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Brief paper

H_2 control of preview systems ☆

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Abstract

The H_2-optimal controller for systems with preview, in which the knowledge of external input is available in advance for the controller, is derived. The single input case is first considered and solved by transforming the problem into a non-standard LQR problem. Based on the single input result, the multiple inputs case and the multiple preview times case are treated. In every case considered, the controller consists of a static state feedback plus a finite impulse response block. The paper also provides a formula for the optimal H_2-norm that clearly shows how the performance gain owing to the previewed input varies as the preview time increases.

Keywords: H_2-optimal control; Preview systems

1. Introduction

In certain control problems, all or parts of the external input signals are known in advance. An example is a tracking problem where the tracked trajectory is known in advance. Exploiting this knowledge might improve the control system performance. However, most controller design techniques do not take it into account. Control systems that do exploit the advance knowledge of the input are commonly designated as preview control systems. This paper aims to derive the H_2-optimal controller for systems with previewed input. While previewing the input signals may increase the performance of a control system, it also increases the complexity of the controller. For this reason, the performance gain has to be significant enough to justify the increased controller complexity. Therefore, the following questions, which are paraphrased from questions posed in Anderson and Moore (1979) in studying the advantage of smoothing over filtering, are highly relevant. How does the performance gain owing to the previewed input with preview time h vary as h increases? What is the maximum achievable performance gain, i.e., what is the performance gain associated with infinite preview time ($h \to \infty$)? Besides the primary objective of obtaining the H_2-optimal controller, this paper also aims to answer these questions.

Several results have been put forward to incorporate the advance knowledge of the external input signals into H_∞ and H_2 designs. The H_∞ preview control problem was considered in Kojima and Ishijima (2003a, 2003b), Shaked and de Souza (1995) and Cohen and Shaked (1997). Later, a game theoretic solutions in both continuous and discrete time were proposed in the papers of Tadmor and Mirkin (2005a, 2005b). The closely related problem of H_∞ fixed-lag smoothing was treated in Mirkin (2003), Theodor and Shaked (1994), Tadmor and Mirkin (2005a, 2005b) and Mirkin and Tadmor (2004). In particular, the paper (Mirkin & Tadmor, 2004) provides an explicit computation of the optimal cost as a function of the preview time.

In the field of H_2 design, the H_2 fixed-lag smoothing problem has been solved in the 60s (see Anderson & Moore, 1979 and references therein). Lately, there have been several results that treat the H_2 control problem of preview systems. In the discrete time framework, the paper by Mosca and Casavola (1995) solved the linear quadratic tracking problem with preview in the deterministic setting, while the paper of

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Mosca and Zappa (1989) treated the problem in stochastic setting. The latter was extended to cover the minimax version of the problem in Mosca, Casavola, and Giarre (1990). See also Mosca (1995) and Samson (1982). Apart from the discrete time setting used in these publications, the main differences between the problems considered in the aforementioned publications and our problem are that only single preview time cases is considered (in contrast with the multiple preview times case also considered in our problem) and that the mathematical machinery used in most of these publications hinges on Kučera's Diophantine equation (Kučera, 1991), which is completely different from our approach. In the continuous time framework, the paper of Kojima and Ishijima (1999) treated an LQ problem with stored disturbance, while in Kojima (2004) an H_2 preview control problem, which is equivalent to the single input control problem considered in this paper, was solved. However, the result in Kojima (2004) is limited to the single input (and therefore single preview time) case. More recently, the paper of Marro and Zattoni (2005) treated a feedforward disturbance rejection problem. Both Kojima (2004) and Marro and Zattoni (2005) utilized the same ingredient: partitioning the optimization time interval into an infinite horizon problem and a finite horizon problem. In Kojima (2004), the finite horizon part is solved using orthogonal projection arguments. The idea of splitting the optimization interval was first employed in Tadmor (1997) in the context of robust control in the gap and later was also used for solving the H_2 control problem of systems with multiple i/o delays in Moelja and Meinsma (2005).

In this paper, an alternative derivation of the solution of the H_2 preview feedback control problem is provided. As in Kojima (2004), the technique of splitting the optimization time interval into two time intervals with the preview time h as the boundary is used. The problem is effectively split into two parts: a standard infinite horizon LQR problem and a finite horizon LQR problem with a non-standard constraint of a jump in the final state. The standard infinite horizon part results in state feedback part of the optimal controller, while the non-standard finite horizon part is tackled using the Pontryagin minimum principle and results in finite impulse response part of the optimal controller. Both the derivation in this paper and in Kojima (2004) result in the same formulation of the optimal controller for the single input case. The main difference lies in the formula for the optimal H_2-norm. The formula in Kojima (2004) requires solving a differential Riccati equation and does not provide a clear insight in determining the effect of the preview time h on the H_2 performance. The derivation in this paper results in a different formulation that complements the results in Kojima (2004). Not only that the formula derived in this paper appears simpler (it only requires solving a Lyapunov equation), but it also clearly shows the performance gain owing to the previewed input. It also allows the computation of the maximum achievable performance corresponding to the infinite preview time. Furthermore, this paper also treats the multiple inputs case and the multiple preview times case, which are not treated in preceding results. As the optimal solution of the H_2 preview feedback control problem is already known, the main contributions of this paper are the explicit formula of the optimal H_2-norm as a function of the preview time and the solution for the multiple inputs and multiple preview times cases. The paper is organized as follows. After the introduction and some preliminaries, the single input H_2 preview control problem is considered. The result is then extended to the multiple inputs case and multiple preview times case in the subsequent sections. The paper is concluded with a numerical example and concluding remarks.

