\{w^{+}, w^{-}\right\}$where $W^{\prime} \subset W, W^{\prime} \cup\left\{w^{+}\right\}$, $W^{\prime} \cup\left\{w^{-}\right\}$, and $W^{\prime}$, where $W^{\prime} \subseteq W$. The fourth subset $W^{\prime}$ is not equivalent to $W$ because the second default of $D_{w}$ allows the derivation of $\neg p$, which is not derivable from $W$. If $W^{\prime} \subset W$, since $W$ is (classically) irredundant, $W^{\prime} \cup\left\{w^{+}, w^{-}\right\}$allows for the application of the first default of $D_{w}$, deriving $\neg p$; therefore, this subset is not equivalent to the background theory.

The only two other subsets to consider are $W^{\prime} \cup\left\{w^{+}\right\}$and $W^{\prime} \cup\left\{w^{-}\right\}$. In the first subset, only $w \wedge p$ can be generated. In the second subset, only $\neg w \wedge p$ can be generated. From this point on, we have exactly the same processes of $W^{\prime} \cup\{w\}$ and $W^{\prime} \cup\{\neg w\}$ according to $D$. The generated extensions are the same but for the addition of $p$.

These three lemmas together proves that a reduction from QBF to formula redundancy can be "raised" by the addition of an existential quantifier in the front of the QBF.

Lemma 16 If there exists a polynomial reduction from formulae $Q . E$, where $Q$ is a sequence of quantifier of a given class, to the problem of formula redundancy of a default theory in the following form, then there exists a polynomial reduction from formulae of the form $\exists w Q . F$ to the formula redundancy of a theory in the following form.

$$
\left\langle D \cup\left\{\frac{\alpha: \beta \wedge F}{\gamma}\right\}, W\right\rangle
$$

The formulae in $D,\{\alpha, \beta, \gamma\}$, and $W$ do not depend on $F$. The background theory $W$ is classically irredundant.

Proof. Let $Q$ be a sequence of quantifiers so that the validity of the formula $Q . E$ can be reduced to formula redundancy of a default theory of the above form. We show a reduction from the validity of $\exists w Q . F$ to formula redundancy of a default theory of the same form.

By definition, both $Q .\left.F\right|_{w=\text { true }}$ and $Q .\left.F\right|_{w=\text { false }}$ can be reduced to the problem of formula redundancy. These two formulae only differ on their matrixes, which are $\left.F\right|_{w=\text { true }}$ and $\left.F\right|_{w=\text { false. }}$. Therefore, the resulting default theories are:

$$
\begin{aligned}
& \left\langle D \cup\left\{\frac{\alpha:\left.\beta \wedge F\right|_{w=\text { true }}}{\gamma}\right\}, W\right\rangle \\
& \left\langle D \cup\left\{\frac{\alpha:\left.\beta \wedge F\right|_{w=\text { false }}}{\gamma}\right\}, W\right\rangle
\end{aligned}
$$

Since $w$ does not occur anywhere else in the theory, we can replace $\left.F\right|_{w=\text { true }}$ and $\left.F\right|_{w=\text { false }}$ with $F \wedge w$ and $F \wedge \neg w$, respectively. Indeed, justifications are only checked for consistency, and for any formula $R$ not containing $w$, the consistency of $\left.R \cup F\right|_{w=\text { true }}$ is the same as the consistency of $R \cup(F \wedge w)$, and the consistency of $\left.R \cup F\right|_{w=\text { false }}$ is the same as the consistency of $R \cup(F \wedge \neg w)$.

$$
\begin{aligned}
& \left\langle D \cup\left\{\frac{\alpha: \beta \wedge F \wedge w}{\gamma}\right\}, W\right\rangle \\
& \left\langle D \cup\left\{\frac{\alpha: \beta \wedge F \wedge \neg w}{\gamma}\right\}, W\right\rangle
\end{aligned}
$$

