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Hamiltonian discretization of boundary control systems☆

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Abstract

A fundamental problem in the simulation and control of complex physical systems containing distributed-parameter components concerns finite-dimensional approximation. Numerical methods for partial differential equations (PDEs) usually assume the boundary conditions to be given, while more often than not the interaction of the distributed-parameter components with the other components takes place precisely via the boundary. On the other hand, finite-dimensional approximation methods for infinite-dimensional input–output systems (e.g., in semi-group format) are not easily relatable to numerical techniques for solving PDEs, and are mainly confined to linear PDEs. In this paper we take a new view on this problem by proposing a method for spatial discretization of boundary control systems based on a particular type of mixed finite elements, resulting in a finite-dimensional input–output system. The approach is based on formulating the distributed-parameter component as an infinite-dimensional port-Hamiltonian system, and exploiting the geometric structure of this representation for the choice of appropriate mixed finite elements. The spatially discretized system is again a port-Hamiltonian system, which can be treated as an approximating lumped-parameter physical system of the same type. In the current paper this program is carried out for the case of an ideal transmission line described by the telegrapher's equations, and for the two-dimensional wave equation.
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1. Introduction

In previous works, see e.g. van der Schaft and Maschke (1995), Maschke, van der Schaft, and Breedveld (1992), Dalsmo and van der Schaft (1999), van der Schaft (2000a, b), it has been shown how port-based network modelling of complex lumped-parameter physical systems naturally leads to a generalized Hamiltonian formulation of the dynamics. In fact, the Hamiltonian is given by the total energy of the energy-storing elements in the system, while the geometric structure, defining together with the Hamiltonian the dynamics of the system, is given by the power-conserving interconnection structure of the system, and is called a

Dirac structure. Furthermore, energy-dissipating elements may be added by terminating some of the system ports. The resulting class of open dynamical systems has been called "port-Hamiltonian systems" (van der Schaft & Maschke, 1995; van der Schaft, 2000a). The identification of the Hamiltonian structure of the dynamical model is important for various reasons. From a simulation point of view it yields information about the energy function and other conserved quantities in the system, which preferably should be respected in simulation. Furthermore, it is instrumental in finding the most convenient representation of the equations of motion of the system; in the format of purely differential equations or of mixed sets of differential and algebraic equations (DAEs), see e.g. van der Schaft (2000b). From an analysis point of view it allows to use the powerful methods regarding from the theory of Hamiltonian systems. Finally, the Hamiltonian structure may be fruitfully used in the control design, e.g. by the explicit use of the energy function and conserved quantities for the construction of a Lyapunov function (possibly after the connection with another port-Hamiltonian controller system), or by directly modifying by feedback the interconnection and dissipation

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structure and shaping the internal energy. We refer to Ortega, van der Schaft, Mareels, and Maschke (2001), van der Schaft (2000a), Ortega, van der Schaft, Maschke, and Escobar (2002) for various works in this direction.

Recently, the framework of port-Hamiltonian systems has been extended to classes of distributed-parameter systems (van der Schaft & Maschke, 2002; Maschke & van der Schaft, 2000; Talasila, Golo, & van der Schaft, 2002; Golo, van der Schaft, & Stramigioli, 2003), such as Maxwell's equations over a domain with boundary incorporating energy radiation through its boundary, the *n*-dimensional wave equation, compressible ideal fluids, as well as beam models (Golo et al. (2003)). Hereto a special type of infinite-dimensional Dirac structure has been introduced, based on Stokes' theorem. Physically, this Stokes–Dirac structure captures the basic balance laws of the system, like Faraday's and Ampère's law or mass balance. The port-Hamiltonian formulation is a non-trivial extension of the Hamiltonian formulation of partial differential equations (PDEs) by means of Poisson structures (see e.g. Olver, 1993), since in the latter case it is crucially assumed that the boundary conditions are such that the energy-flow through the boundary of the spatial domain is zero. In order to allow a non-zero boundary energy-flow the use of Dirac structures instead of Poisson structures appears to be indispensable (Maschke & van der Schaft, 2000; van der Schaft & Maschke, 2002).

As a result, complex physical systems consisting of components which are either *lumped-parameter* or *distributed-parameter* systems, and which moreover may belong to different physical domains (mechanical, electrical, hydraulic, etc.), can be modelled in a unified way.

From a simulation point of view, however, a fundamental problem concerns the incorporation of the powerful numerical methods for the solution of PDEs, such as finite-element and finite-difference methods, into this framework. Also for control purposes it may be crucial to approximate either the distributed-parameter system with a finite-dimensional system, or the infinite-dimensional controller system by a finite-dimensional one. This is not an easy task. One fundamental problem is the fact that numerical solution methods for PDEs usually assume the boundary conditions to be *given*. On the other hand, more often than not it are precisely the boundary conditions which represent the interaction of the distributed-parameter component with the other components of the system. A typical simple example is an electrical circuit containing a transmission line. The transmission line is usually modeled by PDEs (the telegrapher's equations) with boundary conditions being the values of the voltages and currents at both ends of the cable. Clearly, those voltages and currents cannot be considered to be given, since the transmission line is connected to the other dynamical components of the electrical network. We thus have to approximate the distributed-parameter system with a finite-dimensional system, while retaining the power port-structure of the system (like the boundary voltages and currents in the trans-

mission line example). In this paper we show how this can be done for the examples of the transmission line and the two-dimensional wave equation. Indeed, we will show how the intrinsic Hamiltonian formulation suggests finite element methods which result in finite-dimensional approximations which are again port-Hamiltonian systems. In the case of the transmission line the basic observation is that in the geometric Hamiltonian formulation the state space variables are (electric and magnetic) *densities*, or in geometric language *one-forms*, while the voltages and currents are *functions* on the spatial domain of the system. Furthermore, in the case of the two-dimensional wave equation the state variables are given by *one-forms* (elastic strains) and *two-forms* (kinetic momenta). From a geometric point of view it is natural to use *different* finite-elements for the approximation of functions, one-forms and two-forms. This point of view was already stressed in the work by Bossavit (1991, 1998) on the use of finite elements for the computation of solutions of Maxwell's equations (for given boundary conditions), and was shown to lead to *mixed* (vector) finite elements.

The further outline of this paper is as follows. The definition of port-Hamiltonian systems is recalled in Section 2. In Section 3, we introduce the spatial discretization procedure for the telegrapher's equations, which preserves the port-Hamiltonian structure. We show how this implies that certain physical properties, like the existence of conserved quantities and energy balance, are retained in the finite-dimensional approximation. Furthermore, we discuss the choice of the approximating functions within our discretisation scheme, and its physical interpretation. In Section 4 we apply the discretization procedure to the two-dimensional boundary-controlled wave equation, written in port-Hamiltonian form. Finally, in Section 5 some simulation results regarding the discretization of the transmission line are presented, while Section 6 contains the conclusions.

Preliminary versions of results in this paper have been reported in Talasila et al. (2002), Golo, Talasila and van der Schaft (2002).

1.1. Notation

Although in this paper we consider only the telegrapher's equations and the two-dimensional wave equation, the proposed methodology is applicable to the general class of distributed-parameter port-Hamiltonian systems as treated in van der Schaft and Maschke (2002). In order to stress the generality of our method we throughout employ the differential-geometric framework of differential forms on the spatial domain $\mathscr{Z}$ of the system.

In the context of a *one-dimensional* spatial domain this means that we distinguish between *functions* (also called *zero-forms*) and *one-forms* defined on the interval representing the spatial domain of the transmission line. Basically, functions can be *evaluated* at any point of the interval,

while one-forms can be *integrated* over every sub-interval of the interval. If we consider a spatial coordinate z for the interval $\mathscr{Z}$, then a function f is simply given by the values $f(z) \in \mathbb{R}$ for every coordinate value z in the interval, while a one-form ω is given as $g(z)\,\mathrm{d}z$ for a certain density function g. Physically, in the transmission line example the voltages and currents are functions, while the charge and flux densities correspond to one-forms. (Note that it does not make physical sense to talk about the charge at a certain point of the interval; instead one considers the total charge contained in a part of the transmission line.) We denote the set of zero-forms and one-forms on $\mathscr{Z}$ by $\Omega^0(\mathscr{Z})$, respectively $\Omega^1(\mathscr{Z})$. Given a coordinate z for the spatial domain we obtain by spatial differentiation of a function $f(z)$ the one-form $\omega := \mathrm{d}f/\mathrm{d}z(z)\,\mathrm{d}z$. In coordinate-free language this is denoted as $\omega = \mathrm{d}f$, where d is called the *exterior derivative*, converting functions into one-forms.

In the case of a *two-dimensional* spatial domain $\mathscr{Z}$ (as in the example of the two-dimensional wave equation) we have to distinguish between zero-forms (functions), one-forms and two-forms. Again, functions are objects which can be evaluated at any point in the spatial domain. Furthermore, one-forms are objects which can be integrated along any line-segment in the spatial domain, while two-forms are objects which can be integrated over any open part of the spatial domain. Physically (see Section 4), in the two-dimensional wave equation the elastic strain is a one-form and the kinetic momentum is a two-form. Furthermore, the velocity is a function and the elastic stress is a one-form. Given coordinates z_1, z_2 for the spatial domain, a function is simply given by the values $f(z_1, z_2) \in \mathbb{R}$ for every point (z_1, z_2), while a one-form is expressed as $g_1(z_1, z_2)\,\mathrm{d}z_1 + g_2(z_1, z_2)\,\mathrm{d}z_2$ for certain functions g_1, g_2. Finally, a two-form $\omega \in \Omega^2(\mathscr{Z})$ is given by the infinitesimal area element $k(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2$ for a certain function k. By spatial differentiation of a function $f(z_1, z_2)$ we obtain the one-form $\partial f/\partial z_1(z_1, z_2)\,\mathrm{d}z_1 + \partial f/\partial z_2(z_1, z_2)\,\mathrm{d}z_2$, while spatial differentiation of a one-form $g_1(z_1, z_2)\,\mathrm{d}z_1 + g_2(z_1, z_2)\,\mathrm{d}z_2$ results in the two-form $(\partial g_2/\partial z_1(z_1, z_2) - \partial g_1/\partial z_2(z_1, z_2))\,\mathrm{d}z_1\,\mathrm{d}z_2$.

In geometric, coordinate-free, language both spatial differentiation operations are denoted by the exterior derivative d, transforming zero-forms into one-forms, and one-forms into two-forms. In this geometric setting the main theorem of integral calculus on the interval and Gauss' theorem on $\mathbb{R}^2$ can be generalized to Stokes' theorem stating that $\int_{\mathscr{Z}} \mathrm{d}\omega = \int_{\partial\mathscr{Z}} \omega$, for any $(n-1)$-form ω on a n-dimensional manifold $\mathscr{Z}$ with $(n-1)$-dimensional boundary $\partial\mathscr{Z}$.

