Explicit Stochastic Predictive Control of Combustion Plants Based on Gaussian Process Models

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Abstract

Energy production is one of the largest sources of air pollution. A feasible method to reduce the harmful flue gas emissions and to increase the efficiency is to improve the control strategies of the existing thermoelectric power plants. This makes the Nonlinear Model Predictive Control (NMPC) method very suitable for achieving an efficient combustion control. Recently, an explicit approximate approach for stochastic NMPC based on a Gaussian process model was proposed. The benefits of an explicit solution, in addition to the efficient on-line computations, include also verifiability of the implementation, which is an essential issue in safety-critical applications. This paper considers the application of an explicit approximate approach for stochastic NMPC to the design of an explicit reference tracking NMPC controller for a combustion plant based on its Gaussian process model. The controller brings the air factor (respectively the concentration of oxygen in the flue gas) on its optimal value with every change of the load factor and thus an optimal operation of the combustion plant is achieved.

Keywords: Model predictive control, stochastic systems, probabilistic models, multi-parametric nonlinear programming, power plants.

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1. Introduction.

Energy production is one of the largest sources of air pollution. Therefore a rational and ecological use of energy is the main task of the thermoelectric power plants. A feasible method to reduce the NO_x, CO, CO_2 emissions and to increase the efficiency is to improve the control strategies of existing power plants, i.e. to optimize the combustion process (Čretnik & Gumprecht, 2006). The objectives for the improvement of the power plant combustion process are energy saving, pollution reduction, longer plant lifetime, less downtime and maintenance effort, increased safety in operation, i.e. overall cost reduction. These goals can be achieved through application of modern control algorithms with low on-line computational complexity and high reliability of the implementation. Feedback combustion control is possible since continuous flue gas analyzers are available (Čretnik & Gumprecht, 2006). For control purposes it would be ideal to measure all flue gas components. But the price for such realization would be too high in comparison with the savings achieved. Therefore the control of the oxygen fraction in the flue gas, measured on-line by the well known in-situ ZrO_2 analyzers, is often the best solution (Čretnik, 1994). Based on that, different algorithms for combustion control have been studied in (Slevin, 1984; Čretnik, 1992; Bitenc, Čretnik, Petrovčič & Strmčnik, 1992; Čretnik, 1994; Kocijan, 1997) and more recent in e.g. (Rangaswamy, Shanmugan, Mohammed & Thyagarajan, 2005; Dong, Wang & Zhao, 2005). It should be noted that these methods assume that *the combustion model is known exactly*. However, the mathematical models are only an approximation of the real process and they usually contain some amount of uncertainty (unknown additive disturbances and/or uncertain model parameters). In order to achieve a robust performance of the control system it would be necessary to *take into account the uncertainty* when designing the controller.

Nonlinear Model Predictive Control (NMPC) has become the accepted methodology to solve complex control problems related to process industries (Mayne, Rawlings, Rao & Scokaert, 2000; Allgöwer & Zheng, 2000; Kouvaritakis & Cannon, 2001). It involves the solution at each sampling instant of a finite horizon optimal control problem subject to nonlinear system dynamics and state and input constraints. *Stochastic* NMPC problems are formulated in the applications where the

system to be controlled is described by a stochastic model. Thus, the approaches in (Lee & Cooley, 1998; van Hessem, Scherer & Bosgra, 2001; Yan & Bitmead, 2005) are based on linear state space models with *stochastic* parameters and/or additive noise and they optimize the expected value of the cost function subject to hard input constraints (Lee & Cooley, 1998) or probabilistic constraints (van Hessem, Scherer & Bosgra, 2001; Yan & Bitmead, 2005). In (Kouvaritakis, Cannon & Couchman, 2006; Couchman, Kouvaritakis & Cannon, 2006; Couchman, Cannon & Kouvaritakis, 2006; Couchman, Cannon & Kouvaritakis, 2005), *stochastic* MPC approaches incorporating a probabilistic cost and probabilistic constraints are developed. The method suggested in (Kouvaritakis, Cannon & Couchman, 2006) is based on a moving average (MA) model with random coefficients. It was further extended to linear time-varying MA models (Couchman, Kouvaritakis & Cannon, 2006) and to state space models with stochastic uncertainty in the output or the input map (Couchman, Cannon & Kouvaritakis, 2006; Couchman, Cannon & Kouvaritakis, 2005). It should be noted that the mentioned *stochastic* MPC approaches are based on *parametric* probabilistic models. Alternatively, the stochastic systems can be modeled with *non-parametric* models which can offer a significant advantage compared to the *parametric* models. This is related to the fact that the *non-parametric* probabilistic models provide information about prediction uncertainties which are difficult to evaluate appropriately with the *parametric* models. The Gaussian process model is an example of a *non-parametric* probabilistic black-box model and up to now it has been applied to model mainly static nonlinearities. The use of Gaussian processes in the modelling of dynamic systems is a recent development e.g. (Kocijan, Girard, Banko & Murray-Smith, 2005; Solak, Murray-Smith, Leithead, Leith & Rasmussen, 2003). In (Kocijan & Murray-Smith, 2005; Likar & Kocijan, 2007; Murray-Smith, Sbarbaro, Rasmussen & Girard, 2003), an on-line optimization approach for *stochastic* NMPC based on Gaussian process model is proposed.

It has recently been shown that the feedback solution to linear and quadratic constrained MPC problems has an *explicit* representation as a piece-wise linear (PWL) state feedback defined on a polyhedral partition of the state space (Bemporad, Morari, Dua & Pistikopoulos, 2002). The benefits of an *explicit* solution, in addition to the efficient on-line computations, include also

verifiability of the implementation, which is an essential issue in safety-critical applications. For nonlinear and stochastic MPC the prospects of *explicit* solutions are even higher than for linear MPC, since the benefits of computational efficiency and verifiability are even more important. In (Grancharova, Kocijan & Johansen, 2007), an approach for off-line computation of *explicit* suboptimal *stochastic* NMPC controller for constrained nonlinear systems based on a *Gaussian process model* has been proposed. The approach is based on the multi-parametric Nonlinear Programming (mp-NLP) ideas (Fiacco, 1983) and represents an extension of the approximate methods in (Johansen, 2004; Grancharova, Johansen & Tøndel, 2005).

In this paper, a Gaussian process model of a combustion plant (a steam boiler PK 401 at Cinkarna Celje Company, Celje, Slovenia) is obtained. Then, the approximate mp-NLP approach (Grancharova, Kocijan & Johansen, 2007) is applied to design an explicit reference tracking NMPC controller that brings the air factor of the combustion plant on its optimal value with every change of the load factor. Thus, an efficient on-line optimization of the combustion plant is achieved where both the economic and the environmental aspects are taken into account. Because of the operation security of the considered combustion plant, the results obtained in the paper are based on simulation data. However, the paper shows the potential use of the presented approximate mp-NLP approach to the optimal control of industrial combustion plants.

The paper is structured as follows. In section 2, the techno-economical and environmental viewpoints of combustion plant operation are considered. The modelling of dynamic systems with Gaussian processes is described in section 3, where a Gaussian process model of a specific combustion plant is obtained. In section 4, the approximate approach to explicit stochastic nonlinear predictive control based on Gaussian process models is presented. The design and the closed-loop performance of an explicit reference tracking NMPC controller for the considered combustion plant are described in section 5. The conclusions are gathered in section 6.

The following abbreviation and notation will be used in the paper. The nonlinear model predictive control problem based on Gaussian process model will be referred to as GP-NMPC problem. $A \succ 0$ means that the square matrix A is positive definite. For $x \in \mathbb{R}^n$, the Euclidean

norm is $\|x\| = \sqrt{x^T x}$ and the weighted norm is defined for some symmetric matrix $A \succ 0$ as $\|x\|_A = \sqrt{x^T A x}$. For a random variable y with Gaussian distribution, $\mathcal{N}(\mu(y), \sigma^2(y))$ denotes its probability distribution, and $\mu(y)$ and $\sigma^2(y)$ are respectively its mean and variance.

