Automatica 48 (2012) 699–711

Contents lists available at SciVerse ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Simultaneous global external and internal stabilization of linear time-invariant discrete-time systems subject to actuator saturation[☆]

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ARTICLE INFO

Article history:
Received 16 August 2010
Received in revised form
14 May 2011
Accepted 15 September 2011
Available online 15 March 2012

Keywords:
Simultaneous stabilization
Input saturation
Low-and-high gain feedback

ABSTRACT

Simultaneous external and internal stabilization in a global framework for linear time-invariant discrete-time systems subject to actuator saturation is considered. Internal stabilization is in the sense of Lyapunov while external stabilization is in the sense of ℓ_p stability with different variations, e.g. with or without finite gain, with fixed or arbitrary initial conditions, with or without bias. Several simultaneous external and internal stabilization problems, all in the global framework, are studied in depth. Moreover, we present a design for appropriate adaptive low-and-high gain feedback controllers that achieve the intended simultaneous external and internal stabilization whenever such problems are solvable.

1. Introduction

Most nonlinear systems encountered in practice consist of linear systems and static nonlinear elements. Physical quantities such as speed, acceleration, pressure, flow, current, voltage, and so on, are always limited to a finite range, and saturation nonlinearities are therefore a ubiquitous feature of physical systems. One class of such systems is the class of linear systems subject to actuator saturation as depicted in Fig. 1 along with a feedback controller, where u denotes the control input and d is an external input or disturbance. Our interest in this paper is on simultaneous external and internal stabilization of the type of systems depicted in Fig. 1.

A brief survey helps to motivate our work. Early work on internal stabilization of linear systems subject to actuator saturation started with the seminal work of Fuller (1969) which established that a chain of integrators with order higher than two cannot be globally asymptotically stabilized by any saturating linear control law. Continuing the theme of Fuller, Sontag and Sussmann (1990), Sussmann and Yang (1991), Yang (1993) and Yang, Sontag, and Sussmann (1997) established that, in general, global asymptotic stabilization of linear systems with bounded inputs can only be achieved using nonlinear feedback laws. Moreover, this stabilization can be achieved if and only if the given system in the absence of saturation is stabilizable and critically unstable (equivalently, *asymptotically null controllable with bounded control* (ANCBC)). We note that critically unstable systems are those systems that have all their open-loop poles within the closed left half plane (continuous-time systems) or within the closed unit disc (discrete-time systems). The works of Sontag et al. unleashed a flurry of activity in internally stabilizing linear systems subject to actuator saturation. Along one direction, Teel (1992a,b) proposed certain design methodologies to design appropriate controllers for global stabilization. Megretski (1996) came up with a gain scheduling based nonlinear control law utilizing Riccati equations. Along another direction, Saberi and his students queried as to what can be achieved by utilizing only linear feedback control laws. In this respect, Lin and Saberi (1993a,b, 1995) proposed and emphasized a semi-global rather than global framework for stabilization using bounded controls. All this early work of these authors and others is surveyed in Bernstein and Michel (1995), Saberi and Stoorvogel (1999), Tarbouriech and Garcia (1997), Saberi, Stoorvogel, and Sannuti (2000), Hu and Lin (2001), Kapila and Grigoriadis (2002), and the references therein.

☆ The work of Ali Saberi and Xu Wang is partially supported by NAVY grants ONR KKK777SB001 and ONR KKK760SB0012. The material in this paper was partially presented at the 2011 American Control Conference (ACC 2012), June 29–July 1, 2011, San Francisco, California, USA. This paper was recommended for publication in revised form by Associate Editor Zongli Lin under the direction of Editor Andrew R. Teel.

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doi:10.1016/j.automatica.2012.02.004

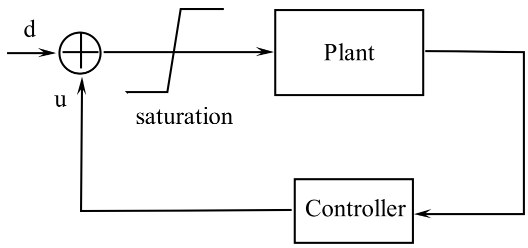

Fig. 1. A linear system subject to actuator saturation.

Following the early phase work outlined above, during the last decade and a half, among others, our research team probed intensely into a number of problems concerning the control of linear systems subject to actuator saturation, and on linear systems subject to input and state constraints, where the controller must guarantee that the output which is a linear combination of inputs and states of a linear system remains in a given set (see, e.g., Saberi, Han, and Stoorvogel (2002) and references therein).

Once the issues related to internal stabilization were resolved, the research was directed towards simultaneous external and internal stabilization. Such simultaneous stabilization has also a long history. A well-known result in linear system theory states that asymptotically stable systems have very good external stability properties. Thus, for linear systems the notions of internal stability and external stability in any sense are highly coupled. However, for general nonlinear systems and in particular for linear systems subject to actuator saturation, these two notions of stability are vastly different. For generic nonlinear systems this was first looked at by Sontag, see for instance Sontag (1990). For the class of linear systems subject to actuator saturations, the concept of simultaneous external and internal stabilization was first studied in Hou, Saberi, and Lin (1997), Hou, Saberi, Lin, and Sannuti (1998) and Lin (1997), Bao, Lin, and Sontag (2000). Subsequent to this work, there exist numerous other works on simultaneous external and internal stabilization (see e.g. Saberi, Hou, and Stoorvogel (2000), Chitour and Lin (2003)). The picture that emerges from all these works is that, for the case when external disturbance is additive to the control input, all the issues associated with simultaneous external and internal stabilization are more or less resolved, but only for continuous-time systems.

Our focus in this paper is on discrete-time linear systems subject to actuator saturation. For *continuous-time* systems, a key result is given in Saberi, Hou et al. (2000). This work, while pointing out all the complexities involved in simultaneous *global* external and *global* internal stabilization, resolves all such issues and develops certain scheduled low-and-high gain design methodologies to achieve the required simultaneous global-global stabilization. Analogous results for discrete-time systems do not exist so far in the literature. Discrete-time has its own peculiarities. High-gain cannot be as freely used as in continuous-time but also almost disturbance decoupling could always be achieved in continuous-time case while in discrete-time case, this is not possible in general. This paper can be thought of as a companion paper to Saberi, Hou et al. (2000) as it resolves fully all the issues for discrete-time systems. In particular, we develop here the necessary and sufficient conditions for simultaneous *global* external and *global* internal stabilization, and furthermore develop also the required design methodologies to accomplish such a stabilization whenever it is feasible.

We organize the paper as follows: In Section 2, we formulate precisely two problems studied in this paper, namely (1) simultaneous global ℓ_p stabilization without finite gain and internal global asymptotic stabilization (G_p/G), and (2) simultaneous global ℓ_p stabilization with finite gain and internal global asymptotic stabilization $(G_p/G)_{fg}$. In Section 3, we describe controller design methodologies, and in Section 4, we establish the solvability conditions for (G_p/G) and $(G_p/G)_{fg}$ and construct an adaptive-low-gain and high-gain controller that solves the two problems by using a parametric Lyapunov equation.

2. Preliminary notations and problem formulation

In this section, after stating certain standard notations, we recall the notions of external stability for a general discrete-time nonlinear system. Based on these notions, we formulate the simultaneous stabilization problems which we study in this paper.

For $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm and x' denotes the transpose of x. For $X \in \mathbb{R}^{n\times m}$, $\|X\|$ denotes its induced 2-norm and X' denotes the transpose of X. $\operatorname{trace}(X)$ denotes the trace of X. If X is symmetric, $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ denote the smallest and largest eigenvalues of X respectively. For a subset $\mathcal{X} \subset \mathbb{R}^n$, $\mathcal{X}^c$ denotes the complement of $\mathcal{X}$. For $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \le k_2$, $\overline{k_1, k_2}$ denotes the integer set $\{k_1, k_1+1, \dots, k_2\}$.

A continuous function $\phi : [0,\infty) \to [0,\infty)$ is said to be a class $\mathcal{K}$ function if

(1) $\phi(0) = 0$;
(2) ϕ is strictly increasing.

The ℓ_p space with $p \in [1,\infty)$ consists of all vector-valued discrete-time signals y from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sum_{k=0}^{\infty} \|y(k)\|^p < \infty.$$

For a signal $y \in \ell_p$, the ℓ_p norm of y is defined as

$$\|y\|_p = \left(\sum_{k=0}^{\infty} \|y(k)\|^p\right)^{\frac{1}{p}}.$$

The ℓ_∞ space consists of all vector-valued discrete-time signals y from $\mathbb{Z}^+ \cup \{0\}$ to $\mathbb{R}^n$ for which

$$\sup_{k\ge 0} \|y(k)\| < \infty.$$

For a signal $y \in \ell_\infty$, the ℓ_∞ norm of y is defined as

$$\|y\|_\infty = \sup_{k\ge 0} \|y(k)\|.$$

The following relationship holds for all ℓ_p spaces: for $1 < p < q < \infty$

$$\ell_1 \subset \ell_p \subset \ell_q \subset \ell_\infty.$$

Moreover, for any $y \in \ell_p$ with $p \in [1,\infty)$, the following properties hold:

(1) $\|y\|_\infty \le \|y\|_p$;
(2) $y(k) \to 0$ as $k \to \infty$.

Next we recall the definitions of external stability. Consider a system

$$\Sigma : \begin{cases} x(k+1) = f(x(k), d(k)), & x(0) = x_0 \\ y(k) = g(x(k), d(k)) \end{cases}$$

with $x(k) \in \mathbb{R}^n$ and $d(k) \in \mathbb{R}^m$. The two classical ℓ_p stabilities are defined as follows.

Definition 1. For any $p \in [1,\infty]$, the system Σ is said to be *ℓ_p stable with fixed initial condition and without finite gain* if for $x(0) = 0$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 2. For any $p \in [1,\infty]$, the system Σ is said to be *ℓ_p stable with fixed initial condition and with finite gain* if for $x(0) = 0$

and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p such that for any $d \in \ell_p$,

$$\|y\|_p \leq \gamma_p \|d\|_p.$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

As observed in Shi, Saberi, and Stoorvogel (2003), the initial condition plays a dominant role in whether achieving ℓ_p stability is possible or not. Hence any definition of external stability must take into account the effect of initial condition. The notion of external stability with arbitrary initial condition was introduced in Shi et al. (2003). We recall these definitions below.

Definition 3. For any $p \in [1, \infty]$, the system Σ is said to be ℓ_p *stable with arbitrary initial condition and without finite gain* if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$.