Furthermore, given a k-form ω_1 and an ℓ-form ω_2 the *wedge product* $\omega_1 \wedge \omega_2$ is an $(k+\ell)$-form. For example, $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ is the two-form commonly denoted as $\mathrm{d}z_1\,\mathrm{d}z_2$ (but note that $\mathrm{d}z_2 \wedge \mathrm{d}z_1 = -\mathrm{d}z_1 \wedge \mathrm{d}z_2$). Finally, we will use the Hodge star operator $*$, converting any k-form ω on a n-dimensional spatial domain $\mathscr{Z}$ to an $(n-k)$-form $*\omega$. The definition of the Hodge star operator relies on the assumption of a Riemannian metric on the spatial domain $\mathscr{Z}$; however, in the present paper this Riemannian metric will simply

be the Euclidean inner product corresponding to a choice of local coordinates on $\mathscr{Z}$. Thus on the one-dimensional spatial domain $\mathscr{Z}$ with spatial coordinate z we simply have $*g(z) = g(z)\,\mathrm{d}z$, $*(g(z)\,\mathrm{d}z) = g(z)$. If $\mathscr{Z}$ is two dimensional with coordinates (z_1, z_2) then $*k(z_1, z_2) = k(z_1, z_2)\,\mathrm{d}z_1\mathrm{d}z_2$ and $*(k(z_1, z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2) = k(z_1, z_2)$, while $*(g_1(z_1, z_2)\,\mathrm{d}z_1 + g_2(z_1, z_2)\,\mathrm{d}z_2) = g_1(z_1, z_2)\,\mathrm{d}z_2 - g_2(z_1, z_2)\,\mathrm{d}z_1$.

2. Dirac structures and port-Hamiltonian systems

The definition of a port-Hamiltonian system is based on grouping the energy-storing elements of the system, leading to a Hamiltonian function given by the total stored energy, and formalizing the power-conserving interconnection between them by the geometric notion of a Dirac structure (see (van der Schaft & Maschke, 1995, 2002; Dalsmo & van der Schaft, 1999; van der Schaft, 1999, 2000a, b, for details).

First we recall the definition of Dirac structures. Let $\mathscr{F}$, $\mathscr{E}$ be real vector spaces whose elements are labeled as f, e, respectively. We call $\mathscr{F}$ the space of *flows*, and $\mathscr{E}$ the space of *efforts*. The product space $\mathscr{P} := \mathscr{F} \times \mathscr{E}$ is assumed to be endowed with a *scalar pairing* $\langle \cdot | \cdot \rangle : \mathscr{F} \times \mathscr{E} \to \mathbb{R}$, formalizing the notion of *power*:

Definition 1. Let $\mathscr{F}$, $\mathscr{E}$ be real vector spaces. A map $\langle \cdot | \cdot \rangle : \mathscr{P} = \mathscr{F} \times \mathscr{E} \to \mathbb{R}$ is called a scalar pairing if it is linear in each argument and it is non-degenerate, that is $\langle e | f \rangle = 0$, $\forall e \in \mathscr{E} \Rightarrow f = 0$ and $\langle e | f \rangle = 0$, $\forall f \in \mathscr{E} \Rightarrow e = 0$.

(In the finite-dimensional case the canonical choice for $\mathscr{E}$ is the *dual* space of $\mathscr{F}$ with scalar pairing defined by the *duality product*.)

$\mathscr{P} = \mathscr{F} \times \mathscr{E}$ is called a multi-dimensional (power) *port*, and the vector pair $p = (f, e)$ is called the vector of *port variables*. On $\mathscr{P}$ there exists a *bilinear form* $\ll, \gg : \mathscr{P} \times \mathscr{P} \to \mathbb{R}$ defined as

$$\langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle := \langle e^1 | f^2 \rangle + \langle e^2 | f^1 \rangle$$

Definition 2. A subspace $\mathscr{D} \subset \mathscr{P}$ is a Dirac structure if $\mathscr{D} = \mathscr{D}^\perp$, where $\perp$ denotes the orthogonal complement with respect to the bilinear form $\ll, \gg$.

Actually, this is the definition of a *constant* Dirac structure on linear spaces as given (in a slightly restricted sense) in Courant (1990), see also Dorfman (1993). The definition can be extended to (non-constant) Dirac structures on manifolds; see Courant (1990), Dorfman (1993), as well as Dalsmo and van der Schaft, 1999. It immediately follows that $\langle e | f \rangle = 0$, $\forall (f, e) \in \mathscr{D}$, formalizing that a Dirac structure represents a *power-conserving* interconnection structure; i.e., the net power in the Dirac structure is zero. The definition of a port-Hamiltonian system can now be stated as follows. Consider the energy-storing elements of the system with energy-variables x living in a total state space $\mathscr{X}$. The space of energy *rate* variables will define the linear

space $\mathscr{F}^x$ of *internal* flows f^x. In general, $\mathscr{X}$ is a manifold but for the purposes of this paper we assume that $\mathscr{X}$ is a linear space (finite-dimensional for a lumped-parameter system and infinite-dimensional for a distributed-parameter system), implying that $\mathscr{F}^x = \mathscr{X}$. Furthermore, consider the space $\mathscr{F}^B$ of external (or *boundary*) flows f^B, modeling the interaction of the system with its environment (through the boundary of the system).

After having specified the total space of flows $\mathscr{F}$ as the product $\mathscr{F}^x \times \mathscr{F}^B$ the space $\mathscr{E}$ of conjugated efforts is constructed as follows. Consider a linear space $\mathscr{E}^x$ of conjugated internal effort variables e^x, together with a scalar pairing $\langle \cdot | \cdot \rangle_x : \mathscr{F}^x \times \mathscr{E}^x \to \mathbb{R}$ as above. Let $\mathscr{E}^B$ be a linear space of boundary effort variables e^B, together with a scalar pairing $\langle \cdot | \cdot \rangle_B : \mathscr{F}^B \times \mathscr{E}^B \to \mathbb{R}$. Define $\mathscr{E} := \mathscr{E}^x \times \mathscr{E}^B$, with $\langle \cdot | \cdot \rangle_x + \langle \cdot | \cdot \rangle_B$ the scalar pairing on $\mathscr{F} \times \mathscr{E}$.

Next we consider a Dirac structure $\mathscr{D}$ relating the internal and boundary flows f^x, f^B and internal and boundary efforts e^x, e^B, that is $\mathscr{D} \subset \mathscr{F}^x \times \mathscr{F}^B \times \mathscr{E}^x \times \mathscr{E}^B$. Furthermore, we consider a Hamiltonian $H : \mathscr{X} \to \mathbb{R}$ representing the *total energy* stored in the system. Under weak smoothness conditions on H

$$H(x + \delta x) - H(x) = \langle \delta_x H | \delta x \rangle_x + o(\delta x), \quad \forall \delta x \quad (1)$$

for a certain conjugated internal effort $\delta_x H \in \mathscr{E}^x$, called the vector of *co-energy variables*. In the finite-dimensional case $\delta_x H$ is just the *gradient* of H (the vector of partial derivatives), while in the infinite-dimensional case $\delta_x H$ will be the *variational derivative*; see Olver (1993) for details.

As said before, the internal flows f^x are the flows connected to the energy-storing elements, and thus are set equal to the rate of the energy variables: $f^x = \dot{x} \in \mathscr{X}$, while the internal efforts e^x are set equal to the vector of co-energy variables: $e^x = \delta_x H$. Finally, the dynamical system defined by the relations

$$(\dot{x}(t), f^B(t), \delta_x H(x(t)), e^B(t)) \in \mathscr{D}, \quad t \in \mathbb{R},$$

is called a *port-Hamiltonian system*. This definition of port-Hamiltonian systems includes in general the occurrence of *algebraic constraints*, whereas in the absence of algebraic constraints the Dirac structure specializes to a *Poisson structure*; see e.g. van der Schaft (2000a), Ortega et al. (2001).

An important class of infinite dimensional port-Hamiltonian systems, modelling a large class of boundary control distributed-parameter systems, is defined as follows (see Maschke & van der Schaft (2000), van der Schaft & Maschke (2002) for details). Let $\mathscr{Z}$ be an n-dimensional smooth manifold with $(n-1)$-dimensional smooth boundary $\partial \mathscr{Z}$. The state space $\mathscr{X}$ is defined as $\mathscr{X} := \Omega^{n^q}(\mathscr{Z}) \times \Omega^{n^p}(\mathscr{Z})$, with $n^q + n^p = n+1$. Here $\Omega^{n^\bullet}(\mathscr{Z})$ stands for the space of exterior $n^\bullet$-forms on $\mathscr{Z}$. An element of the space $\mathscr{X}$ is denoted by $x = [q, p]^T$. The spaces $\mathscr{F}^x$, $\mathscr{E}^x$ are defined as $\mathscr{F}^x := \Omega^{n^q}(\mathscr{Z}) \times \Omega^{n^p}(\mathscr{Z})$ and $\mathscr{E}^x := \Omega^{n-n^q}(\mathscr{Z}) \times \Omega^{n-n^p}(\mathscr{Z})$, and the elements $f^x \in \mathscr{F}^x$, $e^x \in \mathscr{F}^x$ are given as $f^x = [f^q, f^p]^T$ and $e^x = [e^q, e^p]^T$. Furthermore, the spaces $\mathscr{F}^B$, $\mathscr{E}^B$ are

defined as $\mathscr{F}^B := \Omega^{n-n^p}(\partial \mathscr{Z})$, $\mathscr{E}^B := \Omega^{n-n^q}(\partial \mathscr{Z})$ with corresponding boundary flows and efforts denoted by f^B and e^B, respectively. It has been shown in van der Schaft & Maschke (2002) that the following system defines a port-Hamiltonian system:

$$\begin{bmatrix} f^q \\ f^p \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ (-1)^{n^p n^q} \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^q \\ e^p \end{bmatrix}, \quad (2a)$$

$$f^B = e^p|_{\partial \mathscr{Z}}, \quad e^B = -(-1)^{n^q} e^q|_{\partial \mathscr{Z}}, \quad (2b)$$

$$f^q = \frac{\mathrm{d}q}{\mathrm{d}t}, \quad f^p = \frac{\mathrm{d}p}{\mathrm{d}t}, \quad (2c)$$

$$e^q = \delta_q H, \quad e^p = \delta_p H. \quad (2d)$$

Here $|_{\partial \mathscr{Z}}$ denotes the restriction to the boundary $\partial \mathscr{Z}$. The space of all admissible flows end efforts satisfying (2a), (2b) represents a Dirac structure (called Stokes–Dirac structure Maschke & van der Schaft (2000), van der Schaft & Maschke (2002)), with respect to the scalar pairing

$$\int_{\mathscr{Z}} e^q \wedge f^q + \int_{\mathscr{Z}} e^p \wedge f^p + \int_{\partial \mathscr{Z}} e^B \wedge f^B. \quad (3)$$

A large class of boundary control distributed-parameter systems are represented by Eq. (2): transmission line, nD wave equation, uniaxial bar, torsional bar, $1D$ compressible fluid, Maxwell's equations, and so on. Furthermore, port-Hamiltonian systems described by (2) are the basic building blocks for Hamiltonian models of beams, shells, and in general, flexible structures. For later use we recall two basic properties of port-Hamiltonian systems (2) (see van der Schaft & Maschke (2002) for details):

- Energy balance: $\frac{\mathrm{d}H}{\mathrm{d}t}(t) + \int_{\partial \mathscr{Z}} e^B(t) \wedge f^B(t) = 0$,
- Conserved quantities:
  ○ If $n^q = n$ then $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{Z}} q = \int_{\partial \mathscr{Z}} e^p$,
  ○ If $n^p = n$ then $\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathscr{Z}} p = \int_{\partial \mathscr{Z}} e^q$,
  ○ If $n^q < n$ then $\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}q = 0$,
  ○ If $n^p < n$ then $\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{d}p = 0$.