2. Optimal operation of combustion plants.

Fuel composition can be expressed with percentage of carbon C, hydrogen H, oxygen O, nitrogen N, sulphur S, ash A and water W (Čretnik, Strmčnik & Zupančič, 1985):

$$C+H+O+N+S+A+W=100\% \quad (1)$$

Composition of the air is usually expressed only with the percentage of oxygen O_2 and nitrogen N_2:

$$O_2+N_2=21\%+79\%=100\% \quad (2)$$

The combustion process is schematically shown in Fig. 1.

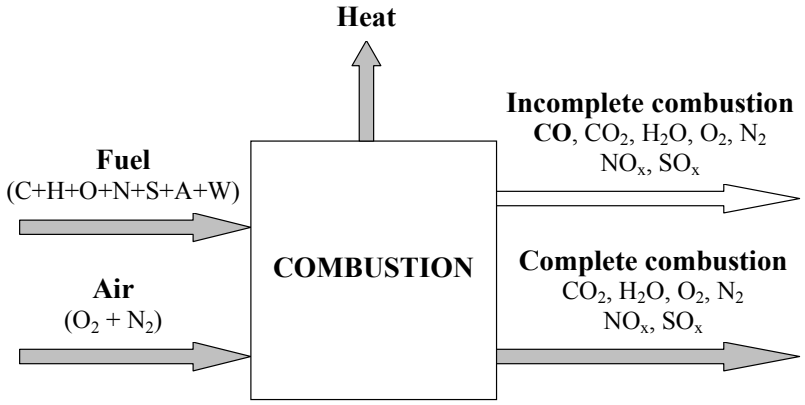

Fig. 1. Input and output flows of the combustion process.

The limited fuel sources, considerable increase in the fuel prices and the enormous environment pollution require decreasing the fuel use, the heat losses and the amount of harmful flue-gas emissions, i.e. to optimize the combustion process (Čretnik, 1992). It has been shown in (Čretnik, 1992) that in order to achieve an optimal operation of the combustion plants, it is necessary to optimize the air factor λ defined as:

$$\lambda = \frac{V_{air}}{V_{air,steh}} \tag{3}$$

where V_{air} is the volume of the air which goes into the combustion chamber and $V_{air,steh}$ is the stehiometrically required volume of the air necessary for complete combustion of 1 kg fuel. The combustion plant is working with air deficiency when $\lambda < 1$, and with air excess when $\lambda > 1$. Fig. 2 from (Čretnik, 1992) shows the aspects of the optimal combustion of fuel. From techno-economical viewpoint, the losses of the combustion can be reduced in two ways: 1) by reducing the quantity of the unburned fuel and 2) by reducing the quantity of the flue gases, i.e. of the heat losses. This leads to the optimal value $\lambda_{opt,t}$ of the air factor (cf. Fig. 2, left). From environmental viewpoint, it is desired to minimize the quantity of the harmful emissions and the corresponding optimal value of the air factor is $\lambda_{opt,e}$ (cf. Fig. 2, left). By taking into account both the techno-economical and the environmental aspects of combustion operation, it follows that the value λ of the air factor should be kept within the interval $[\lambda_{opt,t}; \lambda_{opt,e}]$.

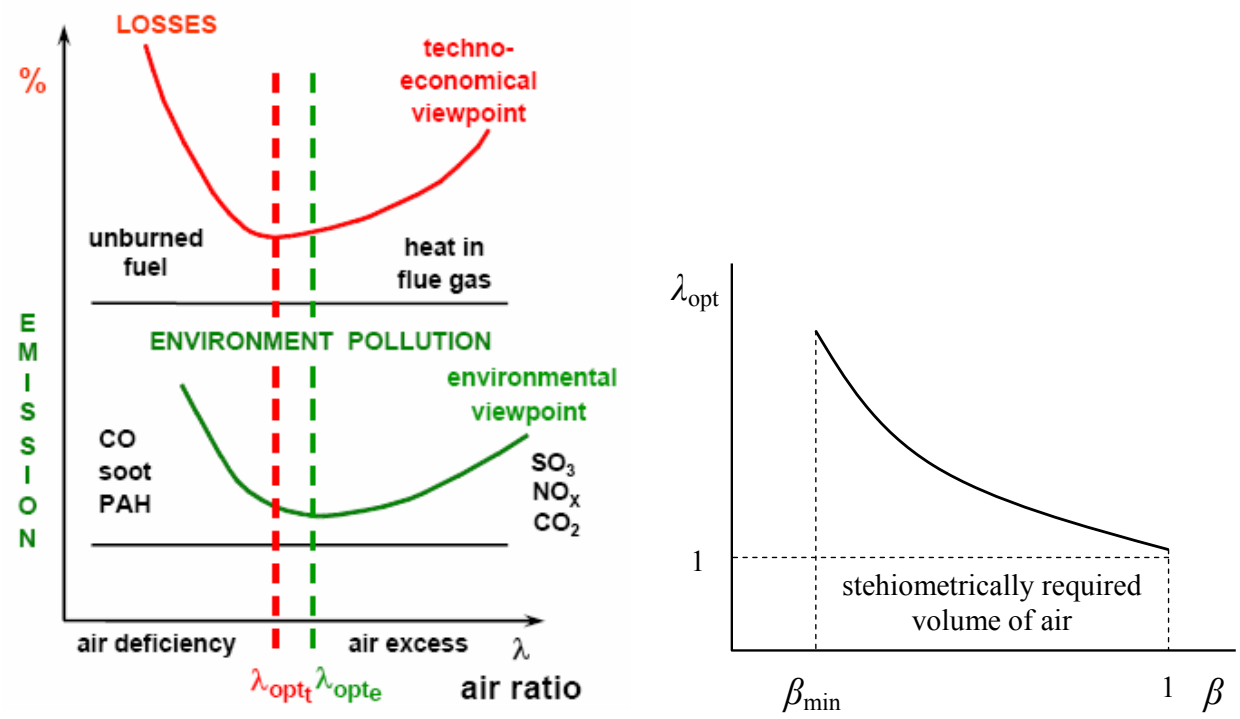

Fig. 2. Left: Techno-economical and environmental viewpoints of combustion (Čretnik, 1992). Right: The dependence of the optimal air factor on the load factor (Čretnik, 1992).

It has been also shown in the praxis that the optimal air factor λ_{opt} depends on the load factor β defined as:

$$\beta = \frac{\Phi_{fuel}}{\Phi_{fuel,\max}} \tag{4}$$

where Φ_{fuel} and $\Phi_{fuel,\max}$ are respectively the current and the maximal allowed fuel flowrate. The relation $\lambda_{opt} = f(\beta)$ is shown in Fig. 2 (right), where it can be seen that the optimal operation of the combustion plant is achieved with an air excess.

Therefore, the goal is to apply control algorithms that will maintain the air factor on its optimal value with every change of the load factor. Due to the importance of the described issue from economic and also environmental aspect, the combustion control is the field of constant development and research. This is also the driver for the development of the modeling and control approaches presented in the next sections.

3. Modelling of combustion plants with Gaussian process models.

3.1. Modelling of dynamic systems with Gaussian processes.

A Gaussian process is an example of the use of a flexible, probabilistic, nonparametric model which directly provides us with uncertainty predictions. Its use and properties for modelling are reviewed in (Rasmussen & Williams, 2006).

