

Finite-Horizon Reliable Control with Randomly Occurring Uncertainties and Nonlinearities Subject to Output Quantization [★]

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Abstract

This paper deals with the finite-horizon reliable \mathcal{H}_∞ output feedback control problem for a class of discrete time-varying systems with randomly occurring uncertainties (ROUs), randomly occurring nonlinearities (RONs) as well as measurement quantizations. Both the deterministic actuator failures and probabilistic sensor failures are considered in order to reflect the reality. The actuator failure is quantified by a deterministic variable varying in a given interval and the sensor failure is governed by an individual random variable taking value on $[0, 1]$. Both the nonlinearities and the uncertainties enter into the system in random ways according to Bernoulli distributed white sequences with known conditional probabilities. The main purpose of the problem addressed is to design a time-varying output feedback controller over a given finite horizon such that, in the simultaneous presence of ROUs, RONs, actuator and sensor failures as well as measurement quantizations, the closed-loop system achieves a prescribed performance level in terms of the \mathcal{H}_∞ -norm. Sufficient conditions are first established for the robust \mathcal{H}_∞ performance through intensive stochastic analysis, and then a recursive linear matrix inequality approach is employed to design the desired output feedback controller achieving the prescribed \mathcal{H}_∞ disturbance rejection level. A numerical example is given to demonstrate the effectiveness of the proposed design scheme.

Key words: Finite-horizon reliable control; Time-varying systems; Actuator and sensor failure; Randomly occurring uncertainties; Randomly occurring nonlinearities.

1 Introduction

For several decades, stochastic control and nonlinear control serve as two of the most active research areas in systems and control that have found successful applications in a variety of engineering systems such as automotive engines, robot manipulators, aircraft and electrical motors. So far, considerable research attention has been devoted to the theoretical research on control problems for nonlinear stochastic systems, see [1, 3, 5, 6, 9, 11, 20, 22, 24, 30] and the references therein. For example, for different kinds of nonlinear stochastic systems, the \mathcal{H}_∞ output feedback control problem has been investigated in [24], the adaptive fuzzy control problem has been proposed in [22], the neural-network-based controller design has been addressed in [23], the adaptive sliding

mode controller has been designed in [4, 10, 17, 18] and the observer-based control problems have been solved in [25], respectively. Among various descriptions of nonlinearities, the so-called randomly occurring nonlinearities (RONs) [26] cater for those randomly changeable nonlinearities in terms of their types and/or intensities governed by stochastic variables. RONs, which typically occur in networked environments, encompass several well-studied nonlinearities in stochastic systems and have thus stirred particular research interests in the past few years.

It is noticeable that, in almost all the aforementioned literature, the components of the control systems have been implicitly assumed to be fully reliable. This assumption is, unfortunately, not always true since the failures of control components (e.g. sensors and actuators) often occur in practical applications due to a variety of reasons, for example, the abrupt changes of working conditions, the erosion caused by severe circumstance, the internal component constraints, the intense external disturbance and the aging of sensors or actuators, etc. Therefore, it is of both practical significance and theoretical importance to design a reliable controller against possible actuator and sensor failures such that the essential performance of the controlled system can be guaranteed [13, 19]. In fact, in the past two decades, the problem of reliable controller design has attracted much research attention and many approaches have been proposed in the literature

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including Hamilton-Jacobi equation approach [14, 15], robust pole region assignment technique [8], algebraic Riccati equation approach [21, 29] and linear matrix inequality approach [12, 16, 27]. Despite the fruitful results on *time-invariant systems* over an *infinite horizon*, it is worth pointing out that the *finite-horizon* reliable control problem for *time-varying* systems has not been thoroughly investigated yet, not to mention the case complicated further by nonlinearity and stochasticity.

As is well known, modeling errors (usually parameter uncertainties) constitute an important kind of complexities for system modeling that has a great impact on the subsequent system analysis and design [28]. In today's networked systems, it is quite common that the network load is randomly fluctuated and the signal transmission suffers from unpredictable networked-induced phenomena owing to limited bandwidth. As such, the occurrence of the parameter uncertainties in a networked environment is often of random nature resulting from abrupt phenomena such as modification of the operating point of a linearized model of nonlinear systems, random failures and repairs of the components, changes in the interconnections of subsystems and sudden environment disturbances, etc. Very recently, in [10], the concept of randomly occurring uncertainties (ROUs) has been introduced to reflect the probabilistic fashion of the network-induced modeling error and the corresponding sliding mode control problem has been considered. However, when it comes to the finite-horizon reliable \mathcal{H}_∞ control problem involving ROUs for nonlinear time-varying stochastic systems, the related results are very few and the situation is even worse when measurement quantizations, actuator and sensor failures are also taken into account. It is, therefore, the aim of this paper to shorten such a gap by addressing the finite-horizon reliable \mathcal{H}_∞ control problem with presence of ROUs, RONs as well as measurement quantizations.

Motivated by the above discussion, in this paper, we launch a major study on the finite-horizon reliable \mathcal{H}_∞ output-feedback control problem for a class of discrete time-varying stochastic systems with simultaneous presence of actuator and sensor failures, ROUs, RONs and measurement quantizations. Some sufficient conditions are established, via intensive stochastic analysis, to guarantee the existence of the desired time-varying output feedback controller gains, and then such controller gains are characterized by solving a set of recursive matrix inequalities. A simulation example is finally presented to illustrate the effectiveness of the proposed design scheme. *The main contributions of this paper are highlighted as follows.* 1) *The problem addressed is new in the sense that this paper represents the first of few attempts to deal with the finite-horizon reliable \mathcal{H}_∞ output feedback control problem against actuator and sensor failures for a class of discrete time-varying stochastic systems.* 2) *The system under consideration is comprehensive to cover time-varying parameters, actuator and sensor failures, ROUs, RONs and measurement quantizations, hence reflecting the reality more closely.* 3) *The algorithm developed is computationally appealing in terms of its recursive nature suitable for online application.*

The rest of this paper is outlined as follows. In Section 2, a class of discrete time-varying stochastic systems is presented with actuator and sensor failures, ROUs,

RONs and measurement quantizations. The problem under consideration is formulated. In Section 3, by employing the stochastic analysis techniques, some sufficient conditions are established to guarantee the desired output feedback controller performances and then the controller gains are obtained by solving a set of recursive matrix inequalities. A simulation example is given in Section 4 to demonstrate the main results obtained. Finally, we conclude the paper in Section 5.

Notation. The notation used here is standard except where otherwise stated. \mathbb{R}^n and $\mathbb{R}^{n \times m}$ denote, respectively, the n -dimensional Euclidean space and the set of all $n \times m$ real matrices. $l[0, N]$ is the space of vector functions over $[0, N]$. The notation $X \geq Y$ (respectively, $X > Y$), where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). M^T represents the transpose of the matrix M . 0 represents zero matrix of compatible dimensions. The n -dimensional identity matrix is denoted as I_n or simply I , if no confusion is caused. $\text{diag}\{\dots\}$ stands for a block-diagonal matrix. $\mathbb{E}\{x\}$ and $\mathbb{E}\{x|y\}$ will, respectively, denote expectation of the stochastic variable x and expectation of x conditional on y . $\text{Prob}\{\cdot\}$ means the occurrence probability of the event " \cdot ". In symmetric block matrices, " $*$ " is used as an ellipsis for terms induced by symmetry. The symbol \otimes denotes the Kronecker product. Matrices, if they are not explicitly specified, are assumed to have compatible dimensions.