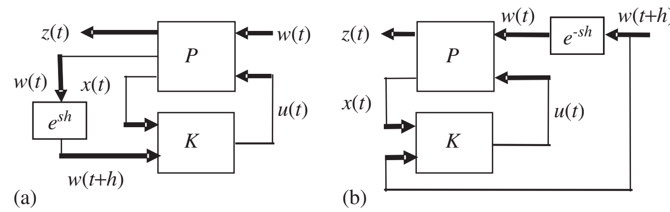

Fig. 1. The preview control setup (a) and its equivalence (b).

Preliminaries: Given a linear time invariant system (LTI) F, its impulse response is denoted by $F(t)$. The squared H_2-norm of a causal LTI system F is equal to $\|F\|_2^2 = \int_0^\infty \operatorname{trace}[F(t)^{\mathrm{T}} F(t)]\,\mathrm{d}t$. Suppose the input and output of the system F are, respectively, denoted by w and z, then the squared H_2-norm may also be defined as

$$\|F\|_2^2 = \sum_{w=(0,\dots,\delta(t),\dots,0)} \int_0^\infty z(t)^{\mathrm{T}} z(t)\,\mathrm{d}t. \tag{1}$$

The unit step function is denoted by $\mathbb{1}(t)$.

2. Problem formulation

The preview control system configuration that is considered is shown in Fig. 1(a). It is very similar to the standard full information control system, in which the controller uses the state x and the external input w as its inputs. The only difference is that the external signal w is available to the controller h time units in advance. This fact is represented in Fig. 1(a) by the negative delay operator e^{sh}. To avoid employing a negative delay operator, the same effect may be achieved by delaying the external input fed to the plant, while the controller receives the non-delayed version. This setting is shown in Fig. 1(b). The control problem itself is formally stated in what follows. Consider the control system of Fig. 1(b) where the dynamics of the plant $P(s)$ are governed by the state space equation:

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t),$$

$$z(t) = C_1 x(t) + D_2 u(t) \tag{2}$$

and the system parameters satisfy the following standard assumptions:

A1 (C_1, A, B_2) is detectable and stabilizable;
A2

$$\begin{bmatrix} A - \mathrm{j}\omega I & B_2 \\ C_1 & D_2 \end{bmatrix}$$

has full column rank $\forall \omega \in \mathbb{R}$.

In addition to the standard assumptions above, to simplify the formulas it is also assumed that

A3 $D_2^{\mathrm{T}} D_2 = I$ and $C_1^{\mathrm{T}} D_2 = 0$.

Assumption A3 will be relaxed later. The problem is to find a stabilizing control K that minimizes the H_2-norm of the transfer function from w to z.

3. Single-input case

In this section the case where B_1 is a column vector, i.e. w is a one-dimensional signal, is considered. To signify the difference, a lower case b_1 is used in place of B_1. The H_2-norm of the transfer function from w to z is equal to the L_2-norm of z provided that $w(t)$ is a delta function. Therefore, by setting $w(t) = \delta(t-h)$, the H_2 optimization problem may be formulated as an LQR problem with a state jump at $t = h$. At this point, the original objective of designing a full information controller that takes $x(t)$ and $w(t+h)$ as inputs is temporarily set aside. Rather the attention is focused on finding the optimal u such that the L_2-norm of z is minimized given $w(t) = \delta(t-h)$. Later it shall be shown that the optimal control law may be implemented by a full information controller with preview as in Fig. 1 and thus producing the desired optimal controller. Given that $B_1 = b_1$ and $w(t) = \delta(t-h)$, the state-space equation (2) becomes

$$\dot{x}(t) = Ax(t) + b_1\delta(t-h) + B_2 u(t),$$
$$z(t) = C_1 x(t) + D_2 u(t), \quad x(0) = 0, \tag{3}$$

and given that $Q = C_1^{\mathrm{T}} C_1$, our objective is

$$\begin{aligned}\min_u J(x_0, u) &= \min_u \int_0^\infty \|C_1 x(t) + D_2 u(t)\|_2^2 \, \mathrm{d}t \\ &= \min_u \int_0^\infty x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} u(t) \, \mathrm{d}t. \end{aligned} \tag{4}$$

The delta function input at $t = h$ in (3) raises the state such that $x(h^+) = x(h^-) + b_1$, so that (3) becomes

$$\dot{x}(t) = Ax(t) + B_2 u(t), \quad z(t) = C_1 x(t) + D_2 u(t),$$
$$x(0) = 0, \quad x(h^+) = x(h^-) + b_1. \tag{5}$$

The state-space equation (5) together with the criterion function (4) constitute an LQR problem. The only difference of the LQR problem (5), (4) and a standard LQR problem is the state jump at $t = h$. One way to circumvent the problem is to use the technique from Tadmor (1997) and Moelja and Meinsma (2005) of dividing the optimization time horizon into two regions with $t = h$ as the boundary so that the state jump can be considered as a boundary condition. It turns out that the optimal control problem in each time region may be solved essentially independent from the other.

