Rethinking Epistemic Logic with Belief Bases

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Abstract

We introduce a new semantics for a logic of explicit and implicit beliefs based on the concept of multi-agent belief base. Differently from existing Kripke-style semantics for epistemic logic in which the notions of possible world and doxastic/epistemic alternative are primitive, in our semantics they are non-primitive but are defined from the concept of belief base. We provide a complete axiomatization and prove decidability for our logic via a finite model argument. We also provide a polynomial embedding of our logic into Fagin & Halpern's logic of general awareness and establish a complexity result for our logic via the embedding.

1 Introduction

Epistemic logic and, more generally, formal epistemology are the areas at the intersection between philosophy [18], artificial intelligence (AI) [10, 26] and economics [23] devoted to the formal representation of epistemic attitudes of agents including belief and knowledge. An important distinction in epistemic logic is between *explicit belief* and *implicit belief*. According to [22], "...a sentence is explicitly believed when it is actively held to be true by an agent and implicitly believed when it follows from what is believed" (p. 198). This distinction is particularly relevant for the design of resource-bounded agents who spend time to make inferences and do not believe all facts that are deducible from their actual beliefs.

The concept of explicit belief is tightly connected with the concept of *belief base* [29, 25, 16, 30]. In particular, an agent's belief base, which is not necessarily closed under deduction, includes all facts that are explicitly believed by the agent. Nonetheless, existing logical formalizations of explicit and implicit beliefs [22, 11] do not clearly account for this connection.

The aim of this paper is to fill this gap by providing a multi-agent logic that precisely articulates the distinction between explicit belief, as a fact in an agent's belief base, and implicit belief, as a fact that is deducible from the agent's explicit beliefs, given the agents' common ground. The concept of *common ground* [31] corresponds to the body of information that the agents commonly believe to be the case and that has to be in the deductive closure of their belief bases. The multi-agent aspect of the logic lies in the fact that it supports reasoning about agents' high-order beliefs, i.e., an agent's explicit (or implicit) belief about the explicit (or implicit) belief of another agent.

Differently from existing Kripke-style semantics for epistemic logic in which the notions of possible world and doxastic/epistemic alternative are primitive, in the semantics of our logic the notion of doxastic alternative is defined from — and more generally grounded on — the concept of belief base.

We believe that an explicit representation of agents' belief bases is crucial in order to facilitate the task of designing intelligent systems such as robotic agents or conversational agents. The problem of extensional semantics for epistemic logic, whose most representative example is the Kripkean

semantics, is their being too abstract and too far from the agent specification. More generally, the main limitation of the Kripkean semantics is that it does not say from where doxastic alternatives come from thereby being ungrounded.¹

The paper is organized as follows. In Section 2, we present the language of our logic of explicit and implicit beliefs. Then, in Section 3, we introduce a semantics for this language based on the notion of multi-agent belief base. We also consider two additional Kripke-style semantics in which the notion of doxastic alternative is primitive. These additional semantics will be useful for proving completeness and decidability of our logic. In Section 4, we show that the three semantics are all equivalent with respect to the formal language under consideration. Then, in Section 5, we provide an axiomatization for our logic of explicit and implicit belief and prove that its satisfiability problem is decidable. In Section 6, we provide a polynomial embedding of our logic into the logic of general awareness by Fagin & Halpern [11], a well-known logic in AI and epistemic game theory. Thanks to this embedding, we will be able to conclude that the satisfiability problem of our logic is PSPACE-complete. After having discussed related work in Section 7, we conclude.

2 A Language for Explicit and Implicit Beliefs

LDA (Logic of Doxastic Attitudes) is a logic for reasoning about explicit beliefs and implicit beliefs of multiple agents. Assume a countably infinite set of atomic propositions $Atm = \{p, q, \ldots\}$ and a finite set of agents $Agt = \{1, \ldots, n\}$.

We define the language of the logic LDA in two steps. We first define the language $\mathcal{L}^0_{LDA}(Atm)$ by the following grammar in Backus-Naur Form (BNF):

$$\alpha ::= p \mid \neg \alpha \mid \alpha_1 \wedge \alpha_2 \mid \mathbf{E}_i \alpha$$

where p ranges over Atm and i ranges over Agt. $\mathcal{L}^0_{\mathsf{LDA}}(Atm)$ is the language for representing explicit beliefs of multiple agents. The formula $\mathsf{E}_i\alpha$ is read "agent i explicitly (or actually) believes that α is true". In this language, we can represent high-order explicit beliefs, i.e., an agent's explicit belief about another agent's explicit beliefs.

The language $\mathcal{L}_{LDA}(Atm)$, extends the language $\mathcal{L}_{LDA}^{0}(Atm)$ by modal operators of implicit belief and is defined by the following grammar:

$$\varphi ::= \alpha \mid \neg \varphi \mid \varphi_1 \wedge \varphi_2 \mid \mathbf{I}_i \varphi$$

where α ranges over $\mathcal{L}^0_{\mathsf{LDA}}(Atm)$. For notational convenience we write $\mathcal{L}^0_{\mathsf{LDA}}$ instead of $\mathcal{L}^0_{\mathsf{LDA}}(Atm)$ and $\mathcal{L}_{\mathsf{LDA}}$ instead of $\mathcal{L}^0_{\mathsf{LDA}}(Atm)$, when the context is unambiguous.

The other Boolean constructions \top , \bot , \lor , \rightarrow and \leftrightarrow are defined from α , \neg and \wedge in the standard way.

For every formula $\varphi \in \mathcal{L}_{LDA}$, we write $Atm(\varphi)$ to denote the set of atomic propositions of type p occurring in φ . Moreover, for every set of formulas $X \subseteq \mathcal{L}$, we define $Atm(X) = \bigcup_{\varphi \in X} Atm(\varphi)$.

The formula $I_i\varphi$ has to be read "agent *i* implicitly (or potentially) believes that φ is true". We define the dual operator \widehat{I}_i as follows:

$$\widehat{\mathsf{I}}_i \varphi \stackrel{\mathsf{def}}{=} \neg \mathsf{I}_i \neg \varphi.$$

 $I_i\varphi$ has to be read " φ is compatible with agent i's implicit beliefs".

¹The need for a grounded semantics for doxastic/epistemic logics has been pointed out by other authors including [24].

3 Formal Semantics

In this section, we present three formal semantics for the language of explicit and implicit beliefs defined above. In the first semantics, the notion of doxastic alternative is not primitive but it is defined from the primitive concept of belief base. The second semantics is a Kripke-style semantics, based on the concept of notional doxastic model, in which an agent's set of doxastic alternatives coincides with the set of possible worlds in which the agent's explicit beliefs are true. The third semantics is a weaker semantics, based on the concept of *quasi*-notional doxastic model. It only requires that an agent's set of doxastic alternatives has to be included in the set of possible worlds in which the agent's explicit beliefs are true. At a later stage in the paper, we will show that three semantics are equivalent with respect to the formal language under consideration.

3.1 Multi-agent belief base semantics

We first consider the semantics based on the concept of multi-agent belief base that is defined as follows.

Definition 1 (Multi-agent belief base) A multi-agent belief base is a tuple $B = (B_1, \dots, B_n, V)$ where:

- for every $i \in Agt$, $B_i \subseteq \mathcal{L}^0_{LDA}$ is agent i's belief base,
- $V \subseteq Atm$ is the actual state.

