Robust H_∞ Finite-Horizon Filtering with Randomly Occurred Nonlinearities and Quantization Effects ⋆

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Abstract

In this paper, the robust H_∞ finite-horizon filtering problem is investigated for discrete time-varying stochastic systems with polytopic uncertainties, randomly occurred nonlinearities as well as quantization effects. The randomly occurred nonlinearity, which describes the phenomena of a nonlinear disturbance appearing in a random way, is modeled by a Bernoulli distributed white sequence with a known conditional probability. A new robust H_∞ filtering technique is developed for the addressed Itô-type discrete time-varying stochastic systems. Such a technique relies on the forward solution to a set of recursive linear matrix inequalities and is therefore suitable for on-line computation. It is worth mentioning that, in the filtering process, the information of both the current measurement and the previous state estimate is employed to estimate the current state. Finally, a simulation example is exploited to show the effectiveness of the method proposed in this paper.

Key words: Stochastic systems; discrete time-varying systems; H_∞ filtering; recursive linear matrix inequalities; randomly occurred nonlinearities; quantization effects.

1 Introduction

Filtering or state estimation problem has long been one of the fundamental problems in control and signal processing areas that has attracted constant research attention. In the past decade, a number of linear/nonlinear filtering techniques have been developed with respect to various filtering performance criteria, such as the H_∞ specification, the minimum variance requirement and the so-called admissible variance constraint. For example, the extended Kalman filters have been designed in [14] for nonlinear deterministic systems and in [13] for nonlinear stochastic systems. The robust filtering problems have been extensively studied in [9, 22, 29, 31] for systems with norm-bounded uncertainties and in [10,15–17] for uncertain systems with integral quadratic constraint. The filters with error variance constraints have been exploited in [22, 29, 31] for systems which are subject to the noises with known statistics. Recently, the H_∞ filtering problems have received particular research interests by means of the linear matrix inequality (LMI) approach, see e.g. [5,8,19,23,24,26,27]. It is worth pointing out that, in most literature mentioned above, the infinite-horizon filtering problem has been considered for time-invariant systems.

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With respect to the time-varying systems, the finite-horizon filtering problems have been paid much research attention due primarily to their application insight. Among others, the recursive Riccati difference equation approach has been widely employed to design the H_∞ filters. For example, the bounded real lemma (BRL) was derived in [28] based on the Riccati difference equation, which is suitable for offline computation since the boundary condition are given for the end of the known interval. In [12], by employing the method of Hilbert adjoins, another version of BRL was obtained, based on which the Riccati difference equation can be solved forward in time, and a reduced order H_∞ filter was designed for the linear discrete-time system. In [9], a robust H_∞ filter with error variance constraints was designed for discrete time-varying uncertain system by forwardly solving a recursive Riccati difference equation. For the Itô-type stochastic systems, the H_2/H_∞ control problem was studied in [32] for discrete time-varying stochastic systems, where a BRL was obtained in terms of a constrained backward difference equation in the stochastic framework. It is worth mentioning that, in [7, 18], a differential/difference linear matrix inequality (DLMI) approach was proposed to obtain a BRL that allows for time-varying matrices in the state-space description and can therefore be applied to various problems involving time-varying systems.

With the rapid development of network technologies, more and more control systems are executed over communication networks, which have many advantages such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability. However, since the network cable is of limited capacity, many challenging issues inevitably emerge, for example, the transmission delay [1, 2, 6, 11, 20, 31], data missing (packet dropouts) [20, 22, 24, 25, 30], signal quantization [5, 21], scheduling confusion, etc. Nevertheless, one interesting problem that has been largely overlooked is the so-called randomly occurred nonlinearities (RONs). As is well known, a wide class of practical systems are influenced by *additive* nonlinear disturbances that are caused by environmental circumstances. Such nonlinear disturbances themselves may be subject to random abrupt changes, for example, random failures and repairs of the components, changes in the interconnections of subsystems, sudden environment changes, modification of the operating point of a linearized model of a nonlinear systems, etc. In other words, the nonlinear disturbances may occur in a probabilistic way and are randomly changeable in terms of their types and/or intensity. Unfortunately, to the best of the authors' knowledge, the *finite-horizon* H_∞ filtering problem for discrete *time-varying Itô-type stochastic systems with RONs* has not been fully investigated, not to mention the case where the systems also involve polytopic uncertainties and quantization effects. It is, therefore, the propose of this paper to shorten such a gap by solving a set of recursive linear matrix inequalities motivated by the DLMI approach developed in [7, 18].

This paper is concerned with the robust H_∞ filtering problem for discrete time-varying stochastic systems with polytopic uncertainties, RONs and quantization effects. In order to take into account the phenomena of nonlinear disturbances appearing in a random way, we make the first attempt to introduce RONs that are modeled by a Bernoulli distributed white sequence with a known conditional probability. Sufficient conditions are derived for the estimation error of the system under consideration to satisfy the H_∞ performance constraint. A robust H_∞ filter is then designed by solving a set of recursive LMIs. The proposed robust H_∞ filtering technique is a recursive algorithm that is suitable for on-line computation by employing more information at and before current time to estimate the current state. Finally, a numerical simulation example is used to demonstrate the effectiveness of the filtering technology presented in this paper.

Notation The notation used here is fairly standard except where otherwise stated. $\mathbb{R}^n$ denotes the n dimensional Euclidean space. $\|A\|$ refers to the norm of a matrix A defined by $\|A\| = \sqrt{\mathrm{trace}(A^TA)}$. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semidefinite (respectively, positive definite). M^T represents the transpose of the matrix M. I denotes the identity matrix of compatible dimension. $\mathrm{diag}\{\cdots\}$ stands for a block-diagonal matrix. Moreover, we may fix a probability space $(\Omega, \mathscr{F}, \mathrm{Prob})$, where Prob, the probability measure, has total mass 1. $\mathbb{E}\{x\}$ stands for the expectation of the random variable x with respect to the given probability measure Prob. The set of all nonnegative integers is denoted by $\mathbb{I}^+$ and the set of all nonnegative real numbers is represented by $\mathbb{R}^+$. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2 Problem formulation and preliminaries

Consider the following class of nonlinear discrete time-varying polytopic uncertain stochastic systems defined on $k \in [0, N]$:

$$\begin{cases} x(k+1) = A^{(\xi)}(k)x(k) + A_1^{(\xi)}(k)x(k)w(k) \\ \qquad\qquad + B^{(\xi)}(k)v(k) + r(k)f(k, x(k)) \\ x(0) = x_0 \\ y(k) = C^{(\xi)}(k)x(k) + D^{(\xi)}(k)v(k) \\ z(k) = M(k)x(k) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $y(k) \in \mathbb{R}^r$ is the measured output vector, $z(k) \in \mathbb{R}^m$ is the state combination to be estimated, and $w(k)$ is a one-dimensional, zero-mean Gaussian white noise sequence on a probability space $(\Omega, \mathscr{F}, \mathrm{Prob})$ with $\mathbb{E}w^2(k) = 1$. Let $\big(\Omega, \mathscr{F}, \{\mathscr{F}_k\}_{k\in[0,N]}, \mathrm{Prob}\big)$ be a filtered probability space where $\{\mathscr{F}_k\}_{k\in[0,N]}$ is the family of sub-σ-algebras of $\mathscr{F}$ generated by $\{w(k)\}_{k\in[0,N]}$. In fact, each $\mathscr{F}_k$ is assumed to be the minimal σ-algebras generated by $\{w(i)\}_{0\leq i\leq k-1}$ while $\mathscr{F}_0$ is assumed to be some given sub σ-algebras of $\mathscr{F}$, independent of $\mathscr{F}_k$ for all $0 \leq k \leq N$ [3], and the initial value x_0 is assumed to belong to $\mathscr{F}_0$. For the exogenous disturbance input $v(k) \in \mathbb{R}^q$, $v = \{v(k)\}_{k\in[0,N]} \in l_2([0, N], \mathbb{R}^q)$ where $l_2([0, N], \mathbb{R}^q)$ is the space of nonanticipatory square-summable stochastic process $v = \{v(k)\}_{k\in[0,N]}$ with respect to $\{\mathscr{F}_k\}_{k\in[0,N]}$ with the norm

$$\|v\|^2_{[0,N]} = \mathbb{E}\left\{\sum_{k=0}^{N} \|v(k)\|^2\right\} = \sum_{k=0}^{N} \mathbb{E}\left\{\|v(k)\|^2\right\}.$$

The nonlinear functions $f : [0, N] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following condition:

$$\|f(k, x(k))\|^2 \leq \alpha(k)\|G(k)x(k)\|^2 \tag{2}$$

for all $k \in [0, N]$, where $\alpha(k) > 0$ is a known positive scalar and $G(k)$ is a known constant matrix.