Definition 4. For any $p \in [1, \infty]$, the system Σ is said to be ℓ_p *stable with arbitrary initial condition with finite gain and with bias* if for any $x_0 \in \mathbb{R}^n$ and any $d \in \ell_p$, we have $y \in \ell_p$ and there exists a γ_p and a class $\mathcal{K}$ function $\phi(\cdot)$ such that for any $d \in \ell_p$

$$\|y\|_p \leq \gamma_p \|d\|_p + \phi(\|x_0\|).$$

The infimum over all γ_p with this property is called the ℓ_p gain of the system Σ.

Now we are ready to formulate the control problems studied in this paper. Consider a linear discrete-time system subject to actuator saturation,

$$x(k+1) = Ax(k) + B\sigma(u(k) + d(k)), \tag{2.1}$$

where state $x \in \mathbb{R}^n$, the output $y = x$, control input $u \in \mathbb{R}^m$, and external input $d \in \mathbb{R}^m$. Here $\sigma(\cdot)$ denotes the standard saturation function defined as

$$\sigma(u) = [\sigma_1(u_1), \ldots, \sigma_1(u_m)]$$

where $\sigma_1(s) = \operatorname{sgn}(s) \min\{|s|, \Delta\}$ for some $\Delta > 0$.

The simultaneous global external and internal stabilization problems studied in this paper are formulated as follows.

Problem 1. For any $p \in [1, \infty]$, the system (2.1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition and without finite gain and globally asymptotically stabilizable via static time invariant state feedback, which we refer to as (G_p/G), if there exists a static state feedback controller $u = f(x)$ such that the following properties hold:

(1) the closed-loop system is ℓ_p stable with fixed initial condition and without finite gain where the output $y = x$.
(2) In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Problem 2. For any $p \in [1, \infty]$, the system (2.1) is said to be simultaneously globally ℓ_p stabilizable with fixed initial condition with finite gain with zero bias and globally asymptotically stabilizable via state feedback, which we refer to as $(G_p/G)_{fg}$, if there exists a static time invariant state feedback controller $u = f(x)$ such that the following properties hold:

(1) the closed-loop system is finite gain ℓ_p stable with fixed initial condition with finite gain and with zero bias where the output $y = x$.
(2) In the absence of external input d, the equilibrium $x = 0$ is globally asymptotically stable.

Note that the notion of global ℓ_p stability with arbitrary initial condition embeds in it the internal stability in some sense. We also formulate below additional external stabilization problems with arbitrary initial conditions.

Problem 3. For any $p \in [1, \infty]$, the system (2.1) is said to be globally ℓ_p stabilizable with arbitrary initial condition and without finite gain via static time invariant state feedback, if there exists a static state feedback controller $u = f(x)$ such that the closed-loop system is ℓ_p stable with arbitrary initial condition and without finite gain where the output $y = x$.

Problem 4. For any $p \in [1, \infty]$, the system (2.1) is said to be globally ℓ_p stabilizable with arbitrary initial condition with finite gain and with bias via state feedback, if there exists a static time invariant state feedback controller $u = f(x)$ such that the closed-loop system is finite gain ℓ_p stable with arbitrary initial condition with finite gain and with bias where the output $y = x$.

Since global asymptotic stabilization is required in all the problems, it is well-known that the following assumption is ***necessary***.

Assumption 1. (1) The pair (A, B) is stabilizable.
(2) A has all its eigenvalues in the closed unit disc.

In fact, as will become clear in the sequel, Assumption 1 is also ***sufficient*** to solve Problems 1–4. To see this, we first note that under Assumption 1, the system (2.1) can be transformed into the form,

$$\begin{pmatrix} x_s(k+1) \\ x_u(k+1) \end{pmatrix} = \begin{pmatrix} A_s & 0 \\ 0 & A_u \end{pmatrix} \begin{pmatrix} x_s(k) \\ x_u(k) \end{pmatrix} + \begin{pmatrix} B_s \\ B_u \end{pmatrix} \sigma(u(k) + d(k)) \tag{2.2}$$

where A_s is Schur stable, A_u has all its eigenvalues on the unit circle and (A_u, B_u) is controllable. Suppose (G_p/G) and/or $(G_p/G)_{f\cdot g}$ of the x_u dynamics can be achieved by a feedback controller $u = f(x_u)$. If B_u has full column rank, it is straightforward to show that $u = f(x_u)$ also achieves (G_p/G) and/or $(G_p/G)_{f\cdot g}$ of the overall system. However, it takes some effort to reach the same conclusion in the general case. We show this in the Appendix under a generic assumption on controller structure.

Therefore, to solve Problems 1–4 for system (2.1), it is sufficient to solve these problems only for the unstable sub-dynamics. In the rest of the paper, we impose the following assumption

Assumption 2. (1) The pair (A, B) is controllable.
(2) A has all its eigenvalues on the unit circle.

3. Controller design

In this section, we would like to present the controller design methodologies which we shall employ to solve the problems formulated in Section 2. The controller design in this paper is based on the classical low-gain and low-and-high-gain feedback design methodologies. The low-gain feedback can be constructed using different approaches such as direct eigenstructure assignment (Lin & Saberi, 1993b), H_2 and H_∞ algebraic Riccati equation based methods (Lin, Stoorvogel, & Saberi, 1996; Teel, 1995), and parametric Lyapunov equation based methods (Zhou, Duan, & Lin, 2008; Zhou, Lin, & Duan, 2009). In this paper, we choose parametric Lyapunov equation method to build the low-gain feedback because of its special properties; as will become clear later on, it greatly simplifies the expressions for our controllers and the subsequent analysis.

Since the low-gain feedback, as indicated by its name, does not allow complete utilization of control capacities, the low-and-high-gain feedback was developed to rectify this drawback and was intended to achieve control objectives beyond stability, such as performance enhancement, robustness and disturbances rejection. The low-and-high gain feedback is composed of a low-gain and a high-gain feedback. As shown in Hou et al. (1998), the solvability of simultaneous global external and internal stabilization problem critically relies on a proper choice of high-gain. In this section, we shall first recall the low-gain feedback design and propose a new high-gain design methodology.

3.1. Low gain state feedback

In this subsection, we review the low-gain feedback design methodology recently introduced in Zhou et al. (2008, 2009) which is based on the solution of a parametric Lyapunov equation. Five key properties of the parametric Lyapunov equation are summarized in the next lemma, where the first three properties are adopted from Zhou et al. (2009).

Lemma 1. *Assume that (A, B) is controllable and A has all its eigenvalues on the unit circle. For any $\varepsilon \in (0, 1)$, the Parametric Algebraic Riccati Equation,*

$$(1-\varepsilon)P_\varepsilon = A'P_\varepsilon A - A'P_\varepsilon B(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon A, \tag{3.1}$$

has a unique positive definite solution $P_\varepsilon = W_\varepsilon^{-1}$ where W_ε is the solution for W of

$$W - \frac{1}{1-\varepsilon}AWA' = -BB'.$$

Moreover, the following properties hold:

(1) *$A_c(\varepsilon) = A - B(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon A$ is Schur stable for any $\varepsilon \in (0, 1)$;*
(2) *$\frac{dP_\varepsilon}{d\varepsilon} > 0$ for any $\varepsilon \in (0, 1)$;*
(3) *$\lim_{\varepsilon \downarrow 0} P_\varepsilon = 0$.*
(4) *There exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$,*

$$\|P_\varepsilon^{\frac{1}{2}} A P_\varepsilon^{-\frac{1}{2}}\| \le \sqrt{2}.$$

(5) *Let ε^* be given by property 4. There exists a M_{ε^*} such that $\|\frac{P_\varepsilon}{\varepsilon}\| \le M_{\varepsilon^*}$ for all $\varepsilon \in (0, \varepsilon^*]$.*

Proof. The existence of the positive definite solution $P_\varepsilon = W_\varepsilon^{-1}$ and properties 1, 2 and 3 were shown in Zhou et al. (2009). Regarding property 4, multiplying by $P_\varepsilon^{-1/2}$ on both sides of (3.1) gives

$$V_\varepsilon'[I - P_\varepsilon^{\frac{1}{2}}B(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon^{\frac{1}{2}}]V_\varepsilon = (1-\varepsilon)I$$

where $V_\varepsilon = P_\varepsilon^{1/2}AP_\varepsilon^{-1/2}$. Since $P_\varepsilon \to 0$ as $\varepsilon \to 0$, there exists an ε^* such that for any $\varepsilon \in (0, \varepsilon^*]$

$$I - P_\varepsilon^{\frac{1}{2}}B(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon^{\frac{1}{2}} \ge \frac{1}{2}I.$$

Hence

$$V_\varepsilon'V_\varepsilon < 2I,$$

or equivalently

$$\|V_\varepsilon\| \le \sqrt{2}.$$

It remains to show property 5. Note that W_ε is a rational matrix in ε and thus P_ε is a rational matrix in ε. Property 3 implies that $P = \varepsilon\bar{P}_\varepsilon$ where $\bar{P}_\varepsilon$ is rational in ε and satisfies $\|\bar{P}_\varepsilon\| < M_{\varepsilon^*}$ for any $\varepsilon \in (0, \varepsilon^*]$. Hence, property 5 holds. This concludes the proof of Lemma 1. □

We define the low-gain state feedback which is a family of parameterized state feedback laws given by

$$u_L(x) = F_L x = -(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax, \tag{3.2}$$

where P_ε is the solution of (3.1). Here, as usual, ε is called the low-gain parameter. From the properties given by Lemma 1, it can be seen that the magnitude of the control input can be made arbitrarily small by choosing ε sufficiently small so that the input never saturates for any, a priori given, set of initial conditions.

3.2. Low-and-high-gain feedback

The low-and-high-gain state feedback is composed of a low-gain state feedback and a high-gain state feedback as

$$u_{LH}(x) = F_{LH}x = F_L x + F_H x \tag{3.3}$$

where $F_L x$ is given by (3.2). The high-gain feedback is of the form,

$$F_H x = \rho F_L x$$

where ρ is called the high-gain parameter.

For continuous-time systems, the high gain parameter ρ can be any positive real number. However, this is not the case for discrete-time systems. In order to preserve local asymptotic stability, this high gain has to be bounded at least near the origin. The existing results in literature on the choice of high-gain parameter for discrete-time system are really sparse. To the best of our knowledge, the only available result is in Lin, Saberi, Stoorvogel, and Mantri (1996, 2000) where the high-gain parameter is a nonlinear function of x as

$$\kappa(x) = \max\{z \in [0, 1] \mid \|F_L x + \alpha z K_H x\|_\infty \le \Delta\}$$

where $K_H = -(B'P_\varepsilon B)^{-1}B'P_\varepsilon(A+BF_L)$ and $\alpha \in [0, 2]$ (assume without loss of generality that B has full rank). This high gain always yields a controller smaller than Δ in magnitude, which lacks the capability of dealing with disturbances. Furthermore, to solve the global external and internal stabilization problem, we need to schedule the high-gain parameter with respect to x. However, this nonlinear high-gain parameter is also not suitable for adaptation since it will make the analysis extremely complicated. Instead, we need a constant high-gain parameter so that the controller (3.3) remains linear. A suitable choice of such a high-gain parameter satisfies

$$\rho \in \left[0, \frac{2}{\|B'P_\varepsilon B\|}\right] \tag{3.4}$$

where P_ε is the solution of parametric Lyapunov equation (3.1).