By Lemma 14, formula redundancy of these two theories corresponds to formula redundancy of the same theories with $w$ or $\neg w$ moved to the background theory. More precisely, the redundancy of the first theory correspond to the existence of a subset $W^{\prime} \subset W$ such that $W^{\prime} \cup\{w\}$ is equivalent to $W \cup\{w\}$ according to the defaults $D \cup\left\{\frac{\alpha: \beta \wedge F}{\gamma}\right\}$. The same holds for the second theory.

$$
\begin{aligned}
& \left\langle D \cup\left\{\frac{\alpha: \beta \wedge F}{\gamma}\right\}, W \cup\{w\}\right\rangle \\
& \left\langle D \cup\left\{\frac{\alpha: \beta \wedge F}{\gamma}\right\}, W \cup\{\neg w\}\right\rangle
\end{aligned}
$$

By Lemma 15, since $W$ is classically irredundant, we have that either the first or the second of the two theories are redundant if and only if the following theory is redundant:

$$
\left\langle D_{w} \cup\left\{\left.\frac{p \wedge \alpha: \beta}{\gamma} \right\rvert\, \frac{\alpha: \beta}{\gamma} \in D\right\}, W \cup\left\{w^{+}, w^{-}\right\}\right\rangle
$$

where $D_{w}$ is defined in the statement of Lemma 15 . As a result, this formula is redundant if and only if either $Q .\left.F\right|_{w=\text { true }}$ is valid or $Q .\left.F\right|_{w=\text { false }}$ is valid, that is, $\exists w Q . F$ is valid.

In order to complete the lemma, we have to show that the background theory of the above theory is classically irredundant, and the theory is in the form specified by the statement of the theorem. Since $W$ is classically irredundant by assumption and $w^{+}$and $w^{-}$are new variables, $W \cup\left\{w^{+}, w^{-}\right\}$is classically irredundant. In the above theory, the matrix $F$ of the QBF is only mentioned in the justification of the default $\frac{p \wedge \alpha: \beta \wedge F}{\gamma}$. Therefore, the theory that results from the reduction is in the form specified by the theorem.

The above lemmas are also valid for consequence-equivalence. In both cases, we have that the hardness of formula redundancy is one level higher in the polynomial hierarchy than clause redundancy.

Theorem 19 Formula redundancy is $\Sigma_{3}^{p}$-hard for faithful equivalence and $\Sigma_{4}^{p}$-hard for consequenceequivalence.

Proof. The reduction shown after Theorem 13 and the reduction used in Theorem 15 are reductions from $\forall \exists \mathrm{QBF}$ and $\forall \exists \forall \mathrm{QBF}$, respectively, into the problem of formula redundancy. These reductions produce a default theory in which the background theory contains a single non-tautological clause, and is therefore irredundant, and the matrix of the QBF only occurs in the justification of a single default. These are the conditions of Lemma 16. As a result, one can reduce an $\exists \forall \exists \mathrm{QBF}$ or an $\exists \forall \exists \forall \mathrm{QBF}$ to the problem of formula redundancy by iteratively applying the modification of Lemma 16 for all variables of the first existential quantifier.

### 4.6 Redundancy of Defaults

The redundancy of a default is defined in the same way as redundancy of clauses.
Definition 12 (Redundancy of a default) A default $d$ is redundant in $\langle D, W\rangle$ if and only if $\langle D \backslash\{d\}, W\rangle$ is equivalent to $\langle D, W\rangle$.

This definition depends on the kind of equivalence used. Therefore, a default can be redundant w.r.t. faithful or consequence-equivalence. The redundancy of defaults is defined as follows.

Definition 13 (Redundancy of a theory) A default theory $\langle D, W\rangle$ is default redundant if and only if there exists $D^{\prime} \subset D$ such that $\left\langle D^{\prime}, W\right\rangle$ is equivalent to $\langle D, W\rangle$.