The first property expresses a basic property of any port-Hamiltonian system: the increase in stored energy (Hamiltonian) is equal to the power supplied by the environment. The second class of properties is strictly related to the definition of the Stokes–Dirac structure, and expresses the basic conservation laws of the system (like conservation of charge in the example of the transmission line; see the next section).

3. Spatial discretization of the transmission line

The spatial domain on which the variables of the transmission line are defined is the interval $\mathscr{Z} = [0, S]$. The telegrapher's equations for the ideal transmission line without

energy dissipation are given by the PDEs

$$\frac{\partial q}{\partial t} = -\frac{\partial I}{\partial z}, \quad \frac{\partial \phi}{\partial t} = -\frac{\partial V}{\partial z}, \tag{4}$$

where $q(t,z)$ denotes the charge density and $\phi(t,z)$ is the flux density, and where the current I and voltage V are given by

$$I(t,z) = \frac{\phi(t,z)}{L(z)}, \quad V(t,z) = \frac{q(t,z)}{C(z)} \tag{5}$$

with $C(z), L(z)$ being, respectively, the distributed capacitance and distributed inductance of the line. Furthermore, the boundary flow and effort variables are given by the voltages and currents at both ends of the line:

$$e^{BL}(t) = V(t,0), \ e^{BR}(t) = V(t,S),$$

$$f^{BL}(t) = I(t,0), \ f^{BR}(t) = I(t,S). \tag{6}$$

These PDEs can be immediately seen to be in port-Hamiltonian form (2), with $n = n^p = n^q = 1$, replacing the letter p by ϕ:

$$f^q = -\mathrm{d}e^\phi, \quad f^\phi = -\mathrm{d}e^q, \tag{7a}$$

$$f_0^B = e^\phi|_{z=0}, \quad f_S^B = e^\phi|_{z=S},$$

$$e_0^B = e^q|_{z=0}, \quad e_S^B = e^q|_{z=S}, \tag{7b}$$

$$H = H^q + H^\phi = \int_{\mathscr{L}} \frac{*q(z)}{2C(z)} q(z)$$

$$+ \int_{\mathscr{L}} \frac{*\phi(z)}{2} L(z)\phi(z), \tag{7c}$$

$$f^q = \frac{\partial q}{\partial t}, \quad f^\phi = \frac{\partial \phi}{\partial t}, \tag{7d}$$

$$e^q = \delta_q H = \frac{*q(z)}{C(z)}, \quad e^\phi = \delta_\phi H = \frac{*\phi(z)}{L(z)}. \tag{7e}$$

The physical meaning of the variables and parameters are summarized in Table 1.

The energy balance for the transmission line takes the form $\mathrm{d}H/\mathrm{d}t(t) - f_0^B e_0^B + f_S^B e_S^B = 0$, while there exist two conserved quantities $\mathrm{d}Q/\mathrm{d}t(t) - f_0^B + f_S^B = 0$ and $\mathrm{d}\Phi/\mathrm{d}t(t) - e_0^B + e_S^B = 0$, where $Q := \int_{\mathscr{L}} q$ and $\Phi := \int_{\mathscr{L}} \phi$ are the total charge and the total flux of the transmission line, respectively.

Remark 1. Note that for a non-quadratic electro-magnetic energy $\mathscr{H}$ we may still define the co-energy variables $e^q = \delta_q H$ and $e^\phi = \delta_\phi H$. In this way we can obtain *non-linear* models for the transmission line (which still are in port-Hamiltonian form); cf. van der Schaft and Maschke (2002).

The spatial discretization procedure as proposed in this section consists of two steps. First, the interconnection structure of the distributed parameter model is spatially discretized, and secondly the constitutive relations of the energy storage part of the transmission line are approximated. In Sections 3.1, 3.2 these two steps are applied to a single discretized lump of the transmission line. Afterwards in

Table 1

Variable	Meaning	Space
q	Infinitesimal charge	$\Omega^1(\mathscr{L})$
ϕ	Infinitesimal flux	$\Omega^1(\mathscr{L})$
f^q	Infinitesimal charge rate	$\Omega^1(\mathscr{L})$
f^ϕ	Infinitesimal flux rate	$\Omega^1(\mathscr{L})$
e^q	Voltage distribution	$\Omega^0(\mathscr{L})$
e^ϕ	Current distribution	$\Omega^0(\mathscr{L})$
f_0^B	Boundary current at left end	$\mathbb{R}$
f_S^B	Boundary current at right end	$\mathbb{R}$
e_0^B	Boundary voltage at left end	$\mathbb{R}$
e_S^B	Boundary voltage at right end	$\mathbb{R}$
$C(z)$	Distributed capacitance	$C^\infty(\mathscr{L})$
$L(z)$	Distributed inductance	$C^\infty(\mathscr{L})$

Section 3.3 we show how to apply the strategy to the transmission line split into multiple parts. Finally, the choice of the approximating functions in order to preserve additional physical properties of the system is discussed in Section 3.4.

3.1. Spatial discretization of the interconnection structure

Consider a part of the transmission line between two points a and b ($0 \leqslant a < b \leqslant S$). The spatial manifold corresponding to this part of transmission line is $Z_{ab} = [a,b]$. The voltage at the point a is denoted by e_a^B and the current at the point is denoted by f_a^B. Similarly, the voltage and current at the point b are denoted by e_b^B and f_b^B, respectively. The relations between the boundary variables f_a^B, e_a^B, f_b^B, e_b^B and the efforts e^q, e^ϕ are by (7b)

$$e_a^B(t) = e^q(t,a), \quad e_b^B(t) = e^q(t,b), \tag{8a}$$

$$f_a^B(t) = e^\phi(t,a), \quad f_b^B(t) = e^\phi(t,b). \tag{8b}$$

Our discretization method is based on the following:

Assumption 1 (Approximation of f^q and f^ϕ). The infinitesimal charge rate f^q and the infinitesimal flux rate f^ϕ are approximated on $\mathscr{L}_{ab}$ as

$$f^q(t,z) = f_{ab}^q(t)\omega_{ab}^q(z), \quad (\text{Thus: } \omega_{ab}^q(z)) \tag{9a}$$

$$f^\phi(t,z) = f_{ab}^\phi(t)\omega_{ab}^\phi(z), \tag{9b}$$

where the one-forms $\omega_{ab}^q, \omega_{ab}^\phi$ satisfy

$$\int_{\mathscr{L}_{ab}} \omega_{ab}^q = 1, \quad \int_{\mathscr{L}_{ab}} \omega_{ab}^\phi = 1. \tag{10}$$

Remark 2. Since the infinitesimal charge q belongs to the same space as the flow f^q it will be approximated as $q(t,z) = Q_{ab}(t)\omega_{ab}^q(z)$. Hence by condition (10) Q_{ab} equals the total charge of the considered lump of the transmission line. Similarly for the total flux.

Assumption 2 (Approximation of e^q and e^ϕ). The co-energy variables voltage $e^q(t,z)$ and current $e^\phi(t,z)$ are approximated as

$$e^q(t,z) = e_a^q(t)\omega_a^q(z) + e_b^q(t)\omega_b^q(z),$$ (11a)

$$e^\phi(t,z) = e_a^\phi(t)\omega_a^\phi(z) + e_a^\phi(t)\omega_b^\phi(z),$$ (11b)

where the zero-forms $\omega_a^q, \omega_b^q, \omega_a^\phi, \omega_b^\phi \in \Omega^0(\mathcal{L}_{ab})$ satisfy

$$\omega_a^q(a) = 1, \quad \omega_a^q(b) = 0, \quad \omega_b^q(a) = 0, \quad \omega_b^q(b) = 1,$$ (12a)

$$\omega_a^\phi(a) = 1, \quad \omega_a^\phi(b) = 0, \quad \omega_b^\phi(a) = 0, \quad \omega_b^\phi(b) = 1.$$ (12b)

Remark 3. Note that the variables $e^q(t,z)$, $e^\phi(t,z)$ at the point a equal $e_a^B(t)$, $f_a^B(t)$, respectively, and at the point b equal $e_b^B(t)$, $f_a^B(t)$. Hence the effort $e^q(t,z)$ cannot be approximated with only one zero-form since otherwise e_a^B and e_b^B are dependent. Similarly for $e^\phi(t,z)$.

Inserting (9a), (11b) and (9b),(11a) into (7a) gives

$$f_{ab}^q(t)\omega_{ab}^q(z) = -e_a^\phi(t)\,d\omega_a^q(z) - e_b^\phi(t)\,d\omega_b^q(z),$$ (13a)

$$f_{ab}^\phi(t)\omega_{ab}^\phi(z) = -e_a^q(t)\,d\omega_a^q(z) - e_b^q(t)\,d\omega_b^q(z).$$ (13b)

Assumption 3 (Compatibility of forms). (1) The one-form ω_{ab}^q and functions ω_a^ϕ, ω_b^ϕ should be chosen in a such way that for every e_a^ϕ, e_b^ϕ we can find f_{ab}^q such that (13a) is satisfied.

(2) The one-form ω_{ab}^ϕ and functions ω_a^q, ω_b^q should be chosen in a such way that for every e_a^q, e_b^q we can find f_{ab}^ϕ such that (13b) is satisfied.

Assumption 3 implies the following relations between the one-forms ω_{ab}^q chosen in Assumption 1 and the functions ω_a^ϕ, ω_b^ϕ chosen in Assumption 2. Take $e_b^\phi = 0$. Then (13a) is true if and only if $d\omega_a^\phi = c\omega_{ab}^q$, for a constant c. Integrating this over $\mathcal{L}_{ab}$ yields $\omega_a^\phi(b) - \omega_a^\phi(a) = c\int_{\mathcal{L}_{ab}}\omega_{ab}^q$. Taking into account relations (12a), (10), one proves that $c = -1$. Therefore

$$d\omega_a^\phi = -\omega_{ab}^q.$$ (14a)

On the other hand, by choosing $e_a^\phi = 0$, one proves that

$$d\omega_b^\phi = \omega_{ab}^q.$$ (14b)

Using a similar argument as above, it can be shown that

$$d\omega_a^q = -\omega_{ab}^\phi, \quad d\omega_b^q = \omega_{ab}^\phi.$$ (14c)

We conclude that as a consequence of Assumption 3 the functions $\omega_a^q, \omega_b^q, \omega_a^\phi, \omega_b^\phi$ are completely *determined* by the one-forms $\omega_{ab}^q, \omega_{ab}^\phi$. For later use we state some additional properties of the basis functions and zero forms.