2 Problem Formulation

Consider the following uncertain discrete time-varying nonlinear stochastic system defined on $k \in [0, N]$:

$$\begin{cases} x(k+1) = (A(k) + \alpha(k)\Delta A(k))x(k) \\ \quad + B_1(k)u(k) + \beta(k)g(k, x(k)) \\ \quad + D(k)w(k) \\ z(k) = M(k)x(k) + B_2(k)u(k) \\ x(0) = \varphi_0 \end{cases} \quad (1)$$

where $x(k) \in \mathbb{R}^{n_x}$ represents the state vector; $u(k) \in \mathbb{R}^{n_u}$ is the control input; $z(k) \in \mathbb{R}^{n_z}$ is the controlled output; $w(k) \in \mathbb{R}^{n_w}$ is the disturbance input which belongs to $l[0, N]$; and φ_0 is a given real initial value. $A(k)$, $B_1(k)$, $B_2(k)$, $D(k)$ and $M(k)$ are known, real, time-varying matrices with appropriate dimensions.

The nonlinear function $g(k, x(k))$ satisfies the following condition:

$$\|g(k, x(k))\|^2 \leq \varepsilon(k)\|E(k)x(k)\|^2 \quad (2)$$

where $\varepsilon(k) > 0$ is a known positive scalar and $E(k)$ is a known time-varying matrix.

The real-valued matrix $\Delta A(k)$ represents the norm-bounded parameter uncertainties of the following structure

$$\Delta A(k) = H_a(k)F(k)N(k), \quad (3)$$

where $H_a(k)$ and $N(k)$ are known time-varying matrices, while $F(k)$ is an unknown time-varying matrix function satisfying the following condition

$$F^T(k)F(k) \leq I. \quad (4)$$

The stochastic variables $\alpha(k) \in \mathbb{R}$ and $\beta(k) \in \mathbb{R}$ are Bernoulli distributed white sequences taking values on either 0 or 1 with

$$\begin{aligned} \text{Prob}\{\alpha(k) = 1\} &= \bar{\alpha}, & \text{Prob}\{\alpha(k) = 0\} &= 1 - \bar{\alpha}, \\ \text{Prob}\{\beta(k) = 1\} &= \bar{\beta}, & \text{Prob}\{\beta(k) = 0\} &= 1 - \bar{\beta}, \end{aligned} \quad (5)$$

where $\bar{\alpha} \in [0, 1]$ and $\bar{\beta} \in [0, 1]$ are known constants.

Remark 1 *The random variables $\alpha(k)$ and $\beta(k)$ are introduced to characterize the phenomena of the ROUs and the RONS, respectively. The uncertainty ΔA and the nonlinearity g enter into the system in random ways according to individual Bernoulli distributions. Such a statistics description is particularly suitable for reflecting parameter/nonlinear variations that are unpredictable but appear in a random way with known probability laws.*

In this paper, the measurement $\bar{y}(k)$ with probabilistic sensor failures is described by

$$\begin{aligned} \bar{y}(k) &= \Xi(k)C(k)x(k) + \bar{E}(k)v(k) \\ &= \sum_{j=1}^{n_y} \rho_j(k)C_j(k)x(k) + \bar{E}(k)v(k) \end{aligned} \quad (6)$$

where $v(k) \in \mathbb{R}^{n_v}$ is the measure noise belonging to $l[0, N]$. $\Xi(k)$ is defined as $\Xi(k) := \text{diag}\{\rho_1(k), \dots, \rho_{n_y}(k)\}$ where $\rho_j(k)$ has the probability density function $\zeta_j(s)$ ($j = 1, \dots, n_y$) on the interval $[0, 1]$ with mathematical expectation $\bar{\mu}_j$ and variance χ_j^2 . $C_j(k)$ is defined by

$$C_j(k) := \text{diag}\{\underbrace{0, \dots, 0}_{j-1}, \underbrace{1, 0, \dots, 0}_{n_y-j}\}C(k).$$

In the sequel, we denote $\bar{\Xi} = \mathbb{E}\{\Xi(k)\} = \text{diag}\{\bar{\mu}_1, \bar{\mu}_2, \dots, \bar{\mu}_{n_y}\}$. Throughout the paper, we assume that $\alpha(k)$, $\beta(k)$ and $\rho_j(k)$ ($j = 1, \dots, n_y$) are unrelated random variables.

In a networked environment, it is often the case that $\bar{y}(k)$ is quantized before being transmitted to the controller. Let us denote the quantizer as $q(\cdot) = [q_1(\cdot) \ q_2(\cdot) \ \dots \ q_{n_y}(\cdot)]^T$ which is symmetric, i.e., $q_j(-v) = -q_j(v)$ ($j = 1, \dots, n_y$). The map of the quantization process is

$$y(k) = [q_1(\bar{y}^{(1)}(k)) \ q_2(\bar{y}^{(2)}(k)) \ \dots \ q_{n_y}(\bar{y}^{(n_y)}(k))]^T$$

where $y(k) \in \mathbb{R}^{n_y}$ and $\bar{y}^{(i)}(k)$ ($i = 1, \dots, n_y$) denotes the i th element of $\bar{y}(k)$. For each $q_j(\cdot)$ ($1 \leq j \leq m$), the

set of quantization levels is described by

$$\begin{aligned} \mathcal{U}_j &= \{\pm \hat{\mu}_i^{(j)}, \hat{\mu}_i^{(j)} = \bar{\chi}_j \hat{\mu}_0^{(j)}, i = 0, \pm 1, \pm 2, \dots\} \cup \{0\}, \\ &0 < \bar{\chi}_j < 1, \hat{\mu}_0^{(j)} > 0 \end{aligned}$$

and each of the quantization level corresponds to a segment such that the quantizer maps the whole segment to this quantization level.

According to [7], the logarithmic quantizer is given by

$$q_j(\bar{y}^{(j)}(k)) = \begin{cases} \hat{\mu}_i^{(j)}, & \frac{1}{1+\delta_j} \hat{\mu}_i^{(j)} \leq \bar{y}^{(j)}(k) \leq \frac{1}{1-\delta_j} \hat{\mu}_i^{(j)} \\ 0, & \bar{y}^{(j)}(k) = 0 \\ -h_j(-\bar{y}^{(j)}(k)), & \bar{y}^{(j)}(k) < 0 \end{cases}$$

where $\delta_j = (1 - \bar{\chi}_j)/(1 + \bar{\chi}_j)$. It can be easily seen from the above definition that $q_j(\bar{y}^{(j)}(k)) = (1 + \Delta_k^{(j)})\bar{y}^{(j)}(k)$ with $|\Delta_k^{(j)}| \leq \delta_j$.