A Gaussian process is a collection of random variables which have a joint multivariate Gaussian distribution. Assuming a relationship of the form $y = f(z)$ between an input $z \in \mathbb{R}^D$ and output $y \in \mathbb{R}$, we have $y(1), y(2), \ldots, y(M) \sim \mathcal{N}(0, \mathbf{K})$, where $K_{pq} = \mathrm{Cov}(y(p), y(q)) = C(z(p), z(q))$ gives the covariance between the output points $y(p)$ and $y(q)$ corresponding to the input points $z(p)$ and $z(q)$. Thus, the mean $\mu(z)$ (usually assumed to be zero) and the covariance function $C(z(p), z(q))$ fully specify the Gaussian process. Note that the covariance function $C(z(p), z(q))$ can be any function with the property that it generates a positive definite covariance matrix. A common choice is the Gaussian covariance function (Williams, 1998;

Rasmussen & Williams, 2006):

$$C(z(p), z(q)) = v_1 \exp\left[-\frac{1}{2} \sum_{i=1}^{D} w_i (z_i(p) - z_i(q))^2 \right] + v_0 \alpha_{pq} \tag{5}$$

where $\Theta = [w_1, \ldots, w_D, v_0, v_1]$ is a vector of parameters called hyperparameters and z_i denotes the *i*-th component of the D-dimensional input vector z. The hyperparameter v_1 controls the magnitude of the covariance and the hyperparameters w_i represent the relative importance of each component z_i of vector z. The part $v_0 \alpha_{pq}$ represents the covariance between outputs due to white noise, where α_{pq} is the Kronecker operator and v_0 is the white noise variance (when assuming different kinds of noise the covariance function should be changed appropriately, e.g. (Gibbs, 1997)). For a given problem, the hyperparameters are learned (identified) using the data at hand. After the learning, one can use the w parameters as indicators of 'how important' the corresponding input components (dimensions) are: if w_i is zero or near zero it means that the inputs in dimension i contain little information and could possibly be removed.

Consider a set of M D-dimensional input vectors $\mathbf{Z} = [z(1), z(2), \ldots, z(M)]^T$ and a vector of output data $Y = [y(1), y(2), \ldots, y(M)]^T$. Based on the data $(\mathbf{Z}, Y)$, and given a new input vector z^*, we wish to estimate the probability distribution of the corresponding output y^*. Unlike other models, there is no model parameter determination as such, within a fixed model structure. With this model, most of the effort consists in *tuning* the parameters of the covariance function. This is done by maximizing the log-likelihood:

$$\mathcal{L}(\Theta) = \log(p(Y \mid \mathbf{Z})) = -\frac{1}{2}\log(|\mathbf{K}|) - \frac{1}{2} Y^T \mathbf{K}^{-1} Y - \frac{M}{2}\log(2\pi) \tag{6}$$

with the vector of hyperparameters Θ and $M \times M$ training covariance matrix $\mathbf{K}$, where the hyperparameters distribution $p(\Theta \mid Y, \mathbf{Z})$ is approximated with their most likely values. The optimization requires the computation of the derivative of $\mathcal{L}$ with respect to each of the parameters:

$$\frac{\partial \mathcal{L}(\Theta)}{\partial \theta_i} = -\frac{1}{2}\mathrm{trace}\left(\mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_i} \right) + \frac{1}{2} Y^T \mathbf{K}^{-1} \frac{\partial \mathbf{K}}{\partial \theta_i} \mathbf{K}^{-1} Y \tag{7}$$

Here, it involves the computation of the inverse of the $M \times M$ covariance matrix $\mathbf{K}$ at every iteration, which can be computationally demanding for large M. The reader is referred to (Rasmussen & Williams, 2006) for a detailed description of the parameter optimization methods.

Given that the hyperparameters are known, we can estimate the probability distribution of the corresponding output y^* at some new input vector z^*:

$$p(y^* \mid Y, \mathbf{Z}, z^*) = \frac{p(Y, y^*)}{p(Y \mid \mathbf{Z})} \tag{8}$$

It can be shown that this distribution is Gaussian with mean and variance (Williams, 1998):

$$\mu(z^*) = k(z^*)^T \mathbf{K}^{-1} Y \tag{9}$$

$$\sigma^2(z^*) = k_0(z^*) - k(z^*)^T \mathbf{K}^{-1} k(z^*) + v_0 \tag{10}$$

where $k(z^*) = [C(z(1), z^*), ..., C(z(M), z^*)]^T$ is the $M \times 1$ vector of covariances between the test input and the training inputs and $k_0(z^*) = C(z^*, z^*)$ is the autocovariance of the test input. The vector $k(z^*)^T \mathbf{K}^{-1}$ in (9) can be interpreted as a vector of smoothing terms which weights the training outputs Y to make a prediction at the test point z^*. If the new input is far away from the data points, the term $k(z^*)^T \mathbf{K}^{-1} k(z^*)$ in (10) will be small, so that the predicted variance $\sigma^2(z^*)$ will be large. Thus, from the system identification point of view equation (9) provides the model prediction and equation (10) its confidence.

Gaussian processes can be used to model static nonlinearities and can therefore be used for modelling of dynamic systems if delayed input and output signals are used as regressors (Kocijan, Girard, Banko & Murray-Smith, 2005). In such cases an autoregressive model is considered, such that the current predicted output depends on previous estimated outputs, as well as on previous control inputs:

$$\begin{aligned} z(t) &= [\hat{y}(t-1), \hat{y}(t-2), \ldots, \hat{y}(t-L), u(t-1), u(t-2), \ldots, u(t-L)]^T \\ \hat{y}(t) &= f(z(t)) + \eta(t) \end{aligned} \tag{11}$$

where t denotes consecutive number of data sample, L is a given lag, and $\eta(t)$ is the prediction error. The quality of the predictions with a Gaussian process model is assessed by computing the

average squared error (ASE):

$$ASE = \frac{1}{M}\sum_{i=1}^{M}[\mu(\hat{y}(i)) - y(i)]^2 \tag{12}$$

and by the log predictive density error (LD) (Kocijan, Girard, Banko & Murray-Smith, 2005):

$$LD = \frac{1}{2}\log(2\pi) + \frac{1}{2M}\sum_{i=1}^{M}\left(\log[\sigma^2(\hat{y}(i))] + \frac{[\mu(\hat{y}(i)) - y(i)]^2}{\sigma^2(\hat{y}(i))}\right) \tag{13}$$

In (12), (13), $\mu(\hat{y}(i))$ and $\sigma^2(\hat{y}(i))$ are the prediction mean and variance, $y(i)$ is the system's output and M is the number of the training points.

The Gaussian process model now not only describes the dynamic characteristics of the non-linear system, but at the same time provides information about the confidence in the predictions. The Gaussian process can highlight areas of the input space where prediction quality is poor, due to the lack of data, by indicating the higher variance around the predicted mean.

3.2. Gaussian process model of a combustion plant.

The system under investigation is a process of combustion in a steam boiler PK 401 at Cinkarna Celje Company, Celje, Slovenia. Since it was not possible to perform experiments on this plant because of the security of plant operation, the Gaussian process model identification was based on simulation data generated by adding a Gaussian disturbance to the analytical model developed in (Čretnik, Strmčnik & Zupančič, 1985).

The fuel composition is expressed with the percentages of carbon C, hydrogen H, oxygen O, nitrogen N, sulphur S, ash A and water H_2O (denoted respectively with x_C^{fuel}, x_H^{fuel}, x_O^{fuel}, x_N^{fuel}, x_S^{fuel}, x_A^{fuel}, $x_{H_2O}^{fuel}$):

$$x_C^{fuel} + x_H^{fuel} + x_O^{fuel} + x_N^{fuel} + x_S^{fuel} + x_A^{fuel} + x_{H_2O}^{fuel} = 100\% \tag{14}$$