Proposition 1. $\omega_a^q, \omega_b^q, \omega_a^\phi, \omega_b^\phi, \omega_{ab}^q, \omega_{ab}^\phi$ *satisfy*

(i) $\omega_a^q(z) + \omega_b^q(z) = 1,$

(ii) $\omega_a^\phi(z) + \omega_b^\phi(z) = 1,$

(iii) $\int_{\mathcal{L}_{ab}} \omega_a^q(z)\omega_{ab}^q(z) + \int_{\mathcal{L}_{ab}} \omega_b^q(z)\omega_{ab}^q(z) = 1,$

(iv) $\int_{\mathcal{L}_{ab}} \omega_a^\phi(z)\omega_{ab}^\phi(z) + \int_{\mathcal{L}_{ab}} \omega_b^\phi(z)\omega_{ab}^\phi(z) = 1,$

(v) $\int_{\mathcal{L}_{ab}} \omega_a^q(z)\omega_{ab}^q(z) + \int_{\mathcal{L}_{ab}} \omega_a^\phi(z)\omega_{ab}^\phi(z) = 1.$

Proof.

(i) $d(\omega_a^q + \omega_b^q) \underset{(14c)}{=} 0 \Rightarrow \omega_a^q(z) + \omega_b^q(z)$

$$= \omega_a^q(a) + \omega_b^q(a) \underset{(12a)}{=} 1,$$

(ii) $d(\omega_a^\phi + \omega_b^\phi) \underset{(14a),(14b)}{=} 0 \Rightarrow \omega_a^\phi(z) + \omega_b^\phi(z)$

$$= \omega_a^\phi(0) + \omega_b^\phi(0) \underset{(12b)}{=} 1,$$

(iii) $\int_{\mathcal{L}_{ab}} \omega_a^q(z)\omega_{ab}^q(z) + \int_{\mathcal{L}_{ab}} \omega_b^q(z)\omega_{ab}^q(z)$

$$= \int_{\mathcal{L}_{ab}} (\omega_a^q(z) + \omega_b^q(z))\omega_{ab}^q(z) \underset{(i)}{=} \int_{\mathcal{L}_{ab}} \omega_{ab}^q \underset{(10)}{=} 1,$$

(iv) $\int_{\mathcal{L}_{ab}} \omega_a^\phi(z)\omega_{ab}^\phi(z) + \int_{\mathcal{L}_{ab}} \omega_b^\phi(z)\omega_{ab}^\phi(z)$

$$= \int_{\mathcal{L}_{ab}} (\omega_a^\phi(z) + \omega_b^\phi(z))\omega_{ab}^\phi(z) \underset{(ii)}{=} \int_{\mathcal{L}_{ab}} \omega_{ab}^\phi \underset{(10)}{=} 1,$$

(v) $\int_{\mathcal{L}_{ab}} \omega_a^q(z)\omega_{ab}^q(z) + \int_{\mathcal{L}_{ab}} \omega_a^\phi(z)\omega_{ab}^\phi(z)$

$$\underset{(14a),(14c)}{=} -\int_{\mathcal{L}_{ab}} d(\omega_a^q\omega_a^\phi) = \omega_a^q(a)\omega_a^\phi(a)$$

$$- \omega_a^q(b)\omega_a^\phi(b) \underset{(12)}{=} 1. \qquad \square$$

Remark 4. The simplest choice for the functions $\omega_a^q, \omega_b^q, \omega_a^\phi, \omega_b^\phi$ and one-forms $\omega_{ab}^q, \omega_{ab}^\phi$ satisfying all the conditions are *linear splines*. Indeed, we may take $\omega_a^q, \omega_b^q, \omega_a^\phi, \omega_b^\phi$ to be the linear splines defined by (12), and the one-forms $\omega_{ab}^q, \omega_{ab}^\phi$ to be the one-forms with corresponding density function equal to $1/(b - a)$, and zero outside. From a differential-geometric point of view this corresponds to the Whitney zero-forms and one-forms, see Bossavit (1991, 1998).

The spatially discretized interconnection structure corresponding to the considered part of the transmission line is obtained as follows. Inserting (14a), (14b) into (13a) gives $f_{ab}^q(t)\omega_{ab}^q(z) = e_a^\phi(t)\omega_{ab}^q(z) - e_b^\phi(t)\omega_{ab}^q(z)$. Integrating this over $\mathcal{L}_{ab}$ gives

$$f_{ab}^q(t) = e_a^\phi(t) - e_b^\phi(t).$$ (15a)

Similarly using (13b) one proves that

$$f_{ab}^\phi(t) = e_a^q(t) - e_b^q(t). \tag{15b}$$

For the sake of clarity, the argument t is omitted in the rest of this subsection. The relations describing the spatially discretized interconnection structure of the part of transmission line are thus given by (see Remark 3 and Eqs. (8) and (8b))

$$\begin{bmatrix} e_a^B \\ e_b^B \\ f_a^B \\ f_b^B \\ f_{ab}^q \\ f_{ab}^\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} e_a^q \\ e_b^q \\ e_a^\phi \\ e_b^\phi \end{bmatrix}. \tag{16}$$

The net power of the considered part of the transmission line is

$$\int_{\mathscr{Z}_{ab}} e^q(z) f^q(z) + \int_{\mathscr{Z}_{ab}} e^\phi(z) f^\phi(z) - e_a^B f_a^B + e_b^B f_b^B. \tag{17}$$

Inserting (9a), (11) into (17), where the properties (iii), (iv), (v) of Proposition 1 are taken into account, gives

$$P_{ab}^{net} = [\alpha_{ab} e_a^q + (1 - \alpha_{ab}) e_b^q] f_{ab}^q + [(1 - \alpha_{ab}) e_a^\phi$$
$$+ \alpha_{ab} e_b^\phi] f_{ab}^\phi - e_a^B f_a^B + e_b^B f_b^B, \tag{18}$$

where $\alpha_{ab} := \int_{\mathscr{Z}_{ab}} \omega_a^q(z) \omega_{ab}^q(z)$. Expression (18) is used for the identification of the port variables of the discretized interconnection structure. The third term and the fourth term on the right side of (18) imply that the port variables of the incoming port are (f_a^B, e_a^B) and that the port variables of the outgoing port are (f_b^B, e_b^B). The first term, the power supplied to or taken from the electrical part of the part of the transmission line, implies that the flow variable of the electric port is f_{ab}^q and that the effort variable is $\alpha_{ab} e_a^q + (1 - \alpha_{ab}) e_b^q$. Similarly, the second term implies that the port variables of the magnetic port are $(f_{ab}^\phi, (1 - \alpha_{ab}) e_a^\phi + \alpha_{ab} e_b^\phi)$. Thus by defining

$$e_{ab}^q := \alpha_{ab} e_a^q + (1 - \alpha_{ab}) e_b^q,$$
$$e_{ab}^\phi := (1 - \alpha_{ab}) e_a^\phi + \alpha_{ab} e_b^\phi \tag{19}$$

the expression for P_{ab}^{net} becomes

$$P_{ab}^{net} = \langle e_{ab} | f_{ab} \rangle = f_{ab}^q e_{ab}^q + f_{ab}^\phi e_{ab}^\phi - e_a^B f_a^B + e_b^B f_b^B, \tag{20}$$

where $f_{ab} = [f_{ab}^q, f_{ab}^\phi, f_a^B, f_b^B]^T$, $e_{ab} = [e_{ab}^q, e_{ab}^\phi, e_a^B, e_b^B]^T$.

Remark 5. Observe that in contrast to (20) the expression (18) does not define a scalar pairing as in Definition 1 since

$P_{ab}^{net} = 0$ for every $(f_{ab}^q, f_{ab}^\phi, f_a^B, f_b^B)$ does not imply that the vector $(e_a^q, e_b^q, e_a^\phi, e_b^\phi, e_a^B, e_b^B)$ is zero.

Elimination of $e_a^q, e_b^q, e_a^\phi, e_b^\phi$ from (16), (19) gives

$$\underbrace{\begin{bmatrix} -1 & 0 & \alpha_{ab} & \alpha_{ba} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{E_{ab}} \begin{bmatrix} e_{ab}^q \\ e_{ab}^\phi \\ e_a^B \\ e_b^B \end{bmatrix}$$
$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{ba} & \alpha_{ab} \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{F_{ab}} \begin{bmatrix} f_{ab}^q \\ f_{ab}^\phi \\ f_a^B \\ f_b^B \end{bmatrix} = 0, \tag{21}$$

where $\alpha_{ba} = 1 - \alpha_{ab}$. Eq. (21) represents the spatially discretized interconnection structure, which we abbreviate to

$$\mathscr{D}_{ab} = \{(f_{ab}, e_{ab}) \in \mathbb{R}^8 : E_{ab} e_{ab} + F_{ab} f_{ab} = 0\}. \tag{22}$$

Proposition 2. *The subspace $\mathscr{D}_{ab}$ is a Dirac structure with respect to the bilinear form*

$$\langle\langle (f_{ab}^1, e_{ab}^1), (f_{ab}^2, e_{ab}^2) \rangle\rangle = \langle e_{ab}^1, f_{ab}^2 \rangle + \langle e_{ab}^2, f_{ab}^1 \rangle. \tag{23}$$

Proof. $\mathscr{D}_{ab}$ is a Dirac structure with respect to the bilinear form given by (23) if and only if (van der Schaft & Maschke, 1995; Dalsmo & van der Schaft, 1999; van der Schaft, 2000a):
  (i) Rank condition: $rank[F_{ab} E_{ab}] = 4$.
  (ii) Zero net power: $F_{ab} R E_{ab}^T + E_{ab} R F_{ab}^T = 0$, where the matrix $R = diag\{1, 1, -1, 1\}$ represents the signs in the expression for the net power P_{ab}^{net} described by (20).
  Both conditions are easily checked. □

3.2. Approximation of the energy part of the transmission line

After the spatial discretization of the interconnection structure, the next step is to discretize the constitutive relations of the energy storage. Recall that in the port-Hamiltonian representation the system is specified by its Dirac structure—the interconnection structure-, together with its Hamiltonian—the constitutive relations of the energy storage.

Both the flow variables f^q and f^ϕ and the energy variables q and ϕ are one-forms. Hence, since f^q and f^ϕ are approximated by (9), and they are related to q and ϕ by (7d), it is consistent to approximate q and ϕ on $\mathscr{Z}_{ab}$ in the

same way by

$$q(t,z) = Q_{ab}(t)\omega^q_{ab}(z), \quad \phi(t,z) = \Phi_{ab}(t)\omega^\phi_{ab}(z), \quad (24)$$

where

$$\frac{dQ_{ab}(t)}{dt} = f^q_{ab}(t), \quad \frac{d\Phi_{ab}(t)}{dt} = f^\phi_{ab}(t). \quad (25)$$

Observe that Q_{ab} represents the total amount of charge of the considered part of the transmission line and Φ_{ab} represents the total amount of flux of the same part (see also Remark 2).