Defining $\Delta_k := \text{diag}\{\Delta_k^{(1)}, \dots, \Delta_k^{(n_y)}\}$, the measurements with quantization effect and sensor failures can be expressed as

$$y(k) = (I + \Delta_k)\bar{y}(k). \quad (7)$$

Furthermore, by defining $\bar{\Delta} := \text{diag}\{\delta_1, \dots, \delta_{n_y}\}$ and $\bar{F}(k) := \Delta_k \bar{\Delta}^{-1}$, we can see that $\bar{F}(k)$ is an unknown real-valued time-varying matrix satisfying $\bar{F}(k)\bar{F}^T(k) \leq I$.

When the actuators experience failures, we use $u^F(k)$ to describe the control signal sent from actuators and the control input of actuator fault can therefore be described as follows:

$$u^F(k) = G(k)u(k) \quad (8)$$

where $G(k)$ is the actuator fault matrix with

$$\begin{cases} G(k) = \text{diag}\{g_1(k), \dots, g_{n_u}(k)\} \\ \underline{G}(k) \leq G(k) \leq \bar{G}(k) \\ \underline{G}(k) = \text{diag}\{\underline{g}_1(k), \dots, \underline{g}_{n_u}(k)\} \geq 0 \\ \bar{G}(k) = \text{diag}\{\bar{g}_1(k), \dots, \bar{g}_{n_u}(k)\} \leq I \end{cases} \quad (9)$$

and the variables $g_i(k)$ ($i = 1, 2, \dots, n_u$) quantify the failures of the actuators. $\underline{g}_i(k)$ and $\bar{g}_i(k)$ serve as the lower and the upper bounds on $g_i(k)$, respectively.

Defining

$$G_0(k) = \text{diag}\{\hat{g}_1(k), \dots, \hat{g}_{n_u}(k)\} := \frac{\underline{G}(k) + \bar{G}(k)}{2}$$

$$= \text{diag}\left\{\frac{\underline{g}_1(k) + \bar{g}_1(k)}{2}, \dots, \frac{\underline{g}_{n_u}(k) + \bar{g}_{n_u}(k)}{2}\right\} \quad (10)$$

$$\tilde{G}(k) = \text{diag}\{\tilde{g}_1(k), \dots, \tilde{g}_{n_u}(k)\} := \frac{\bar{G}(k) - \underline{G}(k)}{2}$$

$$= \text{diag}\left\{\frac{\bar{g}_1(k) - \underline{g}_1(k)}{2}, \dots, \frac{\bar{g}_{n_u}(k) - \underline{g}_{n_u}(k)}{2}\right\} \quad (11)$$

the matrix $G(k)$ can be rewritten as

$$G(k) = G_0(k) + \Delta_G(k)$$

$$= G_0(k) + \text{diag}\{\tau_1(k), \dots, \tau_{n_u}(k)\} \quad (12)$$

where $\tau_i(k)$ ($i = 1, 2, \dots, n_u$) are certain scalars satisfying $|\tau_i(k)| \leq \tilde{g}_i(k)$.

In this paper, we adopt the following time-varying output-feedback controller for system (1):

$$\begin{cases} x_c(k+1) = A_c(k)x_c(k) + B_c(k)y(k) \\ u(k) = C_c(k)x_c(k) \end{cases} \quad (13)$$

where $x_c(k) \in \mathbb{R}^{n_c}$ is the controller state, $A_c(k)$, $B_c(k)$ and $C_c(k)$ are the controller parameters to be designed.

Letting $\eta(k) = [x^T(k) \ x_c^T(k)]^T$ and $\varpi(k) = [w^T(k) \ v^T(k)]^T$, we have the following closed-loop system to be investigated:

$$\begin{cases} \eta(k+1) = (\mathcal{A}(k) + \tilde{\alpha}(k)\Delta\mathcal{A}(k) + \tilde{B}_c(k))\eta(k) \\ \quad + (\tilde{\beta} + \tilde{\beta}(k))\mathcal{G}(k, x(k)) + \mathcal{D}(k)\varpi(k) \\ z(k) = \mathcal{M}(k)\eta(k) \end{cases} \quad (14)$$

where

$$\begin{aligned} \mathcal{A}(k) &= \bar{\mathcal{A}}(k) + \Delta\bar{\mathcal{A}}(k) + \bar{\alpha}\Delta\mathcal{A}(k), \\ \Delta\bar{\mathcal{A}}(k) &= \bar{\mathcal{H}}_a(k)\bar{F}(k)\bar{E}_c(k), \quad \tilde{\alpha}(k) = \alpha(k) - \bar{\alpha}, \\ \Delta\mathcal{A}(k) &= \mathcal{H}_a(k)F(k)\mathcal{N}(k), \quad \tilde{\beta}(k) = \beta(k) - \bar{\beta}, \\ \bar{\mathcal{A}}(k) &= \begin{bmatrix} A(k) & B_1(k)G(k)C_c(k) \\ B_c(k)\bar{\Xi}C(k) & A_c(k) \end{bmatrix}, \\ \bar{E}_c(k) &= [\bar{\Delta}\bar{\Xi}C(k) \ 0], \quad \bar{\mathcal{H}}_a(k) = [0 \ B_c^T(k)]^T, \\ \mathcal{H}_a(k) &= [H_a^T(k) \ 0]^T, \quad \tilde{B}_c(k) = \tilde{B}_{c0}(k) + \Delta\tilde{B}_{c0}(k), \\ \mathcal{D}(k) &= \bar{\mathcal{D}}(k) + \Delta\bar{\mathcal{D}}(k), \quad \tilde{\Xi}(k) = \Xi(k) - \bar{\Xi}, \\ \Delta\tilde{B}_{c0}(k) &= \bar{\mathcal{H}}_a(k)\bar{F}(k)\bar{E}_c(k), \quad \bar{E}_E(k) = [0 \ \bar{\Delta}\bar{E}(k)], \\ \tilde{B}_{c0}(k) &= \begin{bmatrix} 0 & 0 \\ B_c(k)\tilde{\Xi}(k)C(k) & 0 \end{bmatrix}, \quad \mathcal{N}(k) = [N(k) \ 0], \\ \bar{E}_c(k) &= [\bar{\Delta}\bar{\Xi}(k)C(k) \ 0], \\ \bar{\mathcal{D}}(k) &= \text{diag}\{D(k), B_c(k)\bar{E}(k)\}, \\ \Delta\bar{\mathcal{D}}(k) &= \bar{\mathcal{H}}_a(k)\bar{F}(k)\bar{E}_E(k), \\ \mathcal{M}(k) &= [M(k) \ B_2(k)G(k)C_c(k)], \\ \mathcal{G}(k, x(k)) &= [g^T(k, x(k)) \ 0]^T. \end{aligned}$$

The objective of this paper is to find a sequence of reliable controller parameters $A_c(k)$, $B_c(k)$ and $C_c(k)$ such that the closed-loop system (14) satisfies the following performance requirement:

$$J := \mathbb{E} \left\{ \sum_{k=0}^{N-1} \left(\|z(k)\|^2 - \gamma^2 \|\varpi(k)\|_R^2 \right) - \gamma^2 \eta^T(0) S \eta(0) \right\} < 0 \quad (15)$$

where $\|\varpi(k)\|_R^2 = \varpi^T(k)R\varpi(k)$, and R and S are known positive definite weighted matrices.