Lemma 2. *Consider system (2.1) satisfying Assumption 2. Let P_ε be the solution of (3.1). For any a priori given compact set $\mathcal{X}$, there exists an ε^* such that for any $\varepsilon \in [0, \varepsilon^*]$ and ρ satisfying (3.4), the origin of the interconnection of (2.1) with the low-and-high-gain feedback*

$$u_{LH} = -(1+\rho)(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax$$

is locally asymptotically stable with domain of attraction containing $\mathcal{X}$.

Proof. Let c be such that

$$c = \sup_{\substack{\varepsilon \in (0,\varepsilon^*] \\ x \in \mathcal{X}}} x'P_\varepsilon x.$$

where ε^* is given by Property (4) and (5) in Lemma 1. Define a Lyapunov function $V(x) = x'P_\varepsilon x$ and a level set $\mathcal{V}(c) = \{x \mid V(x) \le c\}$. We have $\mathcal{X} \subset \mathcal{V}(c)$. From Lemma 1, there exists an $\varepsilon_1 \le \varepsilon^*$ such that for any $\varepsilon \in (0, \varepsilon_1]$ and $x \in \mathcal{V}(c)$,

$$\|(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon Ax\| \le \Delta.$$

Define $\mu = \|B'P_\varepsilon B\|$. We evaluate $V(k+1) - V(k)$ along the trajectories as

$$\begin{aligned}
V(k+1) - V(k)&\\
&= -\varepsilon V(k) - \sigma(u_{LH})'\sigma(u_{LH}) + [\sigma(u_{LH}) - u_L]' \\
&\quad \times (I+B'PB)[\sigma(u_{LH}) - u_L] \\
&\le -\varepsilon V(k) - \sigma(u_{LH})'\sigma(u_{LH}) \\
&\quad + (1+\mu)[\sigma(u_{LH}) - u_L]'[\sigma(u_{LH}) - u_L] \\
&= -\varepsilon V(k) - \frac{1+\mu}{\mu}\|u_L\|^2 + \mu\left\|\sigma(u_{LH}) - \frac{1+\mu}{\mu}u_L\right\|^2
\end{aligned}$$

where we abbreviated $u_{LH}(k)$ and $u_L(k)$ by u_{LH} and u_L respectively. Since $\|u_L\| \leq \Delta$ and ρ satisfies (3.4), we have

$$\|u_L\| \leq \|\sigma(u_{LH})\| \leq \left(1+\frac{2}{\mu}\right)\|u_L\|.$$

This implies that

$$\left\|\sigma(u_{LH}) - \frac{1+\mu}{\mu}u_L\right\| \leq \frac{1}{\mu}\|u_L\|,$$

and thus,

$$\mu\left\|\sigma(u_{LH}) - \frac{1+\mu}{\mu}u_L\right\|^2 - \frac{1}{\mu}\|u_L\|^2 \leq 0.$$

Combining the above, we get for any $x(k) \in \mathcal{V}(c)$,

$$V(k+1) - V(k) \leq -\varepsilon V(k).$$

We conclude local asymptotic stability of the origin with a domain of attraction containing $\mathcal{X}$. □

Remark 1. We would like to explain the role played by the high-gain parameter ρ in the controller design. For semi-global asymptotic stabilization, the domain of attraction is basically determined by the low-gain parameter ε provided that ρ lies in a proper range. When ρ is too large, stabilization is not possible. This is different from continuous-time systems for which the high gain parameter ρ does not have any impact on internal stability. But like continuous-time systems, ρ plays a dominant role in issues other than internal stability such as external stabilization, robust stabilization and disturbance rejection.

3.3. Scheduling of low-gain parameter

In the semi-global framework, with controller (3.2), the domain of attraction of the closed-loop system is determined by the low-gain parameter ε. In order to solve the global stabilization problem, this ε can be scheduled with respect to the state. This has been done in the literature, see for instance Hou et al. (1998).

A family of scheduled low-gain feedback controllers for global stabilization is given by

$$u_L(x) = F_{\varepsilon(x)}x = -(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax \tag{3.5}$$

where $P_{\varepsilon(x)}$ is the solution of (3.1) with ε replaced by $\varepsilon(x)$. In general, the scheduled parameter $\varepsilon(x)$ should satisfy the following properties:

(1) $\varepsilon(x) : \mathbb{R}^n \to (0, \varepsilon^*]$ is continuous and piecewise continuously differentiable where ε^* is a design parameter.
(2) There exists an open neighborhood $\mathcal{O}$ of the origin such that $\varepsilon(x) = 1$ for all $x \in \mathcal{O}$.
(3) For any $x \in \mathbb{R}^n$, we have $\|F_{\varepsilon(x)}x\| \leq \delta$.
(4) $\varepsilon(x) \to 0$ as $\|x\| \to \infty$.
(5) $\{\, x \in \mathbb{R}^n \mid x'P_{\varepsilon(x)}x \leq c \,\}$ is a bounded set for all $c > 0$.

Because of the specific problem facing us, we use the scheduling given in Hou et al. (1998) which not only satisfies the above conditions but also yields an adaptive low-gain parameter with certain properties that are fundamental to our design,

$$\varepsilon(x) = \max\left\{r \in (0, \varepsilon^*] \mid (x'P_r x)\,\mathrm{trace}(P_r) \leq \frac{\Delta^2}{b}\right\} \tag{3.6}$$

where $\varepsilon^* \in (0, 1)$ is any a priori given constant and $b = 2\,\mathrm{trace}(BB')$ while P_r is the unique positive definite solution of parametric Lyapunov equation (3.1) with $\varepsilon = r$.

Note that the scheduled low-gain controller (3.5) with (3.6) satisfies

$$\|(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax\| \leq \Delta.$$

To see this, observe that

$$\begin{aligned}
&\|(B'P_{\varepsilon(x)}B + I)^{-1}B'P_{\varepsilon(x)}Ax\|^2 \\
&\quad \leq \|B'P_{\varepsilon(x)}Ax\|^2 \\
&\quad \leq \|B'\|^2\|P_{\varepsilon(x)}\|\;\|P_{\varepsilon(x)}^{1/2}AP_{\varepsilon(x)}^{-1/2}\|^2\|P_{\varepsilon(x)}^{1/2}x\|^2 \\
&\quad \leq 2\|BB'\|\;\|P_{\varepsilon(x)}\|x'P_{\varepsilon(x)}x \\
&\qquad \text{(where we use property 4 of Lemma 1)} \\
&\quad \leq 2\,\mathrm{trace}(BB')\,\mathrm{trace}(P_{\varepsilon(x)})\,x'P_{\varepsilon(x)}x \\
&\quad \leq \Delta^2.
\end{aligned}$$

3.4. Scheduling of high-gain parameter

As emphasized earlier, the high gain parameter plays a crucial role in dealing with external inputs/disturbances. In order to solve the simultaneous external and internal stabilization problems for continuous-time systems, different methods of schedulings of high-gain parameter have been developed in the literature Hou et al. (1998), Lin (1997) and Saberi, Hou et al. (2000). Unfortunately, none of them carry over to discrete-time case because the high gain has to be restricted near the origin. In this subsection, we introduce a new scheduling of the high-gain parameter with which we shall solve the (G_p/G) and $(G_p/G)_{fg}$ problems as formulated in Section 2.

Our scheduling depends on the specific control objective. If one is not interested in finite gain, the following scheduled high gain suffices to solve (G_p/G) problem,

$$\rho_0(x) = \frac{1}{\|B'P_{\varepsilon(x)}B\|}. \tag{3.7}$$

Clearly, this high gain satisfies the constraints that

$$\rho_0(x) \leq \frac{2}{\|B'P_{\varepsilon(x)}B\|}.$$

We observe that this high-gain parameter is radially unbounded. However, if we further pursue finite gain ℓ_p stabilization, the rate of growth of $\rho(x)$ with respect to $\|x\|$ as given in (3.7) is not sufficient for us. The scheduled high-gain parameter must rise quickly enough to overwhelm any disturbances in ℓ_p before the state is steered so large that it actually prevents finite gain. Therefore, we shall introduce a different scheduling of high-gain parameter. In order to do so, we need the following lemma.

Lemma 3. *Assume that $2p \geq 1$. For any $\eta > 1$ there exists a $\beta > 0$ such that*

$$(u+v)^p \leq u^p + \eta u^p + \beta v^p \tag{3.8}$$

for all $u, v \geq 0$.

Proof. The lemma is a known result for $p \geq 1$; see, for instance, Shi et al. (2003). For $p \in [\frac{1}{2}, 1)$, we have $2p \geq 1$ and then

$$(\sqrt{u+v})^{2p} \leq (\sqrt{u} + \sqrt{v})^{2p} \leq u^{2p} + \eta u^{2p} + \beta v^{2p}$$

where we use the lemma with p replaced by $2p$ which is the known case. □

Let ε^* and M_{ε^*} be given by Lemma 1 and let P^* be the solution of (3.1) with $\varepsilon = \varepsilon^*$. The scheduled high gain parameter is given by:

$$\rho_f(x) = \begin{cases} \rho_0(x) = \dfrac{1}{\|B'P_{\varepsilon(x)}B\|}, & x'P_\varepsilon(x)x \leq c \\[2ex] \dfrac{8\rho_1(x)}{\varepsilon(x)\lambda_{\min}P_{\varepsilon(x)}}, & \text{otherwise} \end{cases} \tag{3.9}$$

with

$$\rho_1(x) = \frac{\lambda_{\max} P_{\varepsilon(x)}}{\lambda_{\min} P_{\varepsilon_1(x)}} \rho_2(x) \tag{3.10}$$

where

$$\rho_2(x) = \begin{cases} 1 & p = \infty \\ \left[\dfrac{\rho_p \beta(\varepsilon(x))}{1 - \left(1 - \frac{\varepsilon_1(x)}{4(1+L_{\varepsilon_1(x)})}\right)^{p/2}} + 1 \right]^{2/p}, & p \in [1, \infty) \end{cases}$$

where ρ_p is a positive constant to be determined later and c, $\varepsilon_1(x)$ and L_s are given by

$$c = \Delta^2 \max\{4M_{\varepsilon^*} b, 4(1 + \|B' P^* B\|)\}, \tag{3.11}$$

$$\varepsilon_1(x) = \max\left\{ r \in (0, \varepsilon^*] \mid 2x' P_r x \operatorname{trace}(P_r) \le \frac{\Delta^2}{b} \right\},$$

$$L_s = \frac{\operatorname{trace}(P^*)}{\lambda_{\min} P_s}.$$

Finally, in order to define $\beta(\varepsilon) > 1$ we first define $\eta(\varepsilon)$ satisfying

$$\left[1 - \frac{\varepsilon}{4(1 + L_\varepsilon)}\right]^{p/2} \le (1 + \eta(\varepsilon)) \left[1 - \frac{\varepsilon}{2(1 + L_\varepsilon)}\right]^{p/2} < 1.$$

Next we choose $\beta(\varepsilon) > 1$ such that Lemma 3 holds for $\eta = \eta(\varepsilon)$. In other words, $\beta(\varepsilon)$ is such that for a given $p > 1/2$, ε and $\eta(\varepsilon)$

$$(u + v)^p \le (1 + \eta(\varepsilon)) u^p + \beta(\varepsilon) v^p$$

for all $u > 0$, $v > 0$.