A similar concept is used in belief merging [20] in which each agent is identified with her belief base. Our concept of multi-agent belief base also includes the concept of actual state, as the set of true atomics facts.

The sublanguage $\mathcal{L}^0_{\mathsf{LDA}}(Atm)$ is interpreted with respect to multi-agent belief bases, as follows.

Definition 2 (Satisfaction relation) Let B =

 (B_1,\ldots,B_n,V) be a multi-agent belief base. Then:

$$\begin{aligned} B &\models p &\iff p \in V \\ B &\models \neg \alpha &\iff B \not\models \alpha \\ B &\models \alpha_1 \land \alpha_2 &\iff B \models \alpha_1 \text{ and } B \models \alpha_2 \\ B &\models \mathbf{E}_i \alpha &\iff \alpha \in B_i \end{aligned}$$

The following definition introduces the concept of doxastic alternative.

Definition 3 (Doxastic alternatives) Let $B = (B_1, ..., B_n)$

 B_n, V) and $B' = (B'_1, \dots, B'_n, V')$ be two multi-agent belief bases. Then, $B\mathcal{R}_i B'$ if and only if, for every $\alpha \in B_i$, $B' \models \alpha$.

 $B\mathcal{R}_iB'$ means that B' is a doxastic alternative for agent i at B (i.e., at B agent i considers B' possible). The idea of the previous definition is that B' is a doxastic alternative for agent i at B if and only if, B' satisfies all facts that agent i explicitly believes at B.

A multi-agent belief model (MAB) is defined to be a multi-agent belief base supplemented with a set of multi-agent belief bases, called *context*. The latter includes all multi-agent belief bases that are compatible with the agents' common ground [31], i.e., the body of information that the agents commonly believe to be the case.

Definition 4 (Multi-agent belief model) A multi-agent belief model (MAB) is a pair (B, Cxt), where B is a multi-agent belief base and Cxt is a set of multi-agent belief bases. The class of MABs is denoted by MAB.

Note that in the previous definition we do not require $B \in Cxt$. Let us illustrate the concept of MAB with the aid of an example.

Example 1 Let $Agt = \{1, 2\}$ and $\{p, q\} \subseteq Atm$. Moreover, let (B_1, B_2, V) be such that:

$$B_1 = \{p, E_2 p\},\$$

 $B_2 = \{p\},\$
 $V = \{p, q\}.$

Suppose that the agents have in their common ground the fact $p \to q$. In other words, they commonly believe that p implies q. This means that:

$$Cxt = \{B' : B' \models p \rightarrow q\}.$$

The following definition generalizes Definition 2 to the full language $\mathcal{L}_{LDA}(Atm)$. Its formulas are interpreted with respect to MABs. (Boolean cases are omitted, as they are defined in the usual way.)

Definition 5 (Satisfaction relation (cont.)) Let $(B, Cxt) \in MAB$. Then:

$$(B, Cxt) \models \alpha \iff B \models \alpha$$

 $(B, Cxt) \models I_i \varphi \iff \forall B' \in Cxt : if BR_iB' then$
 $(B', Cxt) \models \varphi$

Let us go back to the example.

Example 2 It is to check that the following holds:

$$(B, Cxt) \models I_1(p \land q) \land I_2(p \land q) \land I_1I_2(p \land q).$$

Indeed, we have:

$$\mathcal{R}_1(B) \cap Cxt = \{B' : B' \models p \land \mathbb{E}_2 p \land (p \to q)\},$$

$$\mathcal{R}_2(B) \cap Cxt = \{B' : B' \models p \land (p \to q)\},$$

and, consequently,

$$(\mathcal{R}_1 \circ \mathcal{R}_2(B)) \cap Cxt = \{B' : B' \models p \land (p \to q)\},\$$

where \circ is the composition operation between binary relations and $\mathcal{R}_i(B) = \{B' : B\mathcal{R}_i B'\}$.

Here, we consider consistent MABs that guarantee consistency of the agents' belief bases. Specifically:

Definition 6 (Consistent MAB) (B, Cxt) is a consistent MAB (CMAB) if and only if, for every $B' \in Cxt \cup \{B\}$, there exists $B'' \in Cxt$ such that $B'\mathcal{R}_iB''$. The class of CMABs is denoted by CMAB.

Let $\varphi \in \mathcal{L}$, we say that φ is valid for the class of CMABs if and only if, for every $(B, Cxt) \in$ CMAB, we have $(B, Cxt) \models \varphi$. We say that φ is satisfiable for the class of CMABs if and only if $\neg \varphi$ is not valid for the the class of CMABs.

3.2 Notional doxastic model semantics

Let us now consider the semantics for LDA based on the concept of notional doxastic model (NDM). It is defined in the next Definition 7, together with the satisfaction relation for the formulas of the language $\mathcal{L}_{LDA}(Atm)$.

Definition 7 (Doxastic model) A notional doxastic model (NDM) is a tuple $M = (W, \mathcal{D}, \mathcal{N}, \mathcal{V})$ where:

- W is a set of worlds,
- $\mathcal{D}: Agt \times W \longrightarrow 2^{\mathcal{L}_{LDA}^0}$ is a doxastic function,
- $\mathcal{N}: Agt \times W \longrightarrow 2^W$ is a notional function, and
- $\mathcal{V}: Atm \longrightarrow 2^W$ is a valuation function,

and that satisfies the following conditions for all $i \in Agt$ and $w \in W$:

(C1)
$$\mathcal{N}(i, w) = \bigcap_{\alpha \in \mathcal{D}(i, w)} ||\alpha||_M$$
, and

(C2) there exists $v \in W$ such that $v \in \mathcal{N}(i, w)$,

with:

$$(M, w) \models p \iff w \in \mathcal{V}(p)$$

$$(M, w) \models \neg \varphi \iff (M, w) \not\models \varphi$$

$$(M, w) \models \varphi \land \psi \iff (M, w) \models \varphi \text{ and } (M, w) \models \psi$$

$$(M, w) \models \mathbf{E}_{i}\alpha \iff \alpha \in \mathcal{D}(i, w)$$

$$(M, w) \models \mathbf{I}_{i}\varphi \iff \forall v \in \mathcal{N}(i, w) : (M, v) \models \varphi$$

and

$$||\alpha||_M = \{v \in W : (M, v) \models \alpha\}.$$

The class of notional doxastic models is denoted by NDM.

We say that a NDM $M=(W,\mathcal{A},\mathcal{D},\mathcal{N},\mathcal{V})$ is *finite* if and only if $W,\mathcal{D}(i,w)$ and $\mathcal{V}^{-1}(w)$ are finite sets for every $i \in Agt$ and for every $w \in W$, where \mathcal{V}^{-1} is the inverse function of \mathcal{V} .

For every agent i and world w, $\mathcal{D}(i, w)$ denotes agent i's set of explicit beliefs at w.

The set $\mathcal{N}(i, w)$, used in the interpretation of the implicit belief operator I_i , is called agent i's set of notional worlds at world w. The term 'notional' is taken from [7, 8] (see, also, [21]): an agent's notional world is a world in which all the agent's explicit beliefs are true. This idea is clearly expressed by the Condition C1. According to the Condition C2, an agent's set of notional worlds must be non-empty. This guarantees consistency of the agent's implicit beliefs.

