Envy-Free Allocations Respecting Social Networks^{*}

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Abstract

Finding an *envy-free* allocation of indivisible resources to agents is a central task in many multiagent systems. Often, non-trivial envy-free allocations do not exist, and, when they do, finding them can be computationally hard. Classical envy-freeness requires that every agent likes the resources allocated to it at least as much as the resources allocated to *any* other agent. In many situations this assumption can be relaxed since agents often do not even know each other. We enrich the envy-freeness concept by taking into account (directed) social networks of the agents. Thus, we require that every agent likes its own allocation *at least* as much as those of all its (out)neighbors. This leads to a "more local" concept of envy-freeness. We also consider a "strong" variant where every agent must like its own allocation *more than* those of all its (out)neighbors.

We analyze the classical and the parameterized complexity of finding allocations that are complete and, at the same time, envy-free with respect to one of the variants of our new concept. To this end, we study different restrictions of the agents' preferences and of the social network structure. We identify cases that become easier (from $\Sigma_2^{\rm P}$ -hard or NPhard to polynomial-time solvable) and cases that become harder (from polynomial-time solvable to NP-hard) when comparing classical envy-freeness with our graph envy-freeness. Furthermore, we spot cases where graph envy-freeness is easier to decide than strong graph envy-freeness, and vice versa. On the route to one of our fixed-parameter tractability results, we also establish a connection to a directed and colored variant of the classical SUBGRAPH ISOMORPHISM problem, thereby extending a known fixed-parameter tractability result for the latter.

Keywords: computational social choice; fair allocation; indivisible goods; social networks; additive utility functions; parameterized complexity; exact algorithms; directed, colored subgraph isomorphism

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1 Introduction

Modern management strategies emphasize the role of teams and team-work. To have an effective team one has to motivate the team members in a proper way. One method of motivating team members is to reward them for achieving a milestone. On the one hand, it is crucial that every member of a team feels rewarded fairly. On the other hand, in every team there are hierarchical or personal relations, which one should take into account in the rewarding process. Since, according to a recent labor statistics in the US [Bureau of Labor Statistics, U.S. Department of Labor, 2017], the average cost of employee benefits (excluding legally required ones) is around 25% of the whole cost of labor, it is important to effectively use rewarding instruments. It is tempting to follow a simplistic belief that tangible incentives motivate best and thus reward employees with cash bonuses and pay raises. However, it has been shown that to keep the employee satisfaction high, an employer should also honor the employees with non-financial rewards [Haider et al., 2015].

We propose a model for the fair distribution of indivisible goods which can be used to find an allocation of non-financial rewards¹ such that each team member is satisfied with her rewards and, at the same time, is not worse off compared to any other peer whom she is in relation with. Besides the rewarding scenario, our model has numerous further potential applications, just to mention a few, targeting marketing strategies (giving non-monetary bonuses to loyal customers), allocating physical resources to virtual resources in virtualization technologies (both network and machine virtualization), and sharing charitable donations between cities or communities which may envy each other.

Returning to our initial example of reward management, it is a well-established fact that team members evaluate the fairness of rewarding based on comparisons with their peers. This phenomenon, first described seventy years ago by social psychologist Festinger [1954], is probably one of the reasons for the popularity of fair allocation (division) problems in computer science. Naturally, when evaluating the subjective fairness of rewards, every team member tends to compare herself to similar peers, neglecting those who differ substantially in position, abilities, or other aspects. This has already been reflected by one of Festinger's hypotheses; however, so far, most research in computer science has focused on fairness notions based on "global" comparisons, that is, pairwise comparisons between all members of society.

In this work we aim at incorporating "local" comparisons into the fair allocation scenario. Having a collection of indivisible resources, we look for a way to distribute them fairly among a group of agents which, prior to the distribution, shared their opinions on how they appreciate the resources. For example, imagine that a company is to reward a team of three employees responsible for a successful project. The team consists of a key account manager (KAM) being the chief of the group, an internet sales manager (ISM), and a business-to-business (B2BSM) sales manager. The company intends some non-financial rewards to recognize the employees' performances. The rewards are 'participating in a language course', 'being the company's

¹Financial rewards can be interpreted as divisible resources while we focus on indivisible resources.

	KAM	B2BSM	ISM
language course		Ø	Ø
TV episode		Ø	Ø
high-end office	Ø	Ø	Ø
employee-of-the-month award	Ø	\checkmark	Ø

Table 1: The results of a survey concerning employees' (0/1) preferences over the possible rewards. Checked boxes indicate the approved rewards of a particular person.

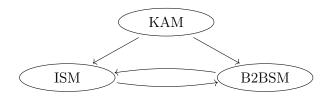


Figure 1: An illustration of who compares to whom for the introductory example. Every node represents an employee and arcs represent directions of comparisons. For instance, if an arc points from the key account manager to the internet sales manager, then the former compares herself to the latter.

representative for an episode of a documentary program', 'moving to a new high-end office', and 'receiving an employee-of-the-month award'. The employees (agents) were surveyed for their favorite rewards, yielding the results given in Table 1.

Each employee considers a rewarding unfair if after exchanging all her rewards with all rewards of some peer, the employee would get more approved rewards. According to the company's rewarding policy, all rewards must be handed out. Considering the standard model of resource allocation, where each employee can compare herself to every other peer, the company cannot find a fair reward allocation. At least one employee has to get two rewards. As a consequence, two employees have at most one reward. However, a rewarding policy in the company assumes that a team's chief is always a basis of team success and thus deserves a better reward. Hence, both sales managers do not compare their rewards to the ones of their boss. Naturally, the key account manager's reward should be at least as good as the ones of the others. To illustrate these relations, we use the directed graph depicted in Figure 1. In this case, the company can reward the key account manager with the office and the employee-of-the-month award, and distribute the two remaining rewards equally to the internet and business-to-business managers. Doing so, the company achieves a fair rewarding. The key account manager has two favorite rewards and there is no incentive to exchange them. The remaining team members do not compare themselves to their boss, so they do not envy her. Finally, both the business-to-business and internet managers have one favorite reward, so there is no envy. Thus, by introducing the graph of relations between the employees, we are able to represent social comparisons.

Related Work In 1948, Steinhaus [1948] asked how to fairly distribute a continuous resource, a "cake," among a set of agents with (possibly different) heterogeneous valuations of the resource. From this first mathematical model of fair allocation, two main research directions evolved. The difference lies in the nature of the resources—divisible or indivisible. The former type yields the so-called cake cutting problem. We refer to the books [Brams and Taylor, 1996, Robertson and Webb, 1998, Moulin, 2003] and recent surveys [Procaccia, 2013, 2016, Bouveret et al., 2016, Markakis, 2017] on fair division problems. Next, we discuss literature specifically related to our setting.

Abebe et al. [2017] and Bei et al. [2017] introduced social networks of agents into the fair division context. They defined (local) fairness concepts based on social networks and then compared them to the classic fairness notions and designed new protocols to find envy-free allocations. Although their models defined local envy-freeness, it significantly differs from our concept by considering divisible resources.

Strongly related to our model is an independent work of Aziz et al. [2018] analyzing relations of different notions of envy-freeness in the context of partial knowledge of agents (extending similar work of Bouveret and Lemaître [2016] classifying fairness concepts in the case of full knowledge) and introducing *epistemic envy-freeness*. More importantly in the context of our work, Aziz et al. [2018] presented a general framework for fairness concepts which also captures our model. However, their main goal was to provide the framework, so they did not study the model itself.

Recently, Beynier et al. [2019] published a study on finding local envy-free allocations in so-called *housing markets*, where an allocation assigns at most one resource per agent. However, this restriction, together with their assumptions that a given social network is undirected and that preferences are ordinal, makes their model substantially different to ours. Notably, they also studied the impact of different graph classes on the computational complexity of the problem, showing that even for very simple graphs the problem is NP-hard.

Lange and Rothe [2019], among other things, used our model to study other efficiency concepts such as Nash welfare and egalitarian social welfare, showing several NP-hardness results, yet also identifying few computationally tractable cases. Very recently, Eduard et al. [2020], motivated by the conference version of this paper, conducted a thorough study of the influence of "tree-likeness" of the agent social network and its density on the computational complexity. It turned out that even for, simultaneously, few *distinguishable* resources, few *distinguishable* agents, a small number of resources in the agent bundles, and social networks similar to trees, the problem remains hard.

An allocation, instead of being executed by a central mechanism, might emerge from a sequence of trades between the agents initially endowed with some resources; this setting gives birth to the problem of *distributed* allocation of indivisible resources. Gourvés et al. [2017] studied this problem of embedding agents into a social network describing the possible agent interactions in housing markets. They addressed the computational hardness of several questions such as existence of a Pareto-efficient allocation, reachability of a particular allocation, or reachability of a resource for a candidate. They proved that answering these questions is NP- hard in general, but it is polynomial-time solvable for some constrained cases. Their model has been studied further by Saffidine and Wilczynski [2018] who analyzed the problems constraining the number of changes and by Bentert et al. [2019] who answered several questions left open by Gourvés et al. [2017]. Chevaleyre et al. [2017] enriched the distributed allocation problem with monetary payments for the trades. They defined a version of graph envy-freeness which takes into account both allocations of resources and the payments of agents. They showed several results describing convergence of trades with respect to a fair allocation. Additionally, they proved that the problem of finding a deal reducing unfairness among the agents is NP-hard.

Recently, a somewhat orthogonal model where relations of resources, instead of agents, are described by a graph has also been studied [Bouveret et al., 2017, Bilò et al., 2019, Igarashi and Peters, 2019, Suksompong, 2017, Bei et al., 2019]. The main focus of this line of work is to study allocations that assign to agents only bundles that form connected components with respect to the given graph.

Our model is also related to a work of Gourvès et al. [2018] who introduced and studied the computational complexity of the SUBSET SUM WITH DIGRAPH CONSTRAINTS problem. Their main motivation lied in applications in job scheduling and the issue of updating modular software. By adding solution constraints encoded as a directed acyclic graph, they generalized standard SUBSET SUM obtaining a variant similar to our model for the case of identical preferences.

In contrast to some previously studied models, in our work we assume that allocations are computed by a central authority and that each agent can obtain more than one resource. Furthermore, the resources we study are not subject to monetary payments to agents as a compensation for not obtaining a resource.

Our Contributions Our work follows the recent trend of combining fair allocation with social networks. We introduce social relations into the area of fair allocation of *indivisible* resources *without monetary payments*. Making use of a greater model flexibility resulting from embedding agents into a social network, we define two new versions of the classical envy-freeness property; namely, (weak) graph envy-freeness and strong graph envy-freeness. Even though Chevaleyre et al. [2017] also introduced a property called graph-envy-freeness, their version differs from ours significantly because, instead of being a property of an allocation, it describes a particular state of the negotiations between the agents, including monetary payments (which has the flavor of divisible resources) paid to the agents so far.

We study problems of finding (weakly/strongly) graph-envy-free and efficient allocations. We mainly focus on completeness as an efficiency criterion. We assume that the agents' preferences over the resources are cardinal, additive, and monotonic. We look beyond the general case (with no further constraints on agents' preferences and an arbitrary social network), and we analyze our problems with respect to social networks being directed acyclic graphs or strongly connected components, and with respect to identical or 0/1 preferences over the resources. As a result, we explore a wide and diverse landscape of the classical computational complexity of the introduced problems. Our results reveal that in comparison to classical envy-freeness,

our model sometimes simplifies the task of finding a proper allocation and sometimes makes it harder. Similarly, we identify cases where finding a (weakly) graph-envy-free allocation is easier than finding a strongly graph-envy-free allocation but also cases where the opposite is true. Additionally, our work assesses the parameterized computational complexity of several cases with respect to a few natural parameters such as the number of agents, the number of resources, and the maximum number of neighbors of an agent. On the route to one of our fixed-parameter tractability results (regarding the case of identical preferences), we also spot a novel fixed-parameter tractability result for the DIRECTED COLORED SUBGRAPH ISOMOR-PHISM problem. We complement our work by showing how our results regarding classical computational complexity for fair and complete allocations generalize to other efficiency concepts like Pareto-efficiency and utilitarian social welfare maximization. Each of our main sections also contains a table surveying the corresponding results.

Organization In the following sections, after covering necessary preliminaries (Section 2), we formally introduce our new model, discuss it, and present the corresponding computational problems (Section 3). Then, we analyze the problem of finding (weakly) graph-envy-free allocations that are complete (Section 4) followed by a study on seeking strongly graph-envy-free allocations that are complete (Section 5). Next, basing on the results regarding (weakly/strong-ly) graph-envy-free and complete allocations, we study the classical computational complexity of finding (weakly/strongly) graph-envy-free allocations that are either Pareto-efficient or maximize the utilitarian social welfare (Section 6). We end with conclusions and suggestions for future work (Section 7).

2 Basic Definitions

We start with basic concepts for describing graphs, which we use to model relations between agents. For a directed graph G = (V, E), consisting of a set V of vertices and a set E of arcs, by N(v) we denote the outneighborhood of vertex $v \in V$, i.e., the set $W \subset V$ of vertices such that for each vertex $w \in W$ there exists an arc $e = (v, w) \in E$, i.e., arc e is directed from v to w. Where needed, we complement our notation by using a subscript indicating the graph we consider.

Definition 1. A condensation of a directed graph \mathcal{G} is a directed graph in which every strongly connected component in \mathcal{G} is contracted to a single vertex.

We continue with defining some standard concepts for allocation problems needed to formally introduce our problems.

Definition 2. An allocation of a set \mathcal{R} of resources to a set \mathcal{A} of agents is a mapping $\pi : \mathcal{A} \to 2^{\mathcal{R}}$ such that $\pi(a)$ and $\pi(a')$ are disjoint whenever $a \neq a'$. For any agent $a \in \mathcal{A}$, we call $\pi(a)$ the bundle of a under π .

Measuring fairness requires a possibility to compare how much agents like different bundles. There are different ways to model preferences of agents over resources. A well-established (and quite flexible) way, which we also focus on, is to express the preferences numerically using so-called utility functions.

Definition 3. For a set \mathcal{R} of resources, a function $u: \mathcal{R} \to \mathbb{Z}$ is called a *utility function* and, for some bundle $X \in \mathcal{R}$ its output is called the *utility* of X.

In our work, we solely focus on utility functions that are additive and monotonic.

Definition 4. A utility function $u: \mathcal{R} \to \mathbb{Z}$ is additive when, for each bundle $X \in \mathcal{R}$, $u(X) = \sum_{r \in X} u(r)$. An additive utility function is *monotonic* if it only outputs non-negative utilities. A utility function is a 0/1 utility function if, for each $r \in \mathcal{R}$, u(r) is either zero or one.

For convenience, throughout this paper, we frequently refer to additive monotonic or 0/1 preferences instead of saying, for instance, "preferences expressed by additive monotonic utility functions." In the problems we study (introduced in Section 3), we always speak about multiple utility functions representing agent preferences (one function per agent) and we, intuitively, call preferences *identical* if every agent has the same utility function.

Having preferences defined, we formally present our graph fairness concepts based on comparisons between neighbors in a social network.