The random variable $r(k) \in \mathbb{R}$, which accounts for the phenomena of RONs, takes values of 1 and 0 with

$$\begin{aligned} &\mathrm{Prob}\{r(k) = 1\} = \delta \\ &\mathrm{Prob}\{r(k) = 0\} = 1 - \delta \end{aligned} \tag{3}$$

where $\delta \in [0, 1]$ is a known constant. Throughout this paper, $r(k)$ is assumed to be independent of $\mathscr{F}_k$ for all $0 \leq k \leq N$.

For all the system matrices in (1), they have appropriate dimensions, where $M(k)$ is a known time-varying matrix, while $A^{(\xi)}(k)$, $A_1^{(\xi)}(k)$, $B^{(\xi)}(k)$, $C^{(\xi)}(k)$ and $D^{(\xi)}(k)$ are unknown time-varying matrices which contain polytopic uncertainties as follows:

$$\begin{aligned} \Xi^{(\xi)} :=& \big(A^{(\xi)}(k), A_1^{(\xi)}(k), B^{(\xi)}(k), C^{(\xi)}(k), D^{(\xi)}(k)\big) \\ \in& \mathfrak{R} \end{aligned} \tag{4}$$

where $\mathfrak{R}$ is a convex polyhedral set described by v vertices

$$\mathfrak{R} := \Big\{\Xi^{(\xi)} | \Xi^{(\xi)} = \sum_{i=1}^{v} \xi_i \Xi^{(i)}, \sum_{i=1}^{v} \xi_i = 1, \quad \xi_i \geq 0, i = 1, 2, \cdots, v\Big\} \tag{5}$$

and $\Xi^{(i)} := (A^{(i)}(k), A_1^{(i)}(k), B^{(i)}(k), C^{(i)}(k), D^{(i)}(k))$ are known matrices for all $i = 1, 2, \cdots, v$.

In this paper, the quantization effects are also taken into account. The quantizer $q(\cdot)$ is defined as

$$\tilde{y}(k) = q(y(k)) = \Big[q_1(y_1(k)) \ \ q_2(y_2(k)) \ \ \cdots \ \ q_r(y_r(k))\Big]^T$$

where $\tilde{y}(k) \in \mathbb{R}^r$ is the signal after quantization then transmitted into the filter. Here, the quantizer $q(\cdot)$ is assumed to be of the logarithmic type. Specifically, for each $q_j(\cdot)$ $(1 \leq j \leq r)$, the set of quantization levels is described by

$$\mathcal{U}_j = \Big\{\pm u_i^{(j)}, u_i^{(j)} = \rho_j^i u_0^{(j)}, i = 0, \ \pm 1, \ \pm 2, \ \cdots\Big\} \cup \{0\}, \quad 0 \leq \rho_j \leq 1, \quad u_0^{(j)} > 0.$$

Each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level. The logarithmic quantizer $q_j(\cdot)$ is defined as

$$q_j(y_j(k)) = \begin{cases} u_i^{(j)}, & \frac{1}{1+\kappa_j} u_i^{(j)} \leq y_j(k) \leq \frac{1}{1-\kappa_j} u_i^{(j)} \\ 0, & y_j(k) = 0 \\ -q_j(-y_j(k)), & y_j(k) \leq 0 \end{cases}$$

with $\kappa_j = (1-\rho_j)/(1+\rho_j)$. It follows from [4] that $q_j(y_j(k)) = (1+\Delta_j(k))y_j(k)$ such that $|\Delta_j(k)| \leq \kappa_j$. Denoting $\Delta(k) = \mathrm{diag}\{\Delta_1(k), \cdots, \Delta_r(k)\}$, the measurements after quantization can be expressed as

$$\tilde{y}(k) = (I + \Delta(k))y(k). \tag{6}$$

Therefore, the quantizing effects have been transformed into sector bound uncertainties. In fact, defining $\Lambda = \mathrm{diag}\{\kappa_1, \cdots, \kappa_r\}$ and $F(k) = \Delta(k)\Lambda^{-1}$, we can obtain an unknown real-valued time-varying matrix $F(k)$ satisfying $F(k)F^T(k) = F^T(k)F(k) \leq I$.

Remark 1 *From (1), it can be seen that the nonlinearity $f(k, x(k))$ enters the system in a probabilistic way described by the random variable $r(k)$, which is the RON as mentioned in the introduction. The phenomena of RONs are ubiquitous in networked systems but have been largely overlooked in the area. The polytopic uncertainties and quantization effects, on the other hand, are two other typically sources that may deteriorate the performance of the networked systems. Therefore, it makes practical sense to consider the polytopic uncertainties, quantization effects as well as RONs within a unified framework.*

We adopt the following time-varying filter for system (1)

$$\begin{cases} \hat{x}(k+1) = F_f(k)\hat{x}(k) + G_f(k)\tilde{y}(k) \\ \hat{z}(k) = M(k)\hat{x}(k), \qquad \hat{x}(0) = 0 \end{cases} \tag{7}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the state estimate, $\hat{z}(k) \in \mathbb{R}^m$ is the estimated output, and $F_f(k)$, $G_f(k)$ $(0 \leq k \leq N)$ are filter parameters to be determined.

Letting estimation error be $e(k) = x(k) - \hat{x}(k)$, then error dynamics can be obtained from (1) and (7) as follows:

$$\begin{cases} e(k+1) = f_e^{(\xi)}(k, e(k), \hat{x}(k)) + S_e^{(\xi)}(k)v(k) \\ \qquad + g_e^{(\xi)}(k, e(k), \hat{x}(k))w(k) \\ \qquad + (r(k) - \delta)f(k, e(k) + \hat{x}(k)) \\ \tilde{z}(k) = z(k) - \hat{z}(k) = M(k)e(k) \end{cases} \tag{8}$$

where

$$\begin{aligned} & f_e^{(\xi)}(k, e(k), \hat{x}(k)) = \delta f(k, e(k) + \hat{x}(k)) \\ & +(A^{(\xi)}(k) - G_f(k)(I + \Delta(k))C^{(\xi)}(k))e(k) \\ & +(A^{(\xi)}(k) - F_f(k) - G_f(k)(I + \Delta(k))C^{(\xi)}(k))\hat{x}(k) \\ & g_e^{(\xi)}(k, e(k), \hat{x}(k)) = A_1^{(\xi)}(k)e(k) + A_1^{(\xi)}(k)\hat{x}(k) \\ & S_e^{(\xi)}(k) = B^{(\xi)}(k) - G_f(k)(I + \Delta(k))D^{(\xi)}(k). \end{aligned}$$

Remark 2 *In the model (1), two kinds of widely studied disturbance inputs are considered, one is the exogenous (additive and deterministic) input $v(k)$ and the other is the state-dependent (multiplicative and stochastic) input $w(k)$. In the case that the disturbance coming with the nonlinearities is neither additive nor multiplicative, the filtering problem becomes more complicated, which gives an interesting topic for our future research.*