3 Main results

In this section, we investigate both the controller analysis and controller design problems for the discrete time-varying nonlinear stochastic system (1) with randomly occurring uncertainties, nonlinearities, actuator and sensor failures subject to output quantization. In the following theorem, we present the following analysis results with known parameter matrix describing the actuator failures.

Theorem 1 Consider the closed-loop system (14) with known actuator failure parameter matrix $G(k)$. Let the disturbance attenuation level $\gamma > 0$, families of positive scalars $\{\varepsilon(k)\}_{0 \leq k \leq N-1} > 0$, the positive definite matrices $R > 0$, $S > 0$ and the controller feedback gain matrices $\{A_c(k)\}_{0 \leq k \leq N-1}$, $\{B_c(k)\}_{0 \leq k \leq N-1}$ and

$\{C_c(k)\}_{0 \leq k \leq N-1}$ be given. The \mathcal{H}_∞ performance requirement defined in (15) is achieved for all nonzero $\varpi(k)$ if, with the initial condition $P(0) \leq \gamma^2 S$, there exist families of positive definite matrices $\{P(k)\}_{0 \leq k \leq N} > 0$, $\{Q(k)\}_{0 \leq k \leq N} > 0$ and families of positive scalars $\{\lambda(k) > 0\}_{0 \leq k \leq N-1}$ satisfying the following recursive matrix inequalities

$$\Omega(k) = \begin{bmatrix} \Omega_{11}(k) & * & * \\ 0 & \Omega_{22}(k) & * \\ \Omega_{31}(k) & \Omega_{32}(k) & \Omega_{33}(k) \end{bmatrix} < 0 \quad (16)$$

with the parameters updated by

$$P(k+1) = Q^{-1}(k+1) \quad (17)$$

where

$$\begin{aligned} \Omega_{11}(k) &= -P(k) + \lambda(k)\varepsilon(k)H^T E^T(k)E(k)H, \\ \Omega_{22}(k) &= \text{diag}\{-\lambda(k)H_I^T H_I, -\gamma^2 R\}, \quad H = \begin{bmatrix} I & 0 \end{bmatrix}, \\ \Omega_{31}(k) &= \begin{bmatrix} \mathcal{A}^T(k) & \underline{B}_c^T(k) & \mathcal{M}^T(k) & \Lambda_\alpha \Delta \mathcal{A}^T(k) & 0 \end{bmatrix}^T, \\ \bar{E}_{ci}(k) &= \begin{bmatrix} \bar{\Delta} C_i(k) & 0 \end{bmatrix}, \quad \Omega_{32}(k) = \begin{bmatrix} \Omega_{321}(k) & \Omega_{322}(k) \end{bmatrix}, \\ \Omega_{321}(k) &= \begin{bmatrix} \bar{\beta} I & 0 & 0 & 0 & \sqrt{\bar{\beta}(1-\bar{\beta})} I \end{bmatrix}^T, \quad H_I = \begin{bmatrix} I & I \end{bmatrix}, \\ \Delta \bar{B}_{ci}(k) &= \bar{\mathcal{H}}_a(k) \bar{F}(k) \bar{E}_{ci}(k), \quad \Lambda_\alpha = \sqrt{\bar{\alpha}(1-\bar{\alpha})}, \\ \Omega_{322}(k) &= \begin{bmatrix} \mathcal{D}^T(k) & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\ \Omega_{33}(k) &= -\text{diag}\{Q(k+1), \hat{Q}(k+1), I, \tilde{Q}(k+1)\}, \\ Q(k+1) &= P^{-1}(k+1), \quad \hat{Q}(k+1) = I_{n_y} \otimes Q(k+1), \\ \hat{B}_{ci}(k) &= \begin{bmatrix} \chi_1 \bar{B}_{c1}^T(k) & \chi_2 \bar{B}_{c2}^T(k) & \cdots & \chi_{n_y} \bar{B}_{c_{n_y}}^T(k) \end{bmatrix}^T, \\ \Delta \hat{B}_{ci}(k) &= \begin{bmatrix} \chi_1 \Delta \bar{B}_{c1}^T(k) & \cdots & \chi_{n_y} \Delta \bar{B}_{c_{n_y}}^T(k) \end{bmatrix}^T, \\ \bar{B}_{ci}(k) &= \begin{bmatrix} 0 & 0 \\ B_c(k) C_i(k) & 0 \end{bmatrix}, \quad \tilde{Q}(k+1) = I_2 \otimes Q(k+1), \\ \underline{B}_c(k) &= \hat{B}_{ci}(k) + \Delta \hat{B}_{ci}(k). \end{aligned}$$

Proof: Defining

$$J(k) = \eta^T(k+1)P(k+1)\eta(k+1) - \eta^T(k)P(k)\eta(k),$$

we obtain that

$$\begin{aligned} &\mathbb{E}\{J(k)\} \\ = &\mathbb{E}\{\xi^T(k)\bar{\Omega}(k)\xi(k) - z^T(k)z(k) + \gamma^2 \varpi^T(k)R\varpi(k)\} \end{aligned}$$

where

$$\begin{aligned} \bar{\Omega}(k) &= \begin{bmatrix} \bar{\Omega}_{11}(k) & * & * \\ \bar{\beta}P(k+1)\mathcal{A}(k) & \bar{\Omega}_{22}(k) & * \\ \bar{\Omega}_{31}(k) & \bar{\Omega}_{32}(k) & \bar{\Omega}_{33}(k) \end{bmatrix}, \\ \xi(k) &= \begin{bmatrix} \eta^T(k) & \mathcal{G}^T(k, x(k)) & \varpi^T(k) \end{bmatrix}^T, \\ \bar{\Omega}_{22}(k) &= \bar{\beta}^2 P(k+1) + \bar{\beta}(1-\bar{\beta})P(k+1), \\ \bar{\Omega}_{11}(k) &= \mathcal{A}^T(k)P(k+1)\mathcal{A}(k) + \bar{\alpha}(1-\bar{\alpha})\Delta \mathcal{A}^T(k) \\ &\quad \times P(k+1)\Delta \mathcal{A}(k) + \left(\hat{B}_{ci}(k) + \Delta \hat{B}_{ci}(k)\right)^T \\ &\quad \times (I_{n_y} \otimes P(k+1)) \left(\hat{B}_{ci}(k) + \Delta \hat{B}_{ci}(k)\right) \\ &\quad + \mathcal{M}^T(k)\mathcal{M}(k) - P(k), \\ \bar{\Omega}_{31}(k) &= \mathcal{D}^T(k)P(k+1)\mathcal{A}(k), \\ \bar{\Omega}_{32}(k) &= \bar{\beta}\mathcal{D}^T(k)P(k+1), \\ \bar{\Omega}_{33}(k) &= \mathcal{D}^T(k)P(k+1)\mathcal{D}(k) - \gamma^2 R. \end{aligned}$$