Lemma 1. *Consider the LQR problem corresponding to the state-space equation* (5) *and the objective* (4). *Let M be the stabilizing solution of the LQR Riccati equation*

$$Q + A^{\mathrm{T}} M + MA - MB_2 B_2^{\mathrm{T}} M = 0. \tag{6}$$

Define

$$u_{2,\mathrm{opt}}(t) = -B_2^{\mathrm{T}} M x(t) \tag{7}$$

and let $u_{1,\mathrm{opt}}$ be the solution of the LQR problem corresponding to the state-space equation

$$\dot{x}(t) = Ax(t) + B_2 u_1(t), \quad x(0) = 0, \tag{8}$$

with the objective

$$\min_{u_1} \left((x(h) + b_1)^{\mathrm{T}} M (x(h) + b_1) + \int_0^h (x^{\mathrm{T}} Q x + u_1^{\mathrm{T}} u_1) \, \mathrm{d}t \right). \tag{9}$$

Then the solution of the LQR problem (5), (4) *is given by*

$$u_{\mathrm{opt}}(t) = [\mathbb{1}(t) - \mathbb{1}(t-h)] u_{1,\mathrm{opt}}(t) + \mathbb{1}(t-h) u_{2,\mathrm{opt}}(t) \tag{10}$$

and the optimal cost is given by (9).

Proof. Consider the state-space equation (5). Assume temporarily that the optimal state at $t = h^-$, denoted by $x_{\mathrm{opt}}(h^-)$, is known. It follows that $x_{\mathrm{opt}}(h^+) = x_{\mathrm{opt}}(h^-) + b_1$. For $t \in [h^+, \infty]$, (5) becomes

$$\dot{x} = Ax + B_2 u, \quad x(h^+) = x_{\mathrm{opt}}(h^-) + b_1, \tag{11}$$

while the cost over this time region is given by

$$J_{[h^+,\infty]} = \int_h^\infty (x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} u(t)) \, \mathrm{d}t. \tag{12}$$

The problem of minimizing (12) given (11) is a standard infinite horizon LQR problem, the solution of which is the state feedback

$$u_{\mathrm{opt}}(t) = -B_2^{\mathrm{T}} M x(t), \quad t \in [h^+, \infty], \tag{13}$$

while the optimal cost is

$$\begin{aligned} J_{[h^+,\infty],\mathrm{opt}} &= x(h^+)^{\mathrm{T}} M x(h^+) \\ &= [x_{\mathrm{opt}}(h^-) + b_1]^{\mathrm{T}} M [x_{\mathrm{opt}}(h^-) + b_1], \end{aligned} \tag{14}$$

where M is the solution of the Riccati equation (6). Hence, it is proved that for $t \in [h^+, \infty]$ the optimal input is indeed given by the state feedback (7). It is also clear that the optimal cost contribution over $t = [h^+, \infty]$, which is given by (14), depends solely on $x_{\mathrm{opt}}(h^-)$. It follows that the infinite horizon LQR problem of minimizing (4) is equivalent to minimizing the finite horizon cost function

$$\min_u \left(\int_0^h (x^{\mathrm{T}} Q x + u^{\mathrm{T}} u) \, \mathrm{d}t + [x(h) + b_1]^{\mathrm{T}} M [x(h) + b_1] \right),$$

from which the optimal input for $t \in [0, h^-]$ may be obtained. □

Lemma 1 gives a partial solution to the LQR problem (5), (4). It is now ascertained that for $t \in [h, \infty]$ the optimal input is a state feedback given by (7). What is left is to solve the finite horizon LQR problem (8), (9). The solution is summarized in the following lemma.

Lemma 2. *Consider the LQR problem corresponding to the state-space equation* (8) *with the objective* (9) *where* M *is the stabilizing solution of the Riccati equation* (6). *Define* $A_p = A - B_2 B_2^{\mathrm{T}} M$, *then the optimal input of the LQR problem* (8), (9) *is given by*

$$u_{1,\mathrm{opt}}(t) = -B_2^{\mathrm{T}} M x(t) - B_2^{\mathrm{T}} \mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M b_1. \tag{15}$$

Proof. We begin by applying the minimum principle to the optimal control problem (8), (9). It may be shown (see, for example, Appendix C of Anderson & Moore, 1989), that the optimal input is given by

$$u_{1,\mathrm{opt}}(t) = B_2^{\mathrm{T}} p(t), \tag{16}$$

where the co-state p and the optimal state x satisfy the following equation

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A & B_2 B_2^{\mathrm{T}} \\ Q & -A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{17}$$

with the boundary condition

$$x(0) = 0, \quad p(h) = -Mx(h) - Mb_1. \tag{18}$$

Notice that except for the boundary condition, the equations are similar to the standard case where $b_1 = 0$. Furthermore, using similar arguments as in the standard case, it may be shown that the differential equation (17), (18) has a unique solution. To obtain the solution of the differential equation (17), (18), define the state transformation

$$q(t) = Mx(t) + p(t). \tag{19}$$

With the state transformation and keeping in mind that M is the solution of the Riccati equation (6), the differential equation (17) is simplified to