Our aim in this paper is to design a filter (7) for the system (1) such that estimation error output $\tilde{z}(k)$ satisfies the H_∞ performance constraint, namely:

$$\|\tilde{z}\|_{[0,N]}^2 \leq \gamma^2 \Big\{\|v\|_{[0,N]}^2 + \mathbb{E}\{e^T(0)Se(0)\}\Big\} \tag{9}$$

for the given disturbance attenuation level $\gamma > 0$ and the positive definite matrix $S = S^T > 0$. Setting $\eta(k) = [e^T(k) \ \ 1]^T$, we subsequently obtain an augmented system as follows:

$$\begin{cases} \eta(k+1) = \mathcal{A}_e^{(\xi)}(k)\eta(k) + \delta\mathcal{F}(k, \eta(k)) \\ \qquad + \mathcal{B}_e^{(\xi)}(k)v(k) + \mathcal{A}_1^{(\xi)}(k)\eta(k)w(k) \\ \qquad + (r(k) - \delta)\mathcal{F}(k, \eta(k)) \\ \tilde{z}(k) = \mathcal{M}(k)\eta(k) \end{cases} \tag{10}$$

where

$$\mathcal{A}_e^{(\xi)}(k) = \begin{bmatrix} A^{(\xi)}(k) - G_f(k)(I+\Delta(k))C^{(\xi)}(k) & \big(A^{(\xi)}(k) - F_f(k) - G_f(k)(I+\Delta(k))C^{(\xi)}(k)\big)\hat{x}(k) \\ 0 & 1 \end{bmatrix}$$
$$= \mathcal{A}^{(\xi)}(k) + \mathcal{S}(k)F(k)\mathcal{T}_1^{(\xi)}(k),$$
$$\mathcal{B}_e^{(\xi)}(k) = \begin{bmatrix} B^{(\xi)}(k) - G_f(k)(I+\Delta(k))D^{(\xi)}(k) \\ 0 \end{bmatrix}$$
$$= \mathcal{B}^{(\xi)}(k) + \mathcal{S}(k)F(k)\mathcal{T}_2^{(\xi)}(k),$$
$$\mathcal{A}_1^{(\xi)}(k) = \begin{bmatrix} A_1^{(\xi)}(k) & A_1^{(\xi)}(k)\hat{x}(k) \\ 0 & 0 \end{bmatrix}, \mathcal{M}(k) = \begin{bmatrix} M(k) & 0 \end{bmatrix},$$
$$\mathcal{F}(k,\eta(k)) = \begin{bmatrix} f(k, e(k) + \hat{x}(k)) \\ 0 \end{bmatrix}, \mathcal{S}(k) = \begin{bmatrix} G_f(k) \\ 0 \end{bmatrix},$$
$$\mathcal{A}^{(\xi)}(k) = \begin{bmatrix} A^{(\xi)}(k) - G_f(k)C^{(\xi)}(k) & \big(A^{(\xi)}(k) - F_f(k) - G_f(k)C^{(\xi)}(k)\big)\hat{x}(k) \\ 0 & 1 \end{bmatrix},$$
$$\mathcal{B}^{(\xi)}(k) = \begin{bmatrix} B^{(\xi)}(k) - G_f(k)D^{(\xi)}(k) \\ 0 \end{bmatrix},$$
$$\mathcal{T}_1^{(\xi)}(k) = \begin{bmatrix} -\Lambda C^{(\xi)}(k) & -\Lambda C^{(\xi)}(k)\hat{x}(k) \end{bmatrix},$$
$$\mathcal{T}_2^{(\xi)}(k) = -\Lambda D^{(\xi)}(k). \tag{11}$$

From assumption (2), the nonlinear function $\mathcal{F}(k,\eta(k))$ satisfies

$$\mathcal{F}^T(k,\eta(k))\mathcal{F}(k,\eta(k)) \le \alpha(k)\eta^T(k)\mathcal{G}^T(k)\mathcal{G}(k)\eta(k) \tag{12}$$

where $\mathcal{G}(k) = \begin{bmatrix} G(k) & G(k)\hat{x}(k) \end{bmatrix}$. To this end, the filtering problem for (1) is now converted into the one of designing a filter (7) such that the augmented system (10) satisfies the H_∞ performance constraint (9).

3 Performance analysis of H_∞ filter

In this section, we will give a BRL for the augmented system (10) to satisfy the H_∞ performance constraint (9) for all nonlinearities $\mathcal{F}(k,\eta(k))$ subject to (12).

To derive the BRL for the augmented system (10), we introduce the following lemma.

Lemma 1 *For a given scalar $\gamma > 0$, the augmented system (10) has the l_2-gain not greater than γ, i.e., the following H_∞ criterion is satisfied*

$$\|\tilde{z}\|^2_{[j,k-1]} \le \mathbb{E}\{V_j(\eta(j))\} + \gamma^2\|v\|^2_{[j,k-1]} \tag{13}$$

for all $0 \le j \le k$ and for all $v \in \mathbb{R}^q$, if and only if there exist a family of positive real-valued functions $V_k : \mathbb{R}^{n+1} \times \mathbb{I}^+ \to \mathbb{R}^+$ ($V_k(0) = 0$ for all $k \in \mathbb{I}^+$) satisfying the following Hamilton-Jacobi inequality

$$V_k(\eta) \ge \sup_{v\in\mathbb{R}^q} \Big\{ \|\tilde{z}(k)\|^2 - \gamma^2\|v\|^2 + \mathbb{E}_{(\omega(k),r(k))}\{V_{k+1}[\mathcal{H}^{(\xi)}(k,\eta,v,r(k),\omega(k))]\} \Big\} \tag{14}$$

for all $\eta \in \mathbb{R}^{n+1}$, where $\mathcal{H}^{(\xi)}(k,\eta,v,r(k),\omega(k)) = \mathcal{A}_e^{(\xi)}(k)\eta(k) + \delta\mathcal{F}(k,\eta(k)) + \mathcal{B}_e^{(\xi)}(k)v(k) + \mathcal{A}_1^{(\xi)}(k)\eta(k)w(k) + (r(k) - \delta)\mathcal{F}(k,\eta(k))$ and $\mathbb{E}_y\{\cdot\}$ is defined similarly to the one in [3].

Proof: The proof follows directly from that of Theorem 2 in [3] and is therefore omitted.

Remark 3 *Lemma 1 is a BRL for general stochastic systems with a random variable $r(k)$. Similar results have been derived by Berman and Shaked in [3] for the stochastic system without involving the random variable $r(k)$.*

The following lemma will be used in deriving our main results.

Lemma 2 *Let $W_0(x)$, $W_1(x)$, $\cdots$, $W_l(x)$ be quadratic functions of $x \in \mathbb{R}^n$, i.e.*

$$W_i(x) = x^T Q_i x, \quad i = 0, 1, \cdots, l, \tag{15}$$

with $Q_i^T = Q_i$. If there exist ρ_1, ρ_2, $\cdots$, $\rho_l \ge 0$ such that

$$Q_0 - \sum_{i=1}^{l} \rho_i Q_i \ge 0, \tag{16}$$

then the following is true

$$W_1(x) \ge 0, \cdots, W_l(x) \ge 0 \Longrightarrow W_0(x) \ge 0. \tag{17}$$

In the following theorem, a sufficient condition is given to guarantee that the augmented system (10) satisfies the H_∞ performance constraint (9) for all nonlinearities $\mathcal{F}(k,\eta(k))$ subject to (12).