4. Main results

In this section, we shall solve the simultaneous external and internal stabilization problems as formulated in Section 2 using the proposed low-and-high-gain controller in Section 3. We first study the simultaneous stabilization without finite gain as formulated in Problems 1 and 3. Then we will solve Problems 2 and 4.

The theorem given below solves the global ℓ_p stabilization with arbitrary initial condition and without finite gain as formulated in Problem 3.

Theorem 1. *Consider the system (2.1) satisfying Assumption 2. For any $p \in [1, \infty]$, the ℓ_p stabilization with arbitrary initial conditions and without finite gain as formulated in Problem 3 can be solved by the adaptive-low-gain and high-gain controller,*

$$u = -(1 + \rho_0(x))(I + B' P_{\varepsilon(x)} B)^{-1} B' P_{\varepsilon(x)} A x, \tag{4.1}$$

where $P_{\varepsilon(x)}$ is the solution of (3.1), $\varepsilon(x)$ is determined adaptively by the scheduling (3.6) and $\rho_0(x)$ is determined by (3.7).

Theorem 1 immediately yields the following result.

Corollary 1. *Consider the system (2.1) satisfying Assumption 2. For any $p \in [1, \infty]$, the (G_P/G) as formulated in Problem 1 can be solved by the same adaptive-low-gain and high-gain controller (4.1).*

Proof of Theorem 1. In this proof, we denote $\varepsilon(x(k))$, $\rho_0(x(k))$, and $P_{\varepsilon(x(k))}$ by $\varepsilon(k)$, $\rho_0(k)$, and $P(k)$ respectively. This abbreviation should not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1} B'P(k)Ax(k)$, $u(k) = v(k) + \rho_0(k)v(k)$ and $\mu(k) = \|B'P(k)B\|$. We have shown that (3.6) implies that $\|v(k)\|_\infty < \Delta$.

We proceed now to show global asymptotic stability. In the absence of d, we can evaluate the increment of $V(k)$ along the trajectory as:

$$\begin{aligned} V(k+1) - V(k) &= x(k+1)'[P(k+1) - P(k)]x(k+1) \\ &\quad - \varepsilon(k)V(k) - \|\sigma(u(k))\|^2 + [\sigma(u(k)) - v(k)]' \\ &\quad \times (I + B'P(k)B)[\sigma(u(k)) - v(k)] \\ &\le x(k+1)'[P(k+1) - P(k)]x(k+1) \\ &\quad - \varepsilon(k)V(k) - \|\sigma(u(k))\|^2 \\ &\quad + (1 + \mu(k))[\sigma(u(k)) - v(k)]'[\sigma(u(k)) - v(k)] \\ &= x(k+1)'[P(k+1) - P(k)]x(k+1) - \varepsilon(k)V(k) \\ &\quad - \frac{1 + \mu(k)}{\mu(k)}\|v(k)\|^2 + \mu(k)\left\|\sigma(u(k)) - \frac{1 + \mu(k)}{\mu(k)}v(k)\right\|^2. \end{aligned}$$

As noted before, $\|v(k)\| \le \Delta$ for all $k \ge 0$, and therefore

$$\|v(k)\| \le \|\sigma(u(k))\| \le \left(1 + \frac{1}{\mu(k)}\right)\|v(k)\|.$$

This implies that

$$\left\|\sigma(u(k)) - \frac{1 + \mu(k)}{\mu(k)}v(k)\right\| \le \frac{1}{\mu(k)}\|v(k)\|,$$

and thus,

$$\begin{aligned} &\mu(k)\left\|\sigma(u(k)) - \frac{1 + \mu(k)}{\mu(k)}v(k)\right\|^2 - \frac{1 + \mu(k)}{\mu(k)}\|v(k)\|^2 \\ &\quad \le -\|v(k)\|^2. \end{aligned}$$

Finally, we get

$$\begin{aligned} V(k+1) - V(k) \le &-\varepsilon(k)V(k) + x(k+1)' \\ &\times [P(k+1) - P(k)]x(k+1). \end{aligned} \tag{4.2}$$

Our scheduling (3.6) implies that $V(k+1) - V(k)$ and $x(k+1)'[P(k+1) - P(k)]x(k+1)$ cannot have the same sign. To see this, assume that $V(k+1) > V(k)$ and $P(k+1) > P(k)$. This implies that

$$\varepsilon(k) < \varepsilon^*.$$

If $V(k)\operatorname{trace}(P(k)) < \frac{\Delta^2}{b}$, then (3.6) implies that $\varepsilon(k) = \varepsilon^*$, which yields a contradiction. If $V(k)\operatorname{trace}(P(k)) = \frac{\Delta^2}{b}$, then $V(k+1)\operatorname{trace}(P(k+1)) > \frac{\Delta^2}{b}$ since by assumption $V(k+1) > V(k)$ and $P(k+1) > P(k)$. But this is impossible by our scheduling (3.6). A similar argument can be used to establish that $V(k+1) - V(k) < 0$ and $P(k+1) - P(k) < 0$ cannot happen simultaneously either.

Using this property, (4.2) then implies that for all $x \ne 0$,

$$V(k+1) - V(k) < 0.$$

This concludes the global asymptotic stability.

What remains is to show ℓ_p stability. Similar to our earlier development, we have

$$\begin{aligned} V(k+1) - V(k) \le &-x(k+1)'[P(k) - P(k+1)]x(k+1) \\ &- \varepsilon(k)V(k) - \frac{1}{\mu(k)}\|v(k)\|^2 \\ &+ \mu(k)\left\|\sigma(u(k) + d(k)) - \frac{1 + \mu(k)}{\mu(k)}v(k)\right\|^2. \end{aligned}$$

Let $d_i(k)$, $v_i(k)$ and $u_i(k)$ denote the ith element of $d(k)$, $v(k)$ and $u(k)$ respectively.

If $|d_i(k)| \le \frac{1}{\mu(k)}|v_i(k)|$, recalling that $|v_i(k)| \le \Delta$, we have

$$|v_i(k)| \le |\sigma_1(u_i(k) + d_i(k))| \le \left(1 + \frac{2}{\mu(k)}\right)|v_i(k)|.$$

Hence

$$\mu(k)\left|\sigma_1(u_i(k)+d_i(k))-\frac{1+\mu(k)}{\mu(k)}v_i(k)\right|^2-\frac{1}{\mu(k)}|v_i(k)|^2\le 0.$$

If $|d_i(k)|\ge \frac{1}{\mu(k)}|v_i(k)|$, we have

$$\begin{aligned}&-\frac{1}{\mu(k)}|v_i(k)|^2+\mu(k)\left|\sigma_1(u_i(k)+d_i(k))-\frac{1+\mu(k)}{\mu(k)}v_i(k)\right|^2\\&\quad\le \mu(k)[(1+\mu(k))d_i(k)+d_i(k)+(1+\mu(k))d_i(k)]^2\\&\qquad+\mu(k)|d_i(k)|^2\\&\quad\le a\mu(k)|d_i(k)|^2,\end{aligned}$$

where $a=(2\mu^*+3)^2+1$, $\mu^*=\|B'P^*B\|$ and P^* is the solution of (3.1) with $\varepsilon=\varepsilon^*$. Therefore, we conclude that

$$\begin{aligned}V(k+1)-V(k)\le\ & x(k+1)'[P(k+1)-P(k)]x(k+1)\\&-\varepsilon(k)V(k)+a\mu(k)\|d(k)\|^2.\end{aligned}\tag{4.3}$$

Note that this implies that

$$V(k+1)-V(k)\le\max\{-\varepsilon(k)V(k)+a\mu(k)\|d(k)\|^2,0\}\tag{4.4}$$

since $V(k+1)-V(k)$ and $x(k+1)'[P(k+1)-P(k)]x(k+1)$ can not have the same sign. Let us first address the case of $p=\infty$. We will show that there exists a c_1 such that $V(k)\le c_1$ for all $k\ge 0$ with $V(0)=0$.

If

$$V(k)\ge a\frac{\mu(k)}{\varepsilon(k)}\|d(k)\|^2,\tag{4.5}$$

we have

$$V(k+1)-V(k)\le 0.\tag{4.6}$$

Property 5 of Lemma 1 yields that there exists a M_{ε^*} independent of k and d such that $V(k)\ge abM_{\varepsilon^*}\|d\|_\infty^2$ implies that (4.5) is satisfied and therefore $V(k+1)-V(k)\le 0$, where, as defined earlier, $b=2\,\mathrm{trace}(BB')$.

On the other hand, we have

$$V(k+1)-V(k)\le a\mu(k)\|d(k)\|^2\le a\mu^*\|d\|_\infty^2.$$

We conclude that

$$V(k)\le V(0)+abM_{\varepsilon^*}\|d\|_\infty^2+a\mu^*\|d\|_\infty^2.\tag{4.7}$$

Property 5 of our scheduling then implies that $x(k)$ is bounded for all $k\ge 0$. This shows ℓ_∞ stability of the closed-loop system with arbitrary initial condition.

We proceed now with the case of $p\in[1,\infty)$. First of all, due to the fact that $\|d\|_\infty\le\|d\|_p$, (4.7) implies that $V(k)$ is bounded for all $k\ge 0$. Hence by our scheduling, there exists an ε_0 such that $\varepsilon(k)\ge\varepsilon_0$ for all $k\ge 0$.

