arXiv:1202.5407v1 [math.OC] 24 Feb 2012

Stabilization of an arbitrary profile for an ensemble of half-spin systems ⋆

Karine Beauchard [a], Paulo Sérgio Pereira da Silva [b], Pierre Rouchon [c]

[a] *CMLA, ENS Cachan, CNRS, UniverSud, 61, avenue du Président Wilson, F-94230 Cachan, FRANCE.*

[b] *University of São Paulo, Escola Politécnica – PTC — Av. Luciano Gualberto trav. 03, 158, 05508-900 – São Paulo – SP BRAZIL*

[c] *Mines ParisTech, Centre Automatique et Systèmes, Unité Mathématiques et Systèmes, 60 Bd Saint-Michel, 75272 Paris cedex 06, FRANCE*

Abstract

We consider the feedback stabilization of a variable profile for an ensemble of non interacting half spins described by the Bloch equations. We propose an explicit feedback law that stabilizes asymptotically the system around a given arbitrary target profile. The convergence proof is done when the target profile is entirely in the south hemisphere or in the north hemisphere of the Bloch sphere. The convergence holds for initial conditions in a H^1 neighborhood of this target profile. This convergence is shown for the weak H^1 topology. The proof relies on an adaptation of the LaSalle invariance principle to infinite dimensional systems. Numerical simulations illustrate the efficiency of these feedback laws, even for initial conditions far from the target profile.

Key words: nonlinear systems, Lyapunov stabilization, LaSalle invariance, quantum systems, Bloch equations, ensemble controllability, infinite dimensional system.

1 Introduction

Ensemble controllability as introduced in Li and Khaneja (2009) is an interesting control theoretic notion well adapted to nuclear magnetic resonance (NMR) systems (see, e.g., Li and Khaneja (2006) and the reference herein). In Beauchard et al. (2010) some controllability issues of such NMR systems are investigated using open-loop controls involving Dirac-combs. In Beauchard et al. (2011) such open-loop Dirac-combs are combined with Lyapunov stabilizing feedback to ensure closed-loop convergence towards a target profile that is one of the two steady-states, the south and north poles of the Bloch sphere. In this note, we extend this Lyapunov design to arbitrary target profiles and prove its local convergence for weak H^1 topology when the target profile lies entirely in the south hemisphere or in the north hemisphere.

We consider an ensemble of non interacting half-spins in a static field $(0,0,B_0)^t$ in $\mathbb{R}^3$, subject to a transverse radio frequency field $(\tilde{u}_1(t),\tilde{u}_2(t),0)^t$ in $\mathbb{R}^3$ (the control input). The ensemble of half-spins is described by the magnetization vector $M\in\mathbb{R}^3$ depending on time t but also on the Larmor frequency $\omega=-\gamma B_0$ (γ is the gyromagnetic ratio). It obeys to the Bloch equation:

$$\frac{\partial M}{\partial t}(t,\omega)=(\tilde{u}_1(t)e_1+\tilde{u}_2(t)e_2+\omega e_3)\wedge M(t,\omega), \quad (1)$$

where $-\infty<\omega_*<\omega^*<+\infty$, $\omega\in(\omega_*,\omega^*)$, (e_1,e_2,e_3) is the canonical basis of $\mathbb{R}^3$, $\wedge$ denotes the wedge product on $\mathbb{R}^3$. The equation (1) is an infinite dimensional bilinear control system. The state is the ω-profile M, where, for every $\omega\in(\omega_*,\omega^*)$, $M(t,\omega)\in\mathbb{S}^2$ (the unit sphere of $\mathbb{R}^3$). The two control inputs $\tilde{u}_1$ and $\tilde{u}_2$ are real valued.

We propose here a first answer to the local stabilization of an arbitrary profile: given an arbitrary target profile $M_f:(\omega_*,\omega^*)\to\mathbb{S}^2$, define an explicit control law

⋆ Corresponding author P. Rouchon Tel. +33 1 40 51 91 15. Fax +33 1 40 51 91 65.

Email addresses: Karine.Beauchard@cmla.ens-cachan.fr (Karine Beauchard), paulo@lac.usp.br (Paulo Sérgio Pereira da Silva), pierre.rouchon@mines-paristech.fr (Pierre Rouchon).