Theorem 1 *Given the disturbance attenuation level $\gamma > 0$, the initial positive definite matrix $S = S^T > 0$ and the filter parameters $\{F_f(k)\}_{0\le k\le N}$, $\{G_f(k)\}_{0\le k\le N}$. If there exist a family of positive scalars $\{\rho(k)\}_{0\le k\le N}$ and a family of positive definite matrices $\{P(k)\}_{0\le k\le N+1}$ satisfying the initial condition $\eta^T(0)P(0)\eta(0) \le \gamma^2 e^T(0)Se(0)$ and the following time-varying LMIs:*

$$\begin{bmatrix} \Upsilon_{11}^{(\xi)}(k) & 0 & \mathcal{A}_e^{(\xi)T}(k)P(k+1) & 0 \\ * & \Upsilon_{22}(k) & \delta P(k+1) & 0 \\ * & * & -P(k+1) & P(k+1)\mathcal{B}_e^{(\xi)}(k) \\ * & * & * & -\gamma^2 I \end{bmatrix} \le 0 \tag{18}$$

for all $0 \le k \le N$, where

$$\Upsilon_{11}^{(\xi)}(k) = \mathcal{A}_1^{(\xi)T}(k)P(k+1)\mathcal{A}_1^{(\xi)}(k) + \mathcal{M}^T(k)\mathcal{M}(k) - P(k) + \rho(k)\alpha(k)\mathcal{G}^T(k)\mathcal{G}(k)$$
$$\Upsilon_{22}(k) = \delta(1-\delta)P(k+1) - \rho(k)I$$

then the augmented system (10) satisfies the H_∞ performance constraint (9) for all nonlinearities $\mathcal{F}(k,\eta(k))$ subject to (12).

Proof: Let $V_k(\eta)=\eta^T P(k)\eta$, where $\{P(k)\}_{0\le k\le N+1}$ are the solutions to the time-varying LMIs (18). It can be calculated that

$$\begin{aligned}&\sup_{v\in\mathbb{R}^q}\Big\{\|\tilde{z}(k)\|^2-\gamma^2\|v\|^2\\&\quad+\mathbb{E}_{(\omega(k),r(k))}\{V_{k+1}[\mathcal{H}^{(\xi)}(k,\eta,v,r(k),\omega(k))]\}\Big\}\\&=\sup_{v\in\mathbb{R}^q}\Big\{-v^T\big(\gamma^2I-\mathcal{B}_e^{(\xi)T}(k)P(k+1)\mathcal{B}_e^{(\xi)}(k)\big)v\\&\quad+2\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)^TP(k+1)\mathcal{B}_e^{(\xi)}(k)v\\&\quad+\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)^TP(k+1)\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)\\&\quad+\eta^T\mathcal{A}_1^{(\xi)T}(k)P(k+1)\mathcal{A}_1^{(\xi)}(k)\eta+\eta^T\mathcal{M}^T(k)\mathcal{M}(k)\eta\\&\quad+\delta(1-\delta)\mathcal{F}^T(k,\eta)P(k+1)\mathcal{F}(k,\eta)\Big\}.\end{aligned}\tag{19}$$

By applying completing squares method, it can be obtained that

$$\begin{aligned}&\sup_{v\in\mathbb{R}^q}\Big\{\|\tilde{z}(k)\|^2-\gamma^2\|v\|^2\\&\quad+\mathbb{E}_{(\omega(k),r(k))}\{V_{k+1}[\mathcal{H}^{(\xi)}(k,\eta,v,r(k),\omega(k))]\}\Big\}\\&=\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)^TP(k+1)\mathcal{B}_e^{(\xi)}(k)\\&\quad\times\big(\gamma^2I-\mathcal{B}_e^{(\xi)T}(k)P(k+1)\mathcal{B}_e^{(\xi)}(k)\big)^{-1}\mathcal{B}_e^{(\xi)T}(k)\\&\quad\times P(k+1)\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)+\big(\mathcal{A}_e^{(\xi)}(k)\eta\\&\quad+\delta\mathcal{F}(k,\eta)\big)^TP(k+1)\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)\\&\quad+\eta^T\mathcal{A}_1^{(\xi)T}(k)P(k+1)\mathcal{A}_1^{(\xi)}(k)\eta+\eta^T\mathcal{M}^T(k)\mathcal{M}(k)\eta\\&\quad+\delta(1-\delta)\mathcal{F}^T(k,\eta)P(k+1)\mathcal{F}(k,\eta)\end{aligned}\tag{20}$$

when maximizing $v(k)=\big(\gamma^2I-\mathcal{B}_e^{(\xi)T}(k)P(k+1)\mathcal{B}_e^{(\xi)}(k)\big)^{-1}\mathcal{B}_e^{(\xi)T}(k)P(k+1)\big(\mathcal{A}_e^{(\xi)}(k)\eta+\delta\mathcal{F}(k,\eta)\big)$. Hence, one can have

$$\begin{aligned}&\sup_{v\in\mathbb{R}^q}\Big\{\|\tilde{z}(k)\|^2-\gamma^2\|v\|^2\\&\quad+\mathbb{E}_{(\omega(k),r(k))}\{V_{k+1}[\mathcal{H}^{(\xi)}(k,\eta,v,r(k),\omega(k))]\}\Big\}-V_k(\eta)\\&=\begin{bmatrix}\eta^T & \mathcal{F}^T(k,\eta)\end{bmatrix}\left\{\begin{bmatrix}\mathcal{A}_e^{(\xi)T}(k)\\ \delta I\end{bmatrix}\Omega_1\begin{bmatrix}\mathcal{A}_e^{(\xi)}(k) & \delta I\end{bmatrix}+\Omega_2\right\}\\&\quad\times\begin{bmatrix}\eta\\ \mathcal{F}(k,\eta)\end{bmatrix}\end{aligned}\tag{21}$$

where

$$\begin{aligned}\Omega_1&=P(k+1)\mathcal{B}_e^{(\xi)}(k)\big(\gamma^2I-\mathcal{B}_e^{(\xi)T}(k)P(k+1)\mathcal{B}_e^{(\xi)}(k)\big)^{-1}\\&\quad\times\mathcal{B}_e^{(\xi)T}(k)P(k+1)+P(k+1)\end{aligned}\tag{22}$$

$$\Omega_2=\begin{bmatrix}\mathcal{A}_1^{(\xi)T}(k)P(k+1)\mathcal{A}_1^{(\xi)}(k)+\mathcal{M}^T(k)\mathcal{M}(k)-P(k) & 0\\ 0 & \delta(1-\delta)P(k+1)\end{bmatrix}.\tag{23}$$

It remains to show (14) holds in virtue of Lemma 2. For this purpose, we rewrite (12) as

$$\begin{bmatrix}\eta^T & \mathcal{F}^T(k,\eta)\end{bmatrix}\begin{bmatrix}-\alpha(k)\mathcal{G}^T(k)\mathcal{G}(k) & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}\eta\\ \mathcal{F}(k,\eta)\end{bmatrix}\le 0.\tag{24}$$

From Lemma 2, it is easy to know that (24) implies

$$\begin{bmatrix}\eta^T & \mathcal{F}^T(k,\eta)\end{bmatrix}\left\{\begin{bmatrix}\mathcal{A}_e^{(\xi)T}(k)\\ \delta I\end{bmatrix}\Omega_1\begin{bmatrix}\mathcal{A}_e^{(\xi)}(k) & \delta I\end{bmatrix}+\Omega_2\right\}\begin{bmatrix}\eta\\ \mathcal{F}(k,\eta)\end{bmatrix}\le 0\tag{25}$$

if the following inequality

$$\begin{bmatrix}\mathcal{A}_e^{(\xi)T}(k)\\ \delta I\end{bmatrix}\Omega_1\begin{bmatrix}\mathcal{A}_e^{(\xi)}(k) & \delta I\end{bmatrix}+\Omega_2+\begin{bmatrix}\rho(k)\alpha(k)\mathcal{G}^T(k)\mathcal{G}(k) & 0\\ 0 & -\rho(k)I\end{bmatrix}\le 0\tag{26}$$

holds for one positive scalar $\rho(k)$. It follows from (18) and Schur complement that (26) is true. Then, taking $j=0$ and $k=N+1$, we obtain directly from Lemma 1 that

$$\|\tilde{z}\|^2_{[0,N]}\le\mathbb{E}\{V_0(\eta(0))\}+\gamma^2\|v\|^2_{[0,N]},\tag{27}$$

from which the H_∞ performance constraint (9) can be guaranteed by noting the initial condition $\eta^T(0)P(0)\eta(0)\le\gamma^2e^T(0)Se(0)$, and therefore the proof of this theorem is complete.