Let $\varphi \in \mathcal{L}$, we say that φ is valid for the class of NDMs if and only if, for every $M = (W, \mathcal{A}, \mathcal{D}, \mathcal{N}, \mathcal{V}) \in \mathbf{NDM}$ and for every $w \in W$, we have $(M, w) \models \varphi$. We say that φ is satisfiable for the class of NDMs if and only if $\neg \varphi$ is not valid for the class of NDMs.

3.3 Quasi-model semantics

In this section we provide an alternative semantics for the logic LDA based on a more general class of models, called quasi-notional doxastic models (quasi-NDMs). This semantics will be fundamental for proving completeness of LDA.

Definition 8 (Quasi-notional doxastic model) A quasi-notional doxastic model (quasi-NDM) is a tuple $M = (W, \mathcal{D}, \mathcal{N}, \mathcal{V})$ where $W, \mathcal{D}, \mathcal{N}$ and \mathcal{V} are as in Definition 7 except that Condition C1 is replaced by the following weaker condition, for all $i \in Agt$ and $w \in W$:

$$(CI^*) \mathcal{N}(i, w) \subseteq \bigcap_{\alpha \in \mathcal{D}(i, w)} ||\alpha||_M.$$

The class of quasi-notional doxastic models is denoted by QNDM. Truth conditions of formulas in \mathcal{L} relative to this class are the same as truth conditions of formulas in \mathcal{L} relative to the class NDM. Validity and satisfiability of a LDA formula φ for the class of quasi-NDMs are defined in the usual way.

As for NDMs, we say that a quasi-NDM $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ is *finite* if and only if $W,\mathcal{D}(i,w)$ and $\mathcal{V}^{-1}(w)$ are finite sets for every $i\in Agt$ and for every $w\in W$.

4 Equivalences between semantics

The present section is devoted to present equivalences between the different semantics for the language $\mathcal{L}_{LDA}(Atm)$. The results of the section are summarized in Figure 1.

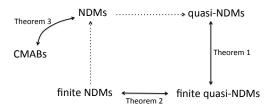


Figure 1: Relations between semantics. An arrow means that satisfiability relative to the first class of structures implies satisfiability relative to the second class of structures. Dotted arrows denote relations that follow straightforwardly given the inclusion between classes of structures.

The figure highlights that the five semantics for the language $\mathcal{L}_{LDA}(Atm)$ defined in the previous section are all equivalent, as from every node in the graph we can reach all other nodes.

Equivalence between quasi-NDMs and finite quasi-NDMs We use a filtration argument to show that if a formula φ of the language \mathcal{L} is true in a (possibly infinite) quasi-NDM then it is true in a finite quasi-NDM.

Let $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ be a (possibly infinite) quasi-NDM and let $\Sigma\subseteq\mathcal{L}_{\mathsf{LDA}}$ be an arbitrary finite set of formulas which is closed under subformulas. (Cf. Definition 2.35 in [6] for a definition of subformulas closed set of formulas.) Let the equivalence relation \equiv_{Σ} on W be defined as follows. For all $w,v\in W$:

$$w \equiv_{\Sigma} v \text{ iff } \forall \varphi \in \Sigma : (M, w) \models \varphi \text{ iff } (M, v) \models \varphi.$$

Let $|w|_{\Sigma}$ be the equivalence class of the world w with respect to the equivalence relation \equiv_{Σ} .

We define W_{Σ} to be the filtrated set of worlds with respect to Σ :

$$W_{\Sigma} = \{ |w|_{\Sigma} : w \in W \}.$$

Clearly, W_{Σ} is a finite set.

Let us define the filtrated valuation function \mathcal{V}_{Σ} . For every $p \in Atm$, we define:

$$\mathcal{V}_{\Sigma}(p) = \{|w|_{\Sigma} : (M, w) \models p\}$$
 if $p \in Atm(\Sigma)$
$$\mathcal{V}_{\Sigma}(p) = \emptyset$$
 otherwise

The next step in the construction consists in defining the filtrated doxastic function. For every $i \in Agt$ and for every $|w|_{\Sigma} \in W_{\Sigma}$, we define:

$$\mathcal{D}_{\Sigma}(i,|w|_{\Sigma}) = \Big(\bigcap_{w \in |w|_{\Sigma}} \mathcal{D}(i,w)\Big) \cap \Sigma.$$

Finally, for every $i \in Agt$ and for every $|w|_{\Sigma} \in W_{\Sigma}$, we define agent i's set of notional worlds at $|w|_{\Sigma}$ as follows:

$$\mathcal{N}_{\Sigma}(i, |w|_{\Sigma}) = \{ |v|_{\Sigma} : v \in \mathcal{N}(i, w) \}.$$

We call the model $M_{\Sigma} = (W_{\Sigma}, \mathcal{A}_{\Sigma}, \mathcal{D}_{\Sigma}, \mathcal{N}_{\Sigma}, \mathcal{V}_{\Sigma})$ the filtration of M under Σ . We can state the following filtration lemma.

Lemma 1 Let
$$\varphi \in \Sigma$$
 and let $w \in W$. Then, $(M, w) \models \varphi$ if and only if $(M_{\Sigma}, |w|_{\Sigma}) \models \varphi$.

PROOF. The proof is by induction on the structure of φ . For the ease of exposition, we prove our result for the language \mathcal{L} in which the "diamond" operator $\widehat{\mathbf{I}}_i$ is taken as primitive and the "box" operator \mathbf{I}_i is defined from it. Since the two operators are inter-definable, this does not affect the validity of our result.

The case $\varphi=p$ is immediate from the definition of \mathcal{V}_{Σ} . The boolean cases $\varphi=\neg\psi$ and $\varphi=\psi_1\wedge\psi_2$ follow straightforwardly from the fact that Σ is closed under subformulas. This allows us to apply the induction hypothesis.

Let us prove the case $\varphi = \mathbf{E}_i \alpha$.

- (\Rightarrow) Suppose $(M, w) \models \mathbf{E}_i \alpha$ with $\mathbf{E}_i \alpha \in \Sigma$. Thus, $\alpha \in \mathcal{D}(i, w)$. Hence, by definition of $\mathcal{D}_{\Sigma}(i, |w|_{\Sigma})$, the fact that Σ is closed under subformulas and the fact that if $\mathbf{E}_i \alpha \in \Sigma$ and $\alpha \in \mathcal{D}(i, w)$ then $\alpha \in \bigcap_{w \in |w|_{\Sigma}} \mathcal{D}(i, w)$, we have $\alpha \in \mathcal{D}_{\Sigma}(i, |w|_{\Sigma})$. It follows that $(M_{\Sigma}, |w|_{\Sigma}) \models \mathbf{E}_i \alpha$.
- (\Leftarrow) For the other direction, suppose $(M_{\Sigma}, |w|_{\Sigma}) \models \mathbb{E}_i \alpha$ with $\mathbb{E}_i \alpha \in \Sigma$. Thus, $\alpha \in \mathcal{D}_{\Sigma}(i, |w|_{\Sigma})$. Hence, by definition of $\mathcal{D}_{\Sigma}(i, |w|_{\Sigma})$, $\alpha \in \mathcal{D}(i, w)$.