### 4.6.1 Making Defaults Irredundant

The following lemma is the version of Theorem 11 to the case of default redundancy rather than clause redundancy. It proves that some defaults can be made irredundant while not changing the redundancy status of the other ones.

Lemma 17 For every default theory $\langle D, W\rangle$, set of defaults $D_{I} \subseteq D$, and $D_{1}, D_{2}, D_{3}$ defined as follows:

$$
\begin{aligned}
& D_{1}=\{d+, d-\} \\
& \text { where: } \\
& d+=\frac{: p \wedge q}{p \wedge q} \\
& d-=\frac{: \neg p \wedge q}{\neg p \wedge q} \\
& D_{2}=\left\{\left.\frac{q \wedge(\neg p \vee \alpha): \neg p \vee \beta}{\left(p \vee v_{i}\right) \wedge(\neg p \vee \gamma)} \right\rvert\, \frac{\alpha: \beta}{\gamma} \in D_{I}\right\} \\
& D_{3}=\left\{\frac{q \wedge p \wedge \alpha: \beta}{\gamma} \left\lvert\, \frac{\alpha: \beta}{\gamma} \in D \backslash D_{I}\right.\right\}
\end{aligned}
$$

if $\langle D, W\rangle$ has extensions and $W$ is consistent, it holds that:

1. the processes of $\left\langle D_{1} \cup D_{2} \cup D_{3}, W\right\rangle$ are (modulo the transformation of the defaults) the same of $\langle D, W\rangle$ with $d+$ added to the front and a number of processes composed of $d-$ and a sequence containing all defaults of $D_{2}$;
2. the extension of $\left\langle D_{1} \cup D_{2} \cup D_{3}, W\right\rangle$ are the same of $\langle D, W\rangle$ with $\{p, q\}$ added plus the single extension $\{\neg p, q\} \cup\left\{v_{i}\right\}$;
3. a subset of $D_{1} \cup D_{2} \cup D_{3}$ is equivalent to it if and only if it contains $D_{1} \cup D_{2}$ and the set of original defaults corresponding to those of $D_{2} \cup D_{3}$ is equivalent to $D$.

Proof. Since all defaults of $D_{2} \cup D_{3}$ have $q$ as a precondition, they are not applicable from $W$. The only defaults that are applicable to $W$ are therefore $d+$ and $d-$, which are mutually exclusive.

Let us consider the processes with $d$ - in first position. Since $d$ - generates $\neg p$, the defaults of $D_{3}$ are not applicable. We prove that $[d-] \cdot \Pi_{2}$ is a successful process, where $\Pi_{2}$ is an arbitrary sequence containing all defaults of $D_{2}$. The preconditions of all defaults of $D_{2}$ are entailed by $q \wedge \neg p$. The union of the justifications and consequences of all defaults of this process is $\{\neg p, q\} \cup\left\{\neg p \vee \beta, p \vee v_{i}, \neg p \vee \gamma\right\}$, which is equivalent to $\{\neg p, q\} \cup\left\{v_{i}\right\}$. This set is consistent with the background theory, which does not contain the variables $p, q$, and $v_{i}$.

If a subset of $D_{1} \cup D_{2} \cup D_{3}$ does not contain $d-$, the literal $\neg q$ cannot derived because no other default has $\neg q$ as a conclusion. If a subset of $D_{1} \cup D_{2} \cup D_{3}$ does not contain a default of $D_{2}$, the corresponding variable $v_{i}$ is not in this extension. As a result, every subset of $D_{1} \cup D_{2} \cup D_{3}$ that is equivalent to it contains $\{d-\} \cup D_{2}$.

Let us now consider the processes with $d+$ in first position. Such a process cannot contain $d-$. Since $p$ and $q$ are generated, the defaults of $D_{2} \cup D_{3}$ can be simplified to $\frac{\alpha: \beta}{\gamma}$ by removing all clauses containing $p$ or $q$ and all literals $\neg p$ and $\neg q$ from the clauses containing them. As
a result, the processes having $d+$ in first position correspond to the processes of the original theory.