Next, we consider two possible cases:

Case 1. For $V(k+1)-V(k)\ge 0$, (4.4) implies that

$$V(k+1)-V(k)\le-\varepsilon(k)V(k)+a\mu(k)\|d(k)\|^2.\tag{4.8}$$

Case 2. For $V(k+1)-V(k)\le 0$, our scheduling implies that

$$x(k+1)'[P(k+1)-P(k)]x(k+1)\ge 0.$$

But this implies that $\varepsilon(k)\le\varepsilon(k+1)\le\varepsilon^*$, and thus

$$V(k+1)\,\mathrm{trace}(P(k+1))\le V(k)\,\mathrm{trace}(P(k)).$$

Hence

$$\begin{aligned}&[V(k+1)-V(k)]\,\mathrm{trace}(P(k+1))\\&\quad\le -V(k)\,\mathrm{trace}[P(k+1)-P(k)].\end{aligned}$$

Then we have

$$\begin{aligned}&|x(k+1)'[P(k+1)-P(k)]x(k+1)|\\&\quad\le|\,\mathrm{trace}(P(k+1)-P(k))|\cdot\|x(k+1)\|^2\\&\quad\le\frac{\mathrm{trace}(P(k+1))}{V(k)}\cdot|V(k+1)-V(k)|\cdot\|x(k+1)\|^2\\&\quad\le\frac{V(k+1)\,\mathrm{trace}(P(k+1))}{V(k)\lambda_{\min}P(k+1)}\cdot|V(k+1)-V(k)|\\&\quad\le\frac{\mathrm{trace}(P^*)}{\lambda_{\min}P(k)}\cdot|V(k+1)-V(k)|\\&\quad\le L(k)\cdot|V(k+1)-V(k)|\end{aligned}$$

where $L(k)=\frac{\mathrm{trace}(P^*)}{\lambda_{\min}(P(k))}$. We have

$$V(k+1)-V(k)\le\frac{-\varepsilon(k)}{1+L(k)}V(k)+a\mu(k)\|d(k)\|^2.\tag{4.9}$$

Given $\varepsilon(k)\in[\varepsilon_0,\varepsilon^*]$ for all $k\ge 0$, (4.8) in case 1 and (4.9) in case 2 ensure that

$$V(k+1)-V(k)\le-\frac{\varepsilon_0}{1+L}V(k)+a\mu^*\|d(k)\|^2,\tag{4.10}$$

where $L=\frac{\mathrm{trace}(P^*)}{\lambda_{\min}P_0}$ and P_0 is the solution of (3.1) with $\varepsilon=\varepsilon_0$. Also, $\varepsilon_0<1$ implies that $\varepsilon_0/(1+L)<1$.

Applying Lemma 3 with η such that

$$(1+\eta)(1-\frac{\varepsilon_0}{1+L})^{p/2}<1,$$

we find that there exists a β such that

$$\begin{aligned}V(k+1)^{p/2}\le\ &(1+\eta)\left(1-\frac{\varepsilon_0}{1+L}\right)^{p/2}V(k)^{p/2}\\&+\beta(a\mu^*)^{p/2}\|d(k)\|^p.\end{aligned}$$

This yields

$$\begin{aligned}&\left[1-(1+\eta)\left(1-\frac{\varepsilon_0}{1+L}\right)^{p/2}\right]\sum_{k=0}^{\infty}V(k)^{p/2}\\&\quad\le\beta(a\mu^*)^{p/2}\|d\|_p^p+V(0)^{p/2}.\end{aligned}$$

Since $\varepsilon(k)\ge\varepsilon_0$ for all k,

$$\begin{aligned}\|x\|_p^p\le&\sum_{k=0}^{\infty}\frac{V(k)^{p/2}}{(\lambda_{\min}P_0)^{p/2}}\\\le&\frac{\beta(a\mu^*)^{p/2}}{(\lambda_{\min}P_0)^{p/2}\left[1-(1+\eta)\left(1-\frac{\varepsilon_0}{1+L}\right)^{p/2}\right]}\|d\|_p^p\\&+\frac{V(0)^{p/2}}{(\lambda_{\min}P_0)^{p/2}\left[1-(1+\eta)\left(1-\frac{\varepsilon_0}{1+L}\right)^{p/2}\right]},\end{aligned}\tag{4.11}$$

we conclude that $d\in\ell_p$ implies that $x\in\ell_p$ for any $x(0)\in\mathbb{R}^n$. This concludes the proof of Theorem 1. □

We observe from (4.7) and (4.11) that as $\|d\|_p$ and $x(0)$ become larger, the ε_0 becomes smaller and the ℓ_p gain becomes larger. In order to pursue finite gain ℓ_p stabilization, it is necessary to modify the high gain parameter. We first consider the case $p=\infty$.

Theorem 2. *Consider the system (2.1) satisfying Assumption 2. For $p=\infty$, ℓ_p stabilization with arbitrary initial condition with finite gain and with bias, as formulated in Problem 4, can be achieved by the adaptive-low-gain and high-gain controller,*

$$u=-(1+\rho_f(x))(I+B'P_{\varepsilon(x)}B)^{-1}B'P_{\varepsilon(x)}Ax,\tag{4.12}$$

where $P_{\varepsilon(x)}$ is the solution of (3.1) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (3.6) and $\rho_f(x)$ is determined by (3.9) and (3.10).

Theorem 2 readily yields the following corollary.

Corollary 2. *Consider the system (2.1) satisfying Assumption 2. For $p = \infty$, the $(G_p/G)_{fg}$ as formulated in Problem 2 can be solved by the same adaptive-low-gain and high-gain controller as (4.12).*

Proof of Theorem 2. For simplicity, we denote $P_{\varepsilon(x(k))}$, $P_{\varepsilon_1(x(k))}$ respectively by $P(k)$ and $P_1(k)$ whenever this does not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1}B'P(k)Ax(k)$ and $u(k) = v(k) + \rho_f(k)v(k)$. We have already shown that the controller (4.1) along with (3.6) satisfies $\|v\|_\infty < \Delta$.

Define the Lyapunov function $V(k) = x(k)'P(k)x(k)$ and a set $\mathcal{V}(c) = \{V(x) \le c\}$ where c is given by (3.11). Owing to property 5 of Lemma 1, it is easy to verify that for $x(k) \in \mathcal{V}(c)^c$, the following inequality holds:

$$\varepsilon(k)V(k) \ge 4\varepsilon(k)M_{\varepsilon^*}b\Delta^2 \ge 8\|B'P(k)B\|\Delta^2. \tag{4.13}$$

In the absence of d, we can evaluate the increment of V along the trajectory as

$$\begin{aligned} V(k+1) - V(k) &= x(k+1)'[P(k+1) - P(k)]x(k+1) \\ &\quad - \varepsilon(k)V(k) - 2v(k)'[\sigma(u(k)) - v(k)] \\ &\quad + [\sigma(u(k)) - v(k)]' \\ &\quad \times B'P(k)B[\sigma(u(k)) - v(k)]. \end{aligned}$$

Also, $\|v(k)\| \le \Delta$ implies that $-2v(k)'[\sigma(u(k)) - v(k)] \le 0$ for any $\rho(k) > 0$. Using this property, we find that for $x(k) \in \mathcal{V}(c)^c$,

$$\begin{aligned} V(k+1) - V(k) &\le x(k+1)'[P(k+1) - P(k)]x(k+1) \\ &\quad - \varepsilon(k)V(k) - 2v(k)'[\sigma(u(k)) - v(k)] \\ &\quad + 4\|B'P(k)B\|\Delta^2 \\ &\le x(k+1)'[P(k+1) - P(k)]x(k+1) \\ &\quad - \frac{\varepsilon(k)}{2}V(k). \end{aligned}$$

The last inequality is owing to (4.13). If

$$x(k+1)'[P(k+1) - P(k)]x(k+1) < 0, \tag{4.14}$$

the last inequality implies that $V(k+1) - V(k) < 0$. But we have argued earlier that (4.14) and $V(k+1) - V(k) < 0$ cannot happen simultaneously by our scheduling (3.6). Therefore $x(k+1)'[P(k+1) - P(k)]x(k+1) \ge 0$. From the proof of Theorem 1,

$$x(k+1)'[P(k+1) - P(k)]x(k+1) \le L(k)[V(k) - V(k+1)].$$

Hence, for $x(k) \in \mathcal{V}(c)^c$,

$$V(k+1) - V(k) < -\frac{\varepsilon(k)}{2(1 + L(k))}V(k).$$

The trajectory will enter $\mathcal{V}(c)$ within finite time. However, for $x(k) \in \mathcal{V}(c)$, we have already proved in the proof of Theorem 1 that

$$V(k+1) - V(k) < 0$$

since in $\mathcal{V}(c)$, $\rho(k) = \rho_0(k) = \frac{1}{\|B'P(k)B\|}$. This proves global asymptotic stability of the origin.

We proceed to show ℓ_∞ stability with arbitrary initial conditions with finite gain with bias. In order to do so, we first find an upper bound of $\frac{V(k)}{\lambda_{\min}P(k)}$ in terms of $\|d\|_\infty$ and then conclude ℓ_∞ stability by observing that $\|x\|_\infty \le \sqrt{\|\frac{V}{\lambda_{\min}P}\|_\infty}$. To this end, we note that the case $V(k+1) - V(k) \le 0$ is not interesting since it is equivalent with

$$\frac{V(k+1)}{\lambda_{\min}P(k+1)} - \frac{V(k)}{\lambda_{\min}P(k)} \le 0$$

due to the fact that $V(k+1) \le V(k)$ implies $\lambda_{\min}P(k+1) \ge \lambda_{\min}P(k)$. Therefore, it will not affect the upper bound of $\frac{V(k)}{\lambda_{\min}P(k)}$. In view of this, throughout the remainder of the proof, we only consider $V(k+1) - V(k) > 0$.

Suppose $V(k+1) - V(k) > 0$, scheduling (3.6) implies that $x(k+1)'[P(k+1) - P(k)]x(k+1) \le 0$. By construction, $\|v(k)\| \le \Delta$. We get

$$\begin{aligned} V(k+1) - V(k) &\le -\varepsilon(k)V(k) - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] \\ &\quad + 4\|B'P^*B\|\Delta^2 \\ &\le 4(1 + \|B'P^*B\|)\Delta^2. \end{aligned}$$

Since $c > 4(1 + \|B'P^*B\|)\Delta^2$, we have

$$V(k+1) - V(k) \le c. \tag{4.15}$$

The above inequality holds for any $x(k) \in \mathbb{R}^n$. Since different high-gains are applied in different regions, we have two possible cases:

Case 1. $x(k) \in \mathcal{V}(c)^c$. Then (4.15) implies that $V(k+1) \le 2V(k)$. But this implies that $\varepsilon_1(k) \le \varepsilon(k+1)$ and $P_1(k) \le P(k+1)$. Let $v_i(k)$ and $d_i(k)$ denote the ith element of $v(k)$ and $d(k)$.