$(\tilde{u}_1(t,M), \tilde{u}_2(t,M))$, a neighborhood U of M_f (in some space of functions to be determined), a diverging sequence of times $(t_n)_{n\in\mathbb{N}}$, such that, for every initial condition $M^0 \in U$, the solution of the closed loop system is uniquely defined and satisfies

$$\lim_{n\to+\infty} \|M(t_n,.) - M_f(.)\|_{L^\infty(\omega_*,\omega^*)} = 0.$$

In this note, the Lyapunov feedback proposed in Beauchard et al. (2011) is adapted to provide a constructive answer to this question. Section 2 is devoted to control design and closed-loop simulations. In section 3 we state and prove the main convergence result, theorem 1.

2 Lyapunov H^1 approach

2.1 Some preliminaries

Let us recall the concept of a solution for (1) when the control input u contains Dirac distributions. When $\tilde{u}_1, \tilde{u}_2 \in L^1_{loc}(\mathbb{R})$, then, for every initial condition $M_0 \in L^2((\omega_*,\omega^*),\mathbb{R}^3)$, the equation (1) has a unique weak solution $M \in C^0([0,+\infty), L^2((\omega_*,\omega^*),\mathbb{R}^3))$. Denote by $\delta(t-a)$ the Dirac distribution located at $t=a$. When $\tilde{u}_1 = \alpha\delta(t-a) + u_1^\sharp$ and $\tilde{u}_2 = u_2^\sharp$ where $u_j^\sharp \in L^1_{loc}(\mathbb{R})$, $\alpha > 0$ and $a \in (0,+\infty)$, then the solution is the classical solution on $[0,a)$ and $(a,+\infty)$, it is discontinuous at the time $t=a$, with an explicit discontinuity given by an instantaneous rotation of angle α around the axis $\mathbb{R}e_1$

$$M(a^+,\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{pmatrix} M(a^-,\omega).$$

The symbol $\|.\|$ (resp. $\langle .,. \rangle$) denotes the Euclidian norm (resp. scalar product) on $\mathbb{R}^3$ and the associated operator norm on $\mathcal{M}_3(\mathbb{R})$.

2.2 Transformation into a driftless system

As in Beauchard et al. (2011) we consider a control with an "impulse-train" structure

$$\tilde{u}_1 = u_1 + \sum_{k=1}^{+\infty} \pi\, \delta(t-kT), \quad \tilde{u}_2 = (-1)^{\epsilon(t)} u_2 \qquad (2)$$

where $\epsilon(t) := E(t/T)$, for some period $T>0$ and $E(\gamma)$ denotes the integer part of the real number γ. The new controls u_1, u_2 belong to $L^1_{loc}(\mathbb{R})$. Considering the change of variable

$$M_1(t,\omega) := P(t)M(t,\omega) \text{ where } P(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \epsilon(t) & 0 \\ 0 & 0 & \epsilon(t) \end{pmatrix} \qquad (3)$$

one gets the following dynamics

$$\frac{\partial M_1}{\partial t}(t,\omega) = [u_1(t)e_1 + u_2(t)e_2 + \epsilon(t)\omega] \wedge M_1(t,\omega). \quad (4)$$

The application of impulses at $t=kT$, by changing the sense of rotation of the null input solution, is expected to reduce the dispersion in the closed loop system. Since $M(t,\omega) = M_1(t,\omega)$ for every $t \in [2kT,(2k+1)T]$, any convergence result on $M_1(t)$ when $t\to+\infty$ provides a convergence result on M.

The first step of the control design consists in putting the system (4) in driftless form. The new function

$$M_2(t,\omega) := \exp[\sigma(t)\omega S] M_1(t,\omega)$$

where

$$\sigma(t) := \int_0^t \epsilon(s)ds, \quad S := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (5)$$

solves

$$\frac{\partial M_2}{\partial t}(t,\omega) = \sum_{i=1}^{2} u_i(t)\Big[\exp(\sigma(t)\omega S)e_i\Big] \wedge M_2(t,\omega). \quad (6)$$

Since $\sigma(2kT) = 0, \forall k \in \mathbb{N}$, any convergence on $M_2(t)$ when $t\to+\infty$ provides a convergence on $M_1(2kT)$ when $k\to+\infty$.