It follows from the constraint (2) that

$$\|H_I \mathcal{G}(k, x(k))\|^2 \leq \varepsilon(k)\|E(k)H\eta(k)\|^2 \quad (18)$$

and therefore we have

$$\begin{aligned} \mathbb{E}\{J(k)\} &\leq \mathbb{E}\left\{\xi^T(k)\bar{\Omega}(k)\xi(k) - \lambda(k)(\|H_I \mathcal{G}(k, x(k))\|^2\right. \\ &\quad \left. - \varepsilon(k)\|E(k)H\eta(k)\|^2)\right\} - \mathbb{E}\left\{z^T(k)z(k) - \gamma^2 \varpi^T(k)R\varpi(k)\right\} \\ &= \mathbb{E}\left\{\xi^T(k)\tilde{\Omega}(k)\xi(k)\right\} - \mathbb{E}\left\{z^T(k)z(k) - \gamma^2 \varpi^T(k)R\varpi(k)\right\} \end{aligned} \quad (19)$$

where

$$\begin{aligned} \tilde{\Omega}(k) &= \bar{\Omega}(k) + \text{diag}\{\lambda(k)\varepsilon(k)H_E, -\lambda(k)H_I^T H_I, 0\}, \\ H_E &= H^T E^T(k)E(k)H. \end{aligned} \quad (20)$$

It follows from (16) that $\tilde{\Omega}(k) < 0$. Summing up (19) on both sides from 0 to $N-1$ with respect to k , we obtain

$$\begin{aligned} &\sum_{k=0}^{N-1} \mathbb{E}\{J(k)\} \\ &= \mathbb{E}\{\eta^T(N)P(N)\eta(N)\} - \eta^T(0)P(0)\eta(0) \\ &\leq \mathbb{E}\left\{\sum_{k=0}^{N-1} \xi^T(k)\tilde{\Omega}(k)\xi(k)\right\} - \mathbb{E}\left\{\sum_{k=0}^{N-1} (z^T(k)z(k) - \gamma^2 \varpi^T(k)R\varpi(k))\right\}. \end{aligned} \quad (21)$$

Hence, the \mathcal{H}_∞ performance index defined in (15) satisfies

$$J \leq \mathbb{E} \left\{ \sum_{k=0}^{N-1} \xi^T(k) \tilde{\Omega}(k) \xi(k) \right\} - \mathbb{E} \left\{ \eta^T(N) P(N) \eta(N) - \eta^T(0) (P(0) - \gamma^2 S) \eta(0) \right\}. \quad (22)$$

Noting that $P(N) > 0$, $\tilde{\Omega}(k) < 0$ and the initial condition $P(0) \leq \gamma^2 S$, it can be obtained that $J < 0$ and the proof is now complete.

Based on the analysis results with the desired output feedback controllers, we are now ready to solve the controller design problem for system (1) in the following theorem. For convenience of later analysis, we denote

$$\begin{aligned} \Gamma_{11}(k) &= \text{diag} \{ -P(k) + \lambda(k) \varepsilon(k) H_E, -\lambda(k) H_I^T H_I, -\gamma^2 R \}, \\ \bar{D}_0(k) &= \text{diag} \{ D(k), 0 \}, D_R(k) = E_I K(k) R_3(k) + \bar{D}_0(k), \\ G_R(k) &= E_I K(k) R_1(k) + R_2(k) G(k) \bar{C}_c(k), E_I = \begin{bmatrix} 0 & I \end{bmatrix}^T, \\ \Gamma_{21}(k) &= \begin{bmatrix} \bar{A}_0(k) + G_R(k) & \bar{\beta} I & D_R(k) \\ (I_{n_y} \otimes E_I K(k)) R_5(k) & 0 & 0 \end{bmatrix}, \\ \Gamma_{22}(k) &= \text{diag} \{ -Q(k+1), -\hat{Q}(k+1) \}, R_2(k) = \begin{bmatrix} B_1(k) \\ 0 \end{bmatrix}, \\ \Gamma_{31}(k) &= \text{diag} \left\{ B_2(k) G(k) \bar{C}_c(k) + \mathcal{M}_0(k), \sqrt{\bar{\beta}(1-\bar{\beta})} I, 0 \right\}, \\ \Gamma_{33}(k) &= \text{diag} \{ -I, -Q(k+1), -Q(k+1) \}, \\ \Gamma_{42}(k) &= \text{diag} \{ (E_I K(k) E_I)^T, (I_{n_y} \otimes E_I K(k) E_I)^T \}, \\ \Gamma_{44}(k) &= \text{diag} \{ -\psi_1(k) I, -\psi_1(k) I \}, \mathcal{M}_0(k) = \begin{bmatrix} M(k) & 0 \end{bmatrix}, \\ \Gamma_{51}(k) &= \begin{bmatrix} \hat{E}_a(k) & 0 & \hat{E}_b(k) \end{bmatrix}, \Gamma_{52}(k) = \begin{bmatrix} \hat{E}_c(k) & 0 \end{bmatrix}, \\ \hat{E}_a(k) &= \begin{bmatrix} \psi_1(k) \bar{E}_c^T(k) & \psi_1(k) \tilde{E}_{ci}^T(k) & 0 & \psi_2(k) \mathcal{N}^T(k) \end{bmatrix}^T, \\ \hat{E}_b(k) &= \begin{bmatrix} \psi_1(k) \bar{E}_E^T(k) & 0 & 0 & 0 \end{bmatrix}^T, \hat{E}_c(k) = \begin{bmatrix} 0 & 0 & \bar{\alpha} \mathcal{H}_a(k) & 0 \end{bmatrix}^T, \\ \Gamma_{55}(k) &= \text{diag} \{ I_2 \otimes (-\psi_1(k) I), I_2 \otimes (-\psi_2(k) I) \}, \\ \Gamma_{53}(k) &= \begin{bmatrix} 0 & 0 & \hat{E}_d(k) \end{bmatrix}, \hat{E}_d(k) = \begin{bmatrix} 0 & 0 & \Lambda_\alpha \mathcal{H}_a(k) & 0 \end{bmatrix}^T, \\ \bar{A}_0(k) &= \text{diag} \{ A(k), 0 \}, R_3(k) = \text{diag} \{ 0, \bar{E}(k) \}, \\ \tilde{E}_{ci}(k) &= \begin{bmatrix} \chi_1 \bar{E}_{c_1}^T(k) & \chi_2 \bar{E}_{c_2}^T(k) & \cdots & \chi_{n_y} \bar{E}_{c_{n_y}}^T(k) \end{bmatrix}^T, \\ R_{1i}(k) &= \begin{bmatrix} 0 & I \\ \bar{\Xi} C(k) & 0 \end{bmatrix}, R_{4i}(k) = \begin{bmatrix} 0 & 0 \\ C_i(k) & 0 \end{bmatrix}, \\ R_5(k) &= \begin{bmatrix} \chi_1 R_{41}^T(k) & \chi_2 R_{42}^T(k) & \cdots & \chi_{n_y} R_{4n_y}^T(k) \end{bmatrix}^T. \quad (23) \end{aligned}$$