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A_p & B_2 B_2^{\mathrm{T}} \\ 0 & -A_p^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \tag{20}$$

with the boundary condition

$$x(0) = 0, \quad q(h) = -Mb_1. \tag{21}$$

It follows that the trajectory of $q(t)$ is given by $q(t) = \mathrm{e}^{-A_p^{\mathrm{T}} t} q(0)$. The initial condition $q(0)$ may be computed by setting $t = h$ in $q(t) = \mathrm{e}^{-A_p^{\mathrm{T}} t} q(0)$ and substituting the boundary condition (21), resulting in,

$$q(0) = -\mathrm{e}^{A_p^{\mathrm{T}} h} M b_1, \tag{22}$$

so that the complete expression for $q(t)$ is obtained: $q(t) = -\mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M b_1$. Using this expression and (19), $p(t)$ may be computed as

$$p(t) = -Mx(t) - \mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M b_1. \tag{23}$$

The optimal input $u_{1,\mathrm{opt}}(t) = B_2^{\mathrm{T}} p(t)$ is then given by (15). □

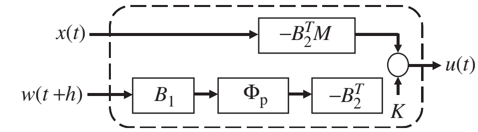

Fig. 2. The optimal controller, single preview time case.

Lemma 1 combined with Lemma 2 provides a complete solution to the infinite horizon LQR problem (5), (4). By Lemmas 1 and 2, the optimal u is given by

$$\begin{aligned} &u_{\mathrm{opt}}(t) \\ &\quad = [\mathbb{1}(t) - \mathbb{1}(t-h)] u_{1,\mathrm{opt}}(t) + \mathbb{1}(t-h) u_{2,\mathrm{opt}}(t) \\ &\quad = -[\mathbb{1}(t) - \mathbb{1}(t-h)] B_2^{\mathrm{T}} (Mx(t) \\ &\qquad + \mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M b_1) - \mathbb{1}(t-h) B_2^{\mathrm{T}} M x(t) \\ &\quad = -B_2^{\mathrm{T}} M x(t) - [\mathbb{1}(t) - \mathbb{1}(t-h)] B_2^{\mathrm{T}} \mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M b_1. \end{aligned} \tag{24}$$

This is the unique optimal u for the control system of Fig. 1 when $w(t) = \delta(t-h)$. Hence, if we find a full information controller with preview that also produces the same input if we set $w(t) = \delta(t-h)$, then we automatically obtain the desired H_2-optimal controller. In the following theorem, the optimal controller is derived.

Theorem 3. *Consider the control system of Fig.* 1(b) *where the plant's dynamics are governed by* (2). *Suppose that* w *is a one-dimensional signal, i.e.* $B_1 = b_1$ *has a single column. Then the optimal controller that minimizes the* H_2*-norm of the transfer function from* w *to* z *is the controller in Fig.* 2, *where* Φ_p *has the following impulse response*:

$$\Phi_p(t) = [\mathbb{1}(t) - \mathbb{1}(t-h)] \mathrm{e}^{-A_p^{\mathrm{T}}(t-h)} M. \tag{25}$$

Here M *is the stabilizing solution of the Riccati equation* (6), *while* $A_p = A - MB_2B_2^{\mathrm{T}}$. *Notice that* Φ_p *has a finite impulse response with support on* $[0, h]$.

Proof. It may be verified that the controller in Fig. 2 generates the optimal u given by (24) when driven by $w(t+h) = \delta(t)$. □

The squared optimal H_2-norm is equal to the optimal cost function (9), which is given in the following theorem.

Theorem 4. *Consider the control system of Fig.* 1(b) *where the plant's dynamics are governed by* (2). *Suppose that* w *is a one-dimensional signal, i.e.* $B_1 = b_1$ *has a single column. Let* M *be the stabilizing solution of the Riccati equation* (6), *while* $A_p = A - MB_2B_2^{\mathrm{T}}$. *Furthermore, let* X *be the solution of the Lyapunov equation*:

$$A_p X + X A_p^{\mathrm{T}} + B_2 B_2^{\mathrm{T}} = 0. \tag{26}$$

Then the squared optimal H_2*-norm of the transfer function from* w *to* z *is*

$$J_{\mathrm{opt}}(h) = b_1^{\mathrm{T}} M b_1 - b_1^{\mathrm{T}} M (X - \mathrm{e}^{A_p h} X \mathrm{e}^{A_p^{\mathrm{T}} h}) M b_1. \tag{27}$$

Proof. It follows from (16), (17) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(p^{\mathrm{T}}x) = p^{\mathrm{T}}\dot{x} + x^{\mathrm{T}}\dot{p} = u_{1,\mathrm{opt}}^{\mathrm{T}}u_{1,\mathrm{opt}} + x^{\mathrm{T}}Qx. \tag{28}$$

Taking the integral of both sides of (28), the optimal value of the integral term in (9) is

$$\int_0^h (x^{\mathrm{T}}Qx + u_{1,\mathrm{opt}}^{\mathrm{T}}u_{1,\mathrm{opt}})\,\mathrm{d}t = p(h)^{\mathrm{T}}x(h) - p(0)^{\mathrm{T}}x(0) = p(h)^{\mathrm{T}}x(h). \tag{29}$$