Provided that the original theory has extensions, every subset of $D_{1} \cup D_{2} \cup D_{3}$ not containing $d+$ lacks these extensions. The defaults of $D_{3}$ are redundant if and only if they are redundant in the original theory. More precisely, a subset $D^{\prime} \subset D_{1} \cup D_{2} \cup D_{3}$ is equivalent to $D_{1} \cup D_{2} \cup D_{3}$ if and only if $D^{\prime}$ contains $D_{1} \cup D_{2}$, and the set of original defaults $D^{\prime \prime}$ corresponding to the defaults of $D^{\prime} \cap\left(D_{2} \cup D_{3}\right)$ is equivalent to $D$.

### 4.6.2 Redundancy of Defaults vs. Sets of Defaults

While a formula is classically redundant if and only if it contains a redundant clause, the same does not happen for default redundancy. The following theorem indeed proves that Reiter and rational default logic do not have the local redundancy property w.r.t. redundancy of defaults.

Theorem 20 There exists a set of defaults $D$ such that, according to Reiter and rational default logic:

1. for any $d \in D$, the theory $\langle D \backslash\{d\}, \emptyset\rangle$ has extensions and $\langle D \backslash\{d\}, W\rangle \not \equiv_{D}^{c}\langle D, \emptyset\rangle$;
2. there exists $D^{\prime} \subset D$ such that $\left\langle D^{\prime}, \emptyset\right\rangle \equiv{ }_{D}^{e}\langle D, \emptyset\rangle$.

Proof. We use a pair of defaults that lead to failure is they are together in the same process. Removing one of them from the default theory leads to a new extension, while removing both of them lead to the original set of extensions. The following defaults are a realization of this idea.

$$
\begin{array}{r}
D=\left\{d_{1}, d_{2}, d_{3}\right\} \\
\text { where: } \\
d_{1}=\frac{: b}{b \wedge c} \\
d_{2}=\frac{: b}{b \wedge \neg c} \\
d_{3}=\frac{: \neg b}{\neg b}
\end{array}
$$

The extensions of some $\left\langle D^{\prime}, \emptyset\right\rangle$, with $D^{\prime} \subseteq D$, are as follows:
$D^{\prime}=D$ we can either apply both $d_{1}$ and $d_{2}$ (leading to a failure) or $d_{3}$ alone; the only extension of this theory is therefore $\neg b$;
$D^{\prime}=\left\{d_{1}, d_{3}\right\}$ both $d_{1}$ and $d_{3}$ can be applied, but not both; that results in two processes having conclusions $\vee \operatorname{Ext}(\langle D, \emptyset\rangle)=(b \wedge \neg c) \vee \neg b \equiv \top$;
$D^{\prime}=\left\{d_{2}, d_{3}\right\}$ same as above: $\vee \operatorname{Ext}(\langle D, \emptyset\rangle)=(b \wedge c) \vee \neg b \equiv \mathrm{~T} ;$
$D^{\prime}=\left\{d_{3}\right\}$ the only selected process is $\left[d_{3}\right]$, which leads to $\vee \operatorname{Ext}\left(\left\langle D^{\prime}, W\right\rangle\right)=\neg b$.
As a result, $\langle D \backslash\{d\}, \emptyset\rangle$ has extensions for every $d \in D$. The default $d_{3}$ is not irredundant, but can be made so by the transformation of Theorem 17, which preserves processes almost exactly; an alternative is to replace $d_{3}$ with $\frac{i \neg b}{b \wedge d}$ and $\frac{i\urcorner b}{b \wedge e}$. The resulting set of default has no redundant default, but has an equivalent subset.

The same result holds for constrained default logic.
Theorem 21 There exists a set of defaults $D$ such that, according to constrained default logic:

1. for any $d \in D$, it holds $\langle D \backslash\{d\}, W\rangle \not \equiv_{D}^{c}\langle D, \emptyset\rangle$;
2. there exists $D^{\prime} \subset D$ such that $\left\langle D^{\prime}, \emptyset\right\rangle \equiv_{D}^{e}\langle D, \emptyset\rangle$.