Definition 5. Let $\mathcal{G} = (\mathcal{A}, \mathcal{E})$ be a directed graph, called an *attention graph*, representing a social network over the agents (i.e., the agents are the vertices of \mathcal{G}). We call allocation π (weakly) graph-envy-free if for each pair of (distinct) agents $a_1, a_2 \in \mathcal{A}$ such that $a_2 \in N(a_1)$ it holds that $u_1(\pi(a_1)) \geq u_1(\pi(a_2))$. By replacing $u_1(\pi(a_1)) \geq u_1(\pi(a_2))$ with $u_1(\pi(a_1)) >$ $u_1(\pi(a_2))$, we obtain the definition of a strongly graph-envy-free allocation.

Naturally, an allocation which gives nothing to every agent is always (weakly) graph-envyfree. To overcome this trivial case, we combine our fairness concepts with different measures of allocation efficiency.

Definition 6. Let π be an allocation of a set \mathcal{R} of resources to a set \mathcal{A} of agents and let $U = \{u_1, u_2, \ldots, u_{|\mathcal{A}|}\}$ be a family of utility functions where function $u_i, i \in |\mathcal{A}|$, represents the preferences of agent a_i . Then, π is *complete* if $\bigcup_{a \in \mathcal{A}} \pi(a) = \mathcal{R}$. Moreover, π is *Pareto-efficient* if there exists no allocation π' that *dominates* π , where dominating means that for all $a_i \in \mathcal{A}$ it holds that $u_i(\pi(a_i)) \leq u_i(\pi'(a_i))$ and for some $a_j \in \mathcal{A}$ it holds that $u_j(\pi(a_j)) < u_j(\pi'(a_j))$.

Assuming that the reader is familiar with basic notions from (classical) computational complexity theory, we now briefly introduce the parameterized viewpoint on the computational complexity. Let ρ be a *parameter* of a problem, that is, a (usually integer) measure that numerically expresses some feature of the problem instances. A problem parameterized by ρ is *fixed-parameter tractable* if it is solvable in $f(\rho) \cdot |I|^{O(1)}$ time for some computable function f and the input size |I| according to the problem's encoding. We also say the problem is solvable in *FPT-time* with respect to ρ . A problem that is W[t]-hard, $t \geq 1$, with respect to parameter ρ is presumably not fixed-parameter tractable with respect to ρ . To show W-hardness of a problem, one needs to employ a so-called *parameterized reduction* in the same way one would apply a polynomial-time many-one reduction to show NP-hardness. In particular, a parameterized reduction from some problem parameterized by ρ to a problem parameterized by ρ' is a many-one reduction computable in FPT-time² with respect to ρ in which the value of parameter ρ' solely depends on the value of parameter ρ . Lastly, we call a problem para-NP-hard if it is NP-hard even for a constant value of the parameter.

Throughout the paper we make heavy use of the graph problem CLIQUE to show our results regarding computational hardness.

Definition 7. In the CLIQUE problem, given an undirected graph and an integer k, the questions is whether there is a clique of size k, i.e., a size-k subset of pairwise adjacent vertices.

CLIQUE is a well-known NP-complete [Garey and Johnson, 1979] problem which is W[1]hard [Downey and Fellows, 1995] when parameterized by the size of the clique.

3 Model and Discussion

This section is devoted to the model we introduce, a discussion on the model and its variants, and a collection of basic observations that provide an initial intuition about problems that we consider.

3.1 Computational Problem

The core of our investigations is a computational problem related to our setting of fair allocation. We define our problems in the form of search problems instead of decision problems. In practical applications in which fair allocation problems are usually found, it is important not only to know that there exists an allocation with particular features, but also to know how it looks like. Clearly, all our problems also have natural decision variants to which we sometimes explicitly refer.

Subsequently, X-(s)GEF-ALLOCATION stands for X-(strongly) graph-envy-free allocation where $X \in \{C, E, W\}$ —'C' referring to complete, 'E' referring to Pareto-efficient, and 'W' referring to utilitarian social welfare. We start with defining our problems with respect to completeness and Pareto-efficiency.

C-GEF-ALLOCATION (resp. C-SGEF-ALLOCATION)

Input: A set \mathcal{A} of n agents, a set \mathcal{R} of m indivisible resources, a family $U = \{u_1, u_2, \ldots, u_n\}$ of non-negative utility functions, one function per agent, and a directed graph $\mathcal{G} = (\mathcal{A}, \mathcal{E})$.

²In this paper, all parameterized reductions can even be computed in polynomial time.

Task: Find a complete and graph-envy-free (resp. strongly graph-envy-free) allocation of \mathcal{R} to \mathcal{A} .

Analogously, we define the problems E-GEF-ALLOCATION and E-SGEF-ALLOCATION, where we seek a Pareto-efficient and (weakly/strongly) graph-envy-free allocation. When considering utilitarian social welfare, we slightly change the task when defining the respective problems W-GEF-ALLOCATION and W-SGEF-ALLOCATION: We look for a (weakly/strongly) graphenvy-free allocation which maximizes the utilitarian social welfare. For reasons of brevity, we usually write C-GEF-A instead of C-GEF-ALLOCATION in the running text; we contract the names of other defined problems analogously.

3.2 Discussion

To study envy-freeness one needs to disallow wasting all resources and, as a result, obtaining a trivial "empty," yet envy-free allocation. In practice, choosing an efficiency concept suitable for an application might be nontrivial and is a problem on its own. Since we do not particularly focus on this issue in our work, we simply require all allocations to be complete while discussing our model in this section.

Our work studies the computational complexity of finding envy-free allocations from two main perspectives. The first one is the nature of preferences that agents report for different resources. The second one is the structure of agent relations in terms of their awareness or knowledge of each other.

Preference Domains. We study cardinal preferences that are additive and monotonic. This type of preferences is considered as a reasonable trade-off between expressive power and elicitation simplicity. However, in the domain of fair allocation, this type of preferences usually leads to (computationally) very challenging problems.

In our work, we consider three constrained types of preferences to track down how the problem's hardness is related to the constraints. To this end, we study identical preferences, 0/1 preferences, and identical 0/1 preferences. At first glance, these constraints might seem too strong to yield a practical model. However, apart from being widespread in the fair allocation literature, they are also practically motivated. Assuming, for example, that agents are humans, it is rather tedious and error-prone for an agent to assign an arbitrarily chosen number to a resource. This, in effect, makes it harder to collect valid utilities. So, it might be desirable to just let the agents choose whether they like or dislike a particular resource, thus making the evaluation process less painful and more reliable; such scenario is modeled by 0/1 preferences. Actually, it even can be impossible to survey all agents. Then, collecting preferences from a sample of all agents, averaging them, and then using the averaged ones for all agents naturally leads to identical preferences.

In the most restricted variant of identical 0/1 preferences, one can in fact think of giving indistinguishable resources to the agents. Such a scenario is natural when there are a lot of

resources of the same type. An extreme case of 0/1 preferences, would be considering a unit of money as a single resource. In this light, the case of identical 0/1 preferences is interesting because it could serve as a fallback each time a set of indivisible resources cannot be allocated fairly. Then, selling the resources and allocating the obtained money makes them somewhat "more divisible," which may allow for a fair allocation. Observe, that this still is not identical to the case of cake-cutting since, clearly, one cannot divide money indefinitely.

Structure of Relations. The concept of graph envy-freeness is a more general version of the standard envy-freeness concept—graph envy-freeness is equivalent to envy-freeness if the given attention graph is a complete graph. In another extreme case, if the attention graph is an edgeless graph, then graph envy-freeness is purposeless, for it does not impose any constraints. In the former case—that is, seeking (complete) envy-free allocations—we know that the corresponding problem is computationally challenging. Obviously, in the latter case the problem boils down to finding any complete allocation and becomes trivial. Hence, the major focus of this work is to nail down the computational complexity of finding complete and (weakly/strongly) graph-envy-free allocations for attention graph structures between these two extremes.

Our motivation, however, is not purely theoretical. Associating the attention graph over agents with their "social relations," "attention relations," or their "knowledge" of each other brings our studies closer to the real world. From this perspective, the graph classes under our consideration—directed acyclic graphs, strongly connected graphs, and general graphs represent different situations that occur in reality. Directed acyclic graphs are suitable to cover different kinds of hierarchical structures, for example, a corporation's employees or departments. Strongly connected graphs model non-scattered or coherent communities, without clearly separated parts like groups of classmates, friends, or teammates. Agent relations can also form structures that are beyond the two above mentioned cases, which justifies analyzing general graphs as attention graphs. Consider, for example, a professional association divided into local branches. Here, probably, there are some attention relations between prominent members of different branches, yet it might not be the case that low-ranked members of a local branch pay attention to those of another branch.

The above interpretation of the attention graph might seem arguable when compared to our choice of the attention graph being directed. However, we find it very likely that, especially for knowledge or attention relations, such a relation might be one-directional. As in our introductory example, it seems natural that subordinates rather do not envy their bosses (at least to a reasonable level). Also, in the case of knowledge, asymmetric information is not uncommon. Consider so-called "social media influencers" who are people highly visible in the social media and who are paid by companies to market their products. The influencers' social-media followers definitely know a lot more about the influencers' personal lives than the other way around.

We point out that our graph envy-freeness concept is designed in a way that an agent totally neglects resources that were not assigned to it and its neighbors in the attention graph (a model where such resources are not neglected is defined and briefly analyzed by Aziz et al. [2018]). As a result, an interesting situation can occur if an agent gets nothing. Such an agent can still be not envious (for example, when the agent has no neighbors in the attention graph), even though it is clear that there are some resources that could have been assigned to the agent such that the agent would be better off. This phenomenon might be considered as a flaw in modeling fairness. However if a central authority that assigns resources is trusted, then even such an agent that gets nothing might feel comfortable. Moreover, the agents might also be very committed and agree that the situation happened for a greater good or they might be "emotionless" (non-human agents).

3.3 Basic Observations

We start with a technical observation saying that graph envy-freeness can be checked in polynomial time. It is enough to compare each agent's own bundle value to the values the agent assigns to its neighbors' bundles.

Observation 1. Given a set \mathcal{R} of resources, a set \mathcal{A} of agents with additive monotonic utility functions over resources in \mathcal{R} , and some allocation $\pi \colon \mathcal{A} \to 2^{\mathcal{R}}$, one can decide in polynomial time whether π is (weakly/strongly) graph-envy-free.

Proof. It suffices to compute the utility every agent associates with its bundle and then compare it to the utilities the agent assigns to the bundles of its neighbors. \Box

Observation 1, in fact, shows containment of the decision variants of C-GEF-A and C-SGEF-A in NP. Hence, every NP-hardness and W[1]-hardness proof in Sections 4 and 5 (in our work every W[1]-hardness proof also yields NP-hardness) also implies NP-completeness of the corresponding decision problem discussed in the proof.

Intuitively, the resources that have no value for each agent are meaningless for the concept of (weakly/strongly) graph-envy-free allocation. In the following observation, we formally show that indeed we can rule them out in the first place.

Observation 2. Consider an instance I of C-GEF-ALLOCATION or C-SGEF-ALLOCATION. Without loss of generality, in I there are only resources to which at least one agent assigns positive utility.

Proof. Assume that I with an agent set \mathcal{A} and a resource set \mathcal{R} contains a resource $r \in \mathcal{R}$ that has utility zero for all agents. Let π be some complete allocation of \mathcal{R} to \mathcal{A} . Consider the (unique) allocation π' with exactly the same bundles as those of π but excluding resource r. Since, for each pair of (not necessarily distinct) agents $a, a' \in \mathcal{A}, u_a(\pi(a')) = u_a(\pi'(a'))$, it must follow that π is (weakly/strongly) graph-envy-free if and only if π' is (weakly/strongly) graph-envy-free.

Observation 2, albeit simple, results in a useful consequence for the case of identical 0/1 preferences; namely, C-GEF-A and C-sGEF-A boil down to distributing a certain number of indistinguishable resources.

Observation 3. Consider an instance of C-GEF-ALLOCATION with m resources, identical utility functions, and graph $\mathcal{G} = (\mathcal{A}, \mathcal{E})$ with some graph-envy-free allocation π . If agent a gets no resource in π , then every agent reachable from a also gets no resource in π .

Proof. Let a be an agent with no resource in π . We give an inductive argument with respect to the distance—measured as the length of the shortest path—from a. Consider the base case of an agent a' reachable from a with distance one (i.e., $(a, a') \in \mathcal{E}$) that gets at least one resource in π . Because the resources are identical and all of them have a positive value (see Observation 2), a must envy a', contradicting that π is graph-envy-free. Next, consider an agent a' at distance k from a. Let b be an agent at distance k-1 from a such that $(b, a') \in \mathcal{G}$. Applying the induction hypothesis, b has no resource and thus, due to the base case of distance one, the same holds for a' proving the hypothesis for distance k.

3.4 ILP Models of the Problems

We present two ILP formulations (out of many possible ones), one for C-GEF-A and one for C-SGEF-A, that we will utilize in order to show fixed-parameter tractability in several subsequent proofs in this work. The two models are almost identical, so we will first provide the model for C-GEF-A and then describe a small change leading to the one for C-SGEF-A.

To devise the ILP model for C-GEF-A, we fix an instance of C-GEF-A consisting of agents $\mathcal{A} = \{a_1, a_2, \ldots, a_n\}$, resources $\mathcal{R} = \{r_1, r_2, \ldots, r_m\}$, a utility functions family $\mathcal{U} = \{u_1, u_2, \ldots, u_n\}$, and an attention graph $\mathcal{G} = \{\mathcal{A}, \mathcal{E}\}$. For some resource r, a type of r is a vector $t_r := (u_{a_1}(r), u_{a_2}(r), \ldots, u_{a_n}(r))$. By $T := \{t_r : r \in \mathcal{R}\}$, we denote the set of all possible types of the resources and, for each $t \in T$, by #t the number of resources of type t.

Our ILP model consists of the following variables. For each agent $a_i \in \mathcal{A}$ and each type $t \in T$, we use an integral non-negative variable x_i^t whose value represents the number of resources of type t given to agent a. We model C-GEF-A using the following ILP program (we are not stating a goal function, since it is enough to find any feasible solution):

$$\forall t \in T: \quad \sum_{i \in [n]} x_i^t = \#t \tag{1}$$

$$\forall (a_i, a_j) \in \mathcal{E} \colon \sum_{t \in T} x_i^t \cdot t[i] \ge \sum_{t \in T} x_j^t \cdot t[j]$$

$$\tag{2}$$

Inequalities (1) ensure that a sought allocation is complete, while Inequalities (2) guarantee weak graph envy-freeness.