The electric energy as a function of the energy variable q is given by (see (7c)) $\int_{\mathscr{L}_{ab}} *(q(t,z)/2C(z))q(t,z)$. Approximation of the infinite-dimensional energy variable q by (24) means that we restrict the infinite-dimensional space of one-forms $\Omega^1(\mathscr{L}_{ab})$ to its one-dimensional subspace spanned by ω^q_{ab}. This leads to the approximation of the electric energy of the considered part of the transmission line by

$$H^q_{ab}(Q_{ab}(t)) = \frac{Q^2_{ab}(t)}{2C_{ab}}, \quad (26a)$$

where

$$C^{-1}_{ab} := \int_{\mathscr{L}_{ab}} * \left(\frac{\omega^q_{ab}(z)}{C(z)} \right) \omega^q_{ab}(z). \quad (26b)$$

Note that this is nothing else than the restriction of the electric energy function to the one-dimensional subspace of $\Omega^1(\mathscr{L}_{ab})$ spanned by ω^q_{ab}. Similarly, the magnetic energy is approximated by

$$H^q_{ab}(\Phi_{ab}(t)) = \frac{\Phi^2_{ab}(t)}{2L_{ab}}, \quad (27a)$$

where

$$L^{-1}_{ab} := \int_{\mathscr{L}_{ab}} * \left(\frac{\omega^\phi_{ab}(z)}{C(z)} \right) \omega^\phi_{ab}(z). \quad (27b)$$

Therefore, the total energy of the considered part of the transmission line is approximated by

$$H_{ab}(Q_{ab}, \Phi_{ab}) = H^q_{ab}(Q_{ab}) + H^\phi_{ab}(\Phi_{ab}) = \frac{Q^2_{ab}}{2C_{ab}} + \frac{\Phi^2_{ab}}{2L_{ab}}. \quad (28)$$

In order to define the discretized dynamics, we equate the discretized effort variables e^q_{ab}, e^ϕ_{ab} of the discretized interconnection structure as defined in (19) with the co-energy variables corresponding to the total approximated energy H_{ab} of the considered part of the transmission line:

$$e^q_{ab}(t) = \frac{\partial H_{ab}(Q_{ab}, \Phi_{ab})}{\partial Q_{ab}}(t) = \frac{Q_{ab}(t)}{C_{ab}},$$

$$e^\phi_{ab}(t) = \frac{\partial H_{ab}(Q_{ab}, \Phi_{ab})}{\partial \Phi_{ab}}(t) = \frac{\Phi_{ab}(t)}{L_{ab}}. \quad (29)$$

Eqs. (21) (interconnection structure) and (25),(29) (constitutive relations of the magnetic and electric ports) represent a finite-dimensional model of the transmission line. As noted before the discretized interconnection structure is a (finite-dimensional) Dirac structure. Hence (21), (25), (29) represents a finite-dimensional port-Hamiltonian system (see e.g. van der Schaft & Maschke, 1995; van der Schaft, 2000a) with Hamiltonian given by (28).

Remark 6. The usual physical approximation of the transmission line by an LC-circuit corresponds to the case α_{ab} equal to 0 or 1, depending on the ordering of the capacitor and the inductor. The corresponding choice of the one-forms ω^q_{ab} and ω^ϕ_{ab} is the Dirac distribution at a, while the choice for the functions ω^q_a, ω^q_b and ω^ϕ_a, ω^ϕ_b is either the square pulse on $\mathscr{L}_{ab}$ with height $1/(b-a)$ or the zero-function.

Note that the approximation as given above can be also performed if the Hamiltonian of the transmission line is *not quadratic* in $q(t,z)$ and $\phi(t,z)$, cf. Remark 1.

3.3. Spatial discretization of the transmission line

The transmission line is split into n parts. The ith part (S_{i-1}, S_i) is discretized as explained in the previous two subsections, where $a = S_{i-1}$ and $b = S_i$. The resulting model consists of n sub-models where each of them represents a port-Hamiltonian system. Since the power connection of port-Hamiltonian systems is again a port-Hamiltonian system (see van der Schaft, 1999) the total discretized system is also a port-Hamiltonian system, whose interconnection structure is given by the composition of the n Dirac structures on (S_{i-1}, S_i), while the total Hamiltonian is given by the sum of the individual Hamiltonians

$$H(Q, \Phi) = \sum_{i=1}^{n} \frac{Q^2_{S_{i-1}, S_i}}{2C_{S_{i-1}, S_i}} + \sum_{i=1}^{n} \frac{\Phi^2_{S_{i-1}, S_i}}{2L_{S_{i-1}, S_i}}.$$

Here $Q = (Q_{S_0, S_1}, Q_{S_1, S_2}, \ldots, Q_{S_{n-1}, S_n})^{\mathrm{T}}$ are the discretized charges and $\Phi = (\Phi_{S_0, S_1}, \Phi_{S_1, S_2}, \ldots, \Phi_{S_{n-1}, S_n})^{\mathrm{T}}$ is the vector of discretized fluxes.

The total discretized model still has two ports. The port $(f^B_{S_0}, e^B_{S_0}) = (f^B_0, e^B_0)$ is incoming and the port $(f^B_{S_n}, e^B_{S_n}) = (f^B_S, e^B_S)$ is outgoing, resulting in the energy balance for the discretized model

$$\frac{dH(Q(t), \Phi(t))}{dt} - e^B_0 f^B_0 + e^B_S f^B_S = 0.$$

Eq. (15a) for the ith part becomes $f^q_{S_{i-1}, S_i}(t) = e^\phi_{S_{i-1}}(t) - e_{S^\phi_i}(t)$. Taking into account (25) and $e^\phi_{S_0} = f^B_0$, $e^\phi_{S_n} = f^B_S$, we have $dQ(t)/dt = f^B_0 - f^B_S$, where $Q := \sum_{i=1}^{n} Q_{S_{i-1}, S_i}$ is the *total charge* of the transmission line. This represents charge conservation. Another conserved quantity (flux conservation) is obtained from Eqs. (15b) and (25), i.e.

$d\Phi(t)/dt = e_0^B - e_S^B$, where $\Phi = \sum_{i=1}^{n} \Phi_{S_{i-1},S_i}$ represents the *total flux* of the transmission line.

3.4. Choice of approximating one-forms

In this section we show how to choose the approximating one-forms satisfying (10) in such a way that additional physical characteristics of the system are taken into account. First we note that by (7e) $q(t,z) = C(z) * e^q(t,z)$. Hence, since $*e^q(t,z) = e^q(t,z)\,dz$, the total charge of the considered part of the line is also given by

$$Q_{ab}(t) = \int_{\mathscr{L}_{ab}} q(t,z) = \int_{\mathscr{L}_{ab}} C(z)e^q(t,z)\,dz. \tag{30}$$

Approximating the effort variable $e^q(t,z)$ on the right-hand side as in Assumption 2 yields

$$\int_{\mathscr{L}_{ab}} C(z)e^q(t,z)$$

$$= \left(\int_{\mathscr{L}_{ab}} C(z)\omega_a^q(z)\,dz \right) e_a^q(t)$$

$$+ \left(\int_{\mathscr{L}_{ab}} C(z)\omega_b^q(z)\,dz \right) e_b^q(t).$$

On the other hand, the left-hand side of (30) is by (29) equal to

$$Q_{ab}(t) = C_{ab}\alpha_{ab}e_a^q(t) + C_{ab}(1 - \alpha_{ab})e_b^q(t). \tag{31}$$

Equating both sides gives

$$C_{ab}\alpha_{ab} = \int_{\mathscr{L}_{ab}} C(z)\omega_a^q(z)\,dz,$$

$$C_{ab}(1 - \alpha_{ab}) = \int_{\mathscr{L}_{ab}} C(z)\omega_b^q(z)\,dz. \tag{32}$$

Similarly, the total flux of the part of the transmission line is given by

$$\Phi_{ab}(t) = \int_{\mathscr{L}_{ab}} \phi(t,z) = \int_{\mathscr{L}_{ab}} L(z)e^\phi(t,z)\,dz \tag{33}$$

and we may impose the following conditions on ω_a^ϕ, ω_b^ϕ

$$L_{ab}\alpha_{ab} = \int_{\mathscr{L}_{ab}} L(z)\omega_a^\phi(z)\,dz,$$

$$L_{ab}(1 - \alpha_{ab}) = \int_{\mathscr{L}_{ab}} L(z)\omega_b^\phi(z)\,dz. \tag{34}$$

Proposition 3. *The conditions* (32) *and* (34) *are satisfied if and only if*

(i) $\alpha_{ab} = \int_{\mathscr{L}_{ab}} \dfrac{C(z)}{C_{ab}}\,dz,$ (ii) $C_{ab} = \int_{\mathscr{L}_{ab}} C(z)\,dz,$

(iii) $\alpha_{ab} = \int_{\mathscr{L}_{ab}} \dfrac{L(z)}{L_{ab}}\,dz,$ (iv) $L_{ab} = \int_{\mathscr{L}_{ab}} L(z)\,dz.$

Proof. Condition (i) is actually a rewritten version of the first part of (32). Condition (ii) is obtained by summing the terms of (32) and taking into account condition (i) of Proposition 1. Conversely, multiplying (i) with C_{ab} and subtracting the obtained result from (ii) gives the second part of (32), since condition (i) of Proposition 1 holds. Similarly (34) implies conditions (iii) and (iv) and vice-versa. □

The following choice of the one-forms ω_{ab}^q, ω_{ab}^ϕ satisfies Assumption 1 and conditions (i)–(iv):

$$\omega_{ab}^q = \frac{C(z)\,dz}{\int_{\mathscr{L}_{ab}} C(z)\,dz}, \quad \omega_{ab}^\phi = \frac{L(z)\,dz}{\int_{\mathscr{L}_{ab}} L(z)\,dz}. \tag{35}$$

Remark 7. The choice (35) takes into account the capacitance and inductance functions $C(z)$ and $L(z)$. If they are constant then ω_{ab}^q and ω_{ab}^ϕ are one-forms corresponding to the constant density function with value $1/(b-a)$ (cf. Remark 6). However, if $C(z)$ and $L(z)$ are not constant then so are ω_{ab}^q and ω_{ab}^ϕ; see Section 5.

4. Two-dimensional case

In this section we indicate that the discretization procedure used for one-dimensional distributed-parameter port-Hamiltonian systems has a natural extension to higher-dimensional cases, by explicitly showing the extension to the two-dimensional wave equation.

4.1. Port-Hamiltonian formulation of the wave equation

Consider the wave equation $\mu\ddot{u} + E\Delta u = 0, u(t,z) \in \mathbb{R}$, $z = (z_1, z_2) \in \mathscr{Z}$, where μ is the mass density, E is Young's modulus, Δ is the (two-dimensional) Laplacian operator, and $\mathscr{Z}$ is a two-dimensional spatial domain with boundary. This equation, for example, models the vertical movement $u(t,z_1,z_2)$ of a vibrating membrane. It can be formulated as a port-Hamiltonian system with boundary port variables as follows (see Talasila et al., 2002, for details, as well as for the port-Hamiltonian formulation of the general n-dimensional wave equation).