Proof. The defaults are the following ones:

$$
\begin{aligned}
& D=\left\{d_{1}, d_{2}, d_{3}\right\} \\
& \text { where: } \\
& d_{1}=\frac{: x}{a} \\
& d_{2}=\frac{: x}{b} \\
& d_{3}=\frac{: \neg x \wedge \neg y}{a b}
\end{aligned}
$$

The theory $\langle D, \emptyset\rangle$ has two selected processes (modulo permutation of defaults): $\left[d_{1}, d_{2}\right]$ and $\left[d_{3}\right]$, both generating the extension $a b$. Removing either $d_{1}$ or $d_{2}$ causes the first process to become $\left[d_{1}\right]$ or $\left[d_{2}\right]$, thus creating a new extension that is either $a$ or $b$. On the other hand, removing both $d_{1}$ and $d_{2}$ makes the only remaining process to be $\left[d_{3}\right]$, which generates the only extension $a b$ of the original theory. The default $d_{3}$ is not redundant, but can be made so by applying the transformation of Theorem 17.

Justified default logic has the local redundancy property w.r.t. default redundancy. This is a combination of two factors: first, justified default logic is failsafe [Lib05e] (every successful process can be made selected by adding some defaults); second, every extension is generated by an unique set of defaults. The proofs requires two lemmas. The first one is about extendibility of processes when new defaults are added to a theory.

Lemma 18 In justified default logic, if $\Pi$ is a selected process of $\left\langle D^{\prime}, W\right\rangle$ and $D^{\prime} \subseteq D$, then there exists a sequence $\Pi^{\prime}$ of defaults of $D \backslash D^{\prime}$ such that $\Pi \cdot \Pi^{\prime}$ is a selected process of $\langle D, W\rangle$.

Proof. Let $\Pi$ be a selected process of $\left\langle D^{\prime}, W\right\rangle$. By definition, it holds $W \cup \operatorname{cons}(\Pi[d]) \models$ $\operatorname{prec}(d)$ and $W \cup \operatorname{cons}(\Pi) \operatorname{Tjust}(d)$ for every $d \in \Pi$. As a result, $\Pi$ is a also a successful process of $\langle D, W\rangle$. Therefore, there exists $\Pi^{\prime}$ such that $\Pi \cdot \Pi^{\prime}$ is a selected process of $\langle D, W\rangle$
because justified default logic is failsafe [Lib05e]. If $\Pi^{\prime}$ contains defaults of $D^{\prime}$, then $\Pi$ would not be a closed process of $\left\langle D^{\prime}, W\right\rangle$.

In order for proving the second lemma, we need an intermediate result, which is already well known.

Lemma 19 In justified default logic, the selected processes of $\langle D, W\rangle$ generating the extension $E$ are composed of exactly the defaults of the following set:

$$
G E N(E, D)=\{d \in D \mid E \models \operatorname{prec}(d) \text { and } E \top \text { just }(d) \cup \operatorname{cons}(d)\}
$$

Proof. Assume that $\Pi$ is a selected process generating $E$ that does not contain a default $d \in G E N(E, D)$. Since $E \models \operatorname{prec}(d)$, $E T$ just $(d) \cup \operatorname{cons}(d)$, and $E=W \cup \operatorname{cons}(\Pi)$, we have that $W \cup \operatorname{cons}(\Pi) \models \operatorname{prec}(d)$ and $W \cup \operatorname{cons}(\Pi) T \operatorname{just}(d) \cup \operatorname{cons}(d)$. As a result, $\Pi \cdot[d]$ is a successful process, contradicting the assumption.