The expression of $p(t)$ is readily available in (23), while the expression for $x(t)$ may be computed from the differential equation (20) and the initial condition (21), (22):

$$x(t) = \Sigma_{11}(t)x(0) + \Sigma_{12}(t)q(0) = -\Sigma_{12}(t)\mathrm{e}^{A_p^{\mathrm{T}}h}Mb_1, \tag{30}$$

$$\Sigma(t) = \begin{bmatrix} \Sigma_{11}(t) & \Sigma_{12}(t) \\ \Sigma_{21}(t) & \Sigma_{22}(t) \end{bmatrix} = \mathrm{e}^{St}, \quad S = \begin{bmatrix} A_p & B_2B_2^{\mathrm{T}} \\ 0 & -A_p^{\mathrm{T}} \end{bmatrix}. \tag{31}$$

Plugging (29) into (9) and using (23), (30) to simplify the expression, the following is obtained:

$$\min_{u_1}\left((x(h)+b_1)^{\mathrm{T}}M(x(h)+b_1) + \int_0^h (x^{\mathrm{T}}Qx + u_1^{\mathrm{T}}u_1)\,\mathrm{d}t\right) = b_1^{\mathrm{T}}Mb_1 - b_1^{\mathrm{T}}M\Sigma_{12}(h)\mathrm{e}^{A_p^{\mathrm{T}}h}Mb_1, \tag{32}$$

where M is the solution of the Riccati equation (6), $\Sigma(t)$ is given by (31), and $A_p = A - B_2B_2^{\mathrm{T}}M$. The formula (32) may be further simplified by finding a simpler expression for Σ_{12}. By defining

$$W = \begin{bmatrix} I & -X \\ 0 & I \end{bmatrix},$$

where X is the solution of the Lyapunov equation (26), it is straightforward to compute $WSW^{-1} = \mathrm{diag}(A_p, -A_p^{\mathrm{T}})$. Note that since A_p is Hurwitz, the Lyapunov equation (26) has a unique solution. Hence, it is straightforward to show that

$$\mathrm{e}^{St} = \begin{bmatrix} \mathrm{e}^{A_pt} & X\mathrm{e}^{-A_p^{\mathrm{T}}t} - \mathrm{e}^{A_pt}X \\ 0 & \mathrm{e}^{-A_p^{\mathrm{T}}t} \end{bmatrix},$$

implying that $\Sigma_{12}(t) = X\mathrm{e}^{-A_p^{\mathrm{T}}t} - \mathrm{e}^{A_pt}X$. Plugging this expression into (32) results in (27). □

Compared to the formula given in Kojima (2004), which involves solving a differential Riccati equation, the formula (27) appears simpler and only requires solving the Lyapunov equation (26) and computing the exponential of a matrix.

Now we investigate the effect of the preview time h on the H_2 performance. It may be shown that the first derivative of the optimal squared H_2-norm with respect to h is $\partial J_{\mathrm{opt}}(h)/\partial h = -b_1^{\mathrm{T}}M\mathrm{e}^{A_ph}B_2B_2^{\mathrm{T}}\mathrm{e}^{A_p^{\mathrm{T}}h}Mb_1 \leqslant 0$. Evidently, the squared optimal H_2-norm as a function of h is non-increasing. In particular, it may be shown that if (C_1, A, B_2) is observable and controllable, $J_{\mathrm{opt}}(h)$ is strictly decreasing. Thus, as the preview time increases, the performance increases as well. Moreover, the first term in the right-hand side of (27) is the optimal squared H_2-norm for $h=0$ (i.e. no preview), so that the second term may be viewed as the performance gain owing to the previewed input. The minimum achievable H_2-norm is obtained if we set $h=\infty$ (i.e. infinite preview), which gives $J_{\mathrm{opt},h=\infty} = b_1^{\mathrm{T}}(M - MXM)b_1$.

4. Multiple inputs case

The optimal controller for the multiple inputs case is a straightforward extension of the single-input result. It turns out that the controller has exactly the same structure as in the single-input case.

Corollary 5. *Consider the control system of Fig.* 1(b) *where the plant's dynamics are governed by* (2). *The optimal controller that minimizes the H_2-norm of the transfer function from w to z is the controller in Fig.* 2. *Here M is the stabilizing solution of the Riccati equation* (6), *while the impulse response of Φ_p is given by* (25). *Moreover, the squared optimal H_2-norm is given by*

$$\mathrm{tr}(B_1^{\mathrm{T}}MB_1 - B_1^{\mathrm{T}}M(X - \mathrm{e}^{A_ph}X\mathrm{e}^{A_p^{\mathrm{T}}h})MB_1), \tag{33}$$

where $A_p = A - B_2B_2^{\mathrm{T}}M$, and X is the solution of the Lyapunov equation (26).