Theorem 2 Consider the closed-loop system (14) with known actuator failure parameter matrix $G(k)$. Let the

disturbance attenuation level $\gamma > 0$, families of positive scalars $\{\varepsilon(k)\}_{0 \leq k \leq N-1} > 0$, the positive definite matrices $R > 0$ and $S > 0$ be given. The \mathcal{H}_∞ performance requirement defined in (15) is achieved for all nonzero $\varpi(k)$ if, with the initial condition $P(0) \leq \gamma^2 S$, there exist families of positive definite matrices $\{P(k)\}_{0 \leq k \leq N} > 0$, $\{Q(k)\}_{0 \leq k \leq N} > 0$, families of real-valued matrices $\{K(k)\}_{0 \leq k \leq N-1}$, $\{\bar{C}_c(k)\}_{0 \leq k \leq N-1}$, families of positive scalars $\{\lambda(k) > 0\}_{0 \leq k \leq N-1}$, $\{\psi_1(k) > 0\}_{0 \leq k \leq N-1}$ and $\{\psi_2(k) > 0\}_{0 \leq k \leq N-1}$ satisfying the following recursive matrix inequalities

$$\Gamma(k) = \begin{bmatrix} \Gamma_{11}(k) & * & * & * & * \\ \Gamma_{21}(k) & \Gamma_{22}(k) & * & * & * \\ \Gamma_{31}(k) & 0 & \Gamma_{33}(k) & * & * \\ 0 & \Gamma_{42}(k) & 0 & \Gamma_{44}(k) & * \\ \Gamma_{51}(k) & \Gamma_{52}(k) & \Gamma_{53}(k) & 0 & \Gamma_{55}(k) \end{bmatrix} < 0 \quad (24)$$

with the parameters updated by

$$P(k+1) = Q^{-1}(k+1) \quad (25)$$

where the other parameters are defined in (23) and Theorem 1. Furthermore, if $(P(k), Q(k+1), K(k), \bar{C}_c(k))$ is the feasible solution of (24), then the output feedback controller parameters in the form of (13) are given as follows:

$$\begin{bmatrix} A_c(k) & B_c(k) \end{bmatrix} = K(k), \quad C_c(k) = \bar{C}_c(k) E_I. \quad (26)$$

Proof: To deal with the parameter uncertainties in (14), we rewrite (16) in the following form:

$$\begin{aligned} & \hat{\Omega}(k) + \tilde{\mathcal{M}}_1(k) \tilde{\mathcal{F}}(k) \tilde{\mathcal{N}}_1(k) + \tilde{\mathcal{N}}_1^T(k) \tilde{\mathcal{F}}^T(k) \tilde{\mathcal{M}}_1^T(k) \\ & + \tilde{\mathcal{M}}_2(k) F(k) \tilde{\mathcal{N}}_2(k) + \tilde{\mathcal{N}}_2^T(k) F^T(k) \tilde{\mathcal{M}}_2^T(k) \\ & < 0 \end{aligned} \quad (27)$$

where

$$\begin{aligned}
\hat{\Omega}(k) &= \begin{bmatrix} \Omega_{11}(k) & * & * \\ 0 & \Omega_{22}(k) & * \\ \hat{\Omega}_{31}(k) & \hat{\Omega}_{32}(k) & \Omega_{33}(k) \end{bmatrix}, \\
\tilde{\mathcal{M}}_1(k) &= \begin{bmatrix} 0 & 0 & 0 & \bar{\mathcal{H}}_a^T(k) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (I_{n_y} \otimes \bar{\mathcal{H}}_a(k))^T & 0 & 0 & 0 \end{bmatrix}^T, \\
\hat{\Omega}_{32}(k) &= \begin{bmatrix} \Omega_{321}(k) & \hat{\Omega}_{322}(k) \end{bmatrix}, \\
\hat{\Omega}_{322}(k) &= \begin{bmatrix} \bar{\mathcal{D}}^T(k) & 0 & 0 & 0 & 0 \end{bmatrix}^T, \\
\tilde{\mathcal{F}}(k) &= \text{diag}\{\bar{F}(k), I_{n_y} \otimes \bar{F}(k)\}, \\
\tilde{\mathcal{N}}_1(k) &= \begin{bmatrix} \bar{E}_c(k) & 0 & \bar{E}_E(k) & 0 & 0 & 0 & 0 & 0 \\ \bar{E}_{ci}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\hat{\Omega}_{31}(k) &= \begin{bmatrix} \bar{\mathcal{A}}^T(k) & \hat{\mathcal{B}}_{ci}^T(k) & \mathcal{M}^T(k) & 0 & 0 \end{bmatrix}^T, \\
\tilde{\mathcal{N}}_2(k) &= \begin{bmatrix} \mathcal{N}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\tilde{\mathcal{M}}_2(k) &= \begin{bmatrix} 0 & 0 & 0 & \bar{\alpha} \mathcal{H}_a^T(k) & 0 & 0 & \sqrt{\bar{\alpha}(1-\bar{\alpha})} \mathcal{H}_a^T(k) & 0 \end{bmatrix}^T.
\end{aligned}$$

It is observed that the following inequality holds:

$$\begin{bmatrix} \hat{\Omega}(k) & * & * & * & * \\ \tilde{\mathcal{M}}_1^T(k) & -\psi_1(k)I & * & * & * \\ \psi_1(k)\tilde{\mathcal{N}}_1(k) & 0 & -\psi_1(k)I & * & * \\ \tilde{\mathcal{M}}_2^T(k) & 0 & 0 & -\psi_2(k)I & 0 \\ \psi_2(k)\tilde{\mathcal{N}}_2(k) & 0 & 0 & 0 & -\psi_2(k)I \end{bmatrix} < 0 \quad (28)$$

In order to avoid partitioning the positive definite matrices $P(k)$ and $Q(k)$, the parameters in (28) are expressed as follows:

$$\begin{aligned}
\bar{\mathcal{A}}(k) &= \bar{\mathcal{A}}_0(k) + E_I K(k) R_1(k) + R_2(k) G(k) \bar{C}_c(k), \\
\bar{\mathcal{D}}(k) &= \bar{\mathcal{D}}_0(k) + E_I K(k) R_3(k), \quad \bar{\mathcal{H}}_a(k) = E_I K(k) E_I, \\
\hat{\mathcal{B}}_{ci}(k) &= (I_{n_y} \otimes E_I K(k)) R_5(k), \quad \bar{\mathcal{B}}_{ci}(k) = E_I K(k) R_{4i}(k) \\
\mathcal{M}(k) &= \mathcal{M}_0(k) + B_2(k) G(k) \bar{C}_c(k) \quad (29)
\end{aligned}$$

where

$$K(k) = \begin{bmatrix} A_c(k) & B_c(k) \end{bmatrix}, \quad \bar{C}_c(k) = \begin{bmatrix} 0 & C_c(k) \end{bmatrix}. \quad (30)$$

Noticing (29) and (30), (24) is obtained by (28) after applying some straightforward algebraic manipulations, and the proof of this theorem is then complete.