2.3 Transformation of the target profile

The second step of the control design consists in transforming a convergence to a variable profile M_f into a convergence to the constant profile $-e_3$, for which we developed tools in the previous work Beauchard et al. (2011). It relies on the following proposition.

Proposition 1 *There exists $C>0$ such that, for all $M_f \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$, there exists $R \in H^1((\omega_*,\omega^*), SO_3(\mathbb{R}))$ satisfying*

$$R(\omega)M_f(\omega) = -e_3, \quad \forall\omega \in [\omega_*,\omega^*], \qquad (7)$$

$$\|R\|_{H^1} \leqslant C\|M_f\|_{H^1}. \qquad (8)$$

Proof: Let $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ and set $f(\omega) := M_f'(\omega) \wedge M_f(\omega)$. Denote by $A(\omega)$ the skew-symmetric operator defined by $\mathbb{R}^3 \ni M \mapsto f(\omega) \wedge M \in \mathbb{R}^3$. Consider the Cauchy problem

$$\frac{d}{d\omega} R = RA(\omega) \text{ on } [\omega_*, \omega^*] \text{ with } R(\omega_*) = R_*$$

where R_* is any rotation sending $M_f(\omega_*)$ to $-e_3$: $R_* M_f(\omega_*) = -e_3$. Since $\omega \mapsto A(\omega)$ is L^2 the solution R is well defined, unique and belongs to $H^1((\omega_*, \omega^*), \mathbb{S}^2)$. Direct computations show that $\frac{d}{d\omega}(RM_f) = 0$. Thus $R(\omega)M_f(\omega) \equiv -e_3$. Moreover, $\|R(\omega)\| = 1$ and $\|R'(\omega)\| = \|A(\omega)\| = \|M_f'(\omega)\|$ for all $\omega \in [\omega_*, \omega^*]$, which proves (8). $\square$.

Let us consider a target profile $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$. Take $R \in H^1((\omega_*, \omega^*), SO_3(\mathbb{R}))$ given by the above proposition. To any solution M_2 of (6), we associate the function

$$N(t, \omega) := R(\omega) M_2(t, \omega), \forall \omega \in [-\omega_*, \omega^*]. \quad (9)$$

This function solves the equation

$$\frac{\partial N}{\partial t}(t, \omega) = \sum_{i=1}^{2} u_i(t) \Big[F(t, \omega) e_i \Big] \wedge N(t, \omega) \quad (10)$$

where

$$F(t, \omega) := R(\omega) \exp(\sigma(t) \omega S). \quad (11)$$

The convergence of $N(t, \omega)$ to $-e_3$ as $t \to +\infty$ is equivalent to the convergence of $M_2(t, \omega)$ to $M_f(\omega)$ as $t \to +\infty$.

2.4 Lyapunov feedback

Let us consider the following Lyapunov-like functional

$$\mathcal{L}(N) := \frac{\|N + e_3\|_{H^1}^2}{2} = \int_{\omega_*}^{\omega^*} \Big(\frac{1}{2} \Big\| \frac{\partial N}{\partial \omega} \Big\|^2 + 1 + \langle N, e_3 \rangle \Big) d\omega. \quad (12)$$

The function $\mathcal{L}$ is defined for any $N \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ and takes its minimal value on this space at the point $N = -e_3$ with $\mathcal{L}(-e_3) = 0$. For any solution of (10), some computations show that

$$\frac{d\mathcal{L}}{dt}[N(t)] = \sum_{i=1}^{2} u_i(t) H_i[t, N(t)]$$

where, for $i = 1, 2$ one has

$$H_i[t, N] := \int_{\omega_*}^{\omega^*} \Bigg[\Big\langle \frac{dN}{d\omega}(\omega), \Big(\frac{\partial F}{\partial \omega}(t, \omega) e_i \Big) \wedge N(\omega) \Big\rangle + \Big\langle e_3, \Big(F(t, \omega) e_1 \Big) \wedge N(\omega) \Big\rangle \Bigg] d\omega.$$

Hence, with the feedback laws

$$u_i(t, N) := -H_i[t, N], \forall i \in \{1, 2\}, \quad (13)$$

it follows that

$$\frac{d\mathcal{L}}{dt}[N(t)] = -u_1(t, N)^2 - u_2(t, N)^2 \leqslant 0. \quad (14)$$

As in Beauchard et al. (2011), we have the following result.