If $|d_i(k)| < \rho_f(k)|v_i(k)|$, then

$$-v_i(k)[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - v_i(k)] \le 0.$$

If $|d_i(k)| \le |\rho_f(k)v_i(k)|$, we have

$$\begin{aligned} &-v_i(k)[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - v_i(k)] \\ &\quad = -v_i(k)[\sigma(v_i(k) + \rho_f(k)v_i(k) + d_i(k)) - \sigma(v_i(k))] \\ &\quad \le \frac{|d_i(k)|}{\rho_f(k)} \cdot |2d_i(k)| \\ &\quad = \frac{2d_i(k)^2}{\rho_f(k)}. \end{aligned}$$

In summary, we find that

$$-2v(k)'[\sigma(u(k) + d(k)) - v(k)] \le \frac{4\|d(k)\|^2}{\rho_f(k)}.$$

This yields

$$\begin{aligned} V(k+1) - V(k) &\le -\frac{\varepsilon(k)}{2}V(k) - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] \\ &\le -\frac{\varepsilon(k)}{2}V(k) + 4\frac{\|d(k)\|^2}{\rho_f(k)} \\ &\le -\frac{\varepsilon(k)\lambda_{\min}P(k)}{2}\left(\|x(k)\|^2 - \frac{\|d(k)\|^2}{\rho_1(k)}\right). \end{aligned}$$

Clearly, $V(k+1) - V(k) \ge 0$ requires that

$$\|x(k)\|^2 \le \frac{\|d(k)\|^2}{\rho_1(k)}.$$

Then

$$\begin{aligned} \frac{V(k+1)}{\lambda_{\min}P(k+1)} &\le \frac{2V(k)}{\lambda_{\min}P_1(k)} \le \frac{2\lambda_{\max}P(k)}{\lambda_{\min}P_1(k)}\|x(k)\|^2 \\ &= 2\rho_1\|x(k)\|^2 \le 2\|d(k)\|^2. \end{aligned} \tag{4.16}$$

Case 2: $x(k) \in \mathcal{V}(c)$. We have $\rho(k) = \rho_0(k)$ and hence the same controller as in Theorem 1 is used. In the proof of Theorem 1, the following two properties have already been shown:

(1) if $V(k) \geq abM_{\varepsilon^*}\|d(k)\|^2$, we have $V(k+1) - V(k) \leq 0$;
(2) $V(k+1) - V(k) \leq a\mu^*\|d(k)\|^2$.

We can immediately draw the conclusion that for $V(k+1)-V(k) > 0$ and $x(k) \in \mathcal{V}(c)$,

$$V(k+1) \leq (abM_{\varepsilon^*} + a\mu^*)\|d(k)\|^2.$$

On the other hand, (4.15) and the fact $V(k) \leq c$ imply that $V(k+1) \leq 2c$. But this implies that there exists a λ_1 independent of d such that

$$\frac{V(k+1)}{\lambda_{\min}P(k+1)} \leq \frac{abM_{\varepsilon^*} + a\mu^*}{\lambda_1}\|d\|_\infty^2. \tag{4.17}$$

In summary, whenever $V(k)$ or, equivalently, $\frac{V(k)}{\lambda_{\min}P(k)}$ is increasing, we have either (4.17) or (4.16) holds depending on $x(k) \in \mathcal{V}(c)$ or not. Therefore,

$$\left\|\frac{V}{\lambda_{\min}P}\right\|_\infty \leq \frac{V(0)}{\lambda_{\min}P(0)} + \max\left\{2, \frac{abM_{\varepsilon^*} + a\mu^*}{\lambda_1}\right\}\|d\|_\infty.$$

Using the fact that $\|x\|_\infty^2 \leq \|\frac{V}{\lambda_{\min}P}\|_\infty$, we have

$$\|x\|_\infty \leq \sqrt{\left\|\frac{V}{\lambda_{\min}P}\right\|_\infty} \leq \sqrt{\frac{V(0)}{\lambda_{\min}P(0)}} + \max\left\{\sqrt{2}, \sqrt{\frac{abM_{\varepsilon^*} + a\mu^*}{\lambda_1}}\right\}\|d\|_\infty. \tag{4.18}$$

Note that $\sqrt{\frac{V(0)}{\lambda_{\min}P(0)}}$ is clearly a class $\mathcal{K}$ function of $\|x(0)\|$. The finite gain ℓ_∞ stability of closed-loop system with arbitrary initial condition and bias follows. □

In Theorem 2, we only need to consider the case that $V(x(k))$ is increasing. However, this does not work when the external input d is in ℓ_p with $p \in [1, \infty)$. The decay rate of $V(x(k))$ when $V(x(k))$ is decreasing definitely has an impact on the ℓ_p norm of x. Therefore, we have to consider both cases and obtain bounds on $\|x\|_p$ in terms of $\|d\|_p$. As will be seen in the next theorem, it requires even more complicated high-gain design and involved analysis.

Theorem 3. *Consider the system (2.1) satisfying Assumption 2. For any $p \in [1, \infty)$, the ℓ_p stabilization with arbitrary initial condition with finite gain with bias problem as formulated in Problem 4 can be solved by the adaptive-low-gain and high-gain controller,*

$$u = -(1 + \rho_f(x))(I + B'P_{\varepsilon(x)}B)^{-1}B'P_{\varepsilon(x)}Ax, \tag{4.19}$$

where $P_{\varepsilon(x)}$ is the solution of (3.1) with $\varepsilon = \varepsilon(x)$, $\varepsilon(x)$ is determined adaptively by (3.6) and $\rho_f(x)$ is determined by (3.9), (3.10) with ρ_p sufficiently large.

Theorem 3 also produces as a special case the solution to $(G_p/G)_{fg}$. This is stated in the following corollary.

Corollary 3. *Consider the system (2.1) satisfying Assumption 2. For any $p \in [1, \infty)$, the $(G_p/G)_{fg}$ as formulated in Problem 2 can be solved by the adaptive-low-gain and high-gain controller (4.19).*

Proof of Theorem 3. For simplicity, we denote $\varepsilon(x(k))$, $\varepsilon_1(x(k))$, $\beta(\varepsilon(x(k)))$, $\rho_f(x(k))$ and $\rho_1(x(k))$ by $\varepsilon(k)$, $\varepsilon_1(k)$, $\beta(k)$, $\rho_f(k)$ and $\rho_1(k)$ respectively and denote $P_{\varepsilon(x(k))}$, $P_{\varepsilon_1(x(k))}$, $L_{\varepsilon_1(x(k))}$ respectively by $P(k)$, $P_1(k)$ and $L_1(k)$. This does not cause any notational confusions.

Define $v(k) = -(I + B'P(k)B)^{-1}B'P(k)Ax(k)$ and $u(k) = v(k)+\rho_f(k)v(k)$. We have already shown that $v(k)$ along with (3.6) satisfies $\|v\|_\infty < \Delta$.

Define the Lyapunov function $V(k) = x(k)'P(k)x(k)$ and a set $\mathcal{V}(c) = \{x \mid V(x) \leq c\}$ with c given by (3.11). As in the proof of Theorem 2, for $x \in \mathcal{V}(c)^c$, the following inequality holds:

$$\varepsilon(k)V(k) \geq 4\varepsilon(k)M_{\varepsilon^*}b\Delta^2 \geq 8\|B'P(k)B\|\Delta^2. \tag{4.20}$$

Using exactly the same argument as used in Theorem 2, we conclude the global asymptotic stability of the origin of the closed-loop system.

It remains to prove global ℓ_p stability with finite gain. The proof proceeds in several steps:

Step 1. Define a function

$$\alpha(s) = \frac{s^{p/2}}{(\lambda_{\min}P_s)^{p/2}\left[1 - \left(1 - \frac{\varepsilon_s}{4(1+L_s)}\right)^{p/2}\right]},$$

where ε_s is a function of s as given by

$$\varepsilon_s = \max\left\{r \in [0, \varepsilon^*] \mid s\,\mathrm{trace}(P_r) \leq \frac{\Delta^2}{b}\right\},$$

and P_s is the solution of (3.1) with $\varepsilon = \varepsilon_s$, $L_s = \frac{\mathrm{trace}(P^*)}{\lambda_{\min}P_s}$. Note that if s is strictly increasing, by the property of our scheduling, ε_s is decreasing and hence $\lambda_{\min}P_s$ is decreasing and L_s is increasing. This implies that $\alpha(s)$ is strictly increasing and is a class $\mathcal{K}$ function.

Define

$$\kappa = \frac{(\lambda_{\min}P^*)^{p/2}\left[1 - \left(1 - \frac{\varepsilon^*}{4(1+L^*)}\right)^{p/2}\right]}{(\lambda_{\min}P_{2c})^{p/2}\left[1 - \left(1 - \frac{\varepsilon_{2c}}{4(1+L_{2c})}\right)^{p/2}\right]},$$

where P^* is the solution of (3.1) with $\varepsilon = \varepsilon^*$ and $L^* = \frac{\mathrm{trace}(P^*)}{\lambda_{\min}P^*}$. Since c is given, ε_{2c}, P_{2c}, L_{2c} and κ are fixed constants. Choose $\rho_p > \max\{1 + \kappa, (\lambda_{\min}P^*)^{p/2}\}$. We have $\rho_f(k) \geq 1$ for any $x(k)$.

We can always divide the whole time horizon into a sequence of successive intervals $\{I_i\}_{i\geq 1}$ with $I_i = \overline{k_i, k_{i+1} - 1}$ such that for each I_i, one of the following cases holds:

(1) For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)^c$ and $V(k+1) - V(k) > 0$.
(2) For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)^c$ and $V(k+1) - V(k) \leq 0$.
(3) For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)$ with $k_{i+1} < \infty$.
(4) For any $k \in I_i$, $x(k) \in \mathcal{V}(2c)$ with $k_{i+1} = \infty$.

Step 2. For case 1, since $V(k+1) - V(k) > 0$, the adaptation (3.6) implies that $x(k+1)'[P(k+1) - P(k)]x(k+1) \leq 0$. As in the proof of Theorem 2, we find

$$V(k+1) - V(k) \leq -\frac{\varepsilon(k)\lambda_{\min}P(k)}{2}\left[\|x(k)\|^2 - \frac{\|d(k)\|^2}{\rho_1(k)}\right].$$

Then, $V(k+1) - V(k) > 0$ implies that

$$\|d(k)\|^2 \geq \rho_1(k)\|x(k)\|^2 \geq \|x(k)\|^2 \tag{4.21}$$

since $\rho_1(k) \geq 1$ by construction.