In Theorem 2, with known actuator failure parameter, the \mathcal{H}_∞ performance requirement defined in (15) is obtained for the closed-loop system (14) and the output feedback controller is designed based on the recursive

matrix inequalities approach. In the following theorem, a design procedure for the desired controller parameters is given in the case that the failure parameter matrix $G(k)$ is unknown but satisfies the constraints (9)–(12).

Theorem 3 Consider the closed-loop system (14). Let the disturbance attenuation level $\gamma > 0$, families of positive scalars $\{\varepsilon(k)\}_{0 \leq k \leq N-1} > 0$, $\{\mu(k)\}_{0 \leq k \leq N-1} > 0$, the positive definite matrices $R > 0$ and $S > 0$ be given. The \mathcal{H}_∞ performance requirement defined in (15) is achieved for all nonzero $\varpi(k)$ if, with the initial condition $P(0) \leq \gamma^2 S$, there exist families of positive definite matrices $\{P(k)\}_{0 \leq k \leq N} > 0$, $\{Q(k)\}_{0 \leq k \leq N} > 0$, families of real-valued matrices $\{K(k)\}_{0 \leq k \leq N-1}$, $\{\bar{C}_c(k)\}_{0 \leq k \leq N-1}$, families of positive scalars $\{\lambda(k) > 0\}_{0 \leq k \leq N-1}$, $\{\psi_1(k) > 0\}_{0 \leq k \leq N-1}$ and $\{\psi_2(k) > 0\}_{0 \leq k \leq N-1}$ satisfying the following recursive matrix inequalities

$$\tilde{\Gamma}(k) = \begin{bmatrix} \hat{\Gamma}(k) & * & * \\ \mu(k)\hat{\mathcal{B}}(k) & -\mu(k)I & * \\ \hat{C}_c(k) & 0 & -\mu(k)(\bar{G}^T(k)\bar{G}(k))^{-1} \end{bmatrix} < 0 \quad (31)$$

with the parameters updated by

$$P(k+1) = Q^{-1}(k+1) \quad (32)$$

where

$$\begin{aligned}
\hat{\mathcal{B}}(k) &= \begin{bmatrix} 0 & 0 & 0 & R_2^T(k) & 0 & B_2^T(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
\hat{C}_c(k) &= \begin{bmatrix} \bar{C}_c(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\Gamma}(k) &= \begin{bmatrix} \Gamma_{11}(k) & * & * & * & * \\ \hat{\Gamma}_{21}(k) & \Gamma_{22}(k) & * & * & * \\ \hat{\Gamma}_{31}(k) & 0 & \Gamma_{33}(k) & * & * \\ 0 & \Gamma_{42}(k) & 0 & \Gamma_{44}(k) & * \\ \Gamma_{51}(k) & \Gamma_{52}(k) & \Gamma_{53}(k) & 0 & \Gamma_{55}(k) \end{bmatrix}, \\
\hat{\Gamma}_{21}(k) &= \begin{bmatrix} \bar{\mathcal{A}}_0(k) + G_{0R}(k) & \bar{\beta}I & D_R(k) \\ (I_{n_y} \otimes E_I K(k))R_5(k) & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\Gamma}_{31}(k) &= \text{diag}\left\{B_G(k), \sqrt{\bar{\beta}(1-\bar{\beta})}I, 0\right\}, \\
B_G(k) &= B_2(k)G_0(k)\bar{C}_c(k) + \mathcal{M}_0(k), \\
G_{0R}(k) &= E_I K(k)R_1(k) + R_2(k)G_0(k)\bar{C}_c(k). \quad (33)
\end{aligned}$$

Furthermore, if $(P(k), Q(k+1), K(k), \bar{C}_c(k))$ is the feasible solution of (31), then the output feedback controller parameters in the form of (13) are given as follows:

$$\begin{bmatrix} A_c(k) & B_c(k) \end{bmatrix} = K(k), \quad C_c(k) = \bar{C}_c(k)E_I. \quad (34)$$

Table 1
The Reliable Controller Design Algorithm

<i>Step 1.</i>	Give the \mathcal{H}_∞ performance index γ , the positive definite matrices R, S and the state initial condition $\eta(0)$. Select the initial value for matrix $P(0)$ which satisfy the condition $P(0) \leq \gamma^2 S$ and set $k = 0$.
<i>Step 2.</i>	For the sampling instant k , solving the recursive matrix inequality (31) to obtain the values of matrix $Q(k+1)$ as well as the desired controller parameters $\{A_c(k), B_c(k), C_c(k)\}$.
<i>Step 3.</i>	Set $k = k + 1$ and obtain $P(k+1)$ by the parameter update formula (32).
<i>Step 4.</i>	If $k < N$, where N is the maximum number of iterations allowed, then go to Step 2, else go to Step 5.
<i>Step 5.</i>	Stop.

Table 2
Recursive process

k	0	1	2	3	...
$A_c(k)$	$\begin{bmatrix} 0.5314 & 0.5314 \\ 0.5314 & 0.5314 \end{bmatrix}$	$\begin{bmatrix} 0.0141 & 0.0141 \\ 0.0141 & 0.0141 \end{bmatrix}$	$\begin{bmatrix} 0.0791 & 0.0110 \\ 0.0241 & 0.0698 \end{bmatrix}$	$\begin{bmatrix} 0.3136 & 0.3136 \\ 0.3136 & 0.3136 \end{bmatrix}$...
$B_c(k)$	$\begin{bmatrix} -2.0606 & 10.4288 \\ -2.0606 & 10.4288 \end{bmatrix}$	$\begin{bmatrix} -0.0785 & 1.0544 \\ -0.0785 & 1.0544 \end{bmatrix}$	$\begin{bmatrix} -0.0414 & 0.5060 \\ -0.0414 & 0.5060 \end{bmatrix}$	$\begin{bmatrix} -0.3578 & 6.4767 \\ -0.3578 & 6.4767 \end{bmatrix}$...
$C_c(k)$	$\begin{bmatrix} 1.8772 & 1.8772 \end{bmatrix}$	$\begin{bmatrix} 0.5637 & 0.5637 \end{bmatrix}$	$\begin{bmatrix} 2.2341 & 2.2341 \end{bmatrix}$	$\begin{bmatrix} 5.2543 & 5.2543 \end{bmatrix}$...

Proof: From (12), we know that $\Gamma(k)$ in Theorem 2 can be rewritten as

$$\Gamma(k) = \hat{\Gamma}(k) + \hat{B}^T(k) \Delta_G(k) \hat{C}_c(k) + \hat{C}_c^T(k) \Delta_G(k) \hat{B}(k). \quad (35)$$

Noticing inequality (12), it is obtained that

$$\Gamma(k) \leq \hat{\Gamma}(k) + \mu(k) \hat{B}^T(k) \hat{B}(k) + \mu^{-1}(k) \hat{C}_c^T(k) \hat{C}_c(k) \times \tilde{G}^T(k) \tilde{G}(k) = \tilde{\Psi}(k). \quad (36)$$

It can be seen that (31) in Theorem 3 implies that $\Gamma(k) \leq \tilde{\Psi}(k) < 0$. This completes the proof.