4 Design of robust H_∞ filters

In this section, the robust H_∞ filter is designed for the nonlinear discrete time-varying stochastic systems subject to RONs as well as the quantization effects in terms of time-varying LMIs.

Lemma 3 *Let Ψ_1, Ψ_2 and F be real matrices of appropriate dimensions with F satisfy $F^TF\le I$. Then, for any scalar $\varepsilon>0$, we have $\Psi_1F\Psi_2+(\Psi_1F\Psi_2)^T\le\varepsilon^{-1}\Psi_1\Psi_1^T+\varepsilon\Psi_2^T\Psi_2$.*

The following theorem provides a recursive LMI approach to the addressed design problem of robust H_∞ filter for the discrete time-varying stochastic system with the stochastic nonlinearities as well as quantization effects.

Theorem 2 *Let the disturbance attenuation level $\gamma > 0$, initial positive definite matrix $S = S^T > 0$ and the quantizer $q(\cdot)$ be given. The robust H_∞ filter (7) can be designed for the stochastic nonlinear system (1) if there exist a family of positive definite matrices $\{P_1(k)\}_{0\le k\le N+1}$, two families of matrices $\{X(k)\}_{0\le k\le N}$, $\{Y(k)\}_{0\le k\le N}$ and three families of positive scalars $\{\varepsilon(k)\}_{0\le k\le N}$, $\{\rho(k)\}_{0\le k\le N}$, $\{P_2(k)\}_{0\le k\le N+1}$ satisfying the initial condition*

$$e^T(0)P_1(0)e(0) + P_2(0) \le \gamma^2 e^T(0)Se(0) \qquad (28)$$

and the recursive LMIs

$$\begin{bmatrix} \Gamma_{11}^{(i)}(k) & 0 & \Gamma_{13}^{(i)}(k) & \Gamma_{14}^{(i)}(k) \\ * & \Gamma_{22}(k) & \delta P(k+1) & 0 \\ * & * & -P(k+1) & \Gamma_{34}^{(i)}(k) \\ * & * & * & \Gamma_{44}^{(i)}(k) \end{bmatrix} \le 0 \qquad (29)$$

$i = 1, 2, \cdots, v$, for all $0 \le k \le N$, where

$$\Gamma_{11}(k) = \begin{bmatrix} \Sigma_1^{(i)}(k) & \Sigma_2^{(i)}(k) \\ \Sigma_2^{(i)T}(k) & \Sigma_3^{(i)}(k) \end{bmatrix}, \Gamma_{22}(k) = diag\{\Sigma_4(k), \Sigma_5(k)\},$$

$$\Gamma_{13}(k) = \begin{bmatrix} \Sigma_6^{(i)}(k) & 0 \\ \Sigma_7^{(i)}(k) & P_2(k+1) \end{bmatrix}, \Gamma_{34}(k) = \begin{bmatrix} \Sigma_8^{(i)}(k) & 0 & Y(k) \\ 0 & 0 & 0 \end{bmatrix},$$

$$\Gamma_{14}(k) = \begin{bmatrix} \varepsilon(k)C^{(i)T}(k)\Lambda^2 D^{(i)}(k) & A_1^{(i)T}(k)P_1(k+1) & 0 \\ \varepsilon(k)\hat{x}^T(k)C^{(i)T}(k)\Lambda^2 D^{(i)}(k) & \hat{x}^T(k)A_1^{(i)T}(k)P_1(k+1) & 0 \end{bmatrix}$$

$$\Gamma_{44}(k) = diag\{\Sigma_9^{(i)}(k), -P_1(k+1), -\varepsilon(k)I\},$$
$$P(k+1) = diag\{P_1(k+1), P_2(k+1)\}$$

with

$$\begin{aligned} \Sigma_1^{(i)}(k) &= M^T(k)M(k) + \varepsilon(k)C^{(i)T}(k)\Lambda^2 C^{(i)}(k) \\ &\quad + \rho(k)\alpha(k)G^T(k)G(k) - P_1(k) \\ \Sigma_2^{(i)}(k) &= \rho(k)\alpha(k)G^T(k)G(k)\hat{x}(k) \\ &\quad + \varepsilon(k)C^{(i)T}(k)\Lambda^2 C^{(i)}(k)\hat{x}(k) \\ \Sigma_3^{(i)}(k) &= \rho(k)\alpha(k)\hat{x}^T(k)G^T(k)G(k)\hat{x}(k) - P_2(k) \\ &\quad + \varepsilon(k)\hat{x}^T(k)C^{(i)T}(k)\Lambda^2 C^{(i)}(k)\hat{x}(k) \\ \Sigma_4(k) &= \delta(1-\delta)P_1(k+1) - \rho(k)I \\ \Sigma_5(k) &= \delta(1-\delta)P_2(k+1) - \rho(k)I \\ \Sigma_6^{(i)}(k) &= A^{(i)T}(k)P_1(k+1) - C^{(i)T}(k)Y^T(k) \\ \Sigma_7^{(i)}(k) &= \hat{x}^T(k)A^{(i)T}(k)P_1(k+1) - \hat{x}^T(k)X^T(k) \\ &\quad - \hat{x}^T(k)C^{(i)T}(k)Y^T(k) \\ \Sigma_8^{(i)}(k) &= P_1(k+1)B^{(i)}(k) - Y(k)D^{(i)}(k) \\ \Sigma_9^{(i)}(k) &= -\gamma^2 I + \varepsilon(k)D^{(i)T}(k)\Lambda^2 D^{(i)}(k) \end{aligned}$$

and $A^{(i)}(k)$, $A_1^{(i)}(k)$, $B^{(i)}(k)$, $C^{(i)}(k)$, $D^{(i)}(k)$ are the matrices at the ith vertex of the polytope. Furthermore, if (28) and (29) are true, the desired filter is given by (7) with the following parameters

$$F_f(k) = P_1^{-1}(k+1)X(k) \qquad (30)$$
$$G_f(k) = P_1^{-1}(k+1)Y(k) \qquad (31)$$

for all $0 \le k \le N$.

Proof: Let us show that the recursive LMIs (29) with the initial condition (28) are sufficient conditions for the augmented system (10) to achieve the H_∞ performance constraint (9). Using the Schur complement, it can be easily shown that (18) is equivalent to

$$\begin{bmatrix} \widehat{\Upsilon}_{11}(k) & 0 & \mathcal{A}_e^{(\xi)T}(k)P(k+1) & 0 & \mathcal{A}_1^{(\xi)T}(k)P(k+1) \\ * & \Upsilon_{22}(k) & \delta P(k+1) & 0 & 0 \\ * & * & -P(k+1) & P(k+1)\mathcal{B}_e^{(\xi)}(k) & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -P(k+1) \end{bmatrix} \le 0 \qquad (32)$$

where

$$\widehat{\Upsilon}_{11}(k) = -P(k) + \mathcal{M}^T(k)\mathcal{M}(k) + \rho(k)\alpha(k)\mathcal{G}^T(k)\mathcal{G}(k). \qquad (33)$$