Let us conclude the proof for the case $\varphi = \widehat{1}_i \psi$. It is easy to check that \mathcal{N}_{Σ} gives rise to the smallest filtration and that the following two properties hold for all $w, v \in W$ and for all $i \in Agt$:

- (i) if $v \in \mathcal{N}(i, w)$ then $|v|_{\Sigma} \in \mathcal{N}_{\Sigma}(i, |w|_{\Sigma})$, and
- (ii) if $|v|_{\Sigma} \in \mathcal{N}_{\Sigma}(i, |w|_{\Sigma})$ then for all $\widehat{\mathbf{I}}_i \varphi \in \Sigma$, if $M, v \models \varphi$ then $M, w \models \widehat{\mathbf{I}}_i \varphi$.
- (⇒) Suppose $(M, w) \models \widehat{\mathbf{1}}_i \psi$ with $\widehat{\mathbf{1}}_i \psi \in \Sigma$. Thus, there exists $v \in \mathcal{N}(i, w)$ such that $(M, v) \models \psi$. By the previous item (i), $|v|_{\Sigma} \in \mathcal{N}_{\Sigma}(i, |w|_{\Sigma})$. Since Σ is closed under subformulas, we have $\psi \in \Sigma$. Thus, by the induction hypothesis, $(M_{\Sigma}, |v|_{\Sigma}) \models \psi$. It follows that $(M_{\Sigma}, |w|_{\Sigma}) \models \widehat{\mathbf{1}}_i \psi$.
- (\Leftarrow) For the other direction, suppose $(M_{\Sigma}, |w|_{\Sigma}) \models \widehat{\mathsf{I}}_{i}\psi$ with $\widehat{\mathsf{I}}_{i}\psi \in \Sigma$. Thus, there exists $|v|_{\Sigma} \in \mathcal{N}_{\Sigma}(i, |w|_{\Sigma})$, such that $(M_{\Sigma}, |v|_{\Sigma}) \models \psi$. Since Σ is closed under subformulas, by the

induction hypothesis, we have $(M,v) \models \psi$. By the item (ii) above, it follows that $(M,w) \models \widehat{1}_i \psi$.

The next step consists in proving that M_{Σ} is the right model construction.

Proposition 1 The tuple $M_{\Sigma} = (W_{\Sigma}, \mathcal{A}_{\Sigma}, \mathcal{D}_{\Sigma}, \mathcal{N}_{\Sigma}, \mathcal{V}_{\Sigma})$ is a finite quasi-NDM.

PROOF. Clearly, M_{Σ} is finite. Moreover, it is easy to verify that it satisfies the Condition C2 in Definition 7. We are going to prove that it satisfies the Condition C1* in Definition 8.

By Lemma 1, if $\alpha \in \left(\bigcap_{w \in |w|_{\Sigma}} \mathcal{D}(i, w)\right) \cap \Sigma$ then $||\alpha||_{M_{\Sigma}} = \{|v|_{\Sigma} : v \in ||\alpha||_{M}\}$. Moreover, as M is a quasi-NDM, we have

$$\mathcal{N}(i,w) \subseteq \bigcap_{\alpha \in \mathcal{D}(i,w)} ||\alpha||_{M} \subseteq \bigcap_{\alpha \in \left(\bigcap_{w \in |w|_{\Sigma}} \mathcal{D}(i,w)\right) \cap \Sigma} ||\alpha||_{M}.$$

Hence, by definitions of $\mathcal{N}_{\Sigma}(i, |w|_{\Sigma})$ and \mathcal{D}_{Σ} ,

$$\mathcal{N}_{\Sigma}(i,|w|_{\Sigma}) \subseteq \bigcap_{\alpha \in \mathcal{D}_{\Sigma}(i,|w|_{\Sigma})} ||\alpha||_{M_{\Sigma}}.$$

The following is our first result about equivalence between the semantics in terms of quasi-NDMs and the semantics in terms of finite quasi-NDMS.

Theorem 1 Let $\varphi \in \mathcal{L}$. Then, if φ is satisfiable for the class of quasi-NDMs, if and only if it is satisfiable for the class of finite quasi-NDMs.

PROOF. The right-to-left direction is obvious. As for the left-to-right direction, let M be a possibly infinite quasi-NDM and let w be a world in M such that $(M,w) \models \varphi$. Moreover, let $sub(\varphi)$ be the set of subformulas of φ . Then, by Lemma 1 and Proposition 1, $(M_{sub(\varphi)}, |w|_{sub(\varphi)}) \models \varphi$ and $M_{sub(\varphi)}$ is a finite quasi-NDM.

Equivalence between finite NDMs and finite quasi-NDMs As the following theorem highlights, the LDA semantics in terms of finite NDMs and the LDA semantics in terms of finite quasi-NDMs are equivalent.

Theorem 2 Let $\varphi \in \mathcal{L}$. Then, φ is satisfiable for the class of finite NDMs if and only if φ is satisfiable for the class of finite quasi-NDMs.

PROOF. The left-to-right direction is obvious. We are going to prove the right-to-left direction. Let $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ be a finite quasi-NDM that satisfies φ , i.e., there exists $w\in W$ such that $(M,w)\models\varphi$. Let

$$\mathcal{T}(M) = \bigcup_{w \in W} \bigcup_{i \in Aat} Atm(\mathcal{D}(i, w))$$

be the *terminology* of model M including all atomic propositions that are in the explicit beliefs of some agent at some world in M. Since M is finite, $\mathcal{T}(M)$ is finite too.

Let us introduce an injective function:

$$f: Agt \times W \longrightarrow Atm \setminus (\mathcal{T}(M) \cup Atm(\varphi))$$

which assigns an identifier to every agent in Agt and world in W. The fact that Atm is infinite while W, $\mathcal{T}(M)$ and $Atm(\varphi)$ are finite guarantees that such an injection exists.

The next step consists in defining the new model $M' = (W', \mathcal{D}', \mathcal{N}', \mathcal{V}')$ with W' = W, $\mathcal{N}' = \mathcal{N}$ and where \mathcal{D}' and \mathcal{V}' are defined as follows.

For every $i \in Agt$ and for every $w \in W$:

$$\mathcal{D}'(i, w) = \mathcal{D}(i, w) \cup \{f(i, w)\}.$$

Moreover, for every $p \in Atm$:

$$\mathcal{V}'(p) = \mathcal{V}(p)$$
 if $p \in \mathcal{T}(M) \cup Atm(\varphi)$, $\mathcal{V}'(p) = \mathcal{N}(i, w)$ if $p = f(i, w)$, $\mathcal{V}'(p) = \emptyset$ otherwise.

It is easy to verify that $\mathcal{N}'(i, w) = \bigcap_{\alpha \in \mathcal{D}'(i, w)} ||\alpha||_{M'}$ for all $i \in Agt$ and for all $w \in W'$ and, more generally, that M' is a finite NDM.

By induction on the structure of φ , we prove that, for all $w \in W$, " $(M, w) \models \varphi$ iff $(M', w) \models \varphi$ ".

The case $\varphi = p$ is immediate from the definition of \mathcal{V}' . By the induction hypothesis, we can prove the boolean cases $\varphi = \neg \psi$ and $\varphi = \psi_1 \wedge \psi_2$ in a straightforward manner.

Let us prove the case $\varphi = \mathbf{E}_i \alpha$.