Proposition 2 *For every initial condition $N_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$, the closed loop system (10), (13) has a unique solution $N \in C^1\left([0, \infty), H^1\left((\omega_*, \omega^*), \mathbb{R}^3\right)\right)$ such that $N(0) = N_0$.*

2.5 Closed-loop simulations

We assume here $\omega_* = 0, \omega^* = 1$ and we solve numerically the T-periodic system (1) with the feedback law $(\tilde{u}_1, \tilde{u}_2)$ given by (2), (13). The closed-loop simulation is performed for $t \in [0, T_f]$, $T_f = 20T$ and $T = 2\pi/(\omega^* - \omega_*)$. The ω-profile $[\omega_*, \omega^*] \ni \omega \mapsto (x(t, \omega), y(t, \omega), z(t, \omega))$ is discretized $\{1, \ldots, N+1\} \ni k \mapsto (x_k(t), y_k(t), z_k(t))$ with a regular mesh of step $\epsilon_N = \frac{\omega^* - \omega_*}{N}$ with $N = 100$. In other words, one has a set of discrete values $\{\omega_i, i = 1, \ldots, N+1\}$, where $\omega_i = \omega_* + (i-1)\epsilon_N$.

We have checked that the closed-loop simulations are almost identical for $N = 100$ and $N = 200$. In the feedback law (16), the integral versus ω is computed assuming that (x, y, z) and (x', y', z') are constant over $](k - \frac{1}{2})\epsilon_N, (k + \frac{1}{2})\epsilon_N[$, their values being (x_k, y_k, z_k) and $\left(\frac{x_{k+1} - x_{k-1}}{2\epsilon_N}, \frac{y_{k+1} - y_{k-1}}{2\epsilon_N}, \frac{z_{k+1} - z_{k-1}}{2\epsilon_N} \right)$. The obtained differential system is of dimension $3(N+1)$. It is integrated via an explicit Euler scheme with a step size $h = T/1000$. We have tested that $h = T/2000$ yields almost the same numerical solution at $t = T_f = 20T$. After each time-step the new values of (x_k, y_k, z_k) are normalized to remain in $\mathbb{S}^2$. The initial ω-profile $M_0(\omega)$ of $(x, y, z) \in \mathbb{S}^2$ is given by $x_0 = 0, y_0 = -\sqrt{1 - z_0^2}$, where $z_0 = -\cos(\frac{\pi}{8}) + 0.05 \left(1 - \cos(\frac{\pi}{8}) \cos(\omega \frac{\pi}{2})\right)$. The desired final profile $M_f(\omega)$ is given by $x_f = -\sqrt{1 - z_f^2}, y_f = 0$, where $z_f = -\cos(\frac{\pi}{16}) + 0.1 \left(1 - \cos(\frac{\pi}{16}) \sin(\omega \frac{\pi}{4})\right)$.

The map $R(\omega)$ is constructed for the discrete set $\{\omega_i, i = 1, \ldots, N+1\}$, in the following way. For

$i = 1$, one takes $r_3(\omega_1) = M_0(\omega_1)$. Now choose a vector θ among the vectors of the canonical basis in a way that $\langle \theta, M_f(\omega_1)\rangle$ is the minimum value. Construct $r_2(\omega_1) = \frac{1}{\|\theta \wedge r_3(\omega_1)\|}(\theta \wedge r_3(\omega_1))$. Then one may take $r_1(\omega_2) = r_2(\omega_2) \wedge r_3(\omega_2)$. Now, for $i = 2, 3, \ldots, N+1$ one chooses $r_3(\omega_i) = M_0(\omega_i)$, $\theta = -r_1(\omega_{i-1})$, $r_2(\omega_i) = \frac{1}{\|\theta \wedge r_3(\omega_i)\|}(\theta \wedge r_3(\omega_i))$ and $r_1(\omega_i) = r_2(\omega_i) \wedge r_3(\omega_i)$, and so on. The orthogonal matrix $\bar{R}(\omega)$ formed by the column vectors r_1, r_2, r_3 is then transposed to obtain $R(\omega)$.

Figures 1 and 2 summarize the main convergence issues for these choices of initial profile M_0 and of the desired final profile M_f. The convergence speed is rapid at the beginning and tends to decrease at the end. We start with $\mathcal{L}(0) \approx 0.1929$. We get $\mathcal{L}(20T) \approx 0.0032$. This numerically observed convergence is confirmed by Theorem 1 here below.