Furthermore, we have already shown that for all $x(k)$, $V(k+1) - V(k) \leq c$. Hence

$$V(k+1) \leq 2V(k).$$

From the definition of $\varepsilon_1(k)$ and $L_1(k)$, this implies that

$$\varepsilon_1(k) \leq \varepsilon(k+1), \qquad L_1(k) \geq L(k+1), \quad \text{and}$$
$$\lambda_{\min}P_1(k) \leq \lambda_{\min}P(k+1). \tag{4.22}$$

Consider specifically $k = k_{i+1} - 1$. We have

$$\begin{aligned}
&\|d(k_{i+1}-1)\|^p - \|x(k_{i+1}-1)\|^p \\
&\geq (\rho_1(k_{i+1}-1)^{p/2} - 1)\|x(k_{i+1}-1)\|^p \\
&\geq \frac{\rho_p \lambda_{\max} P(k_{i+1}-1)^{p/2}\|x(k_{i+1}-1)\|^p}{\lambda_{\min} P_1(k_{i+1}-1)^{p/2}\left[1-\left(1-\frac{\varepsilon_1(k_{i+1}-1)}{4(1+L_1(k_{i+1}-1))}\right)^{p/2}\right]} \\
&\geq \frac{\rho_p V(k_{i+1}-1)^{p/2}}{\lambda_{\min} P_1(k_{i+1}-1)^{p/2}\left[1-\left(1-\frac{\varepsilon(k_{i+1}-1)}{4(1+L(k_{i+1}-1))}\right)^{p/2}\right]} \\
&\geq \frac{(1+\kappa)V(k_{i+1})^{p/2}}{\lambda_{\min} P(k_{i+1})^{p/2}\left[1-\left(1-\frac{\varepsilon(k_{i+1})}{4(1+L(k_{i+1}))}\right)^{p/2}\right]}
\end{aligned}$$

where we use (4.22), $\rho_p > 1+\kappa$ and $V(k_{i+1}-1) > V(k_{i+1})$ in the derivation of the last inequality. We get

$$\|d(k_{i+1}-1)\|^p \geq \|x(k_{i+1}-1)\|^p + (1+\kappa)\alpha(V(k_{i+1})) \tag{4.23}$$

then (4.21) and (4.23) yield

$$\sum_{k=k_i}^{k_{i+1}-1} \|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1} \|d(k)\|^p - (1+\kappa)\alpha(V(k_{i+1})).$$

Step 3. For case 2, the following relationship has been established in the proof of Theorem 1,

$$\begin{aligned}
0 &\leq x(k+1)'[P(k+1) - P(k)x(k+1)] \\
&\leq L(k)(V(k) - V(k+1))
\end{aligned}$$

where $L(k) = \frac{\mathrm{trace}(P^*)}{\lambda_{\min} P(k)}$. Therefore,

$$\begin{aligned}
V(k+1) - V(k) &\leq -\frac{\varepsilon(k)}{2(1+L(k))}V(k) \\
&\quad + \frac{\varepsilon(k)\lambda_{\min}P(k)}{\rho_1(k)(1+L(k))}\|d(k)\|^2 \\
&\leq -\frac{\varepsilon(k)}{2(1+L(k))}V(k) + \frac{\lambda_{\min}P(k)}{\rho_1(k)}\|d(k)\|^2,
\end{aligned}$$

and hence

$$V(k+1) \leq \left[1 - \frac{\varepsilon(k)}{2(1+L(k))}\right]V(k) + \frac{\lambda_{\min}P(k)}{\rho_1(k)}\|d(k)\|^2.$$

Since $V(k)$ is decreasing, we have $\lambda_{\min}P(k+1) \geq \lambda_{\min}P(k)$ and

$$\frac{V(k+1)}{\lambda_{\min}P(k+1)} \leq \left[1 - \frac{\varepsilon(k)}{2(1+L(k))}\right]\frac{V(k)}{\lambda_{\min}P(k)} + \frac{1}{\rho_1(k)}\|d(k)\|^2.$$

By definition of $\beta(k)$,

$$\begin{aligned}
\left(\frac{V(k+1)}{\lambda_{\min}P(k+1)}\right)^{p/2} &\leq \left[1 - \frac{\varepsilon(k)}{4(1+L(k))}\right]^{p/2}\left(\frac{V(k)}{\lambda_{\min}P(k)}\right)^{p/2} \\
&\quad + \beta(k)\frac{\|d(k)\|^p}{\rho_1(k)^{p/2}}.
\end{aligned}$$

Using standard comparison principle, we get for $k \geq k_i$,

$$\begin{aligned}
\left(\frac{V(k)}{\lambda_{\min}P(k)}\right)^{p/2} &\leq \prod_{j=k_i}^{k}\left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}\left(\frac{V(k_i)}{\lambda_{\min}P(k_i)}\right)^{p/2} \\
&\quad + \sum_{j=k_i}^{k-1}\left(\prod_{s=j}^{k-1}\left[1 - \frac{\varepsilon(s)}{4(1+L(s))}\right]^{p/2}\right) \\
&\quad \times \frac{\beta(j)}{\rho_1(j)^{p/2}}\|d(j)\|^p.
\end{aligned}$$

Since $V(k)$ is decreasing, $[1 - \frac{\varepsilon(k)}{4(1+L(k))}]^{p/2}$ is decreasing. Hence,

$$\begin{aligned}
\left(\frac{V(k)}{\lambda_{\min}P(k)}\right)^{p/2} &\leq \left\{\left[1 - \frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}\right\}^{k-k_i}\left(\frac{V(k_i)}{\lambda_{\min}P(k_i)}\right)^{p/2} \\
&\quad + \sum_{j=k_i}^{k-1}\left\{\left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}\right\}^{k-1-j} \\
&\quad \times \frac{\beta(j)}{\rho_1(j)^{p/2}}\|d(j)\|^p.
\end{aligned}$$

We have

$$\begin{aligned}
\sum_{k=k_i}^{k_{i+1}-1}\left(\frac{V(k)}{\lambda_{\min}P(k)}\right)^{p/2} &\leq \frac{1}{1 - \left[1 - \frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}}\left(\frac{V(k_i)}{\lambda_{\min}P(k_i)}\right)^{p/2} \\
&\quad + \sum_{j=k_i}^{k_{i+1}-2}\frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}}\frac{\|d(j)\|^p}{\rho_1(j)^{p/2}}.
\end{aligned}$$

By definition, for any $x(k)$

$$\varepsilon_1(k) \leq \varepsilon(k) \quad \text{and} \quad L_1(k) \geq L(k),$$

and from (3.10)

$$\rho_1(j)^{p/2} \geq \frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon_1(j)}{4(1+L_1(j))}\right]^{p/2}} \geq \frac{\beta(j)}{1 - \left[1 - \frac{\varepsilon(j)}{4(1+L(j))}\right]^{p/2}}.$$

We conclude that

$$\begin{aligned}
\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p &\leq \sum_{k=k_i}^{k_{i+1}-2}\|d(j)\|^p + \alpha(V(k_i)) \\
&\leq \sum_{k=k_i}^{k_{i+1}-1}\|d(j)\|^p + \alpha(V(k_i)).
\end{aligned}$$

Note that $\alpha(V(k_i))$ is increasing. Therefore $\alpha(V(k_i)) \geq \alpha(V(k_{i+1}))$. We can rewrite the above inequality as

$$\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p \leq \sum_{k=k_i}^{k_{i+1}-1}\|d(j)\|^p + (1+\kappa)\alpha(V(k_i)) - \kappa\alpha(V(k_{i+1})).$$

Step 4. For case 3 and 4, if $x(k) \in \mathcal{V}(c)$, from (4.8) and (4.9), we have

$$V(k+1) - V(k) \leq -\frac{\varepsilon(k)}{1+L(k)}V(k) + a\mu^*\|d(k)\|^2.$$

If $x(k) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$ and $V(k+1) - V(k) > 0$, then $x(k+1)'[P(k+1) - P(k)]x(k+1) \leq 0$, we have

$$\begin{aligned}
V(k+1) - V(k) &\leq -\frac{\varepsilon(k)}{2}V(k) - 2v(k)'[\sigma(u(k) + d(k)) - v(k)] \\
&\leq -\frac{\varepsilon(k)}{2}V(k) + 4\frac{\|d(k)\|^2}{\rho_f(k)} \\
&\leq -\frac{\varepsilon(k)}{2}V(k) + 4\|d(k)\|^2.
\end{aligned}$$

If $x(k) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$ and $V(k+1) - V(k) \leq 0$, then $x(k+1)'[P(k+1) - P(k)]x(k+1) \leq L(k)(V(k) - V(k+1))$. We have

$$\begin{aligned}
V(k+1) - V(k) &\leq -\frac{\varepsilon(k)}{2(1+L(k))}V(k) + 4\frac{\|d(k)\|^2}{\rho_f(k)(1+L(k))} \\
&\leq -\frac{\varepsilon(k)}{2(1+L(k))}V(k) + 4\|d(k)\|^2.
\end{aligned}$$

Hence there exists a $\zeta = \max\{4, a\mu^*\}$ such that for all $x(k) \in \mathcal{V}(2c)$, we have

$$V(k+1) - V(k) \le -\frac{\varepsilon(k)}{2(1+L(k))}V(k) + \zeta\|d(k)\|^2.$$

Note that our adaptation (3.6) and the fact that $V(x) \le 2c$ for $k = k_i, \ldots, k_{i+1}-1$ imply that for $k = k_i, \ldots, k_{i+1}-1$, $\varepsilon(k) \ge \varepsilon_{2c}$ and hence

$$-\frac{\varepsilon(k)}{2(1+L(k))} \le -\frac{\varepsilon_{2c}}{2(1+L_{2c})}, \qquad \lambda_{\min}P(k) \ge \lambda_{\min}P_{2c}.$$

Choose η_{2c} such that

$$\left[1 - \frac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2} \le (1+\eta_{2c})\left[1 - \frac{\varepsilon_{2c}}{2(1+L_{2c})}\right]^{p/2} < 1.$$

Applying Lemma 3, there exists a β_{2c} independent of d and k such that

$$V(k+1)^{p/2} \le \left[1 - \frac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2} V(k)^{p/2} + \beta_{2c}\zeta^{p/2}\|d(k)\|^p.$$

Using the same comparison principle as used in case 2, we can find a constant γ_1 dependent on $\varepsilon_{2c}, L_{2c}, \beta_{2c}$ and ζ such that

$$\begin{aligned}
\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p &\le \sum_{k=k_i}^{k_{i+1}-1}\frac{V(k)^{p/2}}{(\lambda_{\min}P_{2c})^{p/2}} \\
&\le \gamma_1\sum_{k=k_i}^{k_{i+1}-2}\|d(k)\|^p \\
&\quad + \frac{V(k_i)^{p/2}}{(\lambda_{\min}P_{2c})^{p/2}\left\{1 - \left[1 - \frac{\varepsilon_{2c}}{4(1+L_{2c})}\right]^{p/2}\right\}} \\
&\le \gamma_1\sum_{k=k_i}^{k_{i+1}-2}\|d(k)\|^p \\
&\quad + \frac{\kappa V(k_i)^{p/2}}{(\lambda_{\min}P(k_i))^{p/2}\left\{1 - \left[1 - \frac{\varepsilon(k_i)}{4(1+L(k_i))}\right]^{p/2}\right\}} \\
&\le \gamma_1\sum_{k=k_i}^{k_{i+1}-2}\|d(k)\|^p + \kappa\alpha(V(k_i)).
\end{aligned}$$