- (\Rightarrow) Suppose $(M, w) \models E_i \alpha$. Then, we have $\alpha \in \mathcal{D}(i, w)$. Hence, by the definition of \mathcal{D}' , $\alpha \in \mathcal{D}'(i, w)$. Thus, $(M', w) \models E_i \alpha$.
- (\Leftarrow) Suppose $(M', w) \models \mathsf{E}_i \alpha$. Then, we have $\alpha \in \mathcal{D}'(i, w)$. The definition of \mathcal{D}' ensures that $\alpha \neq f(i, w)$, since $f(i, w) \notin Atm(\mathsf{E}_i \alpha)$. Thus, $\alpha \in \mathcal{D}(i, w)$ and, consequently, $(M, w) \models \mathsf{E}_i \alpha$.

Let us prove the case $\varphi = \mathbf{I}_i \psi$. $(M, w) \models \mathbf{I}_i \psi$ means that $(M, v) \models \psi$ for all $v \in \mathcal{N}(i, w)$. By induction hypothesis and the fact that $\mathcal{N}(i, w) = \mathcal{N}'(i, w)$, the latter is equivalent to $(M', v) \models \psi$ for all $v \in \mathcal{N}'(i, w)$. The latter means that $(M', w) \models \mathbf{I}_i \psi$.

Since M satisfies φ and " $(M, w) \models \varphi$ iff $(M', w) \models \varphi$ " for all $w \in W$, M' satisfies φ as well.

Equivalence between CMABs and NDMs Our third equivalence result is between CMABs and NDMs.

Theorem 3 Let $\varphi \in \mathcal{L}$. Then, φ is satisfiable for the class of CMABs if and only if φ is satisfiable for the class of NDMs.

PROOF. We first prove the left-to-right direction. Let (B, Cxt) be a CMAB with $B = (B_1, \ldots, B_n, V)$ and such that $(B, Cxt) \models \varphi$. We define the structure $M = (W, \mathcal{D}, \mathcal{N}, \mathcal{V})$ as follows:

- $W = \{w_{B'} : B' \in Cxt \cup \{B\}\},\$
- for every $i \in Agt$ and for every $w_{B'} \in W$, if $B' = (B'_1, \dots, B'_n, V')$ then $\mathcal{D}(i, w_{B'}) = B'_i$,
- for every $i \in Agt$ and for every $w_{B'} \in W$, $\mathcal{N}(i, w_{B'}) = \bigcap_{\alpha \in \mathcal{D}(i, w_{B'})} \{w_{B''} \in W : B'' \models \alpha\}$,

• for every $p \in Atm$, $\mathcal{V}(p) = \{w_{B'} \in W : B' \models p\}$.

One can show that M so defined is a NDM. Moreover, by induction on the structure of φ , one can prove that, for all $w_{B'} \in W$, $M, w_{B'} \models \varphi$ iff $(B', Cxt) \models \varphi$. Thus, $(M, w_B) \models \varphi$.

We now prove the right-to-left direction. Let $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ be a NDM and let w be a world in W such that $(M,w)\models\varphi$. Let us say that a NDM $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ is non-redundant iff there are no $w,v\in W$ such that $\mathcal{V}^{-1}(w)=\mathcal{V}^{-1}(v)$, and, for all $i\in Agt$, $\mathcal{D}(i,w)=\mathcal{D}(i,v)$. It is straightforward to show that if φ is satisfiable for the class of NDMs then φ is satisfiable for the class of non-redundant NDMs. Thus, from the initial model M, we can find a non-redundant NDM $M'=(W',\mathcal{D}',\mathcal{N}',\mathcal{V}')$ and $v\in W'$ such that $(M',v)\models\varphi$. For every $u\in W'$ we define $B^u=(B^u_1,\ldots,B^u_n,V^u)$ such that $B^u_i=\mathcal{D}(i,u)$ for every $i\in Agt$ and $V^u=\mathcal{V}^{-1}(u)$. Moreover, we define the context $Cxt=\{B^u:u\in W'\}$. One can show that, for every $B^u\in Cxt$, (B^u,Cxt) is a CMAB. The fact that M' is non-redundant is essential to guarantee that there is a one-to-one correspondence between W' and Cxt. By induction on the structure of φ , one can prove that, for all $B^u\in Cxt$, $(B^u,Cxt)\models\varphi$ iff $M',u\models\varphi$. Thus, $B^v\models\varphi$.

5 Axiomatics and decidability

This section is devoted to provide an axiomatization and a decidability result for LDA. To this aim, we first provide a formal definition of this logic.

Definition 9 We define LDA to be the extension of classical propositional logic given by the following axioms and rule of inference:

$$(\mathbf{I}_i \varphi \wedge \mathbf{I}_i (\varphi \to \psi)) \to \mathbf{I}_i \psi \tag{\mathbf{K}_{\mathbf{I}_i}}$$

$$\neg(\mathbf{I}_i\varphi\wedge\mathbf{I}_i\neg\varphi)\tag{\mathbf{D}_{\mathbf{I}_i}}$$

$$E_i \alpha \to I_i \alpha$$
 (Int_{E_i,I_i)}

$$rac{arphi}{\mathrm{I}_{i}arphi}$$
 (Nec $_{\mathrm{I}_{i}}$)

We denote that φ is derivable in LDA by $\vdash_{\mathsf{LDA}} \varphi$. We say that φ is LDA-consistent if $\not\vdash_{\mathsf{LDA}} \neg \varphi$.

The logic LDA includes the principles of system KD for the implicit belief operator I_i as well as an axiom Int_{E_i,I_i} relating explicit belief with implicit belief. Note that there is no consensus in the literature about introspection for implicit belief. For instance, in his seminal work on the logics of knowledge and belief [18], Hintikka only assumed positive introspection for belief (Axiom 4) and rejected negative introspection (Axiom 5). Other logicians such as [19] have argued against the use of both positive and negative introspection axioms for belief. Nonetheless, all approaches unanimously assume that a reasonable notion of implicit belief should satisfy Axioms K and D. In this sense, system KD can be conceived as the minimal logic of implicit belief. On this point, see [5].

To prove our main completeness result, we first prove a theorem about soundness and completeness of LDA for the class of quasi-NDMs.

Soundness and completeness for quasi-NDMs To prove completeness of LDA for the class of quasi-NDMs, we use a canonical model argument.

We consider maximally LDA-consistent sets of formulas in \mathcal{L} (MCSs). The following proposition specifies some usual properties of MCSs.

Proposition 2 *Let* Γ *be a MCS and let* φ , $\psi \in \mathcal{L}$. *Then:*

- if $\varphi, \varphi \to \psi \in \Gamma$ then $\psi \in \Gamma$;
- $\varphi \in \Gamma$ or $\neg \varphi \in \Gamma$;
- $\varphi \lor \psi \in \Gamma \text{ iff } \varphi \in \Gamma \text{ or } \psi \in \Gamma.$

The following is the Lindenbaum's lemma for our logic. Its proof is standard (cf. Lemma 4.17 in [6]) and we omit it.

Lemma 2 Let Δ be a LDA-consistent set of formulas. Then, there exists a MCS Γ such that $\Delta \subseteq \Gamma$.

Let the canonical quasi-NDM model be the tuple $M=(W^c,\mathcal{D}^c,\mathcal{N}^c,\mathcal{V}^c)$ such that:

- W^c is set of all MCSs;
- for all $w \in W^c$, for all $i \in Agt$ and for all $\alpha \in \mathcal{L}^0_{LDA}$, $\alpha \in \mathcal{D}^c(i, w)$ iff $E_i \alpha \in w$;
- for all $w, v \in W^c$ and for all $i \in Agt, v \in \mathcal{N}^c(i, w)$ iff, for all $\varphi \in \mathcal{L}$, if $I_i \varphi \in w$ then $\varphi \in v$;
- for all $w \in W^c$ and for all $p \in Atm, w \in \mathcal{V}^c(p)$ iff $p \in w$.