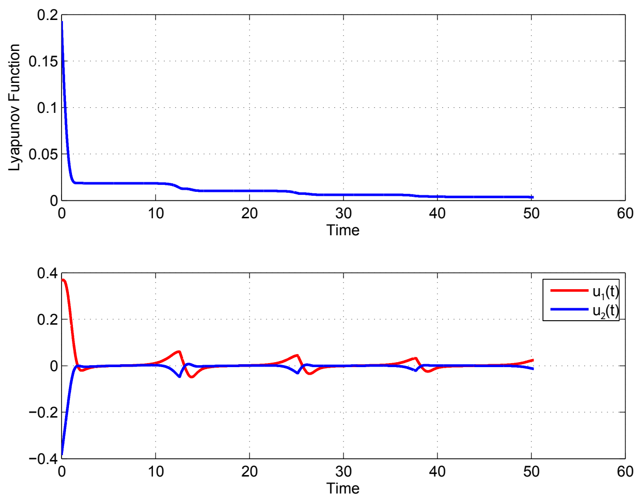

Fig. 1. Lyapunov function $\mathcal{L}(t)$ defined by (12) and the closed-loop control (u_1, u_2) defined by (13)

3 Main Result

3.1 Local stabilization

Theorem 1 *For every $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with*

$$\langle M_f(\omega), e_3\rangle \neq 0, \forall \omega \in [\omega_*, \omega^*], \tag{15}$$

there exists $\delta_1 > 0$ such that, for every $N_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\|N_0 + e_3\|_{H^1} \leq \delta_1$, the solution of the closed loop system (10), (13) with initial condition $N(0,\omega) = N_0(\omega)$ satisfies $N(t) \rightharpoonup -e_3$ weakly in $H^1(\omega_*, \omega^*)$ when $t \to +\infty$.

The above theorem has the following corollary.

Corollary 1 *For every $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with (15), there exists $\delta_2 > 0$ such that, for every $M_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\|M_0 - M_f\|_{H^1} < \delta_2$, the solution of the system (1) with the initial condition $M(0,\omega) = M_0(\omega)$ and the feedback law given by (2), (13) satisfies $M((2kT)^+) \rightharpoonup M_f$ weakly in H^1 when $k \to +\infty$. In particular,*

$$\lim_{k \to +\infty} \|M((2kT)^+, .) - M_f\|_{L^\infty(\omega_*, \omega^*)} = 0.$$

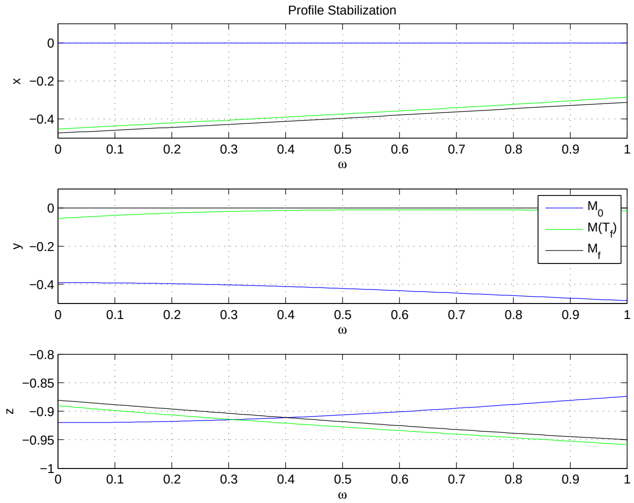

Fig. 2. Initial $(t = 0)$ and final $(t = T_f)$ ω-profiles for x, y and z solutions of the closed-loop system 1 with the feedback (2), (13).

The remaining part of this section is devoted to the proof of Theorem 1

3.2 LaSalle invariant set

The first step of our proof consists in checking that, locally, the invariant set is reduced to $\{-e_3\}$.