The composition of the air is assumed to be 21% oxygen and 79% nitrogen. The equations of the developed analytical model (Čretnik, Strmčnik & Zupančič, 1985) are based on the stoichiometric chemical reactions of combustion:

$$\mathrm{C} + \mathrm{O_2} \rightarrow \mathrm{CO_2} + Q_1 \tag{15}$$

$$\mathrm{C} + \frac{1}{2}\mathrm{O_2} \rightarrow \mathrm{CO} + Q_2 \tag{16}$$

$$2\mathrm{H_2} + \mathrm{O_2} \rightarrow 2\mathrm{H_2O} + Q_3 \tag{17}$$

$$\mathrm{S} + \mathrm{O_2} \rightarrow \mathrm{SO_2} + Q_4 \tag{18}$$

where Q_1, Q_2, Q_3, Q_4 are the heats of the reactions. The composition of the flue gasses, resulting from the combustion process, is expressed in the following way (Čretnik, Strmčnik & Zupančič, 1985):

$$x_{\mathrm{O_2}} + x_{\mathrm{CO}} + x_{\mathrm{CO_2}} + x_{\mathrm{SO_2}} + x_{\mathrm{N_2}} + x_{\mathrm{H_2O}} = 100\% \tag{19}$$

where $x_{\mathrm{O_2}}$, x_{CO}, $x_{\mathrm{CO_2}}$, $x_{\mathrm{SO_2}}$, $x_{\mathrm{N_2}}$ and $x_{\mathrm{H_2O}}$ are the volume percentages of oxygen, carbon monoxide, carbon dioxide, sulphur dioxide, nitrogen and water. Then, the volume balances for the separate components of the flue gasses are described by the following equations (Čretnik, Strmčnik & Zupančič, 1985):

$$\frac{dx_{\mathrm{O_2}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{O_2}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 21\Phi_{air} - 100V_o\Phi_{fuel}\} \tag{20}$$

$$\frac{dx_{\mathrm{CO}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{CO}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 1.866ax_{\mathrm{C}}^{fuel}\Phi_{fuel}\} \tag{21}$$

$$\frac{dx_{\mathrm{CO_2}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{CO_2}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 1.866(1-a)x_{\mathrm{C}}^{fuel}\Phi_{fuel}\} \tag{22}$$

$$\frac{dx_{\mathrm{SO_2}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{SO_2}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 0.699x_{\mathrm{S}}^{fuel}\Phi_{fuel}\} \tag{23}$$

$$\frac{dx_{\mathrm{N_2}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{N_2}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 79\Phi_{air} + 0.8x_{\mathrm{N}}^{fuel}\Phi_{fuel}\} \tag{24}$$

$$\frac{dx_{\mathrm{H_2O}}}{dt} = \frac{1}{V_k}\{-x_{\mathrm{H_2O}}[\Phi_{air} + \Phi_{fuel}(V_d - V_o)] + 11.117x_{\mathrm{H}}^{fuel}\Phi_{fuel} + 1.244x_{\mathrm{H_2O}}^{fuel}\Phi_{fuel}\} \tag{25}$$

In (20)–(25), V_k is the volume of the combustion chamber [$\mathrm{m^3}$], Φ_{fuel} is the normalized total flow of fuel [$\mathrm{kg\ s^{-1}}$], Φ_{air} is the normalized total flow of air [$\mathrm{Nm^3 s^{-1}}$], V_o is the theoretically required

oxygen volume for the combustion of one unit of fuel [Nm^3kg^{-1}], V_d is the theoretically obtained gas volume from one unit of fuel [Nm^3kg^{-1}], a is the relative portion of carbon converted into CO.

The model (20)–(25) enables the simulation of the six flue-gas components. However, for control design purposes only its O_2-part (equation (20)) named also O_2-model is used (Čretnik, 1992; Čretnik, 1994). The input to the O_2-model is the angle position of the damper, which is used to control the air flow Φ_{air}. The model output is the oxygen concentration in the flue gasses. As the damper is a part of the closed-loop, it has to be modeled and added to the O_2-model (20). The dependence of the air flow Φ_{air} on the angle ϕ of the damper is given by the following relation (Čretnik, 1992):

$$\Phi_{air} = \frac{\Phi_{air,\max}}{2} \exp\left(\frac{3(\phi - 45)}{45}\right), \ 0^\circ \le \phi \le 45^\circ \tag{26}$$

$$\Phi_{air} = \frac{\Phi_{air,\max}}{2} \left(2 - \exp\left(\frac{-3(\phi - 45)}{45}\right)\right), \ 45^\circ \le \phi \le 90^\circ \tag{27}$$

where $\Phi_{air,\max}$ is the maximum flow of air.

The O_2-model (20) is a deterministic model, which does not take into account the stochastic disturbances (e.g. change in the fuel composition, change of the humidity of the air flow) that may influence the combustion process. In order to consider the stochastic nature of plant operation, the dynamics of x_{O_2} is represented by the following stochastic discrete-time model:

$$x_{O_2}(t+1) = f(x_{O_2}(t), \Phi_{fuel}(t), \phi(t)) + \xi(t) \tag{28}$$

Here, $\xi(t) \in \mathbb{R}$ is a Gaussian disturbance which represents the additive effect of the unmeasured stochastic disturbances. The sampling time, determined according to system dynamics, was selected to be $T_s = 1$ [s].

The signals ϕ and Φ_{fuel} for identification were generated by random number generators with normal distributions and the signal x_{O_2} was computed from the O_2-model (20). The ϕ signal blocking was $T_\phi = 5T_s$, i.e. it is kept constant for 5 time instants. The Φ_{fuel} signal blocking was

$T_{\Phi_{fuel}} = 100T_s$. The number M of the signals samples used for the identification determines the dimension of the covariance matrix. In our case, $M = 1000$. A Gaussian disturbance ξ with zero mean and variance 0.05 was used. Based on the generated data set, the discrete-time system (28) is approximated with Gaussian process with the following hyperparameters:

$$\Theta = [w_1, w_2, w_3, v_0, v_1] = [0.01346, 0.02847, 0.00036, 0.21984, 55.56554] \quad (29)$$

The maximum likelihood framework was used to determine the hyperparameters. The optimization method applied for identification of the Gaussian process model was the conjugate gradient method with line searches (Girard & Murray-Smith, 2005).

The response of the Gaussian process model to the identification signal is shown in Fig. 3. The associated average squared error and log density error are respectively $ASE = 0.6051$ and $LD = 143.4835$.

The signals ϕ and Φ_{fuel} for validation were generated by random number generator with normal distribution and rate of change that is different from the one used for the identification signals. The response of the Gaussian process model to the validation signals is shown in Fig. 4. The associated prediction errors are $ASE = 0.9177$ and $LD = 188.8626$.

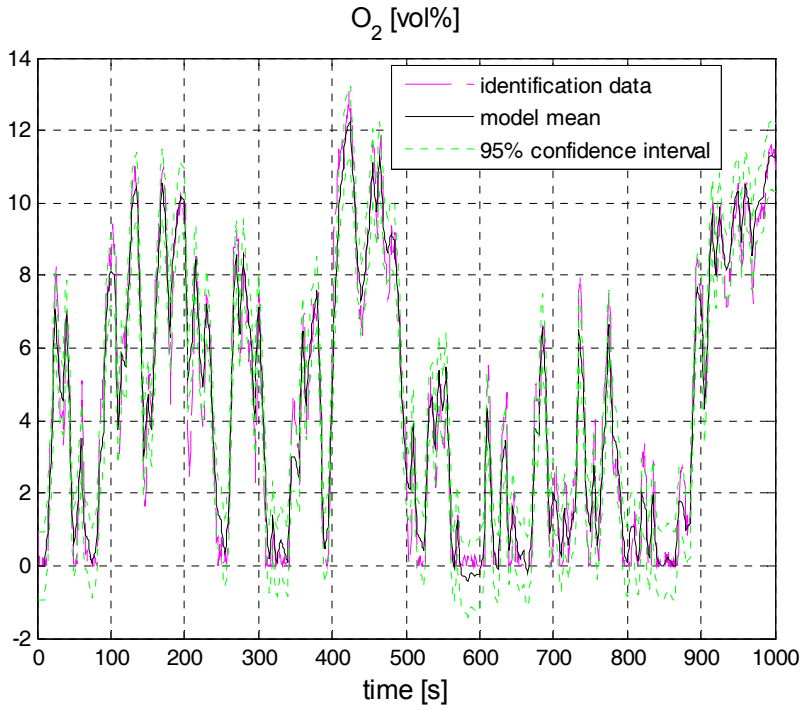

Fig. 3. Response of the Gaussian process model to the excitation signal used for identification.