Let $\Pi$ be a selected process containing a default $d$ not in $G E N(E, D)$. By definition, either $E \not \vDash \operatorname{prec}(d)$ or $E \perp \mathrm{just}(d) \cup \operatorname{cons}(d)$. The first condition implies that $W \cup \operatorname{cons}(\Pi[d]) \not \models d$ whichever the position of $d$ in $\Pi$ is. The second condition implies $W \cup \operatorname{cons}(\Pi) \perp$ just $(d) \cup$ cons $(d)$ : the process $\Pi$ is not successful contrary to the assumption.

The next lemma relates the processes of two theories when they are assumed to have the same extension. In this lemma and in the following theorem, when a process is used in a place where a set of defaults is expected, it means the set of defaults of the process. For example, if $\Pi$ is a sequence of defaults and $D^{\prime}$ a set of defaults, $\Pi \cap D^{\prime}$ is the set of defaults that are both in $\Pi$ and in $D$.

Lemma 20 In justified default logic, if $D^{\prime} \subseteq D,\left\langle D^{\prime}, W\right\rangle \equiv_{D}^{e}\langle D, W\rangle$, and $\Pi$ is a selected process $\langle D, W\rangle$, then there exists a selected process of $\left\langle D^{\prime}, W\right\rangle$ made exactly of the defaults of $\Pi \cap D^{\prime}$ and generating the same extension generated by $\Pi$.

Proof. Let $E=W \cup \operatorname{cons}(\Pi)$ be the extension that is generated by $\Pi$. By the lemma above, it is generated by the defaults in $\operatorname{GEN}(E, D)$. Since $E$ is also an extension of $\left\langle D^{\prime}, W\right\rangle$, it is generated by a process $\Pi^{\prime}$ made exactly of the defaults of $G E N\left(E, D^{\prime}\right)=G E N(E, D) \cap E^{\prime}=$ $\Pi \cap D^{\prime}$.

The main theorem relating the extensions of theories differing for the set of defaults in justified default logic is the following one.

Theorem 22 If $D^{\prime} \subseteq D^{\prime \prime} \subseteq D$ and $\left\langle D^{\prime}, W\right\rangle \equiv^{e}\langle D, W\rangle$ then $\left\langle D^{\prime}, W\right\rangle \equiv^{e}\left\langle D^{\prime \prime}, W\right\rangle$ for justified default logic.

Proof. We first show that every extension of $\left\langle D^{\prime \prime}, W\right\rangle$ is also an extension of $\langle D, W\rangle$, and then show the converse.

Let $E$ be an extension of $\left\langle D^{\prime \prime}, W\right\rangle$. Let $\Pi$ is one its generating processes. By Lemma 18, there exists a sequence $\Pi^{\prime}$ of defaults of $D \backslash D^{\prime \prime}$ such that $\Pi \cdot \Pi^{\prime}$ is a selected process of $\langle D, W\rangle$.

Let $E^{\prime}$ be its generated extension. Since $E$ is generated by $\Pi$ and $E^{\prime}$ is generated by $\Pi \cdot \Pi^{\prime}$, we have $E^{\prime} \models E$. We prove that $E \models E^{\prime}$, which implies $E \equiv E^{\prime}$.

By Lemma 20, since $\Pi$ is a selected process of $\langle D, W\rangle$ and this theory is faithfully equivalent to $\left\langle D^{\prime}, W\right\rangle$, there exists a selected process $\Pi^{\prime \prime}$ of $\left\langle D^{\prime}, W\right\rangle$ made of the defaults of $\left(\Pi \cdot \Pi^{\prime}\right) \cap D^{\prime}$ and generating the extension $E^{\prime}$. Since $\Pi^{\prime}$ is made of defaults of $D \backslash D^{\prime \prime}$ and $D^{\prime} \subseteq D^{\prime \prime}$, we have that $\left(\Pi \cdot \Pi^{\prime}\right) \cap D^{\prime}=\Pi \cap D^{\prime}$. As a result, $\Pi^{\prime \prime}$ is only made of defaults in $\Pi \cap D^{\prime}$. Since $\Pi^{\prime \prime}$ generates $E^{\prime}$ and $\Pi$ generates $E$, we have $E \models E^{\prime}$. We can therefore conclude that $E \equiv E^{\prime}$.