Proposition 3 *For every $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with (15), there exists $\delta > 0$ such that, for every $N_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\|N_0 + e_3\| < \delta$, the map $t \mapsto \mathcal{L}[N(t)]$ is constant on $[0, +\infty)$ if and only if $N_0 = -e_3$.*

Proof: Let us assume that $t \mapsto \mathcal{L}[N(t)]$ is constant. Then $u_1 = u_2 = 0$ and $N(t,\omega) \equiv N_0(\omega)$ (see (14) and (10)). Thus, for every $j \in \{1,2\}$ and $t \in [0, +\infty)$

$$\begin{aligned} 0 = \int_{\omega_*}^{\omega^*} \Big[&\Big\langle N_0'(\omega), \Big(\frac{\partial F}{\partial \omega}(t,\omega) e_j\Big) \wedge N_0(\omega) \Big\rangle \\ &+ \Big\langle e_3, \Big(F(t,\omega) e_j\Big) \wedge N_0(\omega) \Big\rangle \Big] d\omega. \end{aligned} \tag{16}$$

For $t \in [0,T]$, $\sigma(t) = t$ so $F(t,\omega) = \sum_{k=0}^{\infty} \frac{t^k \omega^k}{k!} R(\omega) S^k$ and $\frac{\partial F}{\partial \omega}(t,\omega) = \sum_{k=0}^{\infty} \frac{t^k \omega^k}{k!} R'(\omega) S^k + \sum_{k=1}^{\infty} \frac{t^k \omega^{k-1}}{(k-1)!} R(\omega) S^k$.

Developing (16) in power series expansions of t and using (5), we obtain, for every $j \in \{1,2\}$ and $k \geqslant 1$,

$$\int_{\omega_*}^{\omega^*} \left\langle N_0'(\omega), \left[\left(\tfrac{\omega^k}{k!}R'(\omega) + \tfrac{\omega^{k-1}}{(k-1)!}R(\omega)\right) e_j\right] \wedge N_0(\omega)\right\rangle + \left\langle e_3, \left(\tfrac{\omega^k}{k!}R(\omega)e_j\right) \wedge N_0(\omega)\right\rangle d\omega = 0.$$

By linearity, the following equality holds, for every $Q \in \mathbb{R}[X]$ and $j \in \{1,2\}$

$$\int_{\omega_*}^{\omega^*} \left\langle N_0'(\omega), \Big[\left(Q(\omega)R'(\omega) + Q'(\omega)R(\omega)\right) e_j\Big] \wedge N_0(\omega)\right\rangle + \left\langle e_3, \Big[Q(\omega)R(\omega)e_j\Big] \wedge N_0(\omega)\right\rangle d\omega = 0. \tag{17}$$

Thanks to the density of polynomial functions in $H^1((\omega_*,\omega^*),\mathbb{C})$, the previous equality holds for every $Q \in H^1((\omega_*,\omega^*),\mathbb{R})$. Let us recall the relations $\langle X, Y \wedge Z\rangle = \langle Y, Z \wedge X\rangle$ and $\langle MX, Y\rangle = \langle X, M^\top Y\rangle$, $\forall X,Y,Z \in \mathbb{R}^3$ and $M \in \mathcal{M}_3(\mathbb{R})$, where $M^\top$ denotes the transposed matrix of M. Then, the equality (17) may also we written

$$\begin{aligned} &\int_{\omega_*}^{\omega^*} \left\langle Q(\omega)e_j, R'(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)]\right\rangle \\ &\quad + \left\langle Q'(\omega)e_j, R(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)]\right\rangle \\ &\quad + \left\langle Q(\omega)e_j, R(\omega)^\top[N_0(\omega) \wedge e_3]\right\rangle d\omega = 0. \end{aligned} \tag{18}$$

for every $Q \in H^1((\omega_*,\omega^*),\mathbb{R})$ and $j \in \{1,2\}$. By linearity, we deduce that

$$\int_{\omega_*}^{\omega^*} \left\langle \mathcal{Q}(\omega), R'(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)] + R(\omega)^\top[N_0(\omega) \wedge e_3]\right\rangle + \left\langle \mathcal{Q}'(\omega), R(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)]\right\rangle d\omega = 0 \tag{19}$$

for every $\mathcal{Q} \in H^1((\omega_*,\omega^*),\mathbb{V})$ where $\mathbb{V} := \mathrm{Span}(e_1,e_2)$. Let $\mathbb{P} : \mathbb{R}^3 \to \mathbb{V}$ be the orthogonal projection on $\mathbb{V}$. The previous equality is equivalent to

$$\begin{cases} \mathbb{P}R(\omega)^\top[N_0(\omega) \wedge e_3 - N_0(\omega) \wedge N_0''(\omega)] = 0 \text{ in } H^{-1} \\ R(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)] = 0 \text{ at } \omega = \omega_* \text{ and } \omega^*. \end{cases} \tag{20}$$