For case 3 where $k_{i+1} < \infty$, consider specifically $k = k_{i+1}-1$. Since the states are leaving $\mathcal{V}(2c)$, we have $V(k_{i+1}) - V(k_{i+1}-1) > 0$. Moreover, we have argued that the increment of $V(k)$ for any $x(k)$ is at most c. This implies that $x(k_{i+1}-1) \in \mathcal{V}(c)^c \cap \mathcal{V}(2c)$. Following the same argument as used in case 1, we have

$$\|d(k_{i+1}-1)\|^p \ge \|x(k_{i+1}-1)\|^p + (1+\kappa)\alpha(V(k_{i+1})).$$

Finally, we conclude for $k \in \overline{k_i, k_{i+1}-1}$,

$$\begin{aligned}
\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p &\le \gamma_1\sum_{k=k_i}^{k_{i+1}-1}\|d(k)\|^p \\
&\quad + \kappa\alpha(V(k_i)) - (1+\kappa)\alpha(V(k_{i+1})).
\end{aligned}$$

For case 4 where $k_{i+1} = \infty$, we only have

$$\sum_{k=k_i}^{k_{i+1}}\|x(k)\|^p \le \gamma_1\sum_{k=k_i}^{k_{i+1}}\|d(k)\|^p + \kappa\alpha(V(k_i)).$$

Step 5. In summary of previous steps, we find the following results:

- if I_i belongs to case 1,
$$\sum_{k_i}^{k_{i+1}-1}\|x(k)\|^p \le \sum_{k=k_i}^{k_{i+1}-1}\|d(k)\|^p - (1+\kappa)\alpha(V(k_{i+1})).$$
- if I_i belongs to case 2,
$$\begin{aligned}\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p &\le \sum_{k=k_i}^{k_{i+1}-1}\|d(j)\|^p \\ &\quad + (1+\kappa)\alpha(V(k_i)) - \kappa\alpha(V(k_{i+1})).\end{aligned}$$
- if I_i belongs to case 3,
$$\begin{aligned}\sum_{k=k_i}^{k_{i+1}-1}\|x(k)\|^p &\le \gamma_1\sum_{k=k_i}^{k_{i+1}-1}\|d(k)\|^p \\ &\quad + \kappa\alpha(V(k_i)) - (1+\kappa)\alpha(V(k_{i+1})).\end{aligned}$$
- if I_i belongs to case 4,
$$\sum_{k=k_i}^{k_{i+1}}\|x(k)\|^p \le \gamma_1\sum_{k=k_i}^{k_{i+1}}\|d(k)\|^p + \kappa\alpha(V(k_i)).$$

Note that if I_i belongs to cases 1, 3 and 4, we have either $i = 1$ or I_{i-1} belongs to cases 1, 2 or 3. Then the positive term $\kappa\alpha(V(k_i))$ of I_i can always be canceled by the corresponding negative term of I_{i-1} for $i > 1$.

Similarly, if I_i belongs to case 2, we have either $i = 1$ or I_{i-1} belongs to case 1 or 3. The positive term $(1+\kappa)\alpha(V(k_i))$ can also be canceled by the negative term of I_{i-1} for $i > 1$.

In conclusion, we find that for any $x(0)$ and k,

$$\sum_{k=0}^{k}\|x(k)\|^p \le \max\{1, \gamma_1\}\sum_{k=0}^{k}\|d(k)\|^p + (1+\kappa)\alpha(V(0)).$$

This completes the proof. □

5. Conclusions

It is shown in this paper that (G_p/G) and $(G_p/G)_{fg}$ problems for discrete-time linear systems subject to actuator saturation are solvable if and only if the given linear system is stabilizable and it has all its poles within the unit disc, i.e. if it is ANCBC. We also develop here an adaptive-low-gain and high-gain controller design methodology by using a parametric Lyapunov equation. By utilizing the developed methodology, one can explicitly construct the required state feedback controllers that solve the (G_p/G) and $(G_p/G)_{fg}$ problems whenever they are solvable.

Appendix

We show in this section that for system (2.2) if a feedback controller of the form $u = B'f(x_u)$ achieves (G/G_p) and/or $(G/G_p)_{f\cdot g}$ for the unstable dynamics x_u, it also achieves (G/G_p) and/or $(G/G_p)_{f\cdot g}$ for the overall system.

Let us consider the unstable part of the input-additive case.

$$x_u^+ = Ax_u + B_u\sigma(u+d).$$

Assume we have a feedback $u = B_u'f(x_u)$ such that $x_u \in \ell_p$ and, if possible, with finite gain:

$$\|x_u\|_{\ell_p} \le c_1\|d\|_{\ell_p}.$$

Note that we impose a bit of special structure on the feedback. Namely $u = B_u'f(x_u)$ instead of $u = f(x_u)$ but all our standard controllers satisfy this property which is easily seen if we recall that:

$$-(I+B'P_\varepsilon B)^{-1}B'P_\varepsilon A = B'P_\varepsilon(I+BB'P_\varepsilon)^{-1}A.$$

If we achieve (G/G_p) for the unstable dynamics then it is easily verified that we must have that

$$B_u\sigma(B_u'f(x_u)+d)\in\ell_p$$

while achieving $(G/G_p)_{f\cdot g}$ for the unstable dynamics implies:

$$\|B_u\sigma(B_u'f(x_u)+d)\|_{\ell_p}\le c_2\|d\|_{\ell_p}. \tag{A.1}$$

Now in order to incorporate the stable dynamics we want to establish that:

$$\sigma(B_u'f(x_u)+d)\in\ell_p$$

and ideally with a finite gain:

$$\|\sigma(B_u'f(x_u)+d)\|_{\ell_p}\le c_3\|d\|_{\ell_p}.$$

This implies that for stable dynamics, we shall have

$$\|x_s\|_{\ell_p}\le\gamma\|\sigma(B_u'f(x_u)+d)\|_{\ell_p}\le c_3\gamma\|d\|_{\ell_p}$$

where γ is ℓ_p gain of the pair (A_s,B_s).

We first note that

$$B_u\sigma(B_u'f(x_u)+d)=B_u\sigma(B_u'f(x_u))+B_ud_1$$

with $\|d_1\|_{\ell_p}\le\|d\|_{\ell_p}$. But this implies that

$$\|B_u\sigma(B_u'f(x_u))\|_{\ell_p}\le\|B_u\sigma(B_u'f(x_u)+d)\|_{\ell_p}+\|B_u\|\,\|d\|_{\ell_p}.$$

In other words it is sufficient to prove that

$$\|\sigma(B_u'f(x_u))\|_{\ell_p}\le c_4\|B_u\sigma(B_u'f(x_u))\|_{\ell_p} \tag{A.2}$$

to obtain that:

$$\begin{aligned}\|\sigma(B_u'f(x_u)+d)\|_{\ell_p} &\le \|\sigma(B_u'f(x_u))\|_{\ell_p}+\|d\|_{\ell_p}\\ &\le c_4\|B_u\sigma(B_u'f(x_u))\|_{\ell_p}+\|d\|_{\ell_p}\\ &\le c_4\|B_u\sigma(B_u'f(x_u)+d)\|_{\ell_p}\\ &\quad+(1+c_4\|B_u\|)\|d\|_{\ell_p}\\ &\le (c_4c_2+1+c_4\|B_u\|)\|d\|_{\ell_p}\end{aligned}$$

where we used (A.1).

Remains to verify (A.2) which is implied by the following static inequality:

$$\|\sigma(B_u'v)\|_p\le c_4\|B_u\sigma(B_u'v)\|_p. \tag{A.3}$$

Since this is a static finite-dimensional problem and all finite-dimensional norms are equivalent, it suffices to prove (A.3) for $p=2$.

Note that we can find a matrix S such that:

$$B_u=S\begin{pmatrix}B_{u1}\\0\end{pmatrix}$$

with B_{u1} surjective. Next, we note that it is sufficient to prove that:

$$\|\sigma(B_{u1}'w)\|_2\le c_5\|B_{u1}\sigma(B_{u1}'w)\|_2 \tag{A.4}$$

for some suitably chosen c_5 since for $w=Sv$ we get:

$$\begin{aligned}\|\sigma(B_u'v)\|_2 &\le c_5\|B_{u1}\sigma(B_u'v)\|_2\\ &\le \frac{c_5}{\sigma_{\min}(S)}\left\|S\begin{pmatrix}B_{u1}\\0\end{pmatrix}\sigma(B_u'v)\right\|_2\\ &\le \frac{c_5}{\sigma_{\min}(S)}\|B_u\sigma(B_u'v)\|_2\end{aligned}$$

which yields (A.3) for suitable chosen c_4. It remains to show (A.4). We consider two cases. If $B_{u1}'w$ saturates at least one channel then

$$\begin{aligned}\|B_{u1}\sigma(B_{u1}'w)\|_2 &\ge \langle B_{u1}'w_n,\sigma(B_{u1}'w)\rangle\\ &\ge \|B_{u1}'w_n\|_\infty\\ &\ge \frac{1}{\sqrt{m}}\sigma_{\min}(B_{u1}')\end{aligned}$$

where $w_n=\frac{w}{\|w\|}$ is the normalized vector of w.

In that case:

$$\begin{aligned}\|\sigma(B_{u1}'w)\|_2 &\le \sqrt{m}\,\|\sigma(B_{u1}'w)\|_\infty\\ &=\sqrt{m}\\ &\le \frac{m}{\sigma_{\min}(B_{u1}')}\|B_{u1}\sigma(B_{u1}'w)\|_2.\end{aligned}$$

On the other hand without saturation:

$$\begin{aligned}\|B_{u1}'w\|_2 &\le \|B_{u1}'(B_{u1}'B_{u1}')^{-1}B_{u1}B_{u1}'w\|_2\\ &\le \|B_{u1}'(B_{u1}'B_{u1}')^{-1}\|_2\ \|B_{u1}B_{u1}'w\|_2.\end{aligned}$$

Combining the two cases with and without saturation yields (A.4) for suitable chosen c_5, i.e.

$$c_5\ge\max\left\{\frac{m}{\sigma_{\min}(B_{u1}')},\ \|B_{u1}'(B_{u1}'B_{u1}')^{-1}\|_2\right\}.$$

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