The next step in the proof consists in stating the following existence lemma. The proof is again standard (cf. Lemma 4.20 in [6]) and we omit it.

Lemma 3 Let $\varphi \in \mathcal{L}$ and let $w \in W^c$. Then, if $\widehat{\mathbf{1}}_i \varphi \in w$ then there exists $v \in \mathcal{N}^c(i, w)$ such that $\varphi \in v$.

Then, we prove the following truth lemma.

Lemma 4 Let $\varphi \in \mathcal{L}$ and let $w \in W^c$. Then, $M^c, w \models \varphi$ iff $\varphi \in w$.

PROOF. The proof is by induction on the structure of the formula. The cases with φ atomic, Boolean, and of the form $I_i\psi$ are provable in the standard way by means of Proposition 2 and Lemma 3 (cf. Lemma 4.21 in [6]). The proof for the case $\varphi = E_i\alpha$ goes as follows: $E_i\alpha \in w$ iff $\alpha \in \mathcal{D}^c(i,w)$ iff $M^c, w \models E_i\alpha$.

The last step consists in proving that the canonical model belongs to the class QNDM.

Proposition 3 M^c is a quasi-NDM.

PROOF. Thanks to Axiom (\mathbf{D}_{1_i}) , it is easy to prove that M^c satisfies Condition C2 in Definition 7. Let us prove that it satisfies Condition C1* in Definition 8. To this aim, we just need to prove

that if $\alpha \in \mathcal{D}^c(i, w)$ then $\mathcal{N}^c(i, w) \subseteq ||\alpha||_{M^c}$. Suppose $\alpha \in \mathcal{D}^c(i, w)$. Thus, $\mathbf{E}_i \alpha \in w$. Hence, by Axiom ($\mathbf{Int}_{\mathbf{E}_i, \mathbf{I}_i}$) and Proposition 2, $\mathbf{I}_i \alpha \in w$. By the definition of M^c , it follows that, for all $v \in \mathcal{N}^c(i, w)$, $\alpha \in v$. Thus, by Lemma 4, for all $v \in \mathcal{N}^c(i, w)$, (M^c, v) $\models \alpha$. The latter means that $\mathcal{N}^c(i, w) \subseteq ||\alpha||_{M^c}$.

The following is our first intermediate result.

Theorem 4 The logic LDA is sound and complete for the class of quasi-NDMs.

PROOF. As for soundness, it is routine to check that the axioms of LDA are all valid for the class of quasi-NDMs and that the rule of inference (Nec_{I_i}) preserves validity.

As for completeness, suppose that φ is a LDA-consistent formula in \mathcal{L} . By Lemma 2, there exists $w \in W^c$ such that $\varphi \in w$. Hence, by Lemma 4, there exists $w \in W^c$ such that M^c , $w \models \varphi$. Since, by Proposition 3, M^c is a quasi-NDM, we can conclude that φ is satisfiable for the class of quasi-NDMs.

Soundness and completeness for NDMs and CMABs We can state the two main results of this section. The first is about soundness and completeness for the class of NDMs.

Theorem 5 *The logic LDA is sound and complete for the class of NDMs.*

PROOF. It is routine exercise to verify that LDA is sound for the class of NDMs. Now, suppose that formula φ is LDA-consistent. Then, by Theorems 4 and 1, it is satisfiable for the class of finite quasi-NDMs. Hence, by Theorem 2, it is satisfiable for the class of finite NDMs. Thus, more generally, φ is satisfiable for the class of NDMs.

The second is about soundness and completeness for the class of CMABs.

Theorem 6 The logic LDA is sound and complete for the class of CMABs.

SKETCH OF PROOF. The theorem is provable by means of Theorem 5 and Theorem 3.

Decidability The second main result of this section is decidability of LDA.

Theorem 7 *The satisfiability problem of LDA is decidable.*

PROOF. Suppose φ is satisfiable for the class of NDMs. Thus, by Theorem 5, it is LDA-consistent. Hence, by Theorem 4, it is satisfiable for the class of quasi-NDMs. From the proof of Theorem 1, we can observe that if φ is satisfiable for the class of quasi-NDMs then there exists a quasi-NDM satisfying φ such that (i) its set of worlds contains at most 2^n elements, (ii) the atomic propositions outside $Atm(sub(\varphi))$ are false everywhere in the model, and (iii) the belief base of an agent at a world contains only formulas from $sub(\varphi)$, where n is the size of $sub(\varphi)$. The construction in the proof of Theorem 2 ensures that from this finite quasi-NDM, we can built a finite NDM satisfying φ for which (i) holds and such that (iv) the atomic propositions outside $Atm(sub(\varphi)) \cup X$ are false everywhere in the model, and (v) the belief base of an agent at a world contains only formulas from $sub(\varphi) \cup X$, where X is an arbitrary set of atoms from $Atm \setminus (Atm(\varphi))$ of size at most $2^n \times |Agt|$. Thus, in order to verify whether φ is satisfiable, we fix a X and check satisfiability of φ for all NDMs satisfying (i), (iv) and (v). There are finitely many NDMs of this kind.

6 Relationship with logic of general awareness

This section is devoted to explore the connection between the logic LDA and the logic of general awareness by Fagin & Halpern (F&H) [11]. In particular, we will provide a polynomial embedding of the former into latter and, thanks to this embedding, we will be able to state a complexity result for the satisfiability problem of LDA.

The language of F&H's logic of general awareness LGA, denoted by \mathcal{L}_{LGA} , is defined by the following grammar:

$$\varphi \quad ::= \quad p \mid \neg \varphi \mid \varphi_1 \land \varphi_2 \mid \mathsf{B}_i \varphi \mid \mathsf{A}_i \varphi \mid \mathsf{X}_i \varphi$$

where p ranges over Atm and i ranges over Agt.

The formula $A_i\varphi$ has to be read "agent i is aware of φ ". The operators B_i and X_i have the same interpretations as the LDA operators I_i and E_i . Specifically, $B_i\varphi$ has to be read "agent i has an implicit belief that φ is true", while $X_i\varphi$ has to be read "agent i has an explicit belief that φ is true".

The previous language is interpreted with respect to so-called awareness structures, that is, tuples of the form $M=(S,\mathcal{R}_1,\ldots,\mathcal{R}_n,\mathcal{A}_1,\ldots,\mathcal{A}_n,\pi)$ where every $\mathcal{R}_i\subseteq S\times S$ is a doxastic accessibility relation, every $\mathcal{A}_i:S\longrightarrow 2^{\mathcal{L}_{LDA}}$ is an awareness function and $\pi:Atm\longrightarrow 2^S$ is a valuation function for atomic propositions. In order to relate LGA with LDA, we here assume that every relation \mathcal{R}_i is serial to guarantee that an agent cannot have inconsistent implicit beliefs.