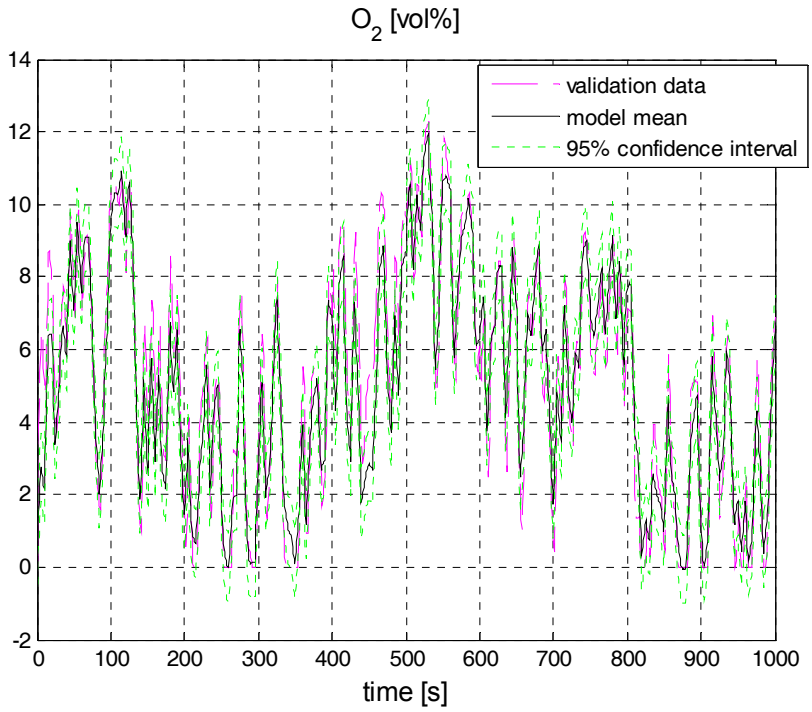

Fig. 4. Response of the Gaussian process model to the excitation signal used for validation.

4. Approximate explicit stochastic nonlinear predictive control based on Gaussian process models

4.1. Formulation of the GP-NMPC problem as an mp-NLP problem.

Consider a stochastic nonlinear discrete-time system:

$$x(t+1) = f(x(t), u(t)) + \xi(t) \tag{30}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and input variables, $\xi(t) \in \mathbb{R}^n$ are Gaussian disturbances, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a nonlinear continuous function. Suppose that a Gaussian process model of the system (30) is obtained by applying the approach described in the previous section. Suppose the initial state $x(t) = x_{t|t}$ and the control inputs $u(t+k) = u_{t+k}$, $k = 0,1, \ldots, N-1$ are given. Then, the probability distribution of the predicted states $x_{t+k+1|t}$, $k = 0,1, \ldots, N-1$ which correspond to the given initial state $x_{t|t}$ and control inputs u_{t+k}, $k = 0,1, \ldots, N-1$ can be obtained (Girard & Murray-Smith, 2005):

$$x_{t+k+1|t} \mid x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t}))$$
$$k = 0,1, \ldots, N-1 \tag{31}$$

The 95% confidence interval of the random variable $x_{t+k+1|t}$ is $[\mu(x_{t+k+1|t}) - 2\sigma(x_{t+k+1|t}); \mu(x_{t+k+1|t}) + 2\sigma(x_{t+k+1|t})]$, where $\sigma(x_{t+k+1|t})$ is the standard deviation.

Here, we consider a reference tracking NMPC problem based on a Gaussian process model (GP-NMPC) where the goal is to have the state vector $x(t)$ track the reference signal $r(t) \in \mathbb{R}^n$. In the problem formulation, the type of the cost function is like the one used in (Bemporad, Morari, Dua & Pistikopoulos, 2002). Suppose that a full measurement of the state $x(t)$ is available at the current time t. For the current $x(t)$, the reference tracking GP-NMPC solves the following optimization problem:

Problem P1:

$$V^*(x(t), r(t), u(t-1)) = \min_U J(U, x(t), r(t), u(t-1)) \tag{32}$$

subject to $x_{t|t} = x(t)$ and:

$$\mu(x_{t+k|t}) - 2\sigma(x_{t+k|t}) \geq x_{\min}, \ k = 1, \ldots, N \tag{33}$$

$$\mu(x_{t+k|t}) + 2\sigma(x_{t+k|t}) \leq x_{\max}, \ k = 1, \ldots, N \tag{34}$$

$$u_{\min} \leq u_{t+k} \leq u_{\max}, \ k = 0,1, \ldots, N-1 \tag{35}$$

$$\Delta u_{\min} \leq \Delta u_{t+k} \leq \Delta u_{\max}, \ k = 0,1, \ldots, N-1 \tag{36}$$

$$\max\{\left\|\mu(x_{t+N|t}) - 2\sigma(x_{t+N|t}) - r(t)\right\|, \left\|\mu(x_{t+N|t}) + 2\sigma(x_{t+N|t}) - r(t)\right\|\} \leq \delta \tag{37}$$

$$\Delta u_{t+k} = u_{t+k} - u_{t+k-1}, \ k = 0,1, \ldots, N-1 \tag{38}$$

$$x_{t+k+1|t} \mid x_{t+k|t}, u_{t+k} \sim \mathcal{N}(\mu(x_{t+k+1|t}), \sigma^2(x_{t+k+1|t}))$$
$$k = 0,1, \ldots, N-1 \tag{39}$$

with $U = [u_t, u_{t+1}, \ldots, u_{t+N-1}]$ and the cost function given by:

$$J(U, x(t), r(t), u(t-1)) = \sum_{k=0}^{N-1} \left[\left\|\mu(x_{t+k|t}) - r(t)\right\|_Q^2 + \left\|\Delta u_{t+k}\right\|_R^2 \right] + \left\|\mu(x_{t+N|t}) - r(t)\right\|_P^2 \tag{40}$$

Here, N is a finite horizon and $P, Q, R \succ 0$. From a stability point of view it is desirable to choose

δ in the terminal constraint (37) sufficiently small (Mayne, Rawlings, Rao & Scokaert, 2000). If the horizon N is large and the Gaussian process model has a small prediction uncertainty, then it is more likely that the choice of a small δ will be possible.

It should be noted that a more general stochastic MPC problem is formulated in (Kouvaritakis, Cannon & Couchman, 2006; Couchman, Kouvaritakis & Cannon, 2006; Couchman, Cannon & Kouvaritakis, 2006; Couchman, Cannon & Kouvaritakis, 2005), where a probabilistic formulation of the cost is introduced that includes the probabilistic bounds of the predicted variable. The stochastic MPC problem considered in this paper (problem P1) is of a more special form since the cost function (40) includes the mean value of the random variable. However, the approximate approach to the explicit solution of problem P1 (presented in section 4.2) can be easily extended to the more general case of stochastic MPC problem formulation where the optimization is performed on the expected value of the cost function.

We introduce an extended state vector:

$$\tilde{x}(t) = [x(t),\, r(t),\, u(t-1)] \in \mathbb{R}^{\tilde{n}},\ \tilde{n} = 2n + m \tag{41}$$

Let $\tilde{x}$ be the value of the extended state at the current time t. Then, the optimization problem P1 can be formulated in a compact form as follows:

Problem P2:

$$V^*(\tilde{x}) = \min_U J(U, \tilde{x}) \quad \text{subject to} \quad G(U, \tilde{x}) \le 0 \tag{42}$$

The GP-NMPC problem defines an mp-NLP, since it is NLP in U parameterized by $\tilde{x}$. An optimal solution to this problem is denoted $U^* = [u_t^*, u_{t+1}^*, ..., u_{t+N-1}^*]$ and the control input is chosen according to the receding horizon policy $u(t) = u_t^*$. Define the set of *N*-step feasible initial states as follows:

$$X_f = \{\tilde{x} \in \mathbb{R}^{\tilde{n}} \mid G(U, \tilde{x}) \le 0 \text{ for some } U \in \mathbb{R}^{Nm}\} \tag{43}$$

If δ in (37) is chosen such that the problem P1 is feasible, then X_f is a non-empty set.