We obtain the model for C-sGEF-Aby adding 1 to the right-hand side of the weak inequality in (2).

	preference type	es		parameterization		
		#agents	#resources	outdegree	# agents + outdegree	#resources + outdegree
type	directed acyclic					
	$additive^{3}$	P (Obs. 4)	P (Obs. 4)	P (Obs. 4)	P (Obs. 4)	P (Obs. 4)
	strongly connected					
Ŀ.	id. $0/1$	P (Cor. 1)	P (Cor. 1)	P (Cor. 1)	P (Cor. 1)	P (Cor. 1)
graph	id.	W[1]-h (Pr. 2)	FPT (Pr. 3)	p-NP-h (Pr. 2)	W[1]-h (Pr. 2)	FPT $(Pr. 3)$
gr	0/1	FPT (Pr. 1)	W[1]-h (Th. 2)	p-NP-h (Th. 2)	FPT (Pr. 1)	W[1]-h (Th. 2)
on	additive	W[1]-h (Pr. 2)	W[1]-h (Th. 2)	p-NP-h (Pr. 2)	W[1]-h (Pr. 2)	W[1]-h (Th. 2)
attention	general					
tte	id. $0/1$	FPT (Pr. 1)	W[1]-h (Th. 1)	p-NP-h (Th. 1)	FPT (Pr. 1)	FPT (Th. 3)
а	id.	W[1]-h (Pr. 2)	W[1]-h (Th. 1)	p-NP-h (Th. 1)	W[1]-h (Pr. 2)	FPT (Th. 3)
	0/1	FPT (Pr. 1)	W[1]-h (Th. 1)	p-NP-h (Th. 1)	FPT (Pr. 1)	W[1]-h (Th. 2)
	additive	$\mathrm{W}[1]\text{-}\mathrm{h}~(\mathrm{Pr.}~2)$	W[1]-h (Th. 1)	p-NP-h (Th. 1)	$\mathrm{W}[1]\text{-}\mathrm{h}~(\mathrm{Pr.}~2)$	$\mathrm{W}[1]\text{-}\mathrm{h}$ (Th. 2)

Table 2: Parameterized complexity of C-GEF-ALLOCATION. The results are grouped by three criteria regarding the problem input: the structure of the attention graph, the preference type, and the parameterization. All cases except for the polynomial-time solvable ones are NP-hard.

4 Finding Weakly Graph-Envy-Free Allocations

We analyze the classical complexity and the parameterized complexity for finding allocations that are complete and (weakly) graph-envy-free. All our results are presented in Table 2. Among others, we identify cases where using our graph-based envy-freeness concept leads to decreased complexity (from NP-hard to P) and cases where it leads to increased complexity (from P to NP-hard), each time comparing to classical envy-freeness.

As a warm-up, we consider the case where the attention graph is acyclic. As mentioned in Section 3, such an attention graph can describe hierarchical dependencies between agents well. In Observation 4, we show that for this scenario C-GEF-A can be solved in linear time. Although the solution is straightforward (allocating all resources to agents without incoming arcs in the attention graph), Observation 4 provides a good starting point for further studies on detecting more polynomial-time solvable cases.

Observation 4. C-GEF-ALLOCATION for monotonic additive preferences and an acyclic input graph is solvable in linear time.

Proof. For an acyclic directed graph \mathcal{G} , there is always a complete and graph-envy-free allocation that allocates all resources to some arbitrary source agent a^* : In such allocation, no agent can envy some out-neighbor because out-neighbors always get the empty bundle. A source agent (without incoming arcs) can be found in linear time.

³The rows referring to more specific utility functions are omitted, as they are subsumed by row "additive."

As the next step, we show that restricting the preferences to identical 0/1 preferences also makes the corresponding variant of C-GEF-A polynomial-time solvable for an attention graph that is strongly connected. Here, because of transitivity of the "greater than or equal to" relation, we obtain a simple tractable case where all agents must obtain the same number of resources. To show this, we start with the following Observation 5.

Observation 5. Let $\pi: \mathcal{A} \to 2^{\mathcal{R}}$ be a graph-envy-free allocation for the case of identical utility functions. Then, for every pair $\{a, a'\}$ of agents that belong to the same strongly connected component and a (universal) utility function u, it holds that (1) $u(\pi(a)) = u(\pi(a'))$, and (2) $|\pi(a)| = |\pi(a')|$ for 0/1 utility functions.

Proof. Consider an input graph which is a cycle over two agents, a_1 and a_2 . For any graphenvy-free allocation π it must be true that $u(\pi(a_1)) \ge u(\pi(a_2))$ and $u(\pi(a_2)) \ge u(\pi(a_1))$. Thus, $u(\pi(a_1)) = u(\pi(a_2))$. By an inductive argument, the equality holds for every pair of agents in a cycle of any length. Moreover, adding new arcs connecting agents being part of a cycle does not change the situation because the relation "greater than or equal to" is transitive and reflexive. Combining this result with Observation 2, we conclude that we need to give every agent the same number of resources.

It is not hard to see that the proof of Observation 5 in fact yields a simple algorithm solving the variant of C-GEF-A in question.

Corollary 1. C-GEF-ALLOCATION for identical 0/1 preferences and an input graph being strongly connected is solvable in linear time.

Proof. Using Observation 5, our algorithm checks whether the number of resources is divisible by the number of agents and returns true if and only if this is the case. \Box

Contrasting the case of an acyclic attention graph (see Observation 4), restricting preferences to identical 0/1 preferences does not guarantee that the corresponding variant of C-GEF-A becomes polynomial-time solvable in general. We obtain Theorem 1, showing that even with identical 0/1 preferences, GEF-ALLOCATION becomes intractable as soon as the attention graph is not strongly connected, by utilizing the second point of Observation 5. This point allows us to view agents from the same strongly connected component as a "uniform block of agents," which then constitutes an important building block of the proof of Theorem 1.

Theorem 1. C-GEF-ALLOCATION for identical 0/1 preferences is NP-hard even if each vertex has out-degree at most two and it is W[1]-hard for the parameter "number of resources."

Proof. We prove Theorem 1 by giving two very similar many-one polynomial-time reductions from CLIQUE. We first show the general scheme of the reduction and prove its correctness. Then, tailoring the scheme to particular cases, we indeed show the NP-hardness when each vertex has out-degree at most two and the W[1]-hardness for the parameter "number of resources."

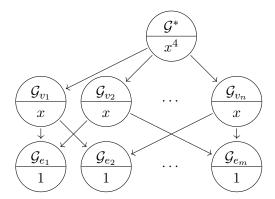


Figure 2: An illustration of the general construction of \mathcal{G} in the proof of Theorem 1. The circles represent strongly connected components. Labels indicate the name (upper part) and the number of agents in the component (lower part). The connections represent arcs between two arbitrarily chosen agents from different components.

Construction Consider a CLIQUE instance formed by an undirected graph G = (V, E) with a set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices and a set $E = \{e_1, e_2, \ldots, e_m\}$ of edges, and clique size k. Without loss of generality, assume that 1 < k < n and $m > \binom{k}{2}$.

We present a polynomial-time many-one reduction from CLIQUE to C-GEF-A using a special variable $x \in \mathbb{N}, x \geq k^2$, that will be defined later in order to show both statements of the theorem. We introduce $x^4 + nx + m$ agents and $x^4 + kx + \binom{k}{2}$ resources which are assigned utility one by each agent. We specify an input graph \mathcal{G} over the agents in two steps. First, we define strongly connected components of \mathcal{G} and then we add arcs connecting them. By connecting two strongly connected components we mean adding an arc between two arbitrarily chosen vertices, one from each connected component. In a first step, we build the following strongly connected components:

- 1. We introduce a root component \mathcal{G}^* which consists of x^4 vertices;
- 2. For each vertex $v \in V$, we introduce a vertex component \mathcal{G}_v which consists of x vertices;
- 3. For each edge $e \in E$, we introduce an *edge component* \mathcal{G}_e which consists of one vertex.

Then, we connect the strongly connected components to form \mathcal{G} of the C-GEF-A instance. Figure 2 depicts graph \mathcal{G} resulting from the following steps:

- 1. For each edge $e = \{v', v''\} \in E$, we connect $\mathcal{G}_{v'}$ and $\mathcal{G}_{v''}$ to edge component \mathcal{G}_e (with an arc pointing to \mathcal{G}_e);
- 2. We connect the root component with every vertex component (with an arc starting at the root component).

For the sake of readability, we extend the concept of envy from agents to sets of agents. We say that a strongly connected component A' envies another strongly connected component A''if there exists an agent from A' that envies an agent from A''. For identical 0/1 preferences, a solution to C-GEF-A has to allocate exactly the same number of resources to every agent being a part of the same strongly connected component (Observation 5 (2)). Thus, we say that we are allocating some number of resources to a strongly connected component (instead of an agent) when we uniformly distribute these resources to the agents that belong to the component.

Correctness We claim that there is a size-k clique in G if and only if there is a complete and graph-envy-free allocation for the constructed C-GEF-A instance. Assume that there is a k-clique $C = (V_C, E_C)$ in graph G. We create a complete and graph-envy-free allocation as follows:

- Give x^4 resources to agents in \mathcal{G}^* , one resource per agent;
- Give x resources to every agent in every vertex component associated with a vertex from V_C , one resource per agent;
- Give one resource to every agent in every edge component associated with an edge from E_C .

The allocation is complete because we assign

$$x^{4} + |V_{C}|x + |E_{C}| = x^{4} + kx + \binom{k}{2}$$

resources. No agent in an edge component has an outgoing arc; hence, by definition, no edge component envies. Every vertex component \mathcal{G}_v , $v \in G$, may envy only edge components it is connected to. If $v \in V_C$, then no vertex in \mathcal{G}_v envies anybody, because every vertex in \mathcal{G}_v has one resource and every vertex of every edge component has at most one resource. If $v \notin V_C$, then v cannot envy because all edge components representing v's incident edges, which are not a part of clique C, have no resource allocated. Finally, the root component does not envy because each of its agents gets one resource and no other agent gets more.

Conversely, assume that there exists a complete and graph-envy-free allocation for the constructed instance of C-GEF-A. Observe that the root component can have no resources if and only if every vertex component has no resources. This in turn is impossible because if each vertex component has no resources, then every edge component also cannot have resources as this would make at least one vertex component envious. Thus, the root component has to get some resources. So, on the one hand, the root component gets at least x^4 resources because it consists of this number of agents. On the other hand, because of a lack of resources, the root component cannot get $2x^4$ resources. This derives from the following calculations using the fact that, by definition, $x \ge k^2 \ge 4$:

$$x^{4} + kx + \binom{k}{2} - x^{4} \le kx + k^{2} \le 2x^{3} < x^{4}.$$

Thus, every agent in the root component gets one resource. Since every agent in the root component might envy all other agents (even all agents in the edge components due to transitivity of the "not less than" relation), every other agent can get at most one resource. Besides the root component's resources, there are still $kx + \binom{k}{2}$ resources left. For every feasible solution there exist exactly k vertex components whose agents have a one-resource bundle. Because

$$(k+1)x = kx + x \ge kx + k^2 > kx + \binom{k}{2},$$

one cannot allocate resources to more than k vertex components. Contrarily, if one allocates x resources to k - 1 vertex components, then there are still $x + \binom{k}{2}$ resources left. However, since each vertex component has an arc to exactly two edge components, there are at most 2(k - 1) < x edge components that can have at most one resource each. Thus, a feasible allocation chooses exactly k vertex components and $\binom{k}{2}$ edge components. Moreover, every vertex component has to be connected to chosen edge components. This exactly corresponds to choosing k distinct vertices and $\binom{k}{2}$ distinct edges such that every edge is incident to two of the chosen vertices.

In the final step of the proof we give concrete values for x in order to show the claims.

- 1. We obtain NP-hardness for C-GEF-A with identical 0/1 preferences and maximum outdegree two setting x := nm. Indeed, since $nm > k\binom{k}{2} = k^2 + \frac{k+1}{2}$, x meets the requirement $x \ge k^2$. In the root component, there are nm^4 agents, which means that it is possible to connect the root component to all vertex components using a different agent from the root component. Thus, the maximum out-degree is two.
- 2. We obtain W[1]-hardness for C-GEF-A with respect to the number of resources by setting $x := k^2$. With such a choice the overall number of resources is depending solely on k, thus the reduction becomes a parameterized reduction and it implies the W[1]-hardness.

Naturally, both choices of x allow for performing the reduction in polynomial time. \Box

The above theorem actually shows more than that identical 0/1 preferences do not guarantee polynomial-time solvability. Indeed, it presents two stronger negative results. C-GEF-A with identical 0/1 preferences presumably cannot be "efficiently" solved even for few resources or even if each agent is allowed to envy at most two other agents.

The remaining hope for positive results (in the form of fixed-parameter tractability) for the case of general attention graphs and identical 0/1 preferences lies in the scenario with few agents. In the following Proposition 1, we show that indeed this variant of C-GEF-A is fixed-parameter tractable with respect to the number of agents. In fact, we show that the fixed-parameter tractability holds also for the case of 0/1 preferences that are not identical.

Proposition 1. C-GEF-ALLOCATION with 0/1 preferences is fixed-parameter tractable with respect to the parameter "number of agents."

Proof. We show that, for the case of 0/1 preferences, the number of variables in the model from Section 3.4 is upper-bounded by a function of the number n of agents in an instance of C-GEF-A. Observe that, for 0/1 preferences, there are at most 2^n different resources types. Thus, the model uses at most $n \cdot 2^n$ variables. Eventually, the result is a consequence of applying the celebrated result of Lenstra [1983] for ILP models with a bounded number of variables.

The intractability results set by Theorem 2 suggest that the restriction that an attention graph must be strongly connected should be kept in further investigations on seeking polynomial-time solvability of C-GEF-A. Thus, we relax the constraints on preferences and we allow for more values than just 1 and 0, turning to the case of identical monotonic additive preferences. However, the first point of Observation 5 opens up a way for a (quite straight-forward) reduction from the NP-hard and W[1]-hard problem EEF EXISTENCE [Bouveret and Lang, 2008]. As a result, the following Proposition 2 shows that C-GEF-A for identical monotonic additive preferences is intractable (in the parameterized sense) for few agents even if each agent has at most one neighbor in the attention graph.

Proposition 2. C-GEF-ALLOCATION for identical monotonic additive preferences is NP-hard and W[1]-hard when parameterized by the number of agents even if the input graph is a cycle.

Proof. We give a polynomial-time many-one reduction from the NP-hard problem EEF EX-ISTENCE with monotonic additive identical preferences studied by Bouveret and Lang [2008]. The problem is to decide whether there exists an envy-free, Pareto-efficient allocation for a given set \mathcal{A} of agents, a set \mathcal{R} of resources, and monotonic additive identical utility functions of the form $u: \mathcal{R} \to \mathbb{N}$. For monotonic additive identical preferences it is enough that an allocation is complete and envy-free to be a solution for EEF EXISTENCE (see Bliem et al. [2016] for a more detailed discussion). To build an instance of GEF-ALLOCATION we take the whole input from the EEF EXISTENCE instance and we add a graph being a cycle over all the agents (in an arbitrary order). To solve EEF-ALLOCATION, every agent has to get an equally valuable bundle which is also the case for the new C-GEF-A instance. The reduction is clearly a parameterized reduction (computable in polynomial time).

Observe that Proposition 2 does not exclude fixed-parameter tractability for few resources (if the preferences are identical). Actually, the following Proposition 3 shows that C-GEF-A is fixed-parameter tractable when parameterized by the number of resources.

Proposition 3. C-GEF-ALLOCATION with identical preferences is fixed-parameter tractable with respect to the number of resources for an input graph being strongly connected.

Proof. Assume an instance of C-GEF-A with identical preferences and a strongly connected envy graph. Let n be the number of agents and m be the number of resources. According to Observation 2, one can withdraw from consideration all zero-valued resources. If there are more agents than there are resources, then the answer for the instance is "no." This is a direct implication of the fact that in the case of a strongly connected attention graph and identical

preferences each agent has to have the same utility and there is at least one resource with a positive value. In the opposite case, we can test all possible n^m allocations. Because $n^m \leq m^m$ and the test for completeness and envy-freeness in a polynomial-time task, we obtain a fixed-parameter algorithm for parameter "number of resources."