By means of Theorem 3, the algorithm for designing the reliable robust controller can be outlined as Table 1.

Remark 2 *In the system model under investigation in this paper, there are mainly six factors that constitute the complexity and complicate the design of reliable controller, which are ROUs, RONs, actuator failure, sensor failures, quantization and time-varying parameters. It can be seen that all these six factors are explicitly reflected in our main results. In Theorem 3, the finite-horizon reliable controller is designed by solving a series of recursive linear matrix inequalities under which both the current system measurement and previous system states are employed to control the current system state. Such a recursive control process is particularly suitable for real-time implementation such as online process control.*

4 An Illustrative Example

In this section, we present a simulation example to illustrate the effectiveness of the proposed reliable controller design scheme for discrete time-varying stochastic

systems with randomly occurring uncertainties, nonlinearities, actuator and sensor failures subject to output quantization. The system data are given as follows:

$$A(k) = \begin{bmatrix} -0.6 & 0.2 \\ 1.1 \sin(5k) & 0.5 \end{bmatrix}, \quad H_a(k) = [0.1 \ 0.3]^T,$$

$$N(k) = [0.2 \ 0], \quad F(k) = \sin(k), \quad B_2(k) = 0.2,$$

$$B_1(k) = \begin{bmatrix} -2 \\ 3 \sin(5k) \end{bmatrix}, \quad D(k) = \begin{bmatrix} 0.1 \sin(3k) \\ -0.3 \end{bmatrix},$$

$$M(k) = [-0.4 \ 0.5 \sin(5k)], \quad \bar{E}(k) = \begin{bmatrix} 0.1 \sin(3k) \\ 0.2 \end{bmatrix},$$

$$C(k) = \begin{bmatrix} -2 + 0.3 \sin(5k) & 0.5 \\ 0 & 1 \end{bmatrix}.$$

The nonlinear function $g(k, x(k))$ is selected as $g(k, x(k)) = 0.5x_1(k)\sin(x_2(k))$. It is easy to see that the constraint (2) is met with $\varepsilon(k) = 1$ and $E(k) = \text{diag}\{0.2, 0.15\}$. The parameters of the logarithmic quantizer are chosen as $\hat{\mu}_0 = 2$, $\bar{\chi}_1 = 0.8$ and $\bar{\chi}_2 = 0.5$.

Let $\bar{\alpha} = 0.9$ and $\bar{\beta} = 0.8$. Assume that the probability density functions of $\rho_1(k)$ and $\rho_2(k)$ in $[0, 1]$ are described by

$$\zeta_1(s_1) = \begin{cases} 0 & s_1 = 0 \\ 0.1 & s_1 = 0.5 \\ 0.9 & s_1 = 1 \end{cases}, \quad \zeta_2(s_2) = \begin{cases} 0 & s_2 = 0 \\ 0.2 & s_2 = 0.5 \\ 0.8 & s_2 = 1 \end{cases}, \quad (37)$$

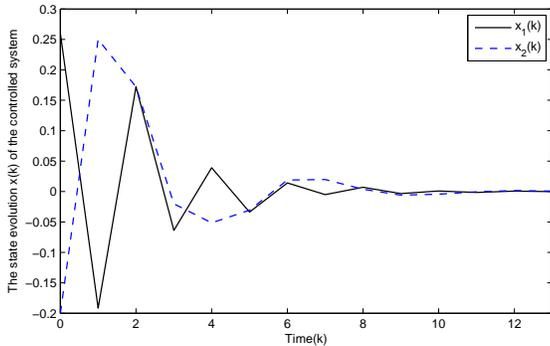


Fig. 1. The state responses of the controlled system

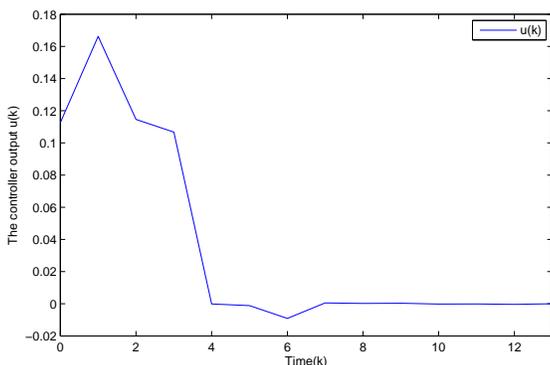


Fig. 2. The controller output $u(k)$

from which the expectations and variances can be easily calculated as $\bar{\mu}_1 = 0.95$, $\bar{\mu}_2 = 0.9$, $\chi_1^2 = 0.15^2$ and $\chi_2^2 = 0.04$.

The actuator fault matrix $G(k)$ is assumed to satisfy $0.85 \leq G(k) \leq 0.9$. Then, we can obtain that $G_0(k) = 0.875$ and $\tilde{G}(k) = 0.025$. In the simulation, let $\mu(k) = 1$. The \mathcal{H}_∞ performance level is chosen as $\gamma = 1$, the initial values of the states are $x_0 = \begin{bmatrix} 0.26 & -0.2 \end{bmatrix}^T$, $\hat{x}_0 = \begin{bmatrix} 0.2 & -0.16 \end{bmatrix}^T$ and the positive definite matrices $R = \text{diag}\{2, 2\}$ and $S = \text{diag}\{2, 2, 2, 2\}$. The exogenous disturbance input is selected as $w(k) = 0.5 \sin(4k)$ and $v(k) = 0.2 \cos(4k)$. According to reliable controller design algorithm, the desired controller parameters in Theorem 3 can be solved recursively subject to given initial conditions and prespecified performance index. Table 2 lists the desired parameters of reliable controller $A_c(k)$, $B_c(k)$ and $C_c(k)$ from the time $k = 0$ to $k = 3$.

The simulation results are shown in Figs. 1-2, where Fig. 1 plots the state simulation results of the closed-loop system (14) and Fig. 2 depicts the controller output. The simulation results confirm that the desired finite-horizon performance is well achieved and the proposed reliable controller design algorithm is indeed effective.

5 Conclusion

In this paper, the finite-horizon reliable \mathcal{H}_∞ output-feedback control problem has been investigated for a class of discrete time-varying systems with ROUs, RONs as well as measurement quantizations. The actuator failures have been quantified by a variable varying in a given interval and the sensor failures have been governed by an individual random variable satisfying a certain probabilistic distribution in the interval $[0, 1]$. Both the RONs and the ROUs have been modeled by the Bernoulli distributed white sequences with known conditional probabilities. In the presence of output quantization, a time-varying output feedback controller has been designed to preserve a guaranteed \mathcal{H}_∞ performance. A simulation example has been utilized to demonstrate the effectiveness of the finite-horizon reliable control techniques presented in this paper. Other possible future research directions include real-time applications of the proposed reliable control theory in telecommunications, and further extensions of the present results to more complex systems with unreliable communication links, such as sampled data systems, bilinear systems, and a more general class of nonlinear stochastic systems.

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