Noting that $\mathcal{A}_e^{(\xi)}(k)$ and $\mathcal{B}_e^{(\xi)}(k)$ contain the uncertainty $F(k)$ from (11), we rewrite (32) as

$$\Pi^{(\xi)}(k) + \mathcal{U}(k)F(k)\mathcal{V}^{(\xi)}(k) + \mathcal{V}^{(\xi)T}(k)F(k)\mathcal{U}^T(k) \le 0 \qquad (34)$$

where

$$\Pi^{(\xi)}(k) = \begin{bmatrix} \widehat{\Upsilon}_{11}(k) & 0 & \mathcal{A}^{(\xi)T}(k)P(k+1) & 0 & \mathcal{A}_1^{(\xi)T}(k)P(k+1) \\ * & \Upsilon_{22}(k) & \delta P(k+1) & 0 & 0 \\ * & * & -P(k+1) & P(k+1)\mathcal{B}^{(\xi)}(k) & 0 \\ * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & -P(k+1) \end{bmatrix}, \qquad (35)$$

$$\mathcal{U}(k) = \begin{bmatrix} 0 & 0 & \mathcal{S}^T(k)P(k+1) & 0 & 0 \end{bmatrix}^T, \qquad (36)$$

$$\mathcal{V}^{(\xi)}(k) = \begin{bmatrix} \mathcal{T}_1^{(\xi)}(k) & 0 & 0 & \mathcal{T}_2^{(\xi)}(k) & 0 \end{bmatrix}. \qquad (37)$$

By applying Lemma 3 together with Schur complement to (34), we know that (34) is true if the following inequality

$$\begin{bmatrix} \Pi^{(\xi)}(k) & \mathcal{U}(k) & \varepsilon(k)\mathcal{V}^{(\xi)T}(k) \\ * & -\varepsilon(k)I & 0 \\ * & * & -\varepsilon(k)I \end{bmatrix} \leq 0 \tag{38}$$

holds for one positive scalar parameter $\varepsilon(k)$. Since the set of system matrices $\Xi^{(\xi)} = (A^{(\xi)}(k), A_1^{(\xi)}(k), B^{(\xi)}(k), C^{(\xi)}(k), D^{(\xi)}(k))$ belongs to the convex polyhedral set $\mathfrak{R}$, there always exist scalars $\xi_i \geq 0$ $(i = 1, 2, \cdots, v)$ such that $\Xi^{(\xi)} = \sum_{i=1}^{v} \xi_i \Xi^i$, $\sum_{i=1}^{v} \xi_i = 1$, where $\Xi^{(i)} = (A^{(i)}(k), A_1^{(i)}(k), B^{(i)}(k), C^{(i)}(k), D^{(i)}(k))$ $(i = 1, 2, \cdots, v)$ are v vertexes of the polytope. Hence, by considering (11) together with (35)-(37), one can easily see that (38) holds if and only if

$$\begin{bmatrix} \Pi^{(i)}(k) & \mathcal{U}(k) & \varepsilon(k)\mathcal{V}^{(i)T}(k) \\ * & -\varepsilon(k)I & 0 \\ * & * & -\varepsilon(k)I \end{bmatrix} \leq 0 \tag{39}$$

for all $i = 1, 2, \cdots, v$. Subsequently, we choose $P(k) = \mathrm{diag}\{P_1(k), P_2(k)\}$ in order to derive the expression of the filter parameters from (11). By noting the relation (30)-(31), it follows that (39) is guaranteed by (29) after using Schur complement and some straightforward algebraic manipulations. In addition, it can be easily seen that the initial condition $\eta^T(0)P(0)\eta(0) \leq \gamma^2 e^T(0)Se(0)$ is implied by (28). Therefore, this theorem follows by Theorem 1.

Remark 4 *Different from the LMI criteria for time-invariant system, the set of recursive LMIs (RLMIs) provided in Theorem 2 are time-varying and non-strict that can be solved via Semi-Definite Programming (SDP), which depend on not only the variable matrices at the current time $P_1(k)$ and $P_2(k)$ but also the variable matrices at the next time $P_1(k+1)$ and $P_2(k+1)$. This makes it possible for us to find a recursive approach to deriving all $P(k)$ from time $k = 0$ to $k = N+1$, and sequentially obtain all desired time-varying filter gains. The RLMIs in Theorem 2 are similar to the DLMIs proposed in [7, 18]. Nevertheless, the RLMIs involve available state estimate and therefore may give rise to less conservative results as more information about the system state is utilized.*

Remark 5 *Recently, the filtering problem in the finite-horizon case has attracted recurring interests due primarily to increasing application of time-vary systems and real-time computation. For example, a forward recursive Riccati difference equation has been derived in [9] for linear systems. However, it is not easy to develop such filtering algorithms for systems with RONs based on the Riccati difference equation. Fortunately, it can be seen from Theorem 2 that a new H_∞ filtering technique is exploited in terms of a set of RLMIs, whose advantage lies mainly in the fact that it is applicable in on-line real-time filtering process for systems involving some nonlinearities such as RONs.*

Based on the condition of Theorem 2, the recursive LMIs algorithm for the design of robust H_∞ filters can be concluded as follows.

The recursive LMI algorithm is given as follows:

Step 1. Give the H_∞ performance index γ, the positive definite matrix S, the initial condition $x(0)$ and its estimate $\hat{x}(0)$, select initial positive definite matrix $P_1(0)$ and positive scalar $P_2(0)$ which satisfy the initial condition (28), and set $k = 0$;

Step 2. Obtain the positive matrix $P_1(k+1)$, positive scalar $P_2(k+1)$, and matrices $X(k)$ and $Y(k)$ by solving the LMIs (29) with known parameters $P_1(k)$, $P_2(k)$ and $\hat{x}(k)$;

Step 3. Derive the filter parameter matrices $F_f(k)$ and $G_f(k)$ by solving (30) and (31), get $\hat{x}(k+1)$ according to (7), and set $k = k+1$;

Step 4. If $k < N$, then go to *Step* 2., else go to *Step* 5.;

Step 5. Stop.

Remark 6 *From the given algorithm, one can see that the state estimate at the time k, i.e., $\hat{x}(k)$, is employed to derive the filter parameter matrices $F_f(k)$ and $G_f(k)$, which means that more current information is used to estimate the state at time $k+1$. It should be pointed out that, in most existing results, only the measured output at the time k is employed to estimate the state at time $k+1$. In this sense, the algorithm can potentially improve the accuracies of the state estimation.*

5 An illustrative example

In this section, a numerical example is presented to demonstrate the effectiveness of the method proposed in this paper.

Consider the following class of nonlinear discrete time-varying polytopic uncertain stochastic system

$$\begin{cases} x(k+1) = \begin{bmatrix} 0 & -0.095+\xi \\ 0.09 & 0.08\sin(6k) \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0.2 \end{bmatrix} v(k) \\ \qquad + \begin{bmatrix} 0 & 0.01 \\ -0.01 & 0.01\sin(6k) \end{bmatrix} x(k)w(k) \\ \qquad + r(k)f(k, x(k)) \\ y(k) = \begin{bmatrix} 0.01\sin(6k) & 0.05 \end{bmatrix} x(k) + v(k) \\ z(k) = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix} x(k) \end{cases} \tag{40}$$

with the initial value $x(0) = \begin{bmatrix} 0.4 & 0 \end{bmatrix}^T$. We choose the nonlinear function $f(k, x(k))$ as

$$f(k, x(k)) = \begin{bmatrix} \frac{0.02x_1(k)}{x_2^2(k)+1} & 0.015x_2(k)\sin(x_1(k)) \end{bmatrix}^T.$$

It can be easily verified that the constraint (2) is met with $\alpha(k) = 1$ and $G(k) = \text{diag}\{0.02, 0.02\}$. The uncertain parameter ξ is unknown but assumed to belong to the known range $[-0.005, 0.005]$.