Proof. According to definition (1), the squared H_2-norm of the control system of Fig. 1(b) may be computed by conducting n_w experiments, with n_w the dimension of w, as described in what follows. For the kth experiment, the kth element of w is set to the delayed delta function $\delta(t-h)$ while the other elements are set to zero. The squared H_2-norm of the closed loop system is then obtained by summing up the squared L_2-norm of the output z for all n_w experiments. Since for each experiment only one element of w is active, for each experiment the control system may be recast into one with scalar external input w. Hence, the single-input results of the previous section applies. Using Theorem 3, it is straightforward to prove that for each experiment, the controller of Fig. 2 generates the optimal input. This implies that the controller is the optimal controller. The squared optimal H_2-norm is obtained by summing up the optimal cost for all n_w experiments, which individually may be computed using (27). □

5. Multiple preview times case

In this section, we treat the general case where each component of the exogenous input $w(t)$ that is fed to controller may have different preview times. The equivalent multiple preview times setup in which $w(t)$ is delayed before being fed to the plant P is shown in Fig. 3. As in the single preview time case, the dynamics of the plant $P(s)$ are governed by (2). The difference is that instead of a single delay operator e^{-sh}, here we have a multiple delay operator $\Lambda(s)$ of the form:

$$\Lambda(s) = \mathrm{diag}(\mathrm{e}^{-sh_1}, \ldots, \mathrm{e}^{-sh_{n_w}}), \tag{34}$$

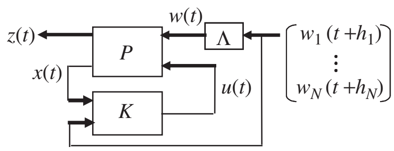

Fig. 3. The multiple preview times case.

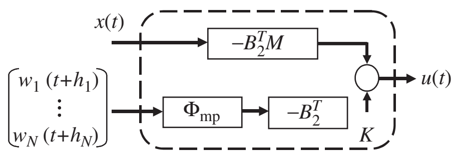

Fig. 4. The optimal controller, multiple preview times case.

where n_w is the dimension of w and $h_k \geqslant 0$, $k = 1, \ldots, n_w$. It turns out that the arguments on which the proof of Corollary 5 is based may also be applied. The result is summarized in the following corollary.

Corollary 6. *Consider the control system of Fig.* 3 *where the plant's dynamics are governed by* (2) *and the delay operator* $\Lambda(s)$ *is given by* (34). *The optimal controller that minimizes the* H_2*-norm of the transfer function from* w *to* z *is the controller in Fig.* 4. *There, the block* Φ_{mp} *is a system with finite impulse response with the kth column of its impulse response, denoted by* $\Phi_{mp,k}(t)$, *is given by*

$$\Phi_{mp,k}(t) = \mathrm{e}^{-A_p^{\mathrm{T}}(t-h_k)} M b_{1,k}(\mathbb{1}(t) - \mathbb{1}(t-h_k)), \tag{35}$$

where M *is the solution of the Riccati equation* (6), $A_p = A - B_2 B_2^{\mathrm{T}} M$, *and* $b_{1,k}$ *denotes the kth column of the matrix* B_1. *Moreover, the squared optimal* H_2*-norm is*

$$\operatorname{tr} B_1^{\mathrm{T}} M B_1 - \sum_{k=1}^{n_w} (b_{1,k}^{\mathrm{T}} M (X - \mathrm{e}^{A_p h_k} X \mathrm{e}^{A_p^{\mathrm{T}} h_k}) M b_{1,k}), \tag{36}$$

where X *is the solution of the Lyapunov equation* (26).

Proof. As in the proof of Corollary 5, the squared H_2-norm of the control system of Fig. 3 may be computed by conducting n_w experiments, as described in what follows. For the kth experiment, the kth element of w is set to the delayed delta function $\delta(t - h_k)$ while the other elements are set to zero. The squared H_2-norm of the closed loop system is then obtained by summing up the squared L_2-norm of the output z for all n_w experiments. As in the multiple inputs single preview time case, for each experiment the problem also reduces to a single-input problem. The only difference is that here the preview time is different for each experiment. Nevertheless, the results from Section 3 still apply for each experiment. Applying the single input results to each experiment related to a particular component of w will result in a particular column of the optimal controller. Using Theorem 3, it is straightforward to ascertain that the controller of Fig. 4, where the FIR block Φ_{mp} is described by (35), generates the optimal input for each experiment. The squared optimal H_2 norm is obtained by summing up the optimal cost of the n_w experiments, resulting in the expression (36). □

6. Relaxing assumption A3

Assumption A3 allows us to formulate the LQR problem (3), (4). The assumption may be relaxed using the well-known method of input substitution. The method works by introducing the state feedback $u(t) = R^{-1/2} v(t) - R^{-1} D_2^{\mathrm{T}} C_1 x(t)$ in (2), where $R = D_2^{\mathrm{T}} D_2$ and v is the new input. With this change of the input, the state equation becomes $\dot{x}(t) = \bar{A} x(t) + b_1 \delta(t-h) + \bar{B}_2 v(t)$, $x(0) = x_0$, while the cost criterion is given by $\min_v \int_0^\infty x(t)^{\mathrm{T}} \bar{Q} x(t) + v(t)^{\mathrm{T}} v(t)\, \mathrm{d}t$ where $\bar{Q} = C_1^{\mathrm{T}} (I - D_2 R^{-1} D_2^{\mathrm{T}}) C$, $\bar{A} = (A - B_2 R^{-1} D_2^{\mathrm{T}} C_1)$, and $\bar{B}_2 = B_2 R^{-1/2}$. The resulting LQR problem is of the same form as (3), (4) and hence may be solved using results from the previous sections.