Here, H^{-1} denotes the dual space of $H^1_0(\omega_*,\omega^*)$ for the L^2-scalar product; the first equation has to be understood in the distribution sens. Thanks to $\|N_0(\omega)\| \equiv 1$, we have $\|R(\omega)^\top[N_0(\omega) \wedge N_0'(\omega)]\| \equiv \|N_0'(\omega)\|$. Thus, the second line of (20) is equivalent to $N_0' = 0$ at ω_* and ω^*. Notice that $\mathbb{P}R(\omega)^\top|_{\mathbb{V}}$ is bijective on $\mathbb{V}$ for every $\omega \in [\omega_*,\omega^*]$. Indeed, thanks to (7) and (15), we have

$$\begin{aligned} \mathrm{Range}[\mathbb{P}R(\omega)^\top|_{\mathbb{V}}]^\perp &= \mathrm{Ker}[\mathbb{P}R(\omega)|_{\mathbb{V}}] \\ &= \{v \in \mathbb{V}; R(\omega) \in \mathbb{R}e_3\} \\ &= \mathbb{V} \cap \mathbb{R}M_f(\omega) = \{0\}. \end{aligned}$$

Moreover, $(\mathbb{P}R(\omega)^\top|_{\mathbb{V}})^{-1} \in H^1$, thus (20) gives

$$\begin{cases} -N_0'' \wedge e_3 + N_0 \wedge e_3 = g \text{ in } H^{-1}((\omega_*,\omega^*),\mathbb{V}), \\ N_0' \wedge e_3 = 0 \text{ at } \omega_*,\omega^*, \end{cases} \tag{21}$$

where

$$g(\omega) := -(\mathbb{P}R(\omega)^\top|_{\mathbb{V}})^{-1}\mathbb{P}R(\omega)^\top[N_0''(\omega) \wedge (N_0(\omega) + e_3)].$$

Therefore, there exists $C_1 = C_1(\omega_*,\omega^*) > 0$ such that

$$\|N_0(\omega) \wedge e_3\|_{H^1} \leqslant C_1\|g\|_{H^{-1}}. \tag{22}$$

Thanks to (8), there exists $C_2 = C_2(\omega_*,\omega^*,\|M_f\|_{H^1}) > 0$ such that

$$\|g\|_{H^{-1}} \leqslant C_2\|N_0 + e_3\|^2_{H^1}.$$

When N_0 is close enough to $-e_3$ in H^1, then $\|N_0 \wedge e_3\|_{H^1}$ and $\|N_0 + e_3\|_{H^1}$ are equivalent norms and then (22) gives

$$\|N_0(\omega) \wedge e_3\|_{H^1} \leqslant C_3\|N_0(\omega) \wedge e_3\|^2_{H^1}$$

for some constant $C_3 = C_3(\omega_*,\omega^*,\|M_f\|_{H^1}) > 0$. This implies $N_0 \wedge e_3 = 0$, i.e. $N_0 = -e_3$. □

Remark 1 *For $M_f \equiv e_1$, any constant function N_0 with values in Span(e_2,e_3) belongs to the invariant set (see (20)). Thus, an assumption of the type (15) is required for our strategy to work.*

3.3 Convergence proof

For the proof of Theorem 1, we need the following result.

Proposition 4 *Take $M_f \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$ and $R \in H^1((\omega_*,\omega^*),SO(3))$ as in Proposition 1. Let $(N_n^0)_{n\in\mathbb{N}}$ a sequence of $H^1((\omega_*,\omega^*),\mathbb{S}^2)$ and $N_\infty^0 \in H^1((\omega_*,\omega^*),\mathbb{S}^2)$ such that $N_n^0 \rightharpoonup N_\infty^0$ weakly in H^1 and $N_n^0 \to N_\infty^0$ strongly in L^2. Let $\alpha \in [0,2T]$ and $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of $[0,2T)$ such that $\tau_n \to \alpha$. Let N_n (resp. N_∞) be the solutions of the closed loop system (10), (13) associated to the initial condition $N_n(\tau_n) = N_n^0$ (resp. $N_\infty(\alpha) = N_\infty^0$). Then, we have $N_n(t) \rightharpoonup N_\infty(t)$ weakly in H^1, $\forall t > \alpha$, and $u_j[t,N_n(t)] \to u_j[t,N_\infty(t)]$, $\forall t > \alpha, \forall j \in \{1,2\}$.*