We continue our investigations on the computational complexity of C-GEF-A considering the last yet unsettled case; namely, the case of 0/1 preferences and few resources. With the classic envy-freeness notion (or \mathcal{G} being complete for C-GEF-A), finding a complete and envyfree allocation can easily be seen to be fixed-parameter tractable with respect to the number of resources (using an analogous technique as used by Bliem et al. [2016, Proposition 1]). For graph envy-freeness however, the following Theorem 2 shows that the problem becomes W[1]hard even for 0/1 preferences and a strongly connected attention graph. This result provides an example where C-GEF-A, which is fixed-parameter tractable (parameterized by the number of resources) for the case of a complete (directed) attention graph, may become intractable if one deletes some arcs from the attention graph.

Theorem 2. C-GEF-ALLOCATION with 0/1 preferences and an input graph being strongly connected is NP-hard; it is W[1]-hard with respect to the combined parameter "number of resources and maximum out-degree"; it is W[1]-hard with respect to the parameter "number of resources"; and it is NP-hard even if the maximum out-degree of the attention graph is three.

Proof. To prove Theorem 2, we give a polynomial-time many-one reduction from CLIQUE to C-GEF-A. To this end, we first fix notation, then describe the construction, and eventually conclude the argument with proving the construction's correctness.

Let I be a CLIQUE instance formed by an undirected graph G = (V, E) with a set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices and a set $E = \{e_1, e_2, \ldots, e_m\}$ of edges, and a clique size k. Without loss of generality, we assume that 2 < k < n and $m > \binom{k}{2}$. The reduction transfers instance I to and instance I' with agent set \mathcal{A} , resource set \mathcal{R} , utilities \mathcal{U} , and attention graph \mathcal{G} .

Construction We build the set \mathcal{A} of agents of all vertices and edges of G, a set \mathcal{D} of dummy agents, and a set C of k^3 constraint agents. Set \mathcal{D} is the union of n + m groups of k^{10} distinct agents each—one group D(v) per each vertex $v \in V$ and one group D(e) per each $e \in E$. Hence, in total, $|\mathcal{A}| = n(k^{10} + 1) + m(k^{10} + 1) + k^3$. We introduce k vertex resources, $\binom{k}{2}$ edge resources, and k^4 constraint resources; we refer to

We introduce k vertex resources, $\binom{k}{2}$ edge resources, and k^4 constraint resources; we refer to these sets as, respectively, \mathcal{R}_v , \mathcal{R}_e , and \mathcal{R}_c . Additionally, we set apart k (arbitrary) constraint resources that we call distinguished constraint resources and denote them by \mathcal{R}_c^* (naturally, $\mathcal{R}_c^* \subset \mathcal{R}_c$). Then, we let $\mathcal{R} := \mathcal{R}_v \cup \mathcal{R}_e \cup \mathcal{R}_c$, and thus we have exactly $k^4 + \binom{k}{2} + k$ resources.

We proceed with constructing the graph \mathcal{G} step by step. We refer to Figure 3 for a better understanding of the big picture of the construction. First, for each group of dummy agents, we create a subgraph called a *separator gadget*. For each agent $x \in V \cup E$, the separator gadget S(x)is a directed cycle over all agents in D(x). Then, using previously defined separator gadgets, we create one part of \mathcal{G} as follows:

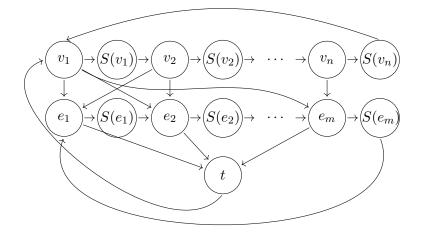


Figure 3: The construction in the proof of Theorem 2.

	V	E	C	\mathcal{D}
\mathcal{R}_{v}	1	0	0	0
$\mathcal{R}_{ ext{e}}$	1	1	0	0
$\mathcal{R}^{st}_{ m c}$	0	1	1	1
$\mathcal{R}_{\mathrm{c}} \setminus \mathcal{R}_{\mathrm{c}}^{\check{*}}$	0	0	1	1

Table 3: The utilities the agents give to the resources in the construction in the proof of Theorem 2.

- 1. for each agent $v_i \in V$, $i \in \{1, 2, ..., n-1\}$, we arbitrarily select two distinct agents x and y from $S(v_i)$ and we create two arcs: (v_i, x) and (y, v_{i+1}) ;
- 2. we select two distinct agents x and y from $S(v_n)$ and add two arcs: (v_n, x) and (y, v_1) .

We proceed analogously with all agents in E. In the next step of constructing the attention graph, for each edge $e = (v_i, v_j) \in E$, we add two arcs to \mathcal{G} : (v_i, e) and (v_j, e) . We conclude the construction by adding a directed cycle over all constraint agents, adding an arc from each $e \in E$ to a distinct, arbitrarily chosen constraint agent, and adding an arc from an arbitrary chosen constraint agent to v_1 .

The final point of the construction deals with the utilities. We use 0/1 utilities as depicted in Table 3.

Correctness We start proving the correctness of the reduction by stating a key lemma about the constraint resources.

Lemma 1. In every graph-envy-free allocation for I', all constraint resources must be given to the constraint agents, one resource per agent.

Proof. We prove the lemma by contradiction. By definition of C-GEF-A, all constraint resources have to be allocated. Towards a contradiction, assume that there exists a graph-envy-free allocation π that gives a constraint resource to some agent $a^* \notin C$. We fix some a^* and we consider the two following cases, both leading to a contradiction.

- 1. Agent $a^* \in \mathcal{D}$. Clearly, a is a part of some separation gadget S(x) that forms a cycle over k^{10} agents. This in turn means that there is an arc $(b, a) \in \mathcal{G}$; thus, agent b has to also get a resource not to envy, which forces another agent, the one preceding b in the cycle, to also get another resource. This "chain reaction" in fact imposes that all k^{10} agents in S(x) need to get a resource. However, there are only $k^4 + {k \choose 2} + k < k^{10}$ resources; hence, we get a contradiction because π cannot be graph-envy-free.
- 2. Agent $a^* \in V \cup E$. Then, there exists some dummy agent b such that there is an arc (b, a). Thus, for b not to envy, it has to get a resource and we arrive at the first case achieving a contradiction.

By our construction, the aforementioned cases are exhaustive and non-overlapping. \Box

Equipped with Lemma 1, we prove the correctness of our reduction. We start with showing that a clique in the original instance implies a "yes"-instance in I' and then we prove that non-existence of a clique in the original existence means that I' is a "no"-instance.

Let $S = \{\hat{v}_1, \hat{v}_2, \dots, \hat{v}_k\}$ be a clique of size k in I, and let $\hat{E} = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_{\binom{k}{2}}\}$ be the set of edges of the clique. Then, a graph-envy-free allocation π for instance I' is constructed as follows.

- 1. The distinguished constraint resources are given to those constraint agents whose incoming arcs come from the agents in \hat{E} ; formally, for each $\hat{e} \in \hat{E}$, if $(\hat{e}, a) \in \mathcal{G}$, $a \in C$, then $\pi(a) = \{r\}, r \in \mathcal{R}^*_{c}$.
- 2. According to Lemma 1, the remaining constraint resources are given to the constraint agents that have not gotten any resource yet, one resource per agent.
- 3. Each agent $\hat{e} \in \hat{E}$ gets a separate edge resource and each agent $\hat{v} \in \hat{V}$ gets a separate vertex resource.

Since the number of edges in clique S and the number of the distinguished constraint resources is the same, according to Steps 1 and 2, allocation π complies with Lemma 1. Observe that the edge agents can only envy agents that got a distinguished constraint resource. Thus, according to Step 1, only edge agents in \hat{E} can be envious. However, in Step 3, each of these agents gets an edge resource; hence, none of them envies. By the construction, an agent $v \in V$ that has no resource envies an agent $e \in E$ such that $\pi(e) \in \mathcal{R}_e$ if and only if $v \in e \in E$; in other words, whenever agent $e = \{v, v'\}$ gets an edge resource, agents v and v' have to get a vertex resource each. Since S is a clique, due to Step 1, it is indeed true that π meets this requirement. Thus, π is complete and graph-envy-free.

Next, assume that I is a "no"-instance, and for the sake of contradiction, suppose that π is a graph-envy-free allocation for I'. Due to Lemma 1 all constraint resources are evenly distributed among the constraint agents. Thus, following from the fact that there are $\binom{k}{2}$ constraint agents with a distinguished constraint resource, there is a set \hat{E} of $\binom{k}{2}$ edge agents that are assigned a single edge resource by π . Let \hat{V} be a set of vertices such that each vertex $\hat{v} \in \hat{V}$ has an outgoing arc pointing to at least one agent in \hat{E} . By the construction of \mathcal{G} , in fact, \hat{V} contains the vertices in G that are incident to edges in \hat{E} . Thus, clearly, $|\hat{V}| \geq k$. However, if π is graph-envy-free, then $|\hat{V}| = k$ because each agent in \hat{V} has to get a vertex resource not to envy and there are k of these resources. In this case, \hat{V} must be a clique, yielding a contraction.

The reduction is computable in polynomial-time. One can easily check that the graph \mathcal{G} is strongly connected and that its maximum out-degree is at most three. Furthermore, the number of resources is a function solely of parameter k.

In our considerations so far, we did not check our results for possible combined parameters. Observe that the negative Theorem 2 in fact proves W[1]-hardness with respect to the combined parameter "number of resources and maximum out-degree" for 0/1 preferences even for a strongly connected attention graph. Similarly, Proposition 2 yields intractability for the combined parameter "number of agents and maximum out-degree" for identical preferences even for a strongly connected attention graph. However, Proposition 1 immediately implies fixed-parameter tractability for all remaining cases for parameter "number of agents and maximum out-degree." Regarding the parameterization by "number of resources and maximum out-degree," Proposition 3 only yields fixed-parameter tractability for strongly connected attention graphs and identical preferences. So, next we cover the remaining cases of general attention graphs and identical or identical 0/1 preferences for this parameterization.

4.1 Few Identical Resources and Small Maximum Out-Degree

We devote this section to present Theorem 3. This positive result intuitively says that, for agents with identical utility functions, if there are few resources and each agent pays attention to relatively few other agents, then C-GEF-A can be solved efficiently. We start with the following Lemma 2^4 about large connected components and then present Theorem 3 together with its proof.

Lemma 2. Consider an instance of C-GEF-ALLOCATION with m identical resources and graph \mathcal{G} . Assume that \mathcal{G} has a strongly connected component \mathcal{C} such that at least one the following holds:

⁴In terms of parameterized algorithmics, Lemma 2 provides a data reduction rule which, however, does not yield a problem kernel. That is, it does not lead to equivalent instances whose sizes can solely be bounded by the parameter (which would directly imply fixed-parameter tractability).

- 1. in the condensation of \mathcal{G} the vertex corresponding to \mathcal{C} has in-degree greater than m or
- 2. component C has more than m agents.

Then, we can compute in polynomial time an equivalent instance of C-GEF-ALLOCATION with attention graph \mathcal{G}' such that \mathcal{G}' is a subgraph of \mathcal{G} and \mathcal{G}' does not contain \mathcal{C} .

Proof. Suppose I is an instance of C-GEF-A that meets the assumptions of Lemma 2. We show that a new, equivalent instance I' of C-GEF-A with a graph \mathcal{G}' that does not contain the strongly connected component \mathcal{C} can be constructed in polynomial time. In order to obtain I', we construct \mathcal{G}' from \mathcal{G} by removing component \mathcal{C} and all strongly connected components reachable with a path from \mathcal{C} .

Due to Observation 5, either all agents in C get at least one resource or none of them gets any. In fact, in both cases of the lemma, all agents in C get no resource. Giving resources to all agents yields an immediate contradiction in Case 2 because of the lack of resources. In Case 1, the lack of resources is also a reason for a contradiction—due to Observation 3, each of the in-neighbors of C has to get at least one resource. So, in both cases no agent in C gets a resource.

Consequently, thanks to non-negative utility values, no agent that has an outgoing arc pointing to an agent in \mathcal{C} can envy anymore; hence, it is safe to eliminate such arcs from \mathcal{G} . Furthermore, again due to Observation 3, no agent reachable from every agent of \mathcal{C} can get a resource. As a result, we can safely remove \mathcal{C} and all components reachable from it. Naturally, such a procedure can be applied in polynomial time, for example, using a modification of breadth-first search.

Theorem 3. C-GEF-ALLOCATION with identical preferences is fixed-parameter tractable with respect to the combined parameter "number of resources and maximum out-degree."

Proof. We give an algorithm that shows the claimed fixed-parameter tractability. A high-level idea of the algorithm is to guess a collection of bundles and the connections of agents that get the bundles. For each such a guess, the algorithm checks whether the guessed situation can be implemented in the input attention graph.

In the proof, we focus on the strongly connected components of an input graph \mathcal{G} , thus we merely use the condensation of an input graph \mathcal{G} . This suffices, since, for identical preferences, the internal structure of strongly connected components does not play any role. Only arcs between distinct strongly connected components are important. So, in the proof, we speak about a *bundle pack*, that is, a collection of bundles given to agents of the same strongly connected components.

We split the algorithm into the following four major steps. Afterwards, we argue about their correctness and running times separately, thus completing the proof. The four steps read as follows.

- 1. Guess a number q of bundle packs and partition the resources into q packs $\{P_1, P_2, \ldots, P_q\}$ (thus, all packs are mutually disjoint and their union contains all resources). We call such a guess a *partial structure* of a solution.
- 2. For each pack P, assign a weight $\rho(P)$ from $\{1, 2, \ldots m\}$ to each of the packs of the partial structure and add arcs between the packs such that the arcs do not create a cycle. We obtain a vertex-weighted directed acyclic graph over the packs that fully describes a solution. Intuitively, each weight represents the number of bundles a pack consists of (and, what is equivalent, the number of agents in a strongly connected component to which the pack is given in the solution). The arcs represent the structure of the strongly connected components to which the packs are given. We call a partial structure with weights and arcs a *structure*.
- 3. Check the sanity of a guessed structure. First, for each pack P in the structure run the algorithm from Proposition 3 feeding it with the resources in P, a clique over $\rho(P)$ agents, and the input (identical) preferences. Then, assuming that the attention graph is the one formed by the structure and that each agent in each strongly connected component of the structure gets a bundle of resources of the same value from the component's pack, check whether the structure describes a graph-envy-free allocation. If at least one of the aforementioned tests fails, then proceed with another guess. Otherwise, continue with the next, final step.
- 4. Check whether there exists a subgraph \mathcal{G}' in the condensation of \mathcal{G} such that \mathcal{G}' has no incoming arcs from vertices outside of \mathcal{G}' and \mathcal{G}' is isomorphic to the graph described by the guessed structure (including weights). If the test is successful, return "yes." Otherwise, proceed with another guess or return "no" if all possible guesses failed.

Correctness. Each structure formed in Steps 1 and 2 describes an allocation where all resources are allocated. Moreover, checking all possible structures exhaustively considers all possible relations (in terms of the attention graph) that can occur between the strongly connected components that are assigned resources.