The energy variables are the 2-form kinetic momentum $p(t,z_1,z_2)$, and the 1-form elastic strain $\varepsilon(t,z_1,z_2)(:= \partial u/\partial z_1\,dz_1 + \partial u/\partial z_2\,dz_2)$. The co-energy variables are the 0-form velocity $v(t,z_1,z_2) = \partial\mathscr{H}/\partial p$ and the 1-form stress $\sigma(t,z_1,z_2) = \partial\mathscr{H}/\partial\varepsilon$. Here $\mathscr{H}$ is the energy density defined as $\mathscr{H}(p,\varepsilon) = \frac{1}{2}(\varepsilon \wedge \sigma + p \wedge v)$, where $\wedge$ is the wedge product, and thus the co-energy variables σ and v are related to the energy variables by the constitutive relations $\sigma = E * \varepsilon, v = 1/\mu * p$, where $*$ denotes the Hodge star operator corresponding to a choice of the Riemannian metric on $\mathscr{Z}$. The wave equation can be represented as the port-Hamiltonian system (note the opposite sign convention

with respect to (2))

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \varepsilon} \\ \frac{\partial \mathcal{H}}{\partial p} \end{bmatrix},$$

$$\begin{bmatrix} v^{\mathrm{B}} \\ \sigma^{\mathrm{B}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \mathcal{H}}{\partial \varepsilon}|_{\partial \mathscr{L}} \\ \frac{\partial \mathcal{H}}{\partial p}|_{\partial \mathscr{L}} \end{bmatrix} \tag{36}$$

with $d\varepsilon = 0$. Indeed, substituting $\varepsilon = du$ in the first part of (36) we obtain $d(\dot{u} - 1/\mu * p) = 0$, and thus $\dot{u} = 1/\mu * p + f(t)$. Without loss of generality we may set $f(t) = 0$ since $f(t)$ is the motion of the membrane as a whole. Next we write the second part of (36) as $* \dot{p} = - * d(E * \varepsilon)$ and substitute $* p = \mu \dot{u}$. This yields (due to $\varepsilon = du$) $\mu \ddot{u} + E(*d * d)u = 0$ which is the geometric version of the wave equation. The second expression in (36) defines the boundary flow variables v^{B} and boundary effort variables σ^{B} as the velocity $v = \partial \mathcal{H} / \partial p$, respectively the stress $\sigma = \partial \mathcal{H} / \partial \varepsilon$, restricted to the boundary.

4.2. Interconnection structure

Passing from the *interval* grid for the one-dimensional case, we move onto the simplest possible grid for the two-dimensional example—the *triangular* grid. We denote by $\mathscr{L}_{abc}$ the triangular grid defined by the three points a, b, c. The edges of the grid are denoted as $\{ab, bc, ca\}$, and the facet as abc.

Assumption 4 (Approximation of flow variables). The two-form rate kinetic momentum f^p and the one-form rate elastic strain f^ε are approximated on $\mathscr{L}_{abc}$ as

$$f^p(t, z) = f_{abc}^p(t) w_{abc}^p(z), \tag{37a}$$

$$f^\varepsilon(t, z) = f_{ab}^\varepsilon(t) w_{ab}^\varepsilon(z) + f_{bc}^\varepsilon(t) w_{bc}^\varepsilon(z) + f_{ca}^\varepsilon(t) w_{ca}^\varepsilon(z), \tag{37b}$$

where the one-forms $w_l^\varepsilon(z)$, $l \in \{ab, bc, ca\}$, and the two-forms $w_{abc}^p(z)$ satisfy the following conditions:

$$\int_{l'} w_l^\varepsilon = \begin{cases} 0 & \text{if } l \neq l', \\ 1 & \text{if } l = l', \end{cases} \qquad \int_{\mathscr{L}_{abc}} w_{abc}^p = 1.$$

Assumption 5 (Approximation of efforts). The effort variables, the zero-form velocity e^v, and the one-form elastic stress e^σ, are approximated as

$$e^v(t, z) = e_a^v(t) w_a^v(z) + e_b^v(t) w_b^v(z) + e_c^v(t) w_c^v(z), \tag{38a}$$

$$e^\sigma(t, z) = e_{ab}^\sigma(t) w_{ab}^\sigma(z) + e_{bc}^\sigma(t) w_{bc}^\sigma(z) + e_{ca}^\sigma(t) w_{ca}^\sigma(z), \tag{38b}$$

where the following conditions need to be satisfied

$$\int_{l'} w_l^\sigma = \begin{cases} 0 & \text{if } l \neq l', \\ 1 & \text{if } l = l', \end{cases} \qquad w_x^v(y) = \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$$

with $l, l' \in \{ab, bc, ca\}$, $x \in \{a, b, c\}$ and $y \in \{a, b, c\}$.

Assumption 6 (Boundary variables). The boundary port variables are the zero-form velocity $f^{\mathrm{B}}(t, z)$ and the one-form stress $e^{\mathrm{B}}(t, z)$. They are approximated on $\partial \mathscr{L}_{abc}$ as

$$f^{\mathrm{B}} = f_a^{\mathrm{B}}(t) w_a^v(z) + f_b^{\mathrm{B}}(t) w_b^v(z) + f_c^{\mathrm{B}}(t) w_c^v(z), \tag{39a}$$

$$e^{\mathrm{B}} = e_{ab}^{\mathrm{B}}(t) w_{ab}^\sigma(z) + e_{bc}^{\mathrm{B}}(t) w_{bc}^\sigma(z) + e_{ca}^{\mathrm{B}}(t) w_{ca}^\sigma(z). \tag{39b}$$

It is clear that in this way Remarks 2 and 3 generalize to the two-dimensional case.

Inserting the corresponding variables into Eq. (36) we obtain the following relations:

$$f_{abc}^p(t) w_{abc}^p(z) = \\ -d(e_{ab}^\sigma(t) w_{ab}^\sigma(z) + e_{bc}^\sigma(t) w_{bc}^\sigma(z) + e_{ca}^\sigma(t) w_{ca}^\sigma(z)), \tag{40a}$$

$$f_{ab}^\varepsilon(t) w_{ab}^\varepsilon(z) + f_{bc}^\varepsilon(t) w_{bc}^\varepsilon(z) + f_{ca}^\varepsilon(t) w_{ca}^\varepsilon(z) \\ = d(e_a^v(t) w_a^v(z) + e_b^v(t) w_b^v(z) + e_c^v(t) w_c^v(z)). \tag{40b}$$

Assumption 7 (Compatibility of forms). The two-form $w_{abc}^p(z)$ and one-forms $w_{ab}^\sigma(z), w_{bc}^\sigma(z), w_{ca}^\sigma(z)$ should be chosen such that for every $e_{ab}^\sigma(t), e_{bc}^\sigma(t), e_{ca}^\sigma(t)$ we can find $f_{abc}^p(t)$ such that (40a) is satisfied.

A similar compatibility is demanded for (40b).

Suppose that $e_{bc}^\sigma(t) = e_{ca}^\sigma(t) = 0$, then (40a) is satisfied for every $e_{ab}^\sigma(t)$ if and only if $w_{abc}^p(z) = \gamma_a \, dw_{ab}^\sigma(z)$. Hence, $\int_{\mathscr{L}_{abc}} w_{abc}^p(z) = \gamma_a \int_{\mathscr{L}_{abc}} dw_{ab}^\sigma(z) = \gamma_a \int_{\partial \mathscr{L}_{abc}} w_{ab}^\sigma(z)$ implying that $\gamma_a = 1$. Therefore,

$$dw_{ab}^\sigma(z) = w_{abc}^p(z) \tag{41a}$$

and similarly

$$dw_{bc}^\sigma(z) = w_{abc}^p(z), \qquad dw_{ca}^\sigma(z) = w_{abc}^p(z). \tag{41b}$$

Inserting (41a), (41b) and integrating over $\mathscr{L}_{abc}$ gives

$$f_{abc}^p(t) = -e_{ab}^\sigma(t) - e_{bc}^\sigma(t) - e_{ca}^\sigma(t). \tag{42}$$

Taking $e_b^v(t) = e_c^v(t) = 0$ Eq. (40b) is satisfied if and only if $dw_a^v(z) = \gamma_{ab} w_{ab}^\varepsilon(z) + \gamma_{bc} w_{bc}^\varepsilon(z) + \gamma_{ca} w_{ca}^\varepsilon(z)$. Thus $\int_{ab} dw_a^v(z) = \int_{ab} (\gamma_{ab} w_{ab}^\varepsilon(z) + \gamma_{bc} w_{bc}^\varepsilon(z) + \gamma_{ca} w_{ca}^\varepsilon(z))$ yielding $\gamma_{ab} = -1$. Similarly integrating along bc and ca gives $\gamma_{bc} = 0$ and $\gamma_{ca} = 1$. Therefore,

$$dw_a^v(z) = w_{ca}^\varepsilon(z) - w_{ab}^\varepsilon(z). \tag{43a}$$

Similarly we can show that

$$dw_b^v(z) = w_{ab}^\varepsilon(z) - w_{bc}^\varepsilon(z),$$

$$dw_c^v(z) = w_{bc}^\varepsilon(z) - w_{ca}^\varepsilon(z). \tag{43b}$$

Substituting (43a), (43b) into (40b) we obtain

$$f_{ab}^{\varepsilon}(t) = e_b^v(t) - e_a^v(t), \quad f_{bc}^{\varepsilon}(t) = e_c^v(t) - e_b^v(t),$$

$$f_{ca}^{\varepsilon}(t) = e_a^v(t) - e_c^v(t). \tag{44}$$

Finally, the boundary conditions imply

$$e_a^v(t)w_a^v(z) + e_b^v(t)w_b^v(z) + e_c^v(t)w_c^v(z)$$

$$= f_a^B(t)w_a^v(z) + f_b^B(t)w_b^v(z) + f_c^B(t)w_c^v(z),$$

$$e_{ab}^\sigma(t)w_{ab}^\sigma(z) + e_{bc}^\sigma(t)w_{bc}^\sigma(z) + e_{ca}^\sigma(t)w_{ca}^\sigma(z)$$

$$= e_{ab}^B(t)w_{ab}^\sigma(z) + e_{bc}^B(t)w_{bc}^\sigma(z) + e_{ca}^B(t)w_{ca}^\sigma(z),$$

which yields

$$e_a^v(t) = f_a^B(t), e_b^v(t) = f_b^B(t), e_c^v(t) = f_c^B(t),$$

$$e_{ab}^\sigma(t) = e_{ab}^B(t), e_{bc}^\sigma(t) = e_{bc}^B(t), e_{ca}^\sigma(t) = e_{ca}^B(t). \tag{45}$$

Combining all these relations we obtain the relations describing the spatially discretized interconnection structure of the cell, just as in (21). As before we compute the net power, and by identifying the port variables we then derive the discretized interconnection structure. The net power of the cell is the sum of the power of the kinetic domain, the elastic potential domain and the boundary, i.e. $P_{abc}^{net} = P_{abc}^p + P_{abc}^\varepsilon + P_{abc}^B$. The power corresponding to the kinetic domain is

$$P_{abc}^p(t) = \int_{\mathscr{L}_{abc}} e^v(t,z) \wedge f^p(t,z)$$

$$= f_{abc}^p(t)[\alpha_{a,abc}e_a^v(t) + \alpha_{b,abc}e_b^v(t) + \alpha_{c,abc}e_c^v(t)], \tag{46}$$

where

$$\alpha_{m,abc} := \int_{\mathscr{L}_{abc}} w_m^v(z) \wedge w_{abc}^p(z), \quad m \in \{a,b,c\}.$$

We identify the ports of the kinetic domain as $(f_{abc}^p(t), e_{abc}^p(t))$ where $e_{abc}^p(t) = \alpha_{a,abc}e_a^v(t) + \alpha_{b,abc}e_b^v(t) + \alpha_{c,abc}e_c^v(t)$. Now we look at the power corresponding to the boundary

$$P_{abc}^B(t) = \int_{\partial \mathscr{L}_{abc}} e^B(t,z) \wedge f^B(t,z)$$

$$= e_{ab}^B(t)[\beta_{a,ab}f_a^B + \beta_{b,ab}f_b^B(t) + \beta_{c,ab}f_c^B(t)]$$

$$+ e_{bc}^B(t)[\beta_{a,bc}f_a^B + \beta_{b,bc}f_b^B(t) + \beta_{c,bc}f_c^B(t)]$$

$$+ e_{ca}^B(t)[\beta_{a,ca}f_a^B + \beta_{b,ca}f_b^B(t) + \beta_{c,ca}f_c^B(t)] \tag{47}$$

with $\beta_{m,l} = \int_{\partial \mathscr{L}_{abc}} w_m^v \wedge w_l^\sigma, m \in \{a,b,c\}, l \in \{ab,bc,ca\}$. We identify the ports on the boundary as $(f_{ab}^B(t), e_{ab}^B(t))$, $(f_{bc}^B(t), e_{bc}^B(t)), (f_{ca}^B(t), e_{ca}^B(t))$, where

$$f_{ab}^B = \beta_{a,ab}f_a^B(t) + \beta_{b,ab}f_b^B(t) + \beta_{c,ab}f_c^B(t),$$

$$f_{bc}^B = \beta_{a,bc}f_a^B(t) + \beta_{b,bc}f_b^B(t) + \beta_{c,bc}f_c^B(t),$$

$$f_{ca}^B = \beta_{a,ca}f_a^B(t) + \beta_{b,ca}f_b^B(t) + \beta_{c,ca}f_c^B(t). \tag{48}$$

We need some properties of the coefficients α, β to compute the elastic potential power.