In this example, the parameters of the logarithmic quantizer $q(\cdot)$ are taken as $u_0 = 3$ and $\rho = 0.6$. The exogenous disturbance input is selected as $v(k) = \exp(-k/35) \times n(k)$ where $n(k)$ is uniformly distributed over $[-0.05, 0.05]$. The probability is assumed to be $\delta = 0.9$.

Setting $\gamma = 0.3162$ and letting $S = \text{diag}\{73, 1\}$, we can find the initial positive definite matrix $P_1(0) = I$ and positive scalar $P_2(0) = 1$ to satisfy the initial condition (28). According to the given recursive LMI algorithm, the time-varying LMIs in Theorem 2 can be solved recursively by Matlab (with the YALMIP 3.0). Table 1 lists the variable matrices $P_1(k)$, $P_2(k)$ and the desired parameters of filter $F_f(k)$, $G_f(k)$ from the time $k = 0$ to $k = 4$.

In the simulation, the uncertain parameter in the system (40) is taken as $\xi = 0$. Simulation results are presented in Figs. 1-5. Fig. 1 plots the measurement without and with quantization, and the latter is actually employed by the robust H_∞ filter. Fig. 2 shows the output $z(k)$ and its estimate $\hat{z}(k)$. The estimation error $\tilde{z}(k)$ is described in the Fig.3. The actual state response $x_1(k)$ and its estimate $\hat{x}_1(k)$ are depicted in Fig.4, and the actual state response $x_2(k)$ and its estimate $\hat{x}_2(k)$ are plotted in Fig. 5. The simulation has confirmed that the designed filter performs very well.

6 Conclusions

In this paper, we have studied the robust H_∞ filtering problem for discrete time-varying stochastic systems with polytopic uncertainties, RONs and quantization effects. The RONs have been modeled by a Bernoulli distributed white sequence with a known conditional probability. Sufficient conditions have been derived for the estimation error of the system under consideration to satisfy the H_∞ performance constraint. A robust H_∞ filter has then been designed by solving a set of recursive LMIs. The proposed robust H_∞ filtering technique is a recursive algorithm that is suitable for on-line computation by employing more information at and before current time to estimate the current state. A numerical simulation example has been used to demonstrate the effectiveness of the filtering technology presented in this paper.

References

[1] M. Basin and D. C. Alvarez, Alternative optimal filter for linear systems with multiple state and observation delays, *Int. J. of Innovative Computing, Information and Control*, Vol. 4, No. 11, pp. 2889-2898, 2008.

[2] M. Basin, E. Sanchez and R. Martinez-Zuniga, Optimal linear filtering for systems with multiple state and observation delays, *Int. J. of Innovative Computing, Information and Control*, Vol. 3, No. 5, pp. 1309-1320, 2007.

[3] N. Berman and U. Shaked, H_∞ control for discrete-time nonlinear stochastic systems, *IEEE Trans. Automatic Control*, Vol. 51, No. 6, pp. 1041-1046, 2006.

[4] M. Fu and L. Xie, The sector bound approach to quantized feedback control, *IEEE Trans. Automatic Control*, Vol. 50, No. 11, pp. 1689-1711, Nov. 2005.

[5] H. Gao and T. Chen, H_∞ estimation for uncertain systems with limited communication capacity, *IEEE Trans. Automatic Control*, Vol. 52, No. 11, pp. 2070-2084, Nov. 2007.

[6] H. Gao, T. Chen and T. Chai, Passivity and passification for networked control systems, *SIAM J. Control and Optimization*, Vol. 46, No. 4, pp. 1299-1322, 2007.

[7] E. Gershon and U. Shaked, H_∞ output-feedback of continuous-time linear systems with state-multiplicative noise, *Proceedings of the 13th Mediterranean Conference on Control and Automation*, Limassol, Cyprus, pp. 1464-1469, Jun. 27-29, 2005.

[8] C. Gong and B. Su, Robust H_∞ filtering of convex polyhedral uncertain time-delay fuzzy systems, *Int. J. of Innovative Computing, Information and Control*, Vol. 4, No. 4, pp. 793-802, 2008.

[9] Y. S. Hung and F. Yang, Robust H_∞ filtering with error variance constraints for discrete time-varying systems with uncertainty, *Automatica*, Vol. 39, pp. 1185-1194, 2003.

[10] M. R. James and I. R. Petersen, Nonlinear state estimation for uncertain systems with an integral constraint, *IEEE Trans. Signal Processing*, Vol. 46, No. 11, pp. 2926-2937, Nov. 1998.

[11] X. Lu, L. Xie, H. Zhang and W. Wang, Robust Kalman filtering for discrete-time systems with measurement delay, *IEEE Trans. Circuits and Systems -II: Express Briefs*, Vol. 54, No. 6, pp. 522-526, June. 2007.

[12] J. L. Rawson, C. S. Hsu and H. Rho, Reduced order H_∞ filters for discrete linear systems, *Proceedings of IEEE conference on decision and control*, San Diego, CA, USA, pp. 3311-3316, 1997.

[13] K. Reif, S. Günther, E. Yaz and R. Unbehauen, Stochastic stability of the discrete-time extended Kalman filter, *IEEE Trans. Automatic Control*, Vol. 44, No. 4, pp. 714-728, Apr. 1999.

[14] K. Reif and R. Unbehauen, The extended Kalman filter as an exponential observer for nonlinear systems, *IEEE Trans. Signal Processing*, Vol. 47, No. 8, pp. 2324-2328, Aug. 1999.

[15] A. V. Savkin and I. R. Petersen, Recursive state estimation for uncertain systems with an integral quadratic constraint, *IEEE Trans. Automatic Control*, Vol. 40, No. 6, pp. 1080-1083, Jun. 1995.

[16] A. V. Savkin and I. R. Petersen, Robust state estimation and model validation for discrete-time uncertain systems with a deterministic description of noise and uncertainty, *Automatica*, Vol. 34, No. 2, pp. 271-274, 1998.

[17] J. Seo, M. Yu, C. G. Park and J. G. Lee, An extended robust H_∞ filter for nonlinear constrained uncertain systems, *IEEE Trans. Signal Processing*, Vol. 54, No. 11, pp. 4471-4475, Nov. 2006.

[18] U. Shaked and V. Suplin, A new bounded real lemma representation for the continuous-time case, *IEEE Trans. Automatic Control*, Vol. 46, No. 9, pp. 1420-1426, Sep. 2001.

[19] P. Shi, M. Mahmoud, S. K. Nguang and A. Ismail, Robust filtering for jumping systems with mode-dependent delays, *Signal Processing*, Vol. 86, pp. 140-152, 2006.

[20] S. Sun, L. Xie, W. Xiao and Y. C. Soh, Optimal linear estimation for systems with multiple packet dropouts, *Automatica*, Vol. 44, pp. 1333-1342, 2008.

[21] E. Tian, D. Yue and C. Peng, Quantized output feedback control for networked control systems, *Information Sciences*, Vol. 178, pp. 2734-2749, 2008.

[22] Z. Wang, D. W. C. Ho and X. Liu, Variance-constrained filtering for uncertain stochastic systems with missing measurements, *IEEE Trans. Automatic Control*, Vol. 48, No. 7, pp. 1254-1258, Jul. 2003.

[23] Z. Wang, Y. Liu and X. Liu, H_∞ filtering for uncertain stochastic time-delay systems with sector-bounded nonlinearities, *Automatica*, Vol. 44, No. 5, pp. 1268-1277, May 2008.

[24] Z. Wang, D. W. C. Ho, Y. Liu and X. Liu, Robust H_∞ control for a class of nonlinear discrete time-delay stochastic systems with missing measurements, *Automatica*, Vol. 45, No. 3, pp. 684-691, Mar. 2009.

[25] Z. Wang, F. Yang, D. W. C. Ho and X. Liu, Robust H_∞ filtering for stochastic time-delay systems with missing measurements, *IEEE Trans. Signal Processing*, Vol. 54, No. 7, pp. 2579-2587, Jul. 2006.