7. Numerical example

In this section, we present an example of the multiple preview times case. Consider the multiple preview times setup of Fig. 3. Let the plant P be governed by the state equation $\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t)$, $z(t) = C_1 x(t) + D_2 u(t)$, where $A = 1$, $B_1 = [b_{1,1}\ b_{1,2}] = [1\ 2]$, $B_2 = \sqrt{3}$, $C_1 = [1\ 0]^{\mathrm{T}}$, $D_2 = [0\ 1]^{\mathrm{T}}$. It may be verified that Assumptions A1, A2, and A3 are satisfied. Furthermore, the multiple delay operator is given by $\Lambda(s) = \operatorname{diag}(\mathrm{e}^{-sh_1}, \mathrm{e}^{-sh_2})$, with $h_1, h_2 \geqslant 0$. Notice that w_1 and w_2 may have different preview times. The optimal controller and the optimal H_2 norm may be computed using Corollary 6. The first step is to compute the stabilizing solution M of the Riccati equation (6), which in this case becomes a quadratic equation $1 + 2M - 3M^2 = 0$. The stabilizing solution of the above Riccati equation is $M = 1$. Next we compute the matrix A_p, which is given by $A_p = A - B_2 B_2^{\mathrm{T}} M = -2$. The optimal controller is shown in Fig. 4, where the impulse response of the FIR block Φ_{mp} may be computed using (35), $\Phi_{mp}(t) = [\Phi_{mp,1}(t)\ \Phi_{mp,2}(t)]$, where $\Phi_{mp,1}(t) = \mathrm{e}^{2(t-h_1)}(\mathbb{1}(t) - \mathbb{1}(t-h_1))$ and $\Phi_{mp,2}(t) = 2\mathrm{e}^{2(t-h_2)}(\mathbb{1}(t) - \mathbb{1}(t-h_2))$. To compute the optimal H_2 norm we need to compute the solution X of the Lyapunov equation (26) which in this case is $X = 0.75$. The squared optimal H_2 norm can then be computed using (36), which results in the expression

$$5 - 0.75(1 - \mathrm{e}^{-4h_1}) - 3(1 - \mathrm{e}^{-4h_2}). \tag{37}$$

The first term of (37) is the squared optimal H_2 norm for zero preview times, while the second and the third terms, i.e. $-\frac{3}{4}(1 - \mathrm{e}^{-4h_1})$ and $-3(1 - \mathrm{e}^{-4h_2})$, are the performance gain owing to the preview times h_1 and h_2, respectively. The maximum achievable squared H_2 norm corresponding to $h_1, h_2 = \infty$ is 1.25. Since the expression (37) is affine in h_1 and h_2, we

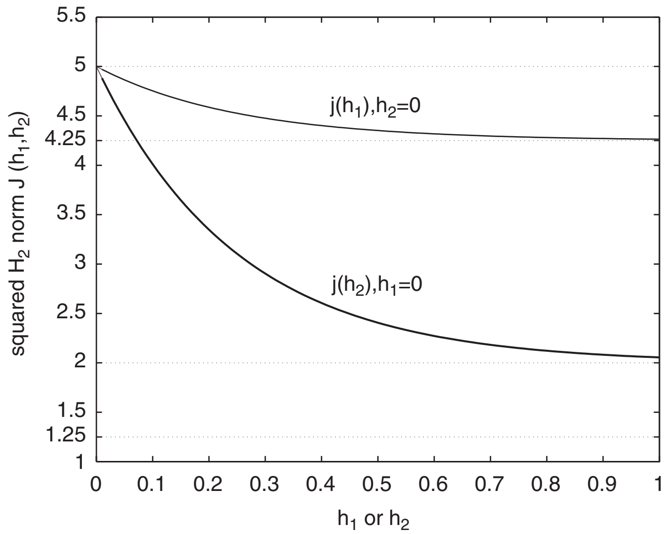

Fig. 5. The squared optimal H_2-norm as a function of the preview times.

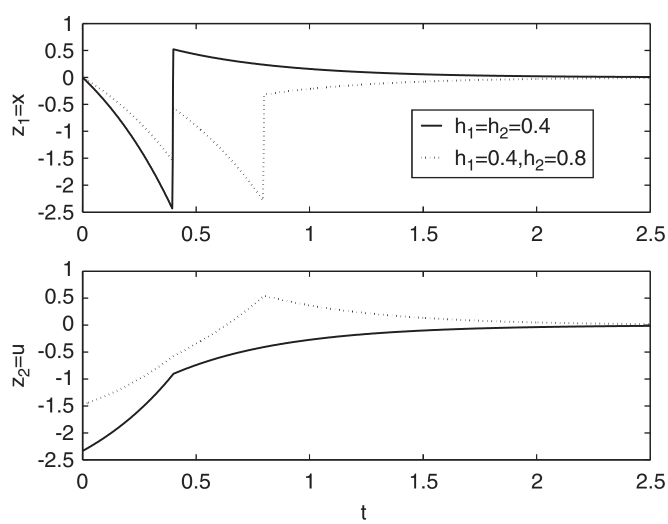

Fig. 6. The output z, top: $z_1 = x$, bottom: $z_2 = u$.