We next show that Step 3 dismisses a guessed structure if and only if the structure does not describe a graph-envy-free allocation. Let us consider some pack P of the structure and an allocation π' that is graph-envy-free for some strongly connected component C(P). We refer to the agents of the connected component C(P) as A(P) and we define $\rho(P) := |A(P)|$. As a direct consequence of Observation 5 we have that (assuming identical preferences) π' is graphenvy-free for C(P) if and only if it would be graph-envy-free if the relations between agents in A(P) were forming a complete graph. Thus, by dismissing a pack P for which there is no graph-envy-free allocation π' for a clique of agents in A(P), we cannot dismiss a correct solution for the whole problem. Moreover, Observation 5 shows that in such an allocation π' each agent in a connected component gets exactly a bundle of the same value—an equal share. This justifies why we can safely dismiss a guessed structure if, for each pack, allocating an equal share of the resources in the pack to every agent of the pack's component does not lead to a graph-envy-free allocation considering only the relation graph described by the structure's arcs.

We reach Step 4 only if a guessed structure describes a graph-envy-free allocation assuming that there are no other arcs than those in the structure. Let S be the (directed acyclic) graph described by the structure, let \mathcal{G}^* be the condensation of \mathcal{G} , and let \mathcal{G}' be a subgraph of \mathcal{G}^* such that \mathcal{G}' is isomorphic to S. We claim that \mathcal{G}' has no incoming arcs from vertices in \mathcal{G}^* that are outside of \mathcal{G}' if and only if graph \mathcal{G}' describes a structure of a graph-envy-free allocation for the input instance. We prove this claim separately for each direction.

 (\Rightarrow) Suppose that \mathcal{G}' has no incoming arcs from vertices in \mathcal{G}^* that are outside of \mathcal{G}' . Because \mathcal{G}' is isomorphic to \mathcal{S} , there is a graph-envy-free allocation π of the resources to all agents of \mathcal{G}' assuming that we ignore all arcs present in \mathcal{G}^* but not in \mathcal{G}' which are incident to vertices of \mathcal{G}' . Observe that, by definition of \mathcal{G}' , these ignored arcs start in some agent in \mathcal{G}' and end in an agent not in \mathcal{G}' . Since every agent outside of \mathcal{G}' has no resources, these arcs cannot introduce envy, which means that π is also graph-envy-free for the whole input instance.

(\Leftarrow) Suppose that \mathcal{G}' describes a structure of a graph-envy-free allocation π for the input instance. Hence, by definition of a structure of an allocation, no agent outside of \mathcal{G}' gets a resource. So, there is no agent from outside of \mathcal{G}' that has an outgoing arc pointing to an agent from \mathcal{G}' because the existence of such an arc would contradict the graph envy-freeness of π .

Running time of Steps 1 to 3. Because in each solution there are at most m packs, the number of all possible partial structures is upper-bounded by m^m . Subsequently, there are at most $m^m \cdot m^4$ structures. The sanity check consists of at most m invocations of the algorithm from Proposition 3, which runs in FPT-time with respect to parameter m, and a single, polynomial check of the graph envy-freeness property as described in Observation 1. Hence, the first three steps of the algorithm run in $f(m) \cdot poly(|I|)$ time.

Running time of Step 4. It remains to show that Step 4 is computable in FPT-time with respect to the number of resources plus the maximum out-degree Δ of vertices in \mathcal{G} . To this end, we use variants of the SUBGRAPH ISOMORPHISM problem.

Definition 8. In the SUBGRAPH ISOMORPHISM problem, given an undirected host graph H and an undirected pattern graph G, the question is whether there is a subgraph H' of H isomorphic to G. When graphs G and H are colored and the isomorphism has to be color-preserving, then we obtain COLORED SUBGRAPH ISOMORPHISM. Additionally, if the graphs are directed, we arrive at DIRECTED (COLORED) SUBGRAPH ISOMORPHISM.

The problem we need to solve in Step 4 is very similar to DIRECTED SUBGRAPH ISOMORPHISM if we take the condensation of \mathcal{G} as a host graph and the guessed structure as a pattern graph. A subtle difference is that we need to ensure that a subgraph \mathcal{G}' being isomorphic to the guessed structure has no arcs incoming from outside of \mathcal{G}' . Although originally DIRECTED SUBGRAPH ISOMORPHISM does not obey this constraint, we simulate it by appropriate coloring of the condensation of \mathcal{G} and the guessed structure followed by solving DIRECTED COLORED SUBGRAPH ISOMORPHISM.

We color each vertex v, representing a pack P of the guessed structure, with in-degree deg⁻(v) with a color $\lambda_{s}(v) := \rho(P)m + \text{deg}^{-}(v)$. Similarly, we color each vertex v with in-degree deg⁻(v) in the condensation graph of \mathcal{G} , representing a strongly connected component with ℓ agents, with a color $\lambda_{c}(v) := \ell m + \text{deg}^{-}(v)$. Then, we solve DIRECTED COLORED SUBGRAPH ISOMOR-PHISM for such transformed graphs. Observe that each guessed structure has at most m packs, we have at most $O(m^2)$ colors in the transformed graphs, and the maximum degree of the condensation of \mathcal{G} is at most ($\Delta + 1$)m. The running time follows from (formally stated and proved below) Lemma 3, which shows fixed-parameter tractability of DIRECTED COLORED SUBGRAPH ISOMORPHISM for our case of directed acyclic graphs (recall that we considered the condensation of \mathcal{G}) with respect to the number of vertices in the pattern graph, the number of colors in the input graphs, and the maximum degree of the host graph.

We are now turning to formally expressing and proving Lemma 3—an important part of the proof of Theorem 3 that exploits a relation between DIRECTED COLORED SUBGRAPH ISO-MORPHISM and SUBGRAPH ISOMORPHISM. This relation allows us to reduce in parameterized sense a specific variant of DIRECTED COLORED SUBGRAPH ISOMORPHISM to SUBGRAPH ISO-MORPHISM and then use known techniques of efficiently (in parameterized sense) solving the latter problem (a paper by Marx and Pilipczuk [2014] outlines a flurry of such techniques;). We point out that a similar idea to ours has been already applied for SUBGRAPH ISOMORPHISM in a different context of fixing images of prescribed vertices (see Lemma 2.6 by Marx and Pilipczuk [2013]⁵). However, we are not aware of any work concerning the specific relation that we present in Lemma 3.

Lemma 3. For directed acyclic graphs, DIRECTED COLORED SUBGRAPH ISOMORPHISM is fixed-parameter tractable with respect to the combined parameter "the number of vertices in the pattern graph, the number of distinct colors in the input graphs, and the maximum degree of the host graph."

Proof. A general strategy of the proof is to show a parameterized many-one reduction from DI-RECTED COLORED SUBGRAPH ISOMORPHISM to SUBGRAPH ISOMORPHISM and then to use an appropriate result for the latter.

Construction Let $G^0 = (V(G^0), E(G^0), \lambda_{G^0})$ and $H^0 = (V(H^0), E(H^0), \lambda_{H^0})$ be directed colored graphs—a pattern graph and a host graph, in that order, with respective vertex-coloring functions λ_G and λ_H —forming an instance I of DIRECTED COLORED SUBGRAPH ISOMOR-PHISM. In addition to describing the following construction, we also illustrate it step-by-step in Figure 4.

We first transfer G^0 and H^0 to undirected graphs subdividing each arc by adding two special colors c_{beg} and c_{end} marking, respectively, the beginning of an arc and the end of an

⁵Only the referenced long version of the paper by Marx and Pilipczuk [2014] contains the relevant lemma.

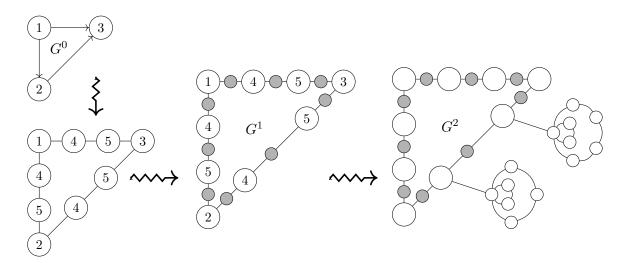


Figure 4: The step-by-step construction of G^2 from G^0 described in the proof of Lemma 3. The unnamed graph shows an auxiliary step helping to visualize the construction. The numbers inside the vertices represent colors: $c_{\text{beg}} = 4$, $c_{\text{end}} = 5$. The dummy vertices are colored gray. For clarity, the bulbs are only demonstrated for two vertices; in fact, every non-dummy vertex has its own bulb constructed.

arc. This transformation allows us to encode the directions of arcs. Specifically, for each arc $e = (u, v) \in E(G^0) \cup E(H^0)$, we add two vertices u' and v', and replace e with three edges $\{u, u'\}, \{u', v'\}, \{v', v\}$, setting the colors of u' and v' to c_{beg} and c_{end} respectively. In the resulting (now undirected) graph, for each edge $e = \{u, v\}$, we add a new vertex x, two edges $\{u, x\}$ and $\{x, v\}$, and delete edge e. We color the new vertex x with a new color c_{\emptyset} that we refer to as the *void color*; we refer to all vertices colored with the void color as the *dummy* vertices and refer to the corresponding vertex-set as V_{\emptyset} . We refer to the resulting graphs and functions as G^1 , H^1 , λ_{G^1} , and λ_{H^1} respectively. For an example of the transformation from G^0 to G^1 see Figure 4.

Second, we transfer G^1 and H^1 to uncolored graphs. Intuitively, we encode each vertex' color with the *bulb* gadget. The bulb gadget of color i > 0 consists of a cycle of length 3 and a cycle of length 3 + i; the two cycles have exactly one common vertex called the *foot*. For each vertex $v \in V(G^1) \cup V(H^1)$ with color $i \neq c_{\emptyset}$, we create a copy of the respective bulb gadget and connect v to the bulb gadget's foot. Note that we did not create any bulb gadgets for the dummy vertices. This final transformation (see Figure 4 for an illustration), together with neglecting the coloring functions, gives us undirected, uncolored graphs G^2 and H^2 forming an instance I' of SUBGRAPH ISOMORPHISM. The reduction runs in polynomial time.

Correctness. We show that there is a directed, colored subgraph $\widehat{H^0}$ of H^0 that is isomorphic to G^0 respecting colors if and only if there is an undirected, non-colored subgraph $\widehat{H^2}$ of H^2

isomorphic to G^2 .

Having $\widehat{H^0}$, we apply the same transformations as those applied to G^0 and to H^0 obtaining $\widehat{H^2}$. It is clear that $\widehat{H^2}$ is a subgraph of H^2 and that it is isomorphic to G^2 .

For the reverse direction, suppose we have $\widehat{H^2}$, which is a subgraph of H^2 and is isomorphic to G^2 via an isomorphism $\eta: V(G^2) \to V(\widehat{H^2})$. In two steps, we will show how to transform $\widehat{H^2}$ to a directed colored $\widehat{H^0}$ which is a subgraph of H^0 and which is (simultaneously) isomorphic to G^0 . We say that a vertex is *adjacent to a bulb gadget* if the vertex is adjacent to the foot of the gadget. Similarly, we say that a vertex is *adjacent to a cycle* if the vertex is adjacent to exactly one vertex of this cycle.

First, we "bring back" colors of $\widehat{H^2}$ obtaining an interim graph $\widehat{H^1}$. Consider some vertex $v \in V(\widehat{H^2}) \cap V(H^1)$ that originally had color $i := \lambda_{H^1}(v)$ and vertex $u \in V(G^2) \cap V(G^1)$ originally colored to $j := \lambda_{G^1}(u)$. We show that $\eta(u) = v$ if and only if i = j. To this end, we distinguish two cases depending on whether $j = c_{\emptyset}$.

Case of $j \neq c_{\emptyset}$. By construction, u is adjacent to its respective bulb gadget in G^2 . We show that if $\eta(u) = v$, then v is also adjacent to a bulb of color j; from this it follows that i = j. First observe that no vertex $x \in V(H^2) \cap V(H^1)$ such that $\lambda_{H^1}(x) \neq c_{\emptyset}$ is part of a cycle of length 3. If it were the case, then there would be an edge connecting two dummy vertices or an edge connecting a dummy vertex with the foot of some bulb gadget; by construction, there are no such edges. Since every dummy vertex has only neighbors that are not dummy vertices (and, what we have just shown, these neighbors cannot be part of a cycle of length three), then no dummy vertex is adjacent to a vertex of a cycle of length three. As a result, v cannot be a dummy vertex and thus v is (by construction) adjacent to a bulb. Note that v's neighbors, by definition, are only some dummy agents and the foot of v's bulb. Since every dummy agent (by construction) has degree two and all feet have degree exactly 5, no dummy agent can play the role of a foot. Thus, v is adjacent only to the foot of its own bulb. If v is mapped to u, then it must be the case that both u and v are connected to a bulb of the same color. This holds true because a cycle of length ℓ is not isomorphic to any cycle of length $k \neq \ell$ (nor its subgraph). Thus, the construction of bulbs implies that the colors of u and v are the same, that is, i = j. Case of $j = c_0$. By assumption, u is a dummy vertex, so, according to the construction, u has two neighbors and none of them is a dummy vertex. We already know that every non-dummy vertex is mapped to another non-dummy vertex with the same color; hence both neighbors of uare mapped correctly. Thus, u has to be mapped to a dummy vertex. So, v is a dummy vertex and $i = j = c_{\emptyset}$. Eventually, since we know that the vertices are mapped correctly with respect to their colors, we get $\widehat{H^1}$ by coloring each vertex in $V(\widehat{H^2}) \cap V(H^1)$ with its respective color and removing all bulbs. Note that $\widehat{H^1}$ is a subgraph of H^1 (which is essentially H^2 with proper colors instead of bulbs) and it is isomorphic to G^1 (being just G^2 with proper colors instead of bulbs).

The final step is to transform $\widehat{H^1}$ to a subgraph $\widehat{H^0}$ of H^0 isomorphic to G^0 . To achieve this, we first remove every dummy vertex in $\widehat{H^1}$ by adding an edge between its neighbors (by the construction each dummy vertex has exactly two neighbors). In the second step substitute all paths of form $\{u, x, y, v\}$ in $\widehat{H^1}$ with x colored to c_{beg} and y colored to c_{end} . Such paths, by our construction, are non-overlapping and encode that there is an arc (u, v) in H^0 . Performing all substitutions we achieve $\widehat{H^0}$. Because the above procedure is well-defined, it is clear that $\widehat{H^1}$ is isomorphic to G^1 if and only if $\widehat{H^0}$ is isomorphic to G^0 .

Let q be the number of different colors the input graphs are colored with. Applying our reduction, we arrive at an instance I' in which the pattern graph G^2 has at most $f(E(G^0), q) := |E(G^0)|(2+3+2q+12)+|V(G^0)|(q+6) \in O(|V(G^0)|^2q)$ vertices. Furthermore, for Δ being the maximum degree of H^0 , we obtain the host graph H^2 having maximum degree at most $\Delta + 1$. Due to a result of Cai et al. [2006, Theorem 1], SUBGRAPH ISOMORPHISM is fixed-parameter tractable for the combined parameter "number of vertices of the pattern graph and maximum degree of the host graph," which yields the result.

The proof of Lemma 3 complements the proof of Theorem 3. We, however, emphasize that the ideas we applied, are only sufficient to provide a classification results. It remains open to further improve the FPT running time in order to obtain a practically relevant efficient algorithm solving the considered case of C-GEF-A.

Theorem 3 concludes our analysis of the computational complexity of C-GEF-A (recall Table 2 for a compact overview of the results we obtained).