Proposition 4. *The coefficients $\alpha_{m,abc}$ and $\beta_{m,l}$ satisfy the following relations, with $l \in \{ab,bc,ca\}$:*

(i) $\alpha_{a,abc} + \alpha_{b,abc} + \alpha_{c,abc} = 1$,
(ii) $\beta_{a,l} + \beta_{b,l} + \beta_{c,l} = 1$,
(iii) $\int_{\mathscr{L}_{abc}} w_l^\sigma \wedge dw_m^v = \alpha_{m,abc} - \beta_{m,l}, m \in \{a,b,c\}$.

Proof.

(i) Eqs. (43a), (43b) imply that $d(w_a^v(z)+w_b^v(z)+w_c^v(z)) = 0 \Rightarrow w_a^v(z)+w_b^v(z)+w_c^v(z) = k, k \in \mathbb{R}$. Since $w_m^v(m) = 1$ and $w_m^v(m') = 0$ for $m \neq m'$ it follows that $w_a^v(z) + w_b^v(z) + w_c^v(z) = 1$. We have

$$\alpha_{a,abc} + \alpha_{b,abc} + \alpha_{c,abc}$$

$$= \int_{\mathscr{L}_{abc}} (w_a^v(z) + w_b^v(z) + w_c^v(z)) \wedge w_{abc}^p(z) = 1.$$

(ii) Condition (ii) follows from

$$\beta_{a,l} + \beta_{b,l} + \beta_{c,l}$$

$$= \int_{\partial \mathscr{L}_{abc}} (w_a^v(z) + w_b^v(z) + w_c^v(z)) \wedge w_l^\sigma = 1.$$

(iii) $w_l^\sigma \wedge dw_m^v = -dw_m^v \wedge w_l^\sigma = w_m^v \wedge dw_l^\sigma - d(w_m^v \wedge w_l^\sigma) = w_m^v \wedge w_{abc}^p(z) - d(w_m^v \wedge w_l^\sigma)$ and hence by Stokes' theorem

$$\int_{\mathscr{L}_{abc}} w_l^\sigma \wedge dw_m^v = \alpha_{m,abc} - \beta_{m,l}. \qquad \square$$

Proposition 5. *The flow $f^\varepsilon(t,z)$ can be rewritten as*

$$f^\varepsilon(t,z) = f_{ab}^\varepsilon(t) dw_b^v(z) - f_{ca}^\varepsilon(t) dw_c^v(z). \tag{49}$$

Proof.

$$f^\varepsilon(t,z) = f_{ab}^\varepsilon(t)w_{ab}^\varepsilon(z) + f_{bc}^\varepsilon(t)w_{bc}^\varepsilon(z) + f_{ca}^\varepsilon(t)w_{ca}^\varepsilon(z)$$

$$= f_{ab}^\varepsilon(t)(w_{ab}^\varepsilon(z) - w_{bc}^\varepsilon(z)) + f_{ca}^\varepsilon(t)(w_{ca}^\varepsilon(z) - w_{bc}^\varepsilon(z))$$

$$+ (f_{ab}^\varepsilon(t) + f_{bc}^\varepsilon(t) + f_{ca}^\varepsilon(t))w_{bc}^\varepsilon(z).$$

Since by (44) $f_{ab}^\varepsilon(t) + f_{bc}^\varepsilon(t) + f_{ca}^\varepsilon(t) = 0$, and using (43b) we have the desired result. $\square$

Finally, we compute the power $P_{abc}^\varepsilon(t)$ in the elastic potential domain as (leaving out the argument t)

$$\int_{\mathscr{L}_{abc}} e^\sigma(z) \wedge f^\varepsilon(z)$$

$$= \int_{\mathscr{L}_{abc}} (e_{ab}^\sigma w_{ab}^\sigma(z) + e_{bc}^\sigma w_{bc}^\sigma(z) + e_{ca}^\sigma w_{ca}^\sigma(z))$$

$$\wedge (f_{ab}^\varepsilon dw_b^v(z) - f_{ca}^\varepsilon dw_c^v(z))$$

$$= [f_{ab}^\varepsilon(\alpha_{b,abc} - \beta_{b,ab})e_{ab}^\sigma + (\alpha_{b,abc} - \beta_{b,ab})e_{bc}^\sigma$$

$$+ (\alpha_{b,abc} - \beta_{b,ab})e_{ca}^\sigma] + f_{ca}^\varepsilon[(\beta_{c,ab} - \alpha_{c,abc})e_{ab}^\sigma$$

$$+ (\beta_{c,bc} - \alpha_{c,abc})e_{bc}^\sigma + (\beta_{c,ca} - \alpha_{c,abc})e_{ca}^\sigma].$$

Now we identify the ports of the elastic potential domain as $(f_{ab}^\varepsilon, e_{ab}^\varepsilon)$ and $(f_{ca}^\varepsilon, e_{ca}^\varepsilon)$ where

$$e_{ab}^\varepsilon = (\alpha_{a,abc} - \beta_{b,ab})e_{ab}^\sigma + (\alpha_{b,abc} - \beta_{b,bc})e_{bc}^\sigma$$

$$+ (\alpha_{b,abc} - \beta_{b,ca})e_{ca}^\sigma, \qquad (50a)$$

$$e_{ca}^\varepsilon = (\beta_{c,ab} - \alpha_{c,abc})e_{ab}^\sigma + (\beta_{c,bc} - \alpha_{c,abc})e_{bc}^\sigma$$

$$+ (\beta_{c,ca} - \alpha_{c,abc})e_{ca}^\sigma. \qquad (50b)$$

Elimination of $e_a^v(t), e_b^v(t), e_c^v(t), e_{ab}^\sigma(t), e_{bc}^\sigma(t), e_{ca}^\sigma(t), f_{ab}^\varepsilon(t),$
$f_{bc}^\varepsilon(t), f_{ca}^\varepsilon(t), f_a^B(t), f_b^B(t), f_c^B(t)$ gives

$$\begin{bmatrix} \alpha_a & 0 & 0 & 0 & 0 & 0 \\ \alpha_b & 0 & 0 & 0 & 0 & 0 \\ \alpha_c & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_{b,abc} - \beta_{b,ab} & \beta_{c,ab} - \alpha_{c,abc} & 1 & 0 & 0 \\ 0 & \alpha_{b,abc} - \beta_{b,bc} & \beta_{c,bc} - \alpha_{c,abc} & 0 & 1 & 0 \\ 0 & \alpha_{b,abc} - \beta_{b,ca} & \beta_{c,ca} - \alpha_{c,abc} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{abc}^p \\ f_{ab}^\varepsilon \\ f_{bc}^\varepsilon \\ f_{ca}^\varepsilon \\ f_{ab}^B \\ f_{bc}^B \\ f_{ca}^B \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -1 & 1 & \beta_{a,ab} & \beta_{a,bc} & \beta_{a,ca} \\ 0 & 1 & 0 & \beta_{b,ab} & \beta_{b,bc} & \beta_{b,ca} \\ 0 & 0 & -1 & \beta_{c,ab} & \beta_{c,bc} & \beta_{c,ca} \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} e_{abc}^p \\ e_{ab}^\varepsilon \\ e_{ca}^\varepsilon \\ e_{ab}^B \\ e_{bc}^B \\ e_{ca}^B \end{bmatrix} = 0. \qquad (51)$$

Proposition 6. *The subspace of admissible flows f_{abc} and efforts e_{abc} defined by $\mathscr{D}_{abc} = \{(f_{abc}, e_{abc}) : F_{abc} f_{abc} + E_{abc} e_{abc} = 0\}$ is a Dirac structure with respect to the scalar pairing (power product) defined by $\langle e_{abc} | f_{abc} \rangle = e_{abc}^T f_{abc}$.*

Proof. It is easy to check that $F_{abc} E_{abc}^T + E_{abc} F_{abc}^T = 0$, and that rank$[F_{abc} \vdots E_{abc}] = 6$. Hence (see van der Schaft & Maschke, 1995; Dalsmo & van der Schaft, 1999) $\mathscr{D}_{abc}$ is a Dirac structure. $\square$

4.3. Approximation of the constitutive relations

The approximation of the *rate* energy variables has been shown in the previous subsection. Now we approximate the energy variables in the same way

$$p(t,z) = p_{abc}(t)w_{abc}^p(z), \qquad (52a)$$

$$\varepsilon(t,z) = \varepsilon_{ab}(t)w_{ab}^\varepsilon(z) + \varepsilon_{bc}(t)w_{bc}^\varepsilon(z) + \varepsilon_{ca}(t)w_{ca}^\varepsilon(z), \qquad (52b)$$

where $dp_{abc}/dt = f_{abc}^p$ and $d\varepsilon_l/dt = f_l^\varepsilon, l \in \{ab, bc, ca\}$. The Hamiltonian of the discretized finite-dimensional model is derived as follows. The kinetic energy of the cell is

$$H_{abc}^p(p) = \frac{1}{2} \int_{\mathscr{Z}_{abc}} * \frac{p(t,z)}{\mu(z_1, z_2)} \wedge p(t,z)$$

$$= \frac{p_{abc}^2(t)}{2} \int_{\mathscr{Z}_{abc}} \frac{*w_{abc}^p(z) \wedge w_{abc}^p(z)}{\mu(z)} = \frac{p_{abc}^2(t)}{2M}$$

$$\text{with } M^{-1} = \int_{\mathscr{Z}_{abc}} \frac{*w_{abc}^p(z) \wedge w_{abc}^p(z)}{\mu(z)}.$$