In LGA, the satisfaction relation is between formulas and pointed models (M, s) where $M = (S, \mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{A}_1, \dots, \mathcal{A}_n, \pi)$ is an awareness structure and $s \in S$ is a state:

$$\begin{split} (M,s) &\models p &\iff p \in \pi(s) \\ (M,s) &\models \neg \varphi &\iff (M,s) \not\models \varphi \\ (M,s) &\models \varphi_1 \land \varphi_2 &\iff (M,s) \models \varphi_1 \text{ and } (M,s) \models \varphi_2 \\ (M,s) &\models \mathsf{B}_i \varphi &\iff \forall s' \in \mathcal{R}_i(s) : (M,s') \models \varphi \\ (M,s) &\models \mathsf{A}_i \varphi &\iff \varphi \in \mathcal{A}_i(s) \\ (M,s) &\models \mathsf{X}_i \varphi &\iff (M,s) \models \mathsf{B}_i \varphi \text{ and } (M,s) \models \mathsf{A}_i \varphi \end{split}$$

There are two important differences between LGA and LDA. First of all, in the semantics of LGA the notion of doxastic alternative is given as a primitive while in the semantics of LDA it is defined from the concept of belief base. Secondly, the LGA ontology of epistemic attitudes is richer than the LDA ontology, as the former includes the concept of awareness which is not included in the latter. We believe these are virtues of LDA compared to LGA. On the one hand, the LDA semantics offers a compact representation of epistemic states in which the concept of belief base plays a central role. This conforms with how epistemic states are traditionally represented in the area of knowledge representation and reasoning (KR). On the other hand, modeling explicit and implicit beliefs without invoking the notion of awareness is a good thing, as the latter is intrinsically polysemic and ambiguous. This aspect is emphasized by F&H, according to whom the notion of awareness is "...open to a number of interpretations. One of them is that an agent is aware of a formula if he can compute whether or not it is true in a given situation within a certain time or space bound" [11, p. 41].

Let us define the following direct translation tr from the LDA language to the LGA language:

$$tr(p) = p \text{ for } p \in Atm$$

$$tr(\neg \alpha) = \neg tr(\alpha)$$

$$tr(\alpha_1 \land \alpha_2) = tr(\alpha_1) \land tr(\alpha_2)$$

$$tr(\neg \varphi) = \neg tr(\varphi)$$

$$tr(\varphi_1 \land \varphi_2) = tr(\varphi_1) \land tr(\varphi_2)$$

$$tr(\mathbf{E}_i \alpha) = \mathbf{X}_i tr(\alpha)$$

$$tr(\mathbf{I}_i \varphi) = \mathbf{B}_i tr(\varphi)$$

As the following theorem highlights the previous translation provides a correct embedding of LDA into LGA.

Theorem 8 Let $\varphi \in \mathcal{L}_{LDA}$. Then, φ is satisfiable for the class of quasi-NDMs if and only if $tr(\varphi)$ is satisfiable for the class of awareness structures.

PROOF. We first prove the left-to-right direction. Let $M=(W,\mathcal{D},\mathcal{N},\mathcal{V})$ be a quasi-NDM and let $w\in W$ such that $(M,w)\models \varphi$. We build the corresponding structure $M'=(S,\mathcal{R}_1,\ldots,\mathcal{R}_n,\mathcal{A}_1,\ldots,\mathcal{A}_n,\pi)$ as follows:

- S = W,
- for every $i \in Agt$ and for every $w \in W$, $\mathcal{R}_i(w) = \mathcal{N}(i, w)$,
- for every $i \in Agt$ and for every $w \in W$, $\mathcal{A}_i(w) = \{ \varphi \in \mathcal{L}_{LGA} : \exists \alpha \in \mathcal{D}(i, w) \text{ such that } \varphi = tr(\alpha) \}$,
- for every $w \in W$, $\pi(w) = \mathcal{V}(w)$.

It is easy to verify that M' is an awareness structure as every relation \mathcal{R}_i is serial.

By induction on the structure of φ , we prove that for all $w \in W$, " $(M, w) \models \varphi$ iff $(M', w) \models tr(\varphi)$ ".

The case $\varphi=p$ and the boolean cases $\varphi=\neg\psi$ and $\varphi=\psi_1\wedge\psi_2$ are clear. Let us now consider the case $\varphi=\mathsf{E}_i\alpha$.

 (\Rightarrow) $(M,w) \models \mathtt{E}_i \alpha$ means that $\alpha \in \mathcal{D}(i,w)$. By definition of \mathcal{A}_i , the latter implies that $tr(\alpha) \in \mathcal{A}_i(w)$ which is equivalent to $(M',w) \models \mathtt{A}_i tr(\alpha)$. Moreover, $(M,w) \models \mathtt{E}_i \alpha$ implies that $\mathcal{N}(i,w) \subseteq ||\alpha||_M$. By induction hypothesis, we have $||\alpha||_M = ||tr(\alpha)||_{M'}$. Thus, by definition of $\mathcal{R}_i(w)$, it follows that $\mathcal{R}_i(w) \subseteq ||tr(\alpha)||_{M'}$. The latter means that $(M',w) \models \mathtt{B}_i tr(\alpha)$. From the latter and $(M',w) \models \mathtt{A}_i tr(\alpha)$, it follows that $(M',w) \models \mathtt{X}_i tr(\alpha)$.

 (\Leftarrow) $(M', w) \models X_i tr(\alpha)$ implies $(M', w) \models A_i tr(\alpha)$ which is equivalent to $tr(\alpha) \in A_i(w)$. By definition of A_i , the latter implies $\alpha \in \mathcal{D}(i, w)$ which is equivalent to $(M, w) \models E_i \alpha$.

Finally, let us consider the case $\varphi = I_i \psi$. By induction hypothesis, we have $||\psi||_M = ||tr(\psi)||_{M'}$. $(M, w) \models I_i \psi$ means that $\mathcal{N}(i, w) \subseteq ||\psi||_M$. By definition of $\mathcal{R}_i(w)$ and $||\psi||_M = ||tr(\psi)||_{M'}$, the latter is equivalent to $\mathcal{R}_i(w) \subseteq ||tr(\psi)||_{M'}$ which in turn is equivalent to $(M', w) \models B_i tr(\psi)$.

Let us prove the right-to-left direction. Let $M = (S, \mathcal{R}_1, \dots, \mathcal{R}_n, \mathcal{A}_1, \dots, \mathcal{A}_n, \pi)$ be an awareness structure. We build the model $M' = (W, \mathcal{D}, \mathcal{N}, \mathcal{V})$ as follows:

- W = S,
- for every $i \in Agt$ and for every $s \in S$, $\mathcal{N}(i, s) = \mathcal{R}_i(s)$,
- for every $i \in Agt$ and for every $s \in S$, $\mathcal{D}(i,s) = \{\alpha \in \mathcal{L}_{LDA} : \mathcal{R}_i(s) \subseteq ||tr(\alpha)||_M \text{ and } tr(\alpha) \in \mathcal{A}_i(s)\},$
- for every $s \in S$, $\mathcal{V}(s) = \pi(s)$.