In parametric programming problems one seeks the solution $U^*(\tilde{x})$ as an explicit function of the parameters $\tilde{x}$ in some set $X \subseteq X_f \subseteq \mathbb{R}^{\tilde{n}}$ (Fiacco, 1983). The explicit solution allows us to replace the computationally expensive real-time optimization with a simple function evaluation.

4.2. Approximate mp-NLP approach to explicit GP-NMPC.

In general, the exact solution of problem P2 can not be found. In this section, the computational method (Grancharova, Kocijan, & Johansen, 2007) for constructing an explicit PWL approximate solution of the reference tracking GP-NMPC problem is described.

4.2.1. Close-to-global solution of mp-NLP.

In general, problem P2 can be non-convex with multiple local minima. Therefore, it would be necessary to apply an efficient initialization of problem P2 so to find a close-to-global solution. One possible way to obtain this is to find a close-to-global solution at a point $v_0 \in X_0$ by comparing the local minima corresponding to several initial guesses and then to use this solution as an initial guess at the neighbouring points $v_i \in X_0$, $i = 1, 2, \ldots, N_1$, i.e. to propagate the solution. The following procedure is used to generate a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$, where $v_i \in X_0$, $i = 0, 1, 2, \ldots, N_1$.

Procedure 1 (generation of set of points):

Consider any hyper-rectangle $X_0 \subseteq X_f$ *with vertices* $\Lambda^0 = \{\lambda_1^0, \lambda_2^0, \ldots, \lambda_{N_\lambda}^0\}$ *and center point* v_0. *Consider also the hyper-rectangles* $X_0^j \subset X_0$, $j = 1, 2, \ldots, N_j$ *with vertices respectively* $\Lambda^j = \{\lambda_1^j, \lambda_2^j, \ldots, \lambda_{N_\lambda}^j\}$, $j = 1, 2, \ldots, N_j$. *Suppose* $X_0^1 \subset X_0^2 \subset \ldots \subset X_0^{N_j}$. *For each of the hyper-rectangles* X_0 *and* $X_0^j \subset X_0$, $j = 1, 2, \ldots, N_j$, *determine a set of points that belongs to its facets and denote this set* $\Phi^j = \{\phi_1^j, \phi_2^j, \ldots, \phi_{N_\phi}^j\}$, $j = 0, 1, 2, \ldots, N_j$. *Define the set of all points* $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$, *where* $v_i \in \left\{\bigcup_{j=0}^{N_j} \Lambda^j\right\} \cup \left\{\bigcup_{j=0}^{N_j} \Phi^j\right\}$, $i = 1, 2, \ldots, N_1$.

The following procedure is applied to find a close-to-global solution at the points $v_i \in V_0$, $i = 0,1,2,\ldots,N_1$:

Procedure 2 (close-to-global solution of problem P2):

Consider any hyper-rectangle $X_0 \subseteq X_f$ *with a set of points* $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$ *determined by applying Procedure 1. Then:*

a). *Determine a close-to-global solution of problem* P2 *at the center point* v_0 *through the following minimization:*

$$U^*(v_0) = \arg \min_{U_i^{local} \in \{U_1^{local},\ldots,U_{N_U}^{local}\}} J(U_i^{local}, v_0), \tag{44}$$

where U_i^{local}, $i = 1,2,\ldots,N_U$ *correspond to local minima of the cost function* $J(U, v_0)$ *obtained for a number of initial guesses* U_i^0, $i = 1,2,\ldots,N_U$.

b). *Determine a close-to-global solution of problem* P2 *at the points* $v_i \in V_0$, $i = 1,2,\ldots,N_1$ *in the following way:*

1. Determine a close-to-global solution of problem P2 *at the center point* v_0 *by solving problem* (44). *Let* $i = 1$.

2. Let $V^s = \{v_0, v_1, v_2, \ldots, v_{N_2}\} \subset V_0$ *be the subset of points at which a feasible solution of problem* P2 *has been already determined.*

3. Find the point $\bar{v} \in V^s$ *that is most close to the point* v_i, *i.e.* $\bar{v} = \arg \min_{v \in V^s} \|v - v_i\|$. *Let the solution at* $\bar{v}$ *be* $U^*(\bar{v})$.

4. Solve problem P2 *at the point* v_i *with initial guess for the optimization variables set to* $U^*(\bar{v})$.

5. If a solution of problem P2 *at the point* v_i *has been found, mark* v_i *as feasible and add it to the set* V^s. *Otherwise, mark* v_i *as infeasible.*

6. Let $i = i + 1$. *If* $i \le N_1$, *go to step 2. Otherwise, terminate.*

4.2.2. Computation of feasible PWL solution.

Definition 1 (Feasibility on a discrete set):

Let $X = \{v_1, v_2, \dots, v_Q\} \subset \mathbb{R}^{\tilde{n}}$ *be a discrete set. A function* $U(\tilde{x})$ *is feasible on* X *if* $G(U(v_i), v_i) \leq 0$, $i \in \{1, 2, \dots, Q\}$.

We restrict our attention to a hyper-rectangle $X \subset \mathbb{R}^{\tilde{n}}$ where we seek to approximate the optimal solution $U^*(\tilde{x})$ to problem P2. We require that the state space partition is orthogonal and can be represented as a $k-d$ tree. The main idea of the approximate mp-NLP approach is to construct a feasible piecewise linear (PWL) approximation $\hat{U}(\tilde{x})$ to $U^*(\tilde{x})$ on X, where the constituent affine functions are defined on hyper-rectangles covering X. In case of convexity, it suffices to compute the solution of problem P2 at the $2^{\tilde{n}}$ vertices of a considered hyper-rectangle X_0 by solving up to $2^{\tilde{n}}$ NLPs. In case of non-convexity, it would not be sufficient to impose the constraints only at the vertices of the hyper-rectangle X_0. One approach to resolve this problem is to include some interior points in addition to the set of vertices of X_0 (Grancharova, Johansen, & Tøndel, 2005). These additional points can represent the vertices and the facets centers of one or more hyper-rectangles contained in the interior of X_0. Based on the solutions at all points, a feasible local linear approximation $\hat{U}_0(\tilde{x}) = K_0\tilde{x} + g_0$ to the optimal solution $U^*(\tilde{x})$, valid in the whole hyper-rectangle X_0, is determined by applying the following procedure:

Procedure 3 (computation of explicit approximate solution):

Consider any hyper-rectangle $X_0 \subseteq X_f$ *with a set of points* $V_0 = \{v_0, v_1, v_2, \dots, v_{N_1}\}$ *determined by applying Procedure 1. Compute* K_0 *and* g_0 *by solving the following NLP:*

Problem P3:

$$\min_{K_0, g_0} \sum_{i=0}^{N_1} \left(J(K_0 v_i + g_0, v_i) - V^*(v_i) + \beta \left\| K_0 v_i + g_0 - U^*(v_i) \right\|_2^2\right) \tag{45}$$

subject to:

$$G\left(K_0 v_i + g_0, v_i\right) \le 0 \ , \ \forall v_i \in V_0 \tag{46}$$

In (45), the parameter $\beta > 0$ is a weighting coefficient.

4.2.3. Estimation of error bounds.

Suppose that a state feedback $\hat{U}_0(\tilde{x})$ that is feasible on $V_0 \subseteq X_0$ has been determined by applying Procedure 3. Then, for the cost function approximation error in X_0 we have:

$$\varepsilon(\tilde{x}) = \hat{V}(\tilde{x}) - V^*(\tilde{x}) \le \varepsilon_0 \ , \ \tilde{x} \in X_0 \tag{47}$$

where $\hat{V}(\tilde{x}) = J(\hat{U}_0(\tilde{x}), \tilde{x})$ is the sub-optimal cost and $V^*(\tilde{x})$ denotes the cost corresponding to the close-to-global solution $U^*(\tilde{x})$, i.e. $V^*(\tilde{x}) = J(U^*(\tilde{x}), \tilde{x})$. The following procedure can be used to obtain an estimate $\hat{\varepsilon}_0$ of the maximal approximation error ε_0 in X_0.