The elastic strain can be rewritten as $\varepsilon(t,z) = \varepsilon_{ab}(t) dw_b^v(z) - \varepsilon_{ca}(t) dw_c^v(z) + (\varepsilon_{ab}(t) + \varepsilon_{bc}(t) + \varepsilon_{ca}(t))w_{bc}^v$. Since $d\varepsilon = 0$, $\int_{\mathscr{Z}_{abc}} d\varepsilon(t,z) = \int_{\partial\mathscr{Z}_{abc}} \varepsilon(t,z) = 0$. Thus $\varepsilon(t,z) = \varepsilon_{ab}(t) dw_b^v(z) - \varepsilon_{ca} dw_c^v(z)$. The potential energy is $H_{abc}^\varepsilon(\varepsilon)$ is approximated as

$$\frac{1}{2} \int_{\mathscr{Z}_{abc}} \frac{*\varepsilon}{Y(z)} \wedge \varepsilon = \frac{1}{2} \left[\frac{\varepsilon_{ab}^2}{Y_1} + \frac{\varepsilon_{ca}^2}{Y_2} - \frac{2\varepsilon_{ab}\varepsilon_{ca}}{Y_3} \right]$$

$$\text{with } Y_1^{-1} := \int_{\mathscr{Z}_{abc}} \frac{*dw_b^v(z) \wedge dw_b^v(z)}{Y},$$

$$Y_2^{-1} := \int_{\mathscr{Z}_{abc}} \frac{*dw_c^v(z) \wedge dw_c^v(z)}{Y},$$

$$Y_3^{-1} := \int_{\mathscr{Z}_{abc}} \frac{*dw_b^v(z) \wedge dw_c^v(z) + *dw_c^v(z) \wedge dw_b^v(z)}{2Y}.$$

Therefore, the total energy of the cell is approximated by

$$H_{abc}(p,\varepsilon) = \frac{p^2(t)}{2M} + \frac{1}{2} \left[\frac{\varepsilon_{ab}^2}{Y_1} + \frac{\varepsilon_{ca}^2}{Y_2} - \frac{2\varepsilon_{ab}\varepsilon_{ca}}{Y_3} \right] \qquad (53)$$

and the discretized co-energy variables $e_{abc}^p, e_{ab}^\varepsilon, e_{ca}^\varepsilon$ are

$$e_{abc}^p = \frac{\partial H_{abc}}{\partial p}(p,\varepsilon) = \frac{p_{abc}(t)}{M}, e_{ab}^\varepsilon = \frac{\partial H_{abc}}{\partial \varepsilon_{ab}}(p,\varepsilon) = \frac{\varepsilon_{ab}}{Y_1} - \frac{\varepsilon_{ca}}{Y_3},$$

$$e_{ca}^\varepsilon = \frac{\partial H_{abc}}{\partial \varepsilon_{ca}}(p,\varepsilon) = \frac{\varepsilon_{ca}}{Y_1} - \frac{\varepsilon_{ab}}{Y_3}.$$

5. Example

Consider a transmission line with length $S = e - 1$. The distributed capacitance and the distributive inductance are given by $C(z) = 1/(1+z), L(z) = 1/(1+z)$. On one side we apply an input voltage $u(t)$ and at the other end the transmission line is terminated by a load of resistance $R = 1$. It is assumed that the initial conditions are zero, i.e. $q(0,z) = 0, \phi(0,z) = 0$ for $z \in \mathscr{Z} = [0, S]$. The *exact solution* for the voltage distribution $e^q(t,z)$ is $e^q(t,z) = u(t - \ln(z+1))$.

Table 2

	Approximation	$C_{S_i,S_{i+1}}, L_{S_i,S_{i+1}}$
Exp. 1	material, non-uniform	$\frac{1}{n}$
Exp. 2	material, uniform	$\log \frac{(i+1)(e-1)+n}{i(e-1)+n}$
Exp. 3	spline, non-uniform	$2\frac{e^{1/n}-1}{e^{1/n}+1}$
Exp. 4	spline, uniform	$2\frac{e-1}{2n+(2i+1)(e-1)}$

The transmission line is split into n parts. Two different grids are considered:

Uniform grid: The points S_i are defined by $S_i = (i(e-1))/n$, $1 < i < n$.

Non-uniform grid: $S_i = e^{i/n} - 1$. The non-uniform grid is chosen in such way that the total capacitance and inductance of any part of the transmission line is equal to $1/n$.

The basis one-forms are chosen as follows.

Spline approximation: The one-forms ω_{ab}^q, ω_{ab}^ϕ are defined by (see Remarks 4 and 6) $\omega_{ab}^q(z) = \omega_{ab}^\phi(z) = \mathrm{d}z/(b-a)$. The zero-forms ω_a^q, ω_b^q, ω_a^ϕ, ω_b^ϕ are determined such that Eq. (12), (4) are satisfied, i.e.

$$\omega_a^q(z) = \omega_a^\phi(z) = \frac{b-z}{b-a}, \; \omega_b^q(z) = \omega_b^\phi(z) = \frac{z-a}{b-a}. \qquad (54)$$

Material approximation: The one-forms ω_{ab}^q, ω_{ab}^ϕ are defined by (see Eq. (35))

$$\omega_{ab}^q(z) = \omega_{ab}^\phi(z) = \frac{\mathrm{d}z}{\ln((b+1)/(a+1))(z+1)}$$

and the zero-forms ω_a^q, ω_b^q, ω_a^ϕ, ω_b^ϕ are given by

$$\omega_b^q(z) = \omega_b^\phi(z) = \frac{\ln((z+1)/(a+1))}{\ln((b+1)/(a+1))}, \qquad (55)$$

while $\omega_a^q(z) = \omega_a^\phi(z) = 1 - \omega_b^q(z)$. For both cases the parameter α_{ab}, cf. (18), is $\alpha_{ab} = \frac{1}{2}$. The number of parts is taken to be $n = 5$. The input voltage is $u(t) = \sin(t)$. Four simulation experiments are carried out (see Table 2). These simulation experiments are performed by means of the 20-Sim simulation package. The integration technique is Runge–Kutta 4 and the integration step size is 0.01s. In all simulation experiments the difference between the exact value of the voltage and the value obtained by numerical simulation is the largest at the final spatial point S. Hence we define the deviation $\delta(t)$ as

$$\delta(t) = \begin{cases} -e^q(t,S), & t \leq 1, \\ \sin(t-1) - e^q(t,S), & t > 1. \end{cases}$$

This deviation $\delta(t)$ for all four experiments is shown in Fig. 1(a). The computation error in all experiments has an oscillatory behavior and the amplitude of the oscillation increases for $t < \ln(1+S) = 1$. For $t > 1$ the amplitude of the oscillation decreases to a constant value and the

frequency of the oscillations coincides with the frequency of the input signal. The normalized deviation defined with respect to the amplitude of the deviation in Experiment 1, $\delta_{\mathrm{nom}}(t) := \delta(t)/0.00331$, is shown in Fig. 1(b) for $2 \leq t \leq 10$. The amplitude of the error is the smallest for Experiment 1 in which the grid is non-uniform and the choice of the approximating one-forms is based on the physical properties of the transmission line (the form of $L(z)$ and $C(z)$ is taken into account). Since the deviation error in Experiment 2 is smaller than in Experiment 3, one concludes that the choice of approximating one-forms plays a more important role for the accuracy than the choice of the grid. Finally, the worst results are obtained in the case when the grid is uniform and the approximating one-forms are splines.

With regard to the accuracy of the proposed method we have observed the following. Experiment 1 has been repeated with the number of parts being doubled, i.e. $n = 10$ (in this case $L_{S_{i-1},S_i} = C_{S_{i-1},S_i} = 0.1$, $1 < i < n$). The amplitude of the error for $n = 5$ is 0.0033 and the amplitude of the error for $n = 10$ is 0.00084. Thus the amplitude of the error decreases almost by a factor four. So we may conjecture that the accuracy of the proposed method is of order $1/n^2$.

6. Conclusions

In this paper we have proposed a general methodology for the spatial discretization of boundary control systems modelled as port-Hamiltonian systems. Key feature of this methodology is that the discretized system is again a port-Hamiltonian system. This has advantages from different point of views. First it allows to transfer certain physical properties, such as energy conservation and other conservation laws, from the infinite-dimensional model to the finite-dimensional approximation. Secondly, one may couple the finite-dimensional approximation of the distributed-parameter component to the other system components in the same way as for the original distributed-parameter model (using the boundary port). Thirdly, the discretization *within* the port-Hamiltonian framework allows the use of control strategies based on the Hamiltonian structure.

In the current paper we have only treated the spatial discretization of the telegrapher's equations and the boundary controlled two-dimensional wave equation. However, the methodology is applicable to higher-dimensional spatial domains. In particular the discretization of Maxwell's equations on a three-dimensional spatial domain with boundary is waiting to be worked out.

Another open problem concerns the systematic adaptation of the approximating differential forms to the physical characteristics of the system, see Section 5.

Finally, it is a major challenge to analyze the approximation properties of the discretized port-Hamiltonian systems; in particular, to obtain bounds on the error between
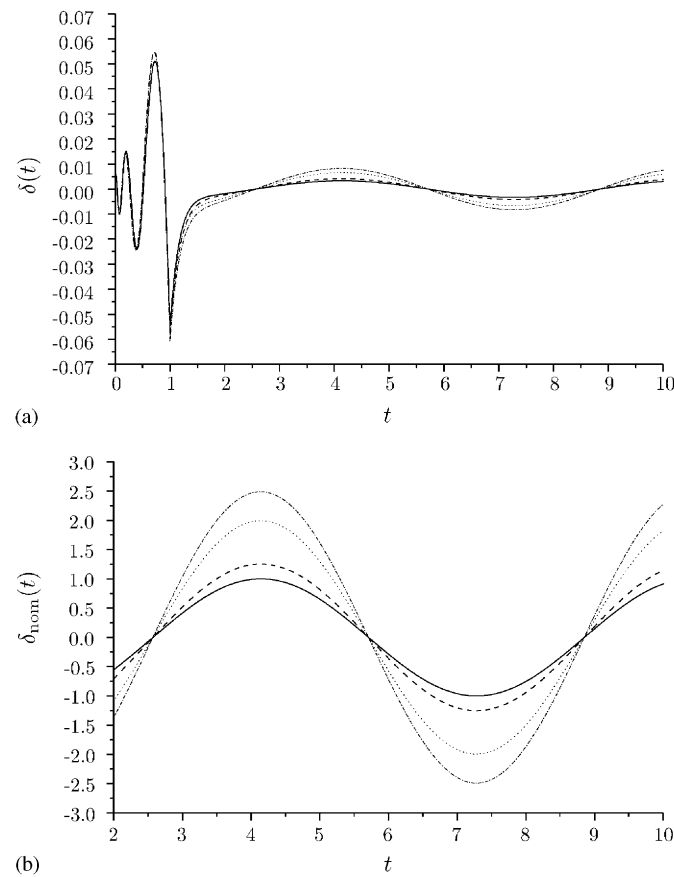
Fig. 1. (a) $\delta(t)$, (b) Magnified plot: solid (Exp. 1), dash (Exp. 2), dot (Exp. 3), dash-dot (Exp. 4).

the distributed-parameter model and its finite-dimensional approximation.

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