Let us now prove the converse: we assume that $E$ is an extension of $\langle D, W\rangle$ and prove that it is also an extension of $\left\langle D^{\prime \prime}, W\right\rangle$. Let $\Pi$ be the process of $\langle D, W\rangle$ that generates $E$. By definition, the following two properties are true:

1. $W \cup \operatorname{cons}(\Pi) \models \operatorname{prec}(d)$ for every $d \in \Pi$;
2. $W \cup \operatorname{cons}(\Pi) \perp$ just $(d) \cup \operatorname{cons}(d)$ for every $d \notin \Pi$.

By Lemma 20, the theory $\left\langle D^{\prime}, W\right\rangle$ has a selected process $\Pi^{\prime}$ that is composed exactly of the defaults of $\Pi \cap D^{\prime}$ and that generates the same extension $E$. Since $W \cup \operatorname{cons}\left(\Pi^{\prime}\right) \equiv$ $W \cup \operatorname{cons}(\Pi)$, the two properties are equivalent to the following two ones:

1. $W \cup \operatorname{cons}\left(\Pi^{\prime}\right) \models \operatorname{prec}(d)$ for every $d \in \Pi$;
2. $W \cup \operatorname{cons}\left(\Pi^{\prime}\right) \perp \mathrm{just}(d) \cup \operatorname{cons}(d)$ for every $d \notin \Pi$.

The first property implies that every default $d \in \Pi \cap\left(D^{\prime \prime} \backslash D^{\prime}\right)$ is applicable to $\Pi^{\prime}$ : this is because the precondition of $d$ is entailed by $W \cup \operatorname{cons}\left(\Pi^{\prime}\right)$ and the process $\Pi^{\prime} \cdot[d]$ is successful because so is $\Pi$, which contains all default of $\Pi^{\prime} \cdot[d]$. The second property implies that no default of $D^{\prime \prime} \backslash \Pi$ is applicable to $\Pi^{\prime}$. As a result, $\Pi^{\prime}$ and the sequence composed of all defaults of $\Pi \cap\left(D^{\prime \prime} \backslash D^{\prime}\right)$ in any order form a selected process of $D^{\prime \prime}$. The extension generated by this process is equivalent to $E$ because this process is composed of a superset of the defaults of $\Pi^{\prime}$ and a subset of the defaults of $\Pi$, and these two processes both generate $E$.

We therefore have as a corollary that justified default logic has the local redundancy property when redundancy of defaults is considered.

Corollary 7 Justified default logic has the local redundancy property w.r.t. redundancy of defaults.

### 4.6.3 Redundancy of Clauses and of Defaults

For Reiter and rational default logic, an upper bound on complexity can be given by showing a reduction from the complexity of clause or formula redundancy to the corresponding problems for defaults. This is possible thanks to the following lemma.

Lemma $21\langle D, W \cup\{\gamma\}\rangle$ has the same Reiter and rational extensions of $\left\langle D \cup\left\{d_{\gamma}\right\}, W\right\rangle$, where $d_{\gamma}=\frac{: T}{\gamma}$.

Proof. Since $d_{\gamma}$ has no precondition and a tautological justification, it is always applicable. Therefore, every process of $\left\langle D \cup\left\{d_{\gamma}\right\}, W\right\rangle$ contains this default, and therefore generates $\gamma$.

This lemma can be iterated for all clauses of $W$, leading to the following result.
Theorem 23 For Reiter and rational default logic, the problems of checking the redundancy of a default or of a default theory are at least as hard as the corresponding problems for clause redundancy.