5 Finding Strongly Graph-Envy-Free Allocations

We move on to the strong variant of our envy-freeness concept and analyze how this stronger notion affects the computational complexity—our findings are summarized in Table 4. We start our discussion with those restrictions of C-sGEF-A that result in efficiently solvable variants of the problem; specifically, we restrict the utility functions to be identical and the attention graph to be strongly connected. At the beginning, it might come a bit as a surprise that the computationally simplest case is not the one of acyclic attention graphs (recall that the case of identical utility functions and strongly connected attention graphs was NP-hard for C-GEF-A) but the one with cyclic attention graphs. Indeed, for identical utility functions and attention graphs containing a cycle, the "greater than" relation required by strong graph envy-freeness forms a cycle in which, by transitivity, we get a paradox: "each agent is required to value its own bundle more than it values its own bundle." By this, we immediately arrive at a trivial impossibility, which we present formally in the observation below.

Observation 6. Let \mathcal{G} be a graph that contains a strongly connected component with more than one vertex. Then, there is no strongly graph-envy-free allocation if the agents have identical preferences.

 $^{^{6}}$ The results for identical 0/1 preferences in this case are subsumed by the results presented in row "id."

	$\operatorname{preference}$ type	es		parameterization	1	
		#agents	#resources	outdegree	$\# { m agents} + { m outdegree}$	#resources + outdegree
	directed acyclic					
gra	id. $0/1$	P (Pr. 4)	P (Pr. 4)	P (Pr. 4)	P (Pr. 4)	P (Pr. 4)
	id.	FPT (Pr. 6)	FPT (Th. 4)	p-NP-h (Pr. 5)	FPT (Pr. 6)	FPT (Th. 4)
	0/1	FPT (Cor. 2)	FPT (Th. 4)	p-NP-h (Pr. 7)	FPT (Cor. 2)	FPT (Th. 4)
	additive	W[1]-h (Th. 5)	FPT (Th. 4)	p-NP-h (Th. 5)	W[1]-h (Th. 5)	FPT (Th. 4)
	strongly connected ⁶					. ,
	id.	O(1) (Obs. 6)	O(1) (Obs. 6)	O(1) (Obs. 6)	O(1) (Obs. 6)	O(1) (Obs. 6)
	0/1	FPT (Cor. 2)	FPT (Th. 4)	p-NP-h (Pr. 7)	FPT (Cor. 2)	FPT (Th. 4)
attention	additive	W[1]-h (Th. 5)	FPT $(Th. 4)$	p-NP-h (Th. 5)	W[1]-h (Th. 5)	FPT $(Th. 4)$
tte	general					
9	id. 0/1 id. 0/1 additive	P (Pr. 4) FPT (Pr. 6) FPT (Cor. 2) W[1]-h (Th. 5)	P (Pr. 4) FPT (Th. 4) FPT (Th. 4) FPT (Th. 4)	P (Pr. 4) p-NP-h (Pr. 5) p-NP-h (Pr. 7) p-NP-h (Th. 5)	P (Pr. 4) FPT (Pr. 6) FPT (Cor. 2) W[1]-h (Th. 5)	P (Pr. 4) FPT (Th. 4) FPT (Th. 4) FPT (Th. 4)

Table 4: Parameterized complexity of C-SGEF-ALLOCATION. The results are grouped by three criteria regarding the problem input: the structure of the attention graph, the preference type, and the parameterization. All hardness results also imply classical NP-hardness.

Proof. By definition, there is a cycle in every strongly connected graph with more than one vertex. Let us arbitrarily choose some agent a from the cycle. Let us call its predecessor as a_p . Now, by the definition of strongly graph-envy-free allocation and transitivity of the "greater than" relation, we have that $u(\pi(a)) > u(\pi(a_p))$ and $u(\pi(a_p)) > u(\pi(a))$ —a contradiction. \Box

Next, we present Algorithm 1 which, applying Observation 6, finds a complete, strongly graph-envy-free allocation for the case of identical 0/1 preferences and arbitrary input graphs.

Proposition 4. C-SGEF-ALLOCATION for identical 0/1 preferences can be solved in linear time.

Proof. By Observation 2, without loss of generality we know that we can assume that there are no resources with value zero. Hence, in Algorithm 1 we safely assume that every resource is assigned utility one by every agent.

Algorithm 1 first checks whether the graph either consists of only one vertex or contains a cycle. If the former is true, then it is enough to give it all the resources to obtain a complete and strongly graph-envy-free allocation. If the input graph contains a cycle, then by Observation 6 no feasible allocation exists.

Giving resources to some agent with zero in-degree cannot break strong graph envy-freeness. Thus, the task reduces to finding a strongly graph-envy-free (possibly incomplete) allocation π that guarantees strong graph envy-freeness for all agents, and then to distribute the remaining Algorithm 1: Let \mathcal{R} be a set of resources, let \mathcal{A} be a set of agents such that every agent assigns the preference value of one to every resource, and let $\mathcal{G} = (\mathcal{A}, E)$ be a directed graph.

 $\begin{array}{l} \mathbf{if} \ |\mathcal{A}| = 1 \ \mathbf{then} \\ | \ \text{Allocate all resources to the single vertex; } \mathbf{return;} \\ \mathbf{if} \ There \ exists \ a \ cycle \ in \ \mathcal{G} \ \mathbf{then} \\ | \ \text{No allocation is possible; } \mathbf{return;} \\ \text{Build a graph } G' = (\mathcal{A} \cup \{v_s\}, E') \ \text{where} \\ E' = \{(u, v) : (v, u) \in E\} \cup \{(v_s, u) : u \in \mathcal{A} \land |N_G(u)| = 0\}; \\ \text{Assign to every vertex } w \in V \ \text{a label } \ell(w) \ \text{being the length of the longest path from } v_s \\ \text{decreased by one;} \\ \mathbf{if} \ |\mathcal{R}| \ge \sum_{w \in W} \ell(w) \ \mathbf{then} \\ | \ \text{Assign the remaining resources from } \mathcal{R} \ \text{to every agent } w \in V; \\ \text{Assign the remaining resources to arbitrary agents with zero in-degree in graph } \mathcal{G}; \\ \mathbf{return;} \\ \text{No allocation is possible; } \mathbf{return;} \\ \end{array}$

resources to agents with zero in-degree. An allocation π should, naturally, use as few resources as possible. Algorithm 1 finds such an allocation π building graph G' and assigning every agent $w \in V$ a label $\ell(w)$. Label $\ell(w)$ is the minimal number of resources that w has to get in a way to achieve a strongly graph-envy-free allocation with the smallest number of resources. We make an inductive argument based on the label value assigned by the algorithm to prove this claim. Let us focus on the input graph \mathcal{G} . Since the agents in \mathcal{A} with label 0 are sinks, it is clear that giving them no resources never violates strong graph envy-freeness. Let us consider some label value x > 0 and an agent $a \in \mathcal{A}$ with $\ell(a) = x$. Because x is the length of a path from a to the furthest sink, there exists an arc (a, a') such that agent a' has label x - 1 and gets a bundle of at least x - 1 resources. Thus, indeed, agent a has to get at least x resources to achieve a strongly graph-envy-free allocation.

Using the breadth-first search, we can assign the labels to the agents and check whether a graph contains a cycle in linear time. Since the same holds for our procedure of building the auxiliary graph G', Algorithm 1 works in linear time.

The efficient solvability settled by Proposition 4 heavily depends on the identical 0/1 preferences. Indeed, in the following Proposition 5 we show that C-sGEF-A becomes intractable for identical preferences in the cases of acyclic attention graphs (which stands in contrast to C-GEF-ALLOCATION that is always solvable in polynomial time if the attention graph is acyclic) and general attention graphs. Reducing from the NP-hard UNARY BIN PACKING [Jansen et al., 2013], we mainly use the fact that in a (directed) path over k agents, the first agent has to get a bundle with utility at least k - 1. **Proposition 5.** C-SGEF-ALLOCATION with identical monotonic additive preferences is NPhard even if the input graph is acyclic and the maximal out-degree is one.

Proof. We give a parameterized reduction from UNARY BIN PACKING [Jansen et al., 2013] where, for a given multiset of integer item sizes encoded in unary, a bin size b, and the maximal number k of bins, the question is whether it is possible to partition the items into at most k bins with capacity b.

Let $I = \{S, b, k\}$ be an instance of UNARY BIN PACKING, where $S = \{s_1, s_2, \ldots, s_n\}$ and $S = \sum_{i=1}^n s_i$. Without loss of generality we assume that $S = k \cdot b$. We create an instance of sGEF-ALLOCATION with the following input: The set of resources is $R = \{r'_1, r'_2, \ldots, r'_b\} \cup \{r_1, r_2, \ldots, r_n\} \cup \{r^*_1, r^*_2, \ldots, r^*_k\}$ and the set of agents is \mathcal{A} , containing bin agents $\{a_1, a_2, \ldots, a_k\}$, dummy agents $\{a'_1, a'_2, \ldots, a'_b\}$, and k special agents $a^*_1, a^*_2, \ldots, a^*_k$. To form the graph G describing the agents' relations, we first build a (directed) path $(a'_b, a'_{b-1}, \ldots, a'_1)$ through the dummy agents. Then, we create an arc from every bin agent to a'_b . Finally, for each $i \in [k]$, we create an arc from a^*_i to a_i . For $i \in [b]$, we set the value of each r'_i to be i - 1. For $i \in [n]$, we set the value of r_i to be s_i . The value of the special resources is set to b + 1.

According to graph G, for an allocation to be strongly graph-envy-free, each dummy agent $a_i, i \in [b]$, has to get resources of total value at least i - 1. This implies that all bin agents have to achieve a utility of b, and each special agent has to get resources valued at least b + 1. This means that the minimal value of allocated resources is exactly $\sum_{i \in [b]} (i-1) + S + k(b+1)$. However, the sum of utilities of all resources is exactly the same. This means that one can only allocate the resources achieving strict \mathcal{G} -envy-freeness if one can allocate resources representing the items to pack \mathcal{S} to bin agents.

The reduction clearly works in polynomial-time, thus proving NP-hardness, and no agent node has out-degree higher than one. $\hfill \Box$

Observe that the NP-hardness from Proposition 5 holds even if each agent can envy at most one another agent. On the positive side, C-sGEF-A with identical utility functions turns out to be fixed-parameter tractable with respect to the number of agents. In the following Proposition 6, we show this by combining a brute-force approach with solving the ILP model of C-sGEF-A.

Proposition 6. C-SGEF-ALLOCATION with identical preferences is fixed-parameter tractable with respect to the number of agents.

Proof. The proof distinguishes two cases depending on whether there are more differently valued resources than agents. It turns out that if this is the case, then we can efficiently brute-force a given instance. Otherwise, we employ the ILP model from Section 3.4 and show that the number of variables is upper-bounded by the number of agents obtaining fixed-parameter tractability [Lenstra, 1983].

Consider an instance of C-SGEF-A with an attention graph \mathcal{G} , a set \mathcal{R} of resources, and the number u_{diff} of different utility values assigned to \mathcal{R} by the agents of the instance. Since checking whether a graph is acyclic can be done in polynomial time, thanks to Observation 5 we can assume without loss of generality that \mathcal{G} has no cycles.

Let us first consider the case in which $u_{\text{diff}} > |\mathcal{A}|$. Since \mathcal{G} is a directed acyclic graph, it has a topological ordering \succ , which can be computed in polynomial time. Furthermore, there clearly exists at least one agent a^* with no incoming arc in \mathcal{G} . These observations allow us to find a strongly graph-envy-free allocation in two, polynomial-time executable, steps. First, using the fact that $u_{\text{diff}} > |\mathcal{A}|$, we select exactly $|\mathcal{A}|$ differently valued resources and distribute them such that if $a \succ a'$, then a gets more valuable resource. The second step is to give all remaining resources to a^* . Because \succ is a topological ordering, for every arc (a, a') it holds that $a \succ a$. Since a has, by definition, a resource with greater value than that of a', a does not envy a'.

Let us first consider the case where $u_{\text{diff}} \leq |\mathcal{A}|$. In the ILP model from Section 3.4 the number of variables is exactly a product of the number of agents and the number of different possible resource types. Recall that a type of some resource is defined as a vector of utility values given to the resource by all agents. Thus, in our case of identical utility functions, a resource type boils down to a single number that is the utility given by the agents to a particular resource. By the assumption of this case, the number of different utilities given to the resources by agents is upper-bounded by $|\mathcal{A}|$. Eventually, the whole number of variables in the ILP model from Section 3.4 is upper-bounded by $|\mathcal{A}|$ which gives us fixed-parameter tractability by Lenstra's result [Lenstra, 1983].

Continuing good news, we provide a positive result for the case with few resources. Specifically, in Theorem 4, we show that C-sGEF-A is fixed-parameter tractable with respect to the number of resources, independently of the attention graph structure and the preference type.

Theorem 4. C-SGEF-ALLOCATION is fixed-parameter tractable with respect to the number of resources.

Proof. At the beginning, let us observe that we can divide vertices of every directed graph into three groups. Group G_{s} consists of *sources*, that is vertices with in-degree zero. Group G_{t} consists of *sinks*, that is vertices with out-degree equals zero. All other vertices belong to group G_{i} . Observe that no vertex belongs to $G_{s} \cap G_{t} \cap G_{i}$ but there might be vertices that belong to $G_{t} \cap G_{s}$.

We denote the number of agents, which are represented by vertices, by n and the number of resources by m. We distinguish five different cases and for each them provide an algorithm to solve it:

- 1. $m \ge n$. Check all possible allocations of the resources to the agents. Since there are at most n^m possible allocations, by the assumption that there are at least as many resources as agents, we obtain that the number of possible allocations is upper-bounded by m^m .
- 2. $m < |(G_{s} \cup G_{i}) \setminus G_{t}|$. There is no strongly graph-envy-free allocation because each agent from $(G_{s} \cup G_{i}) \setminus G_{t}$ has to get at least one resource.

- 3. $m = |(G_{s} \cup G_{i}) \setminus G_{t}|$. Check all possible O(m!) allocations.
- 4. $|(G_{\mathfrak{s}} \cup G_{\mathfrak{i}}) \setminus G_{\mathfrak{t}}| < m < n \text{ and } G_{\mathfrak{s}} \neq \emptyset$. Because there is at least one source (which can get arbitrarily many resources), every sink agent might get no resources. This leads to the possibility to ignore the sink agents and to check all $|(G_{\mathfrak{s}} \cup G_{\mathfrak{i}}) \setminus G_{\mathfrak{t}}|^m$ allocations in FPT-time (with respect to m, as in case 1).
- 5. $|(G_{s} \cup G_{i}) \setminus G_{t}| < m < n$ and $G_{s} = \emptyset$. First, observe that when $G_{s} = \emptyset$, then, actually, the condition for this case gives us that $|G_i| < m$ (recall that $G_i \cap G_t = \emptyset$ by their definitions). In this case, unlike in Case 4, we cannot easily ignore the sink agents because there might be scenarios in which these agents have to get some resources. Consider some sink agent $t \in G_t$ and let N(t) be the set of agents from which there is an arc to t. We want to describe all sinks $t \in G_t$ by their respective sets N(t); hence, for some sink $t \in G_t$, we call the set N(t) a type of t. Observe that there are at most $2^{|G_i|} < 2^m$ different types of sink vertices. This means that without exceeding the requested FPT-time, we can guess which resources will be allocated to each agent in G_i and which resources will be allocated to the sink agents of a certain type (there are, respectively, at most O(m) and $O(m^{2^m})$ such guesses). Clearly, if there is a strongly graph-envy-free allocation, then it is described by such a guess. To see how to proceed with a guess, let us fix an arbitrary one. We show that using this guess, we are able to guess a strongly graph-envy-free allocation (if it exists) in the requested FPT-time. The clue is to correctly allocate the guessed resources to the given sets of sink vertices of different types. Observe that for all sink agents of the same type we can ignore their utility functions, as the sink agents cannot envy (they have no outgoing arcs). Moreover, all sink agents of the same type have incoming arcs from exactly the same inner vertices, which makes the sink agents indistinguishable. Furthermore, there are at most m resources, and thus we have to distribute the guessed resources for the sink agents of the particular type to at most m of these sink agents. As a result, for each type, we can check all possible allocations of the guessed resources for this type to at most m arbitrarily selected sink agents of this type. There are at most 2^m types and for each of them we have at most $O(m^m)$ possible guesses of how to distribute resources within this type.