may display the plots of the performance gain owing to each preview time in a single one-dimensional plot. This plot is shown in Fig. 5. Notice that increasing the preview times beyond around 1 time unit only gives very small performance gain. This can be explained by observing the general expression (36). There, the exponential term $e^{A_p h_k}$ is practically zero if $h_k \geqslant 1/|\mathrm{Re}\,\lambda_{\max}(A_p)|$, where $\lambda_{\max}(A_p)$ denotes the eigenvalue of A_p with the largest real part in absolute value. In our example, $|\mathrm{Re}\,\lambda_{\max}(A_p)| = 2$, so that increasing the preview times beyond several times of $\frac{1}{2}$ is practically useless. Fig. 5 also shows that the plot corresponding to h_2 is steeper than the one corresponding to h_1. This suggests that increasing the preview time of the second channel is more advantageous than increasing the one of the first channel. Thus, it makes sense to use different preview times for different channels. Now suppose we drive the system with an external input of the form $w(t) = [\delta(t-h_1)\ \ \delta(t-h_2)]^{\mathrm{T}}$. The response $z(t) = [x(t)\ \ u(t)]^{\mathrm{T}}$ is shown in Fig. 6 for two values of the pair (h_1, h_2). One plot is for the case where both channels are previewed with the same preview time ($h_1 = h_2 = 0.4$) while the other plot is for the case where the channels are previewed differently ($h_1 = 0.4$, $h_2 = 0.8$).

8. Concluding remarks

In this paper, the H_2 control problem of preview systems is considered. The single input case is first solved. Based on the single input results, the multiple inputs case and the multiple preview times case are treated. The results show that by providing the external input in advance to the controller, the H_2 performance of the control system may be improved. The paper also provides a formula for the optimal H_2-norm that clearly shows how the performance gain owing to the previewed input varies as the preview time increases. The results in this paper are derived using a similar approach to the one used in Moelja and Meinsma (2005) for solving the H_2 control problem for systems with multiple i/o delays. It is therefore possible to solve a combined H_2 preview/delay problem, where the exogenous inputs w is previewed and the internal input u is delayed, in the full information setting using the same approach. This extension will be reported elsewhere.

References

Anderson, B. D. O., & Moore, J. B. (1979). *Optimal filtering*. Englewood Cliffs, NJ: Prentice-Hall.

Anderson, B. D. O., & Moore, J. B. (1989). *Optimal control linear quadratic methods*. Englewood Cliffs, NJ: Prentice-Hall.

Cohen, A., & Shaked, U. (1997). Linear discrete-time H_∞-optimal tracking with preview. *IEEE Transactions on Automatic Control*, *42*, 270–276.

Kojima, A. (2004). H_2 performance on preview feedforward action. In *Proceedings of the 16th MTNS*, Leuven, Belgium.

Kojima, A., & Ishijima, S. (1999). LQ preview synthesis: Optimal control and worst case analysis. *IEEE Transactions on Automatic Control*, *44*, 352–356.

Kojima, A., & Ishijima, S. (2003a). Formulas on preview and delayed H_∞ control. In *Proceedings of the 42nd IEEE CDC* (pp. 6532–6538). Maui, Hawaii, USA.

Kojima, A., & Ishijima, S. (2003b). H_∞ performance of preview control systems. *Automatica*, *39*, 393–701.

Kučera, V. (1991). *Analysis and design of discrete linear control systems*. London: Prentice-Hall.

Marro, G., & Zattoni, E. (2005). H_2-optimal rejection with preview in the continuous-time domain. *Automatica*, *41*, 815–821.

Mirkin, L. (2003). On the H_∞ fixed-lag smoothing: How to exploit the information preview. *Automatica*, *39*, 1495–1504.

Mirkin, L., & Tadmor, G. (2004). Fixed-lag smoothing as a constrained version of the fixed-interval case. In *Proceedings of the 2004 American control conference*.

Moelja, A. A., & Meinsma, G. (2005). H_2-optimal control of systems with multiple i/o delays: Time domain approach. *Automatica*, *41*(7), 1229–1238.

Mosca, E. (1995). *Optimal, predictive, and adaptive control*. Englewood Cliffs, NJ: Prentice-Hall.

Mosca, E., & Casavola, A. (1995). Deterministic LQ preview tracking design. *IEEE Transactions on Automatic Control*, *40*, 1278–1281.

Mosca, E., Casavola, A., & Giarre, L. (1990). Minimax LQ stochastic tracking and servo problems. *IEEE Transactions on Automatic Control*, *35*, 95–97.

Mosca, E., & Zappa, G. (1989). Matrix fraction solution to the discrete-time LQ stochastic tracking and servo problems. *IEEE Transactions Automatic Control*, *34*, 240–242.

Samson, C. (1982). An adaptive LQ controller for non-minimum-phase systems. *International Journal of Control*, *35*, 1–28.

Shaked, U., & de Souza, C. E. (1995). Continuous-time tracking problems in an H_∞ setting: A game theory approach. *IEEE Transactions on Automatic Control*, *40*, 841–852.

Tadmor, G. (1997). Robust control in the gap: A state-space solution in the presence of a single input delay. *IEEE Transactions on Automatic Control*, *42*, 1330–1335.

Tadmor, G., & Mirkin, L. (2005a). H^∞ control and estimation with preview—part I: Matrix ARE solutions in continuous time. *IEEE Transactions on Automatic Control*, *50*, 19–28.

Tadmor, G., & Mirkin, L. (2005b). H^∞ control and estimation with preview—part II: Fixed-size ARE solutions in discrete time. *IEEE Transactions on Automatic Control*, *50*, 29–40.

Theodor, Y., & Shaked, U. (1994). Game theory approach to H_∞-optimal discrete-time fixed point and fixed-lag smoothing. *IEEE Transactions on Automatic Control*, *39*, 1944–1948.

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