Proof. The clause $\gamma$ is redundant in $\langle D, W\rangle$ if and only if $d_{\gamma}$ is redundant in $\left\langle D \cup\left\{d_{\gamma}\right\}, W \backslash\{\gamma\}\right\rangle$. Indeed, $\langle D, W\rangle$ has the same extensions of $\left\langle D \cup\left\{d_{\gamma}\right\}, W \backslash\{\gamma\}\right\rangle$, and removing $\gamma$ from the first theory or removing $d_{\gamma}$ from the second theory lead both to $\langle D, W \backslash\{\gamma\}\rangle$.

The problems of formula redundancy can be reduced to default redundancy by first applying Lemma 21 to all clauses of $W$, and then making all original defaults irredundant using the transformation of Lemma 17.

The complexity of redundancy for defaults can be therefore characterized as follows.
Corollary 8 For Reiter and justified default logic, the problem of redundancy of a default is $\Pi_{2}^{p}$-hard and $\Pi_{3}^{p}$-hard for faithful and consequence-equivalence, respectively; the problem of redundancy of a default theory is $\Sigma_{3}^{p}$-hard and $\Sigma_{4}^{p}$-hard for faithful and consequence-equivalence, respectively.

Equivalence of extensions can be proved to be in $\Pi_{2}^{p}$ even if the defaults or the background theories are not even related.

Theorem 24 Checking whether $\langle D, W\rangle \equiv_{D}^{e}\left\langle D^{\prime}, W^{\prime}\right\rangle$ is in $\Pi_{2}^{p}$ for Reiter and justified default logic.

Proof. The contrary of the statement amounts to checking whether any of the two theories have an extension that the other one does not have. The number of possible extensions, however, is limited by the fact that any extension is generated by the set of consequences of some defaults.

Checking whether $\langle D, W\rangle$ has an extension that $\left\langle D^{\prime}, W^{\prime}\right\rangle$ has not can be done as follows: guess a subset $D^{\prime \prime} \subseteq D$, and let $E=\operatorname{cons}\left(D^{\prime \prime}\right)$; check whether $E$ is an extension of $\langle D, W\rangle$ but is not an extension of $\left\langle D^{\prime}, W^{\prime}\right\rangle$.

Checking whether a formula $E$ is an extension of a default theory can be done with a logarithmic number of satisfiability tests [Ros99, Lib05a]. As a result, the problem can also be expressed as a QBF formula $\exists \forall \mathrm{QBF}$. In order to check whether there exists $D^{\prime \prime}$ such that $E=\operatorname{cons}\left(D^{\prime \prime}\right)$ is in this condition, we only have to add an existential quantifier to the front of this formula. The problem is therefore in $\Pi_{2}^{p}$.

The problem of checking the default redundancy of a theory is obviously in $\Sigma_{3}^{p}$, as it can be solved by guessing a subsets of defaults and then checking equivalence.

Corollary 9 The problem of checking the redundancy of a default or the default redundancy of a theory are $\Pi_{2}^{p}$-complete and $\Sigma_{3}^{p}$-complete, respectively, for Reiter and justified default logic for faithful equivalence.

Consequence equivalence can also be proved to have the same complexity as for the case studied for clauses.

Theorem 25 Checking the consequence-equivalence for Reiter and justified default logic is in $\Pi_{3}^{p}$.

Proof. The converse of the problem can be expressed as: there exists a model $M$ that is a model of an extension of the first theory but not of the second, or vice versa. This corresponds to two quantifications over extensions and a check for whether a formula is an extension. The latter is in $\Delta_{2}^{p}[\log n]$ for the two considered semantics [Ros99, Lib05a]. Therefore, the whole problem is in $\Pi_{3}^{p}$.

As a consequence, the complexity of redundancy for consequence-equivalence is exactly characterized for Reiter and justified default logics.

Corollary 10 The problem of checking the redundancy of a default or the default redundancy of a theory are $\Pi_{3}^{p}$-complete and $\Sigma_{4}^{p}$-complete, respectively, for Reiter default logic and consequence-equivalence.

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