Note that for particular special types of graphs, the algorithm simplifies. If we restrict the input graph to acyclic graphs, then there is always at least one source vertex. For strongly connected graphs there are no sources and sinks, so Case 4 vanishes and Case 5 simplifies greatly. \Box

Propositions 5 and 6 are specific to the case of identical preferences. Thus, they do not cover the general case of 0/1 preferences. Indeed, the following Proposition 7, using a reduction from CLIQUE, shows that C-SGEF-A remains hard also in the case of 0/1 preferences. The result holds even for acyclic attention graphs.

Proposition 7. C-SGEF-ALLOCATION with 0/1 preferences is NP-hard for an input graph being either strongly connected or acyclic even if the maximal out-degree is three.

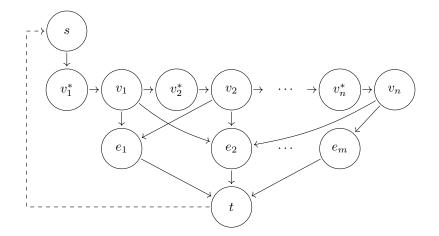


Figure 5: The construction in the proof of Proposition 7. The dashed arc holds for the case of an input graph being strongly connected.

Proof. We prove Proposition 7 by giving a polynomial-time many-one reduction from CLIQUE. We first present the reduction for C-sGEF-A with an acyclic attention graph. Then we massage the reduction to cover the case of an attention graph being strongly connected.

Let I = (G, V, k) = be a CLIQUE instance formed by an undirected graph G = (V, E) with a set $V = \{v_1, v_2, \ldots, v_n\}$ of vertices and a set $E = \{e_1, e_2, \ldots, e_m\}$ of edges, and a target clique size k. Without loss of generality, assume that 1 < k < n and $m > \binom{k}{2}$.

We construct an instance I' of C-sGEF-A associating each vertex and edge of G with an agent, adding *n* separating agents, and the source and sink agents, respectively s, t. Formally, we have $\mathcal{A} := V \cup E \cup \{v_1^*, v_2^*, \ldots, v_n^*\} \cup \{s, t\}$. The following steps describe the construction of graph \mathcal{G} ; it is depicted in Figure 5.

- 1. For each edge $e = \{v, v'\} \in G$, add three arcs, (v, e), (v', e), and (e, t), to \mathcal{G} .
- 2. For each vertex $v_i \in V$, $i \in [|V|]$, add arc (v_i^*, v_i) to \mathcal{G} .
- 3. For each vertex $v_i \in V$, $i \in [|V| 1]$, add arc (v_i, v_{i+1}^*) to \mathcal{G} .
- 4. Add arc (s, v_1^*) .

Then, we construct m edge resources and n + k vertex resources. We arbitrarily split the edge resources into two sets: set \mathcal{R}^- of $m - \binom{k}{2}$ edge resources and set \mathcal{R}^+ of $\binom{k}{2}$ edge resources referring to the latter ones as distinguished edge resources. We denote the set of vertex resources as \mathcal{R}^{\bullet} . Furthermore we add n separating resources, using \mathcal{R}^{\parallel} to refer to them, and a special resource r^{\triangle} . We define the utilities of the resources in Table 5. Naturally, the construction is polynomial-time executable.

To show that the above reduction is correct and finish the proof, we state the following lemma.

Lemma 4. In an allocation π for instance I', the vertex agents, the edge agents, and agent s are not envious if and only if

1.
$$r^{\triangle} \in \pi(s)$$
,

2.
$$\forall i \in [n]: |\pi(v_i^*) \cap \mathcal{R}^{\parallel}| = 1, and$$

3.
$$\forall e \in E : |\pi(e) \cap (\mathcal{R}^- \cup \mathcal{R}^+)| = 1.$$

Proof. We prove the lemma for the two directions separately.

 (\Rightarrow) Claim 1 holds because agent s, which has an outgoing arc, gives a positive utility only to r^{\triangle} . Hence, s has to get r^{\triangle} . Similarly, there is the same number of separating resources and separating agents, which have an outgoing arc each. The separating resources are the only resources to which the separating agents assign positive utility; thus, each separating agent has to get a separating resource, which is exactly what Claim 2 formalizes. The very same argument for the edge agents yields Claim 3.

 (\Leftarrow) Reusing the argumentation from the opposite direction, it is immediate that allocation π does not introduce envy if we neglect the vertex resources. However, observe that the vertex resources are given utility zero by every agent except for the vertex agents. Thus, no matter how we allocate the vertex resources, all non-vertex agents remain unenvious.

In other words, Lemma 4 says that each complete strongly graph-envy-free allocation for I' gives all separating resources to the separating agent, one resource per agent; gives all edge resources to the edge agents, one resource per agent; and gives r^{Δ} to agent s. (Note that the lemma does not specify how to allocate the vertex resources.) Using this convenient (partial) characterization of solutions to I', in the following, we prove correctness of the reduction.

Suppose $C = (V_C, E_C)$ is a clique of size k in G. We construct a complete strongly graphenvy-free allocation π for instance I'. We allocate the special resource to s, the separating resources to the separating agents (a resource per agent), and the edge resources to the edge agents such that each $e \in E_C$ gets a distinguished edge resource. To each agent $v \in V$, we allocate two vertex resources if v belongs to V_C , or else we allocate one vertex resource to v. Clearly, all resources are allocated. Thanks to Lemma 4, it remains to show that no vertex agent is envious. Observe that if $v \in V_C$, then v gives its own bundle value two. As a result, v cannot envy because the only arcs it has are arcs pointing to some edge agents and each of them has a single resource. So, towards a contradiction, let $v \in V \setminus V_C$ be an envious vertex agent. Since v gets one resource valued one, v can only envy an edge agent with a distinguished resource. Thus, let $e \in E_C$ be such an agent to which v is pointing. This means that edge e belongs to clique C but v does not belong to this clique; a contradiction.

For the opposite direction suppose that π is a solution to I'. Due to Lemma 4, we know that there are exactly $\binom{k}{2}$ edge agents with a distinguished resource; we refer to them as *selected edge agents*. By the construction, π has to give each vertex agent at least one vertex resource and every vertex agent that is pointing to a selected edge agent two vertex resources; we call

	s	V	E	t	$v_1^*, \ldots v_n^*$
r^{\bigtriangleup}	1	0	0	0	0
\mathcal{R}^-	0	0	1	0	0
\mathcal{R}^+	0	1	1	0	0
$ \begin{array}{c} \hline r^{\triangle} \\ \mathcal{R}^{-} \\ \mathcal{R}^{+} \\ \mathcal{R}^{\parallel} \\ \mathcal{R}^{\bullet} \end{array} $	0	0	0	0	1
\mathcal{R}^{ullet}	0	1	0	0	0
$r^{ abla}$	0	0	0	1	0

Table 5: Utilities of resources in the proof of Proposition 7. Resource r^{∇} is needed only to prove Proposition 7 for an input graph being strongly connected.

the latter selected vertex agents. To avoid envy, each selected edge agent can only have an incoming arc from two selected vertex agents. Associating selected vertex agents with vertices and selected edge agents with edges, this situation exactly maps to finding a clique in graph G. This concludes the proof for \mathcal{G} being acyclic.

We can easily adapt the aforementioned reduction to the case where the attention graph is strongly connected. We do so by adding an arc (t, s) and one special resource r^{∇} . We let only agent t give utility one to r^{∇} . We claim that these two problems are equivalent; that is, there is a one-to-one mapping between the solutions of the two problems.

Let π be a complete and strongly graph-envy-free allocation for an acyclic attention graph \mathcal{G} . We obtain an allocation π' for a strongly connected graph \mathcal{G}' by copying π and additionally giving r^{∇} to t. Clearly, the only envy that could have emerged, between t and s, is prevented by r^{∇} . Since every agent except for t gives r^{∇} utility zero, resource r^{∇} cannot change the envy state of another agents.

Conversely, if an allocation π' is complete and strongly graph-envy-free for \mathcal{G}' , then $\pi'(t) = \{r^{\nabla}\}$. Thus, we construct an allocation π for the corresponding acyclic attention graph \mathcal{G} by copying π' and removing resource r^{∇} . Since arc (t, s) does not exist in \mathcal{G} , agent t, which gets no resource under π , is not envious under π . Naturally, every other agent was not envious under π' , so it cannot be envious under π .

Similarly to the case of identical preferences, the NP-hardness showed in Proposition 7 can be successfully tackled for the case of few agents of resources. Parameterized tractability of CsGEF-A with respect to the number of resources has already been shown for the general case in Theorem 4. The following Corollary 2, which is a straightforward consequence of Proposition 1, complements the picture by stating that C-sGEF-A with identical utility functions is tractable for few agents.

Corollary 2. C-SGEF-ALLOCATION with 0/1 preferences is fixed-parameter tractable with respect to the parameter "number of agents."

The last remaining question is the parameterized computational complexity of C-sGEF-A with respect to the number of agents in the case of general monotonic additive utility functions.

In Theorem 5 we give a negative answer by showing that for this parameterization C-SGEF-A remains intractable even for acyclic attention graphs.

Theorem 5. If the input graph \mathcal{G} is either a directed path or a cycle, then C-SGEF-ALLOCA-TION for monotonic additive preferences is NP-hard and W[1]-hard when parameterized by the number of agents.

Proof. We give a parameterized reduction from UNARY BIN PACKING [Jansen et al., 2013] where, given a set of integer item sizes encoded in unary, a bin size B, and the maximal number k of bins, the question is whether it is possible to partition items into at most k bins without exceeding their size.

Intuitively, we create a resource for each item and construct an agent to represent every bin. Each of the constructed agents gives every resource the same utility as the size of an item the resource represents. Then, we carefully construct an instance of SGEF-ALLOCATION so that every agent has to get a bundle to which it assigns utility at most B.

To present the reduction formally, let us fix an instance $I = \{S, B, k\}$ of UNARY BIN PACKING, where $S = \{s_1, s_2, \ldots, s_n\}$ and $S = \sum_{i=1}^n s_i$. Without loss of generality, we assume that $S = k \cdot B$. We create a new instance of sGEF-ALLOCATION with with the set $R = \{r_1, \ldots, r_n, r_{n+1}, r_{n+2}, \ldots, r_{n+k}, r^*\}$ of resources and the set $A = \{a_1, a_2, \ldots, a_{k+2}\}$ of agents. Associating nodes with agents, we construct graph \mathcal{G} which is a directed path $(a_1, a_2, \ldots, a_{k+2})$. Agent a_{k+2} gives no utility to every resource. Moreover resource r^* is assigned a non-zero utility only by two agents a_k and a_{k+1} , i.e., $u_{k+1}(r^*) = 1$, $u_k(r^*) = \frac{S}{k}$ and $\forall_{i \in [k]} u_i(r^*) = 0$. For every resource $r_i, 0 < i \le n$, we set the values of the utility functions to be s_i for agents $\{u_1, u_2, \ldots, u_k\}$ and zero otherwise. Resource $r_{n+j}, 0 < j \le k$, is assigned utility zero by all agents but a_j , which assigns utility one.

To show that solving the new instance is equivalent to solving the initial instance of UNARY BIN PACKING, we firstly observe that agent a_{k+1} must have a non-zero utility. The only resource the agent can get to achieve it is r^* . Observe that it implies that a_k has a bundle of utility at least $\frac{S}{k} + 1$. From obtaining the resources $r_{n+1}, r_{n+2}, \ldots, r_{n+k}$, agent a_k can get at most one utility; namely, from resource r_{n+k} . As a result, a_k has to get utility at least $\frac{S}{k}$ by obtaining a selection of the resources r_1, r_2, \ldots, r_n . We can apply this argument iteratively for the remaining agents $a_1, a_2, \ldots, a_{k-1}$. Thus, we get that each agent a_1 to a_k gets a subset of $\{r_1, r_2, \ldots, r_n\}$ that it values to at least $\frac{S}{k}$. Since there are exactly k of these agents, this bound is tight. To fulfill the requirements of strong graph envy-freeness, each agent $a_i \in \{a_1, a_2, \ldots, a_k\}$ also gets resource r_{n+i} . The solution to UNARY BIN PACKING is now formed by sets of elements s_i corresponding to the bundles of agents a_1, a_2, \ldots, a_k .

So far we have shown the proof for the case of a directed path. However, on can add an arc from a_{k+1} to a_1 making the path a cycle. Then, adding one resource, liked only by agent a_{k+1} yields a proof for the case of a cycle.

Our reduction is computable in polynomial time. UNARY BIN PACKING is known to be W[1]hard with respect to the number of bins [Jansen et al., 2013]. There are polynomially many agents with respect to the parameter k of UNARY BIN PACKING, which proves the theorem. \Box

	W-GEF-Allocation $/$ E-GEF-Allocation			W-sGEF-Allocation / E-sGEF-Allocation		
	DAG	SCG	General	DAG	SCG	General
id. 0/1	Р	Р	NP-h	P	O(1) (Obs. 6)	Р
id.	Р	NP-h	NP-h	NP-h	O(1) (Obs. 6) O(1) (Obs. 6)	NP-h
0/1	P (Pr. 9)	NP-h	NP-h	NP-h	NP-h	NP-h
add.	NP-h (Pr. 8) / P (Pr. 9)	NP-h	NP-h	NP-h	NP-h	NP-h

Table 6: Computational complexity of W-GEF-ALLOCATION and E-GEF-ALLOCATION. Unless not explicitly stated, the results follow from Corollary 3. Expect for finding weakly graphenvy-free allocations in case of additive preferences and acyclic graphs G, the complexity is independent of whether we additionally aim for Pareto-efficiency or social welfare maximization in the considered cases.

Settling the computational complexity of the case considered in Theorem 5 finishes our analysis of the (parameterized) computational complexity of C-sGEF-A (recall Table 4 providing an overview of the obtained results).

In the following Figure 6, we provide a concise diagram simplifying comparing the results that we achieved for sGEF-A and C-sGEF-A.

6 A Glimpse on Allocation Efficiency Beyond Completeness

In this section, we briefly discuss to which extent our results from the previous sections transfer to settings where one looks for graph-envy-free allocations that are not necessarily complete but that are Pareto-efficient (E-GEF-A) or that optimize the utilitarian social welfare (W-GEF-A). We show that several of our NP-hardness results for C-GEF-A transfer to E-GEF-A and W-GEF-A without much effort. Additionally, we present a case of E-GEF-A, where the presence of the attendance graph decreases the complexity of a problem from $\Sigma_2^{\rm P}$ -hardness to polynomialtime solvability. Our findings for E-GEF-A and C-GEF-A are summarized in Table 6. We also point out that the complexity of W-GEF-A has already been further explored by Lange and Rothe [2019] motivated by the corresponding section from the conference version of this paper [Bredereck et al., 2018].