Let us prove that M' is a quasi-NDM. It clearly satisfies Condition C2 in Definition 7. In order to prove that it satisfies Condition C1* in Definition 8, we first prove by induction on the structure of α that $||tr(\alpha)||_M = ||\alpha||_{M'}$. The case $\alpha = p$ is clear as well as the boolean cases. Let us prove the case $\alpha = \mathbb{E}_i \alpha'$. We have $M', s \models \mathbb{E}_i \alpha'$ iff $\alpha' \in \mathcal{D}(i, s)$. By definition of $\mathcal{D}(i, s)$, we have $\alpha' \in \mathcal{D}(i, s)$ iff $\mathcal{R}_i(s) \subseteq ||tr(\alpha')||_M$ and $tr(\alpha') \in \mathcal{A}_i(s)$. The latter is equivalent to $M, s \models \mathbb{X}_i tr(\alpha')$.

Suppose that $\alpha \in \mathcal{D}(i,s)$. By definition of $\mathcal{D}(i,s)$, it follows that $\mathcal{R}_i(s) \subseteq ||tr(\alpha)||_M$. Thus, since $||tr(\alpha)||_M = ||\alpha||_{M'}$, we have $\mathcal{R}_i(s) \subseteq ||\alpha||_{M'}$. Hence, by definition of $\mathcal{N}(i,s)$, $\mathcal{N}(i,s) \subseteq ||\alpha||_{M'}$. This shows that M' satisfies Condition C1*.

In the rest of the proof we show that for all $s \in S$, " $(M,s) \models tr(\varphi)$ iff $(M',s) \models \varphi$ ". The proof is by induction on the structure of φ .

The case $\varphi = p$ and the boolean cases are clear. Let us now consider the case $\varphi = \mathbf{E}_i \alpha$. $(M,w) \models tr(\mathbf{E}_i \alpha)$ means that $(M,w) \models \mathbf{X}_i tr(\alpha)$. The latter is equivalent to $\mathcal{R}_i(s) \subseteq ||tr(\alpha)||_M$ and $tr(\alpha) \in \mathcal{A}_i(s)$. By definition of $\mathcal{D}(i,s)$, the latter is equivalent to $\alpha \in \mathcal{D}(i,s)$ which in turn is equivalent to $(M',w) \models \mathbf{E}_i \alpha$.

Let us finally consider the case $\varphi = I_i \psi$. By induction hypothesis, we have $||tr(\psi)||_M = ||\psi||_{M'}$. $(M, w) \models tr(I_i \psi)$ means that $(M, w) \models B_i tr(\psi)$ which in turn means that $\mathcal{R}_i(w) \subseteq$

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||tr(\psi)||_M. By definition of \mathcal{N}(i,w) and ||\psi||_M = ||tr(\psi)||_{M'}, the latter is equivalent to \mathcal{N}(i,w) \subseteq ||\psi||_{M'} which in turn is equivalent to (M',w) \models \mathbf{I}_i \psi.
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The following theorem is a direct consequence of Theorems 1, 2, 3 and 8.

Theorem 9 Let $\varphi \in \mathcal{L}_{LDA}$. Then, φ is satisfiable for the class of CMABs if and only if $tr(\varphi)$ is satisfiable for the class of awareness structures.

In [1], it is proved that, for every $X\subseteq\{$ reflexivity, transitivity, Euclideanity $\}$, the satisfiability problem for the logic of general awareness interpreted over awareness structures whose relations \mathcal{R}_i satisfy all properties in X is PSPACE-complete. To prove PSPACE-membership, an adaptation of the tableau method for multi-agent epistemic logic by [13] is proposed. PSPACE-hardness follows from the PSPACE-hardness of multi-agent epistemic logics proved by [13]. It is easy to adapt Ågotnes & Alechina's method to show that the logic of general awareness interpreted over awareness structures whose relations \mathcal{R}_i are serial is also PSPACE-complete. As a consequence, we can prove the following result.

Theorem 10 The satisfiability problem of LDA is PSPACE-complete.

SKETCH OF PROOF. PSPACE-membership follows from the previous polynomial-time reduction of LDA satisfiability problem to LGA satisfiability problem relative to serial awareness structures and the fact that the latter problem is in PSPACE. PSPACE-hardness follows from the PSPACE-hardness of multi-agent epistemic logics [13].

7 Related work

The present work lies in the area of logics for non-omniscient agents. Purely syntactic approaches to the logical omniscience problem have been proposed in which an agent's beliefs are described either by a set of formulas which is not necessarily closed under deduction [9, 28] or by a set of formulas obtained by the application of an incomplete set of deduction rules [21]. Logics of time-bounded reasoning have also been studied [4, 12], in which reasoning is a represented as a process that requires time due to the time-consuming application of inference rules. Finally, logics of (un)awareness have been studied both in AI [11, 33, 2] and economics [27, 17, 14].

As we have shown in Section 6, our logic LDA is closely related to Fagin & Halpern (F&H)'s logic of general awareness, as there exists a polynomial embedding of the former into the latter. Another related system is the logic of local reasoning also presented in [11] in which the distinction between explicit and implicit beliefs is captured. F&H) use a neighborhood semantics for explicit belief: every agent is associated with a set of sets of worlds, called frames of mind. They define an agent's set of doxastic alternatives as the intersection of the agent's frames of mind. According to F&H's semantics, an agent explicitly believes that φ if and only if she has a frame of mind in which φ is globally true. Moreover, an agent implicitly believes that φ if and only if, φ is true at all her doxastic alternatives. In their semantics, there is no representation of an agent's belief base, corresponding to the set of formulas explicitly believed by the agent. Moreover, differently from our notion of explicit belief, their notion does not completely solve the logical omniscience problem. For instance, while their notion of explicit belief is closed under logical equivalence, our notion is not. Specifically, the following rule of equivalence preserves validity in F&H's logic but not in our logic:

$$\frac{\alpha \leftrightarrow \alpha'}{\mathsf{E}_i \alpha \leftrightarrow \mathsf{E}_i \alpha'}$$

This is a consequence of their use of an extensional semantics for explicit belief. Levesque too provides an extensional semantics for explicit belief with no connection with the notion of belief base [22]. In his logic, explicit beliefs are closed under conjunction, while they are not in our logic LDA.

8 Conclusion

We have presented a logic of explicit and implicit beliefs with a semantics based on belief bases. In the future, we plan to study a variant of this logic in which explicit and implicit beliefs are replaced by truthful explicit and implicit knowledge. At the semantic level, we will move from multi-agent belief bases to multi-agent knowledge bases in which the epistemic accessibility relation \mathcal{R}_i is assumed to be reflexive. The logic will include the following extra-axiom:

$$I_i \varphi \to \varphi$$
 (\mathbf{T}_{I_i})

We also expect to study a variant of our logic with the following extra-axioms of positive and negative introspection for implicit beliefs:

$$I_i \varphi \to I_i I_i \varphi$$
 (4_{Ii})

$$\neg I_i \varphi \to I_i \neg I_i \varphi \tag{5}_{I_i}$$

Moreover, we plan to extend the logic LDA and its epistemic variant by concepts of distributed belief and distribute knowledge.

We also plan to study a dynamic extension of the logic LDA which is in line with existing theories of belief base change. Specifically, we intend to capture different forms of belief base revision in the multi-agent setting offered by LDA including partial meet base revision à la Hansson [15]. The general idea of partial meet base revision, inspired by single-agent partial meet revision for belief sets [3], is that the belief base resulting from the integration of an input formula α should be equal to the intersection of all maximally consistent subsets of the initial belief base including α .

Last but not least, we plan to study the model checking problem for our logic by using the compact representation offered by multi-agent belief models, as defined in Definition 4. As shown by [32], the possibility of using compact models could be beneficial for model checking.

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