Proof: The sequence $(N_n^0)_{n\in\mathbb{N}}$ is bounded in H^1 and $\mathcal{L}[N_n(t)] \leqslant \mathcal{L}[N_n^0]$, for every $t \in [\tau_n,+\infty)$ and $n \in \mathbb{N}$ so there exists $\mathcal{M}_0 > 0$ such that $\|N_n(t)\|_{H^1} \leqslant \mathcal{M}_0$, for every $t \in [\tau_n,+\infty)$ and $n \in \mathbb{N}$. The function $t \in \mathbb{R} \mapsto F(t,.) \in H^1((\omega_*,\omega^*),\mathcal{M}_3(\mathbb{R}))$ defined by (11) is continuous and $2T$-periodic, thus, there exists $\mathcal{M}_1 > 0$ such that $\|F(t,.)\|_{H^1} \leqslant \mathcal{M}_1$, for every $t \in \mathbb{R}$. Thanks to (13), we have $|u_j[t,N_n(t)]| \leqslant 2\mathcal{M}_1\mathcal{M}_0$, for every $n \in \mathbb{N}$ and $t \in [\tau_n,+\infty)$. We deduce from (10) that

$\left\| \frac{\partial N_n}{\partial t}(t) \right\|_{L^2} \leqslant 4\mathcal{M}_1\mathcal{M}_0$ for every $t \in [\tau_n, +\infty)$ and $n \in \mathbb{N}$. The end of the proof is as in Beauchard et al. (2011). □

Proof of Theorem 1: The proof is as in Beauchard et al. (2011). One may replace Barbalat's lemma by the Lebesgue reciprocal theorem, in the following way. Thanks to (14), $t \mapsto u_i[t, N(t)]$ belongs to $L^2(0, +\infty)$, thus, for any diverging sequence of times k_n, the sequence $(t \in (0, +\infty) \mapsto u_i[2k_nT + t, N(2k_nT + t)])_{n\in\mathbb{N}}$ converges to zero in $L^2(0, +\infty)$. Therefore, there exists a subset $\mathcal{N} \subset (0, +\infty)$ with zero Lebesgue measure such that $u_j[t, N(2k_nT + t) \to 0$ for every $t \in (0, +\infty) - \mathcal{N}$ and $j \in \{1, 2\}$. □

4 Concluding remark

Open-loop "impulse-train" control are combined with Lyapunov feedback to steer an initial profile $[\omega_*, \omega^*] \ni \omega \mapsto M(0, \omega)$ of the Bloch-sphere system (1) towards an arbitrary target profile $[\omega_*, \omega^*] \ni \omega \mapsto M_f(\omega)$. Convergence is proved to be local for any target profile belonging either to the south or to the north hemisphere. We guess that our convergence proof could be extended to the case where M_f intersects transversely the equator and thus where M_f is not confined in only one hemisphere.

Acknowledgements

KB and PR were partially supported by the "Agence Nationale de la Recherche" (ANR), Projet Blanc C-QUID number BLAN-3-139579 and Project Blanc EMAQS number ANR-2011-BS01-017-01. PSPS is Partially supported by CNPq – Conselho Nacional de Desenvolvimento Cientifico e Tecnologico, and FAPESP-Fundação de Amparo a Pesquisa do Estado de São Paulo.

References

Beauchard, K., Coron, J.-M., Rouchon, P., 2010. Controllability issues for continuous spectrum systems and ensemble controllability of Bloch equations. Communications in Mathematical Physics 296, 525–557.

Beauchard, K., Pereira da Silva, P. S., Rouchon, P., 2011. Stabilization for an ensemble of half-spin systems. Automatica 48, 68–76, 2012

Li, J., Khaneja, N., 2006. Control of inhomogeneous quantum ensembles. Phys. Rev. A. 73, 030302.

Li, J., Khaneja, N., 2009. Ensemble control of Bloch equations. IEEE Trans. Automatic Control 54 (3), 528–536.