We start with an observation expressing relations between completeness, Pareto-efficiency and maximizing the utilitarian social welfare. There relations show that many results provided so far for C-GEF-A directly transfer to W-GEF-A and E-GEF-A.

Observation 7. Let \mathcal{R} be a set of resources and \mathcal{A} be a set of agents with identical additive monotonic preferences. Then, for a (weakly/strongly) graph-envy-free allocation π the following three statements are equivalent:

1. allocation π is complete,

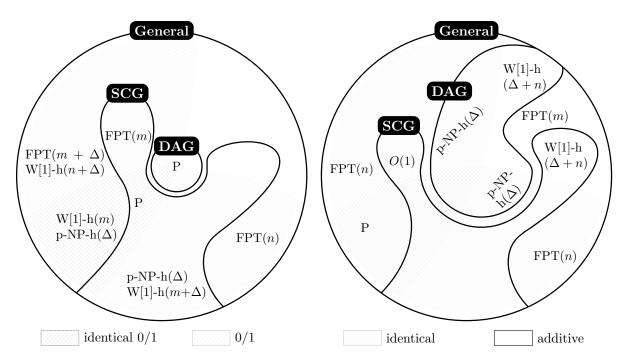


Figure 6: A compact illustration of the computational hardness of C-GEF-A(left) and C-SGEF-A(right), presented also in Tables 2 and 4. Sets represent spaces of all possible problem instances with a particular type of preferences and a particular structure of the attention graph. Types of preferences are indicated by the sets marked with background patters. Different structures of the attention graph are indicated by the blobs tagged with labels. For example, an entry "P" in the set labeled as "SCG" and on checkered background in the left picture, means that C-GEF-A is in P when the attention graph is strongly connected (indicated by set "SCG") and the preferences are identical 0/1 (represented by the checkered background). Similarly, an entry "p-NP-hard" in the set labeled "DAG" on the left-to-right diagonal background on the right picture indicates that C-SGEF-A is para-NP-hard with respect to parameter Δ already when the attention graph is acyclic and directed, and when the utility functions are 0/1. Naturally, this hardness transfer, for example, to C-SGEF-A for the case of general graphs and general utility functions.

2. allocation π is Pareto-efficient, and

3. the utilitarian social welfare \mathcal{W}_{π} of π is maximal, that is, $\mathcal{W}_{\pi} = \sum_{r \in \mathcal{R}} \max_{a \in \mathcal{A}} u_a(r)$.

For 0/1 preferences, it holds that $(3) \Leftrightarrow (2) \Rightarrow (1)$.

Proof. We first provide a proof for the case of identical additive monotonic preferences. The proof follows from the following implication cycle:

$$1 \Rightarrow 3$$
:

Clearly, any complete allocation has the same utilitarian social welfare value, and thus no other allocation can have a higher utilitarian social welfare.

 $3 \Rightarrow 2$:

Assume towards a contradiction that some social-welfare-maximizing and graph-envy-free allocation is not Pareto-efficient. For additive monotonic preferences, an allocation that dominates another allocation must, by definition, have a higher utilitarian social welfare value—a contradiction.

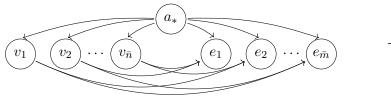
 $2 \Rightarrow 1$:

Assume some Pareto-efficient allocation π' in which at least one resource is not allocated. We obtain a new allocation by giving one unallocated resource to an arbitrarily chosen agent a that has a positive utility towards the resource (agent a exists due to Observation 2). Thus, we increase the utility of a without decreasing the utilities of the other agents, so π' is not Pareto-efficient—a contradiction.

We move on to the case of 0/1 preferences. First, observe that the implication $2 \Rightarrow 1$ also holds for the case of 0/1 preferences. Furthermore, in every Pareto-efficient allocation there is no agent *a* that gets a resource *r* with $u_a(r) = 0$. If this was the case, then giving *r* to some agent *a'* with $u_{a'} = 1$ would give a dominating allocation. Thus, every Pareto-efficient allocation is maximizing the utilitarian social welfare. The opposite direction of this claim is the implication $3 \Rightarrow 2$ proven above.

Observation 7 allows us to transfer all results for W-GEF-A stated for identical additive monotonic preferences. This, however, does not apply to Theorem 2 and Proposition 7, which are about 0/1 preferences, and thus are not fully covered by the above observation. Still, in the respective reductions every resource must be allocated to one of the agents that values it the most in every graph-envy-free and complete allocation. In effect, the same proofs show that the results indeed hold for Pareto-efficiency and utilitarian social welfare. We collect all results for which we can apply Observation 7 or the above discussion in Corollary 3.

Corollary 3. Corollary 1, Theorem 1, Proposition 2 and Theorem 2 hold also when C-GEF-ALLOCATION therein is substituted by W-GEF-ALLOCATION or E-GEF-ALLOCATION. Observation 4, restricted to identical additive preferences, holds for W-GEF-ALLOCATION and E-GEF-ALLOCATION. Proposition 5, Theorem 5, and Proposition 4 hold also when C-SGEF-ALLOCATION therein is substituted by W-SGEF-ALLOCATION or E-SGEF-ALLOCATION.



	a_*	\mathcal{A}_{v}	\mathcal{A}_{e}
r^*	1	0	0
\mathcal{R}_{v}	1	2	0
\mathcal{R}_{e}	1	1	2

(a) General construction of the attention graph with nodes labeled with their names.

(b) Utilities that the agents give to the resources.

Figure 7: Illustrations for the proof of Proposition 8.

Together with Observation 5, Corollary 3 provides an almost complete picture of classical computational complexity lower bounds of the problems we study in this section. It can be observed in the results overview in Table 6 that the only remaining cases are E-GEF-A and W-GEF-A for the case of 0/1 and monotonic additive preferences.

To complete our analysis, we first show that W-GEF-A becomes intractable for directed acyclic graphs and additive monotonic preferences. To prove it, we show a reduction from CLIQUE that uses only utility values 0, 1, and 2.

Proposition 8. W-GEF-ALLOCATION is NP-hard for the input graph being a directed acyclic graph even for three-valued utility functions.

Proof. We show NP-hardness using a reduction from CLIQUE. Let us fix an instance of CLIQUE with a graph $\bar{G} = (\bar{V}, \bar{E})$, and a size-k clique. Specifically, a set $\bar{V} = \{\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{\bar{n}}\}$ is a set of vertices and a set $\bar{E} = \{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{\bar{m}}\}$ is a set of edges. In order to construct an instance of W-GEF-A, we introduce a special agent a_* , vertex agents $\mathcal{A}_{v} = \{v_1, v_2, \ldots, v_{\bar{n}}\}$, and edge agents $\mathcal{A}_{e} = \{e_1, e_2, \ldots, e_{\bar{m}}\}$. We introduce a special resource r^* , a set $\mathcal{R}_{v} = \{r_1, r_2, \ldots, r_k\}$ of vertex resources, and a set $\mathcal{R}_{e} = \{r'_1, r'_2, \ldots, r'_{\binom{k}{2}}\}$ of edge resources. The agents assign utilities to resources as specified in Figure 7b.

We set a social welfare threshold $w = 1 + 2k + 2\binom{k}{2}$. Finally, we build a graph over the agents like depicted in Figure 7a. We connect a_* to every other agent with an arc starting in a_* . For each edge $(\bar{v}_1, \bar{v}_2) \in \bar{E}$, we introduce two arcs pointing from agents v_1 and v_2 to the respective edge.

The social welfare threshold w is the utility of such an allocation which gives every resource to some agent who values it the most. Hence, w is equal to the maximal achievable utility level. As a consequence, every allocation meeting the criterion of social welfare value has to give the special resource to a_* and distribute edge and vertex resources to, respectively, edge and vertex agents. Moreover, every edge and vertex agent has to get at most one respective resource. To support the claim it is enough to observe that because the special agent gets only one resource and likes all the resources equally, giving either vertex or edge agent more than one resource makes the special agent envious. So, in fact, every allocation selects exactly k vertex agents and Algorithm 2: Let \mathcal{R} be a set of resources, let \mathcal{A} be a set of agents with preferences encoded by the utility functions $u_a : \mathcal{R} \to \mathbb{N}, a \in \mathcal{A}$, and let \mathcal{G} be a directed acyclic graph. The sets \mathcal{R} and \mathcal{A} are ordered (arbitrarily).

v	$\textbf{vhile} \; \mathcal{R} \neq \emptyset \; \textbf{do}$
	Remove all agents a with $u_a(r) = 0, \forall r \in \mathcal{R}$ from \mathcal{G} ;
	Allocate the first resource r^* to the first agent a^* with zero in-degree in \mathcal{G} which
	values r^* the most among the agents with zero in-degree in \mathcal{G} ;
	Remove r^* from \mathcal{R} ;

 $\binom{k}{2}$ edge agents. Since selecting some edge agent *e* imposes a selection of vertex agents pointing to *e*, a successful allocation is equivalent to selecting vertices and edges forming a *k*-clique. This argument proves the correctness of our reduction. The fact that the reduction can be performed in polynomial time finally proves NP-hardness of the W-GEF-A problem.

We next provide Algorithm 2, showing that the polynomial-time solvability established for C-GEF-A in the case of directed acyclic graphs and 0/1 preferences also holds for W-GEF-A and E-GEF-A. Moreover, the same algorithm efficiently solves E-GEF-A, thus concluding our study on Pareto-efficient or utilitarian social welfare maximizing envy-free allocations.

Proposition 9. E-GEF-ALLOCATION for acyclic input graphs and monotonic additive preferences and W-GEF-ALLOCATION for acyclic input graphs and 0/1 preferences can be solved in linear time.

Proof. Algorithm 2 arrives at the final allocation after constructing a series of intermediate allocations. We show that every allocation computed by the algorithm is graph-envy-free. Assume towards a contradiction that after some iteration of the while loop, agent a starts envying agent a^* that has just obtained resource r^* . This means that agent a has not been removed yet from \mathcal{G} (otherwise, its value for r^* would be 0). Thus, there is still an arc from a to a^* in \mathcal{G} , contradicting that agent a^* has in-degree zero.

Next, assume towards a contradiction that there is some allocation π' that dominates the allocation π that is provided by our algorithm. We use the same order of the resources and agents as Algorithm 2. Now, let r^* be the first resource and a^* be the first agent with $r^* \in \pi(a^*)$ but $r^* \notin \pi'(a^*)$. It is now easy to verify that our algorithm clearly ensures that $u_{a^*}(\pi(a^*)) > u_{a^*}(\pi'(a^*))$. So, allocation π' cannot dominate π because agent a^* is worse off under π' .

For 0/1 preferences, Algorithm 2 allocates each resource to agent that gives utility 1 to the resource. So, the algorithm maximizes the utilitarian social welfare for 0/1 preferences.

To assess the running time of Algorithm 2, we observe that in each repetition of the while loop, we iterate over the list of agents two times (at most). Hence, we obtain a linear running time. \Box

Let us briefly discuss the surprising computational-complexity separation of polynomialtime solvable E-GEF-A (Proposition 9) and NP-hard W-GEF-A (Proposition 8) for the case of monotonic additive preferences. While, assuming monotonic additive preferences, finding envy-free allocations maximizing the utilitarian social welfare is NP-hard, de Keijzer et al. [2009] have shown that finding a Pareto-efficient and envy-free allocation (EEF) is not only NP-hard but $\Sigma_2^{\rm P}$ -hard. So, Proposition 9 drastically decreases the complexity of E-GEF-ALLO-CATION from $\Sigma_2^{\rm P}$ for general directed graphs to polynomial-time solvability for directed acyclic graphs. The reason for this decrease lies in the following fact. If all resources are allocated to a group of agents in a welfare-maximizing way (considering only these agents), then there is no possibility of taking away any resource from any of these agents without making some agent in the group worse off. Exactly such a "locally welfare-maximizing" allocation (giving resources to source agents only) is computed by Algorithm 2, and thus one cannot construct any dominating allocation. Constructing a dominating allocation, in turn, is the main source of $\Sigma_2^{\rm P}$ -hardness of EEF. Thus, eliminating a possibility of existence of a dominating allocation results in the observed complexity drop.

7 Conclusion

Combining social networks with fairness in the context of resource allocations is a promising line of research. In this growing area, our work provides a first systematic study of (parameterized) computational complexity of finding fair allocations of indivisible resources. Thus, we complement a similar work of Chevaleyre et al. [2017] that, among other things, has a more distributed and (because of considering monetary payments) more divisible-resources flavor.

Our results show that for directed acyclic attention graphs the weak variant of graph envyfreeness is computationally tractable, which is not always true for the strong variant of graph envy-freeness. By way of contrast, the strong variant is mostly computationally easier than the weak one for all other considered families of attention graphs. Specifically, with respect to the considered parameters, we showed fixed-parameter tractability of the strong variant for almost all considered cases (for which we got NP-hardness), while the weak variant is mostly at least W[1]-hard. Exploiting our results for C-GEF-A and C-sGEF-A, we studied the classical computational complexity of finding (weakly/strongly) graph-envy-free allocations that are either Pareto-efficient or maximizing utilitarian welfare. Thereby, we spotted an interesting case of finding fair and Pareto-efficient allocations where the computational complexity dropped from Σ_2^p -hardness in the general case, to polynomial-time solvability in the case of acyclic attention graphs (which correspond to the scenarios with hierarchical "envy relations" structures).

A number of W[1]-hardness results as well as the observed complexity drop described above, motivates a more refined search for islands of tractability concerning practically motivated use cases of our basic models. In this context, there are plenty of opportunities. First, one may study further natural parameters, including the number of resources and maximum utility values. Note, however, that these parameters may need to be combined in order to achieve fixed-parameter tractability results (e.g., a small maximum utility value does not guarantee fixed-parameter tractability). Second, it appears natural to deepen our studies by considering various special graph classes for the underlying social network. In addition, one may move from directed to undirected graphs or one may consider graphs that only consist of small connected components. Again note, however, that the class of bounded-degree graphs (reflected by a parameterization using maximum degree as the parameter) alone, as shown in this work, may not be enough to achieve (fixed-parameter) tractability. Finally, including further fairness concepts beyond the ones we studied appears to be promising as well. In particular combining (weak/strong) graph envy-freeness with a more global concept such as graph epistemic envy-freeness by Aziz et al. [2018] could prevent allocating all resources to topologically-top agents in case of acyclic attention graphs.

Last, but not least, addressing the important need of ways of fairly allocating indivisible resources in real-life scenarios requires an empirical study of graph envy-freeness. An important issue is to propose models of generating data and to collect empirical data. Then, among numerous natural questions, the following seem to stand out. Are the proposed algorithms applicable in practice? Are hard instances likely to appear or are they only carefully tailored special cases used to show theoretical results? Does the empirical data provide some insights leading to new algorithmic approaches or parameterizations? A more application-oriented, empirical approach has already proven successful, resulting in the non-profit web service spliddit.org [Goldman and Procaccia, 2015, Caragiannis et al., 2019] which offers procedures to allocate indivisible goods guaranteeing envy-freeness up to one good.

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