[26] L. Wu, P. Shi, H. Gao and C. Wang, H_∞ filtering for 2D Markovian jump systems, *Automatica*, Vol. 44, No. 7, pp. 1849-1858, Jul. 2008.
[27] L. Wu, J. Lam, W. Paszke, K. Galkowski and E. Rogers, Robust H_∞ filtering for uncertain differential linear repetitive processes, *Int. J. Adaptive Control and Signal Processing*, Vol. 22, No. 3, pp. 243-265, 2008.
[28] L. Xie, C. E. de Souza and Y. Wang, Robust control of discrete time uncertain dynamical systems, *Automatica*, Vol. 29, No. 4, pp. 1133-1137, 1993.
[29] L. Xie, Y. C. Soh and C. E. de Souza, Robust kalman filtering for uncertain discrete-time systems, *IEEE Trans. Automatic Control*, Vol. 39, No. 6, pp. 1310-1314, Jun. 1994.
[30] J. Xiong and J. Lam, Stabilization of linear systems over networks with bounded packet loss, *Automatica*, Vol. 43, No. 1, pp. 80-87, 2007.
[31] F. Yang, Z. Wang, G. Feng and X. Liu, Robust filtering with randomly varying sensor delay: the finite-horizon case, *IEEE Trans. Circuits and Systems -I*, Vol. 56, No. 3, pp. 664-672, Mar. 2009.
[32] W. Zhang, Y. Huang and H. Zhang, Stochastic H_2/H_∞ control for discrete-time systems with state and disturbance dependent noise, *Automatica*, Vol. 43, pp. 513-521, 2007.

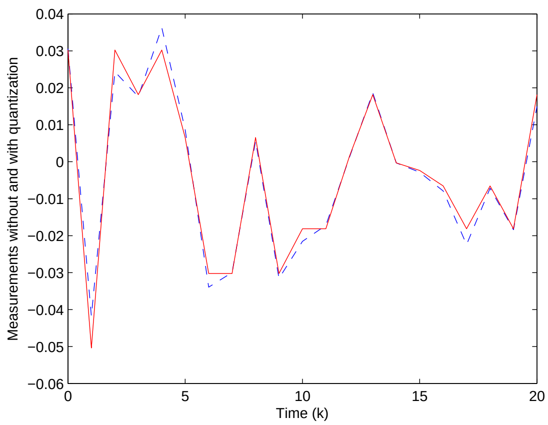

Fig. 1. Measurement without (dashed) and with quantization (solid)

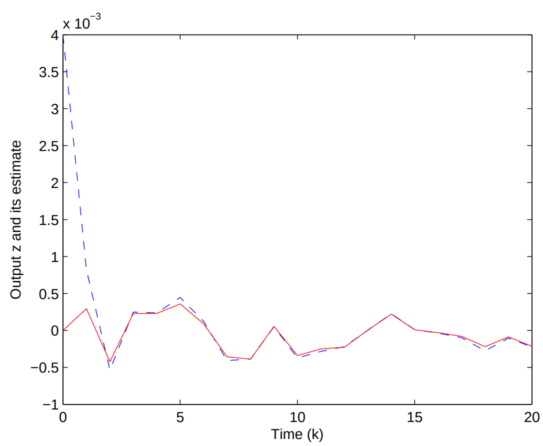

Fig. 2. Output z (dashed) and its estimate $\hat{z}$ (solid)

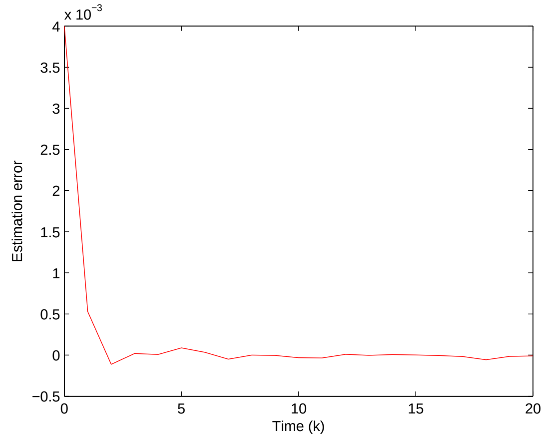

Fig. 3. Estimation error $\tilde{z}$

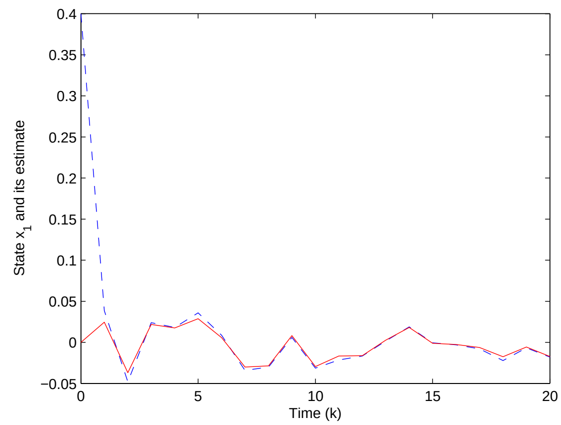

Fig. 4. State x_1 (dashed) and its estimate $\hat{x}_1$ (solid)

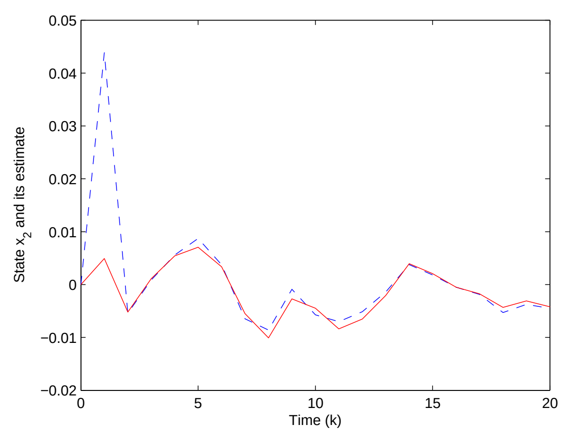

Fig. 5. State x_2 (dashed) and its estimate $\hat{x}_2$ (solid)

Table 1
Recursive process

k	0	1	2	3	4	$\cdots$
$P_1(k)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0.3450 & -0.0284 \\ -0.0284 & 0.4821 \end{bmatrix}$	$\begin{bmatrix} 0.2064 & -0.0153 \\ -0.0153 & 0.2790 \end{bmatrix}$	$\begin{bmatrix} 0.1492 & -0.0111 \\ -0.0111 & 0.2000 \end{bmatrix}$	$\begin{bmatrix} 0.3334 & -0.0679 \\ -0.0679 & 0.6008 \end{bmatrix}$	$\cdots$
$P_2(k)$	1	0.2834	0.1162	0.0705	0.0437	$\cdots$
$F_f(k)$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} -0.0180 & -0.0301 \\ 0.0988 & -0.0712 \end{bmatrix}$	$\begin{bmatrix} -0.0140 & -0.0035 \\ 0.0780 & 0.0409 \end{bmatrix}$	$\begin{bmatrix} -0.0007 & 0.0043 \\ 0.0888 & -0.0155 \end{bmatrix}$	$\begin{bmatrix} -0.0018 & -0.1086 \\ 0.0744 & -0.0240 \end{bmatrix}$	$\cdots$
$G_f(k)$	$\begin{bmatrix} 0.8130 \\ 0.1627 \end{bmatrix}$	$\begin{bmatrix} 0.7219 \\ 0.1443 \end{bmatrix}$	$\begin{bmatrix} 0.7003 \\ 0.1394 \end{bmatrix}$	$\begin{bmatrix} 0.9707 \\ 0.1936 \end{bmatrix}$	$\begin{bmatrix} 0.9744 \\ 0.1945 \end{bmatrix}$	$\cdots$