Procedure 4 (computation of the error bound):

Consider any hyper-rectangle $X_0 \subseteq X_f$ with a set of points $V_0 = \left\{v_0, v_1, v_2, \ldots, v_{N_1}\right\}$ determined by applying Procedure 1. Compute an estimate $\hat{\varepsilon}_0$ of the error bound ε_0 through the following maximization:

$$\hat{\varepsilon}_0 = \max_{i \in \{0,1,2,\ldots,N_1\}} \left(\hat{V}(v_i) - V^*(v_i)\right) \tag{48}$$

4.2.4. Approximate mp-NLP algorithm for explicit GP-NMPC.

Assume the tolerance $\bar{\varepsilon} > 0$ of the cost function approximation error is given. The following algorithm is proposed to design explicit reference tracking GP-NMPC:

Algorithm 1 (explicit reference tracking GP-NMPC)

1. Initialize the partition to the whole hyper-rectangle, i.e. $\Pi = \left\{X\right\}$. Mark the hyper-rectangle X as unexplored.

2. Select any unexplored hyper-rectangle $X_0 \in \Pi$. If no such hyper-rectangle exists, terminate.

3. Compute a solution to problem P2 at the center point v_0 of X_0 by applying Procedure 2a. If problem P2 has a feasible solution, go to step 4. Otherwise, split X_0 into two hyper-rectangles X_1 and X_2 by applying the heuristic rule 2 from (Grancharova, Johansen, & Tøndel, 2005). Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

4. Define a set of points $V_0 = \{v_0, v_1, v_2, \ldots, v_{N_1}\}$ by applying Procedure 1. Compute a solution to problem P2 for $\tilde{x}$ fixed to each of the points v_i, $i = 1, 2, \ldots, N_1$ by applying Procedure 2b. If problem P2 has a feasible solution at all these points, go to step 6. Otherwise, go to step 5.

5. Compute the size of X_0 using some metric. If it is smaller than some given tolerance, mark X_0 infeasible and explored and go to step 2. Otherwise, split X_0 into hyper-rectangles X_1, X_2, …, X_{N_s} by applying the heuristic rule 1 from (Grancharova, Johansen, & Tøndel, 2005). Mark X_1, X_2, …, X_{N_s} unexplored, remove X_0 from Π, add X_1, X_2, …, X_{N_s} to Π, and go to step 2.

6. Compute an affine state feedback $\hat{U}_0(\tilde{x})$ using Procedure 3, as an approximation to be used in X_0. If no feasible solution was found, split X_0 into two hyper-rectangles X_1 and X_2 by applying the heuristic rule 3 from (Grancharova, Johansen, & Tøndel, 2005). Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

7. Compute an estimate $\hat{\varepsilon}_0$ of the error bound ε_0 in X_0 by applying Procedure 4. If $\hat{\varepsilon}_0 \le \bar{\varepsilon}$, mark X_0 as explored and feasible and go to step 2. Otherwise, split X_0 into two hyper-rectangles X_1 and X_2 by applying Procedure 4 from (Grancharova, Johansen, & Tøndel, 2005). Mark X_1 and X_2 unexplored, remove X_0 from Π, add X_1 and X_2 to Π, and go to step 2.

The presented approximate mp-NLP approach is a practical computational method to handle non-convex mp-NLP problems. It does not necessarily lead to guaranteed properties like feasibility and closed-loop stability, but when combined with verification and analysis methods gives a practical tool for development and implementation of explicit NMPC. It also should be noted that in contrast to the conventional MPC based on real-time optimization, the explicit MPC makes the

rigorous verification and validation of the controller performance much easier (Johansen, 2004). Hence, problems due to lack of convexity and numerical difficulties can be addressed during the design and implementation.

5. Explicit stochastic predictive control of combustion plants.

Here, an explicit reference tracking GP-NMPC controller for the combustion plant considered in section 3.2 is designed. The block-scheme of the control system is shown in Fig. 5. The controller brings the air factor (respectively the concentration of oxygen in the flue gas) on its optimal value with every change of the load factor and thus an optimal operation of the combustion plant is achieved.

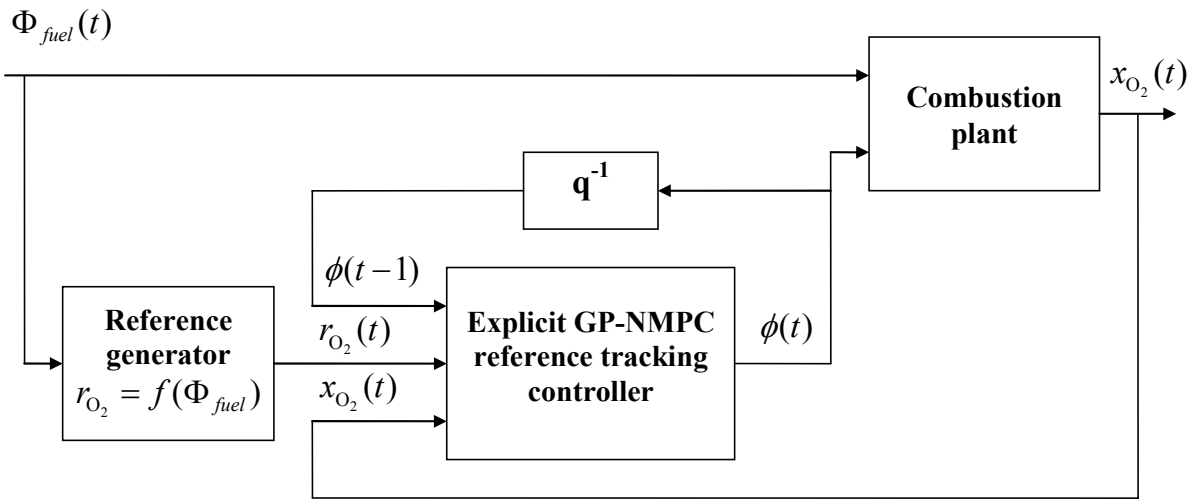

Fig. 5. Block scheme of the control system.

The control input is $u = \phi$ (the angle of the damper for the air flow), the state variable is $x = x_{O_2}$ (the percentage of O_2 in the flue gases), and the reference signal is $r = r_{O_2}$ (the required percentage of O_2 in the flue gases). For this particular combustion plant, the reference values r_{O_2} corresponding to different values of the fuel flowrate Φ_{fuel} have been obtained by experiments and are given in a table form as follows (Čretnik, 1992):

Table 1. Reference values for the percentage of O_2 in the flue gases.

Φ_{fuel} [kg s^{-1}]	0.7	0.8	0.9	1.0	1.1	1.2	1.3
r_{O_2} [vol %]	4.5	4.1	3.7	3.4	3.2	3.0	2.8

In case the fuel flowrate Φ_{fuel} does not take a value from this table, then the reference value r_{O_2} is computed through linear interpolation between the neighbouring points in the table.

The mp-NLP approach described in section 4.2 is applied to design an explicit reference tracking GP-NMPC controller for the combustion plant based on its Gaussian process model obtained in section 3.2:

$$x_{O_2}(t+1) \mid x_{O_2}(t), \Phi_{fuel}(t), \phi(t) \sim \mathcal{N}(\mu(x_{O_2}(t+1)), \sigma^2(x_{O_2}(t+1))) \tag{49}$$

The following control input and rate constraints are imposed on the plant:

$$30^\circ \le \phi \le 60^\circ \; ; \; -3^\circ \le \Delta\phi \le 3^\circ \tag{50}$$

The prediction horizon is $N = 10$ and the terminal constraint is:

$$\mu(x_{O_2}(t+N)) - r_{O_2}(t) \le 0.001 \tag{51}$$

The weighting matrices in the cost function (40) are $Q = 20$, $R = 1$, $P = 20$. The GP-NMPC minimizes the cost function (40) subject to the Gaussian process model (49) and the constraints (50), (51). The formulated GP-NMPC problem results in optimization problem P2 with 10 optimization variables and 41 constraints. One internal region $X_0^1 \subset X_0$ is used in Procedures 1, 2, 3 and 4. This results in problem P3 which has 40 optimization variables and 349 constraints. In (45), it is chosen $\beta = 10$. The extended state vector is $\tilde{x}(t) = [x_{O_2}(t), \Phi_{fuel}(t), \phi(t-1)] \in \mathbb{R}^3$, which leads to a 3-dimensional state space to be partitioned. The latter is defined by $X = [0;7] \times [0.7;1.3] \times [30;60]$. The cost function approximation tolerance is chosen as $\bar{\varepsilon}(X_0) = \max(\bar{\varepsilon}_a, \bar{\varepsilon}_r \min_{\tilde{x} \in X_0} V^*(\tilde{x}))$, where $\bar{\varepsilon}_a = 0.005$ and $\bar{\varepsilon}_r = 0.1$ are the absolute and the relative tolerances, respectively. The partition of the explicit GP-NMPC controller is shown in Fig. 6. It has 513 regions and 12 levels of search. Totally, 18 arithmetic operations are needed in real-time to

compute the control input (12 comparisons, 3 multiplications and 3 additions).

The performance of the closed-loop system was simulated for the following change in the fuel flowrate:

$$\Phi_{fuel}(t) = 1.1[\mathrm{kgs}^{-1}], t \in [0;50];\ \Phi_{fuel}(t) = 1.25[\mathrm{kgs}^{-1}], t \in [51;100];\ \Phi_{fuel}(t) = 1.05[\mathrm{kgs}^{-1}], t \in [101;150] \quad (52)$$

and initial conditions for the state and control variable $x_{O_2}(0) = 3.3\ [\mathrm{vol}\%]$ and $\phi(0) = 46^{\circ}$, respectively. The resulting closed-loop response is depicted in Fig. 7 and Fig. 8.

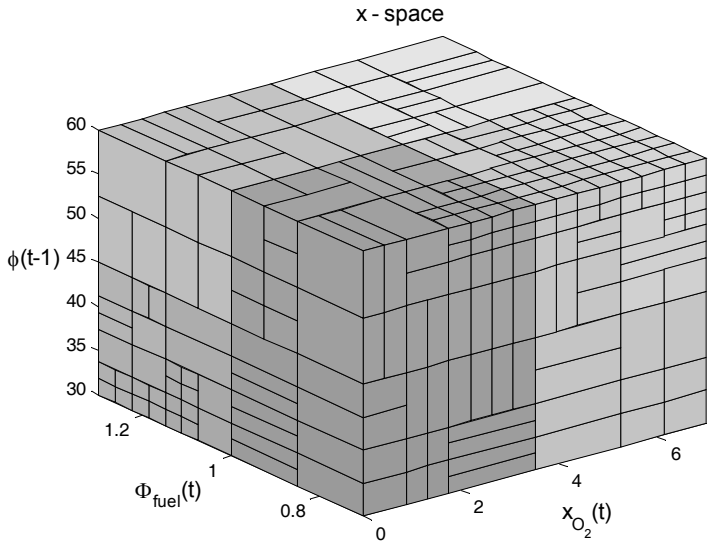

Fig. 6. State space partition of the explicit approximate GP-NMPC controller.

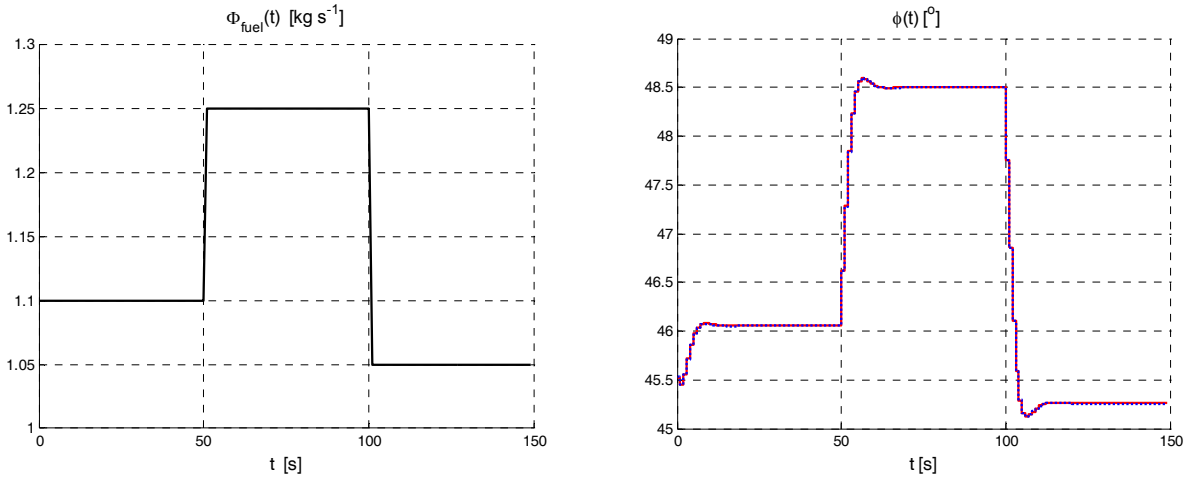

Fig. 7. Left: Change of the fuel flowrate. Right: The control input with the approximate explicit GP-NMPC (the red solid curve) and with the exact GP-NMPC (the blue dotted curve).

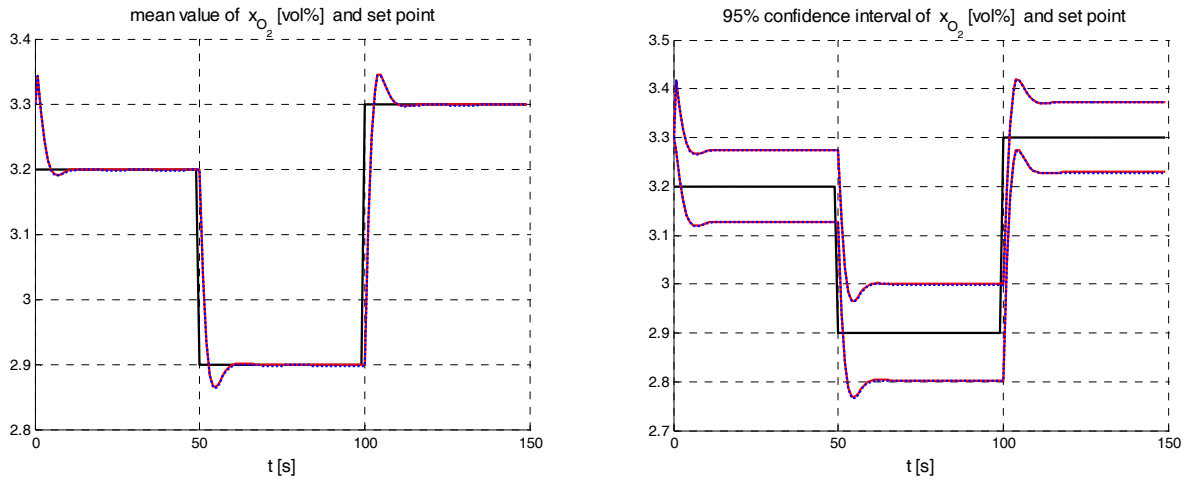

Fig. 8. Left: The mean value of the state variable predicted with the Gaussian process model. Right: The 95% confidence interval of the state variable predicted with the Gaussian process model. The red solid curves are with the approximate explicit GP-NMPC, the blue dotted curves are with the exact GP-NMPC and the black solid curve is the set point.

The results show that the exact and the approximate solutions are almost indistinguishable.

6. Conclusions.

In this paper, an approximate approach for explicit stochastic NMPC is applied to design an explicit reference tracking NMPC controller for a combustion plant based on its Gaussian process model. The controller brings the air factor (respectively the concentration of oxygen in the flue gases) on its optimal value with every change of the load factor and thus an optimal operation of the combustion plant is achieved. Simulations of the closed-loop system show the high quality performance of the explicit stochastic NMPC controller. Although the obtained results are based on simulation data, the paper shows the potential use of the considered approach to the efficient on-line optimization of industrial combustion plants.

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