A Modified Schur Method for Robust Pole Assignment in State Feedback Control *

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Abstract

Recently, a **SCHUR** method was proposed in [8] to solve the robust pole assignment problem in state feedback control. It takes the departure from normality of the closed-loop system matrix A_c as the measure of robustness, and intends to minimize it via the real Schur form of A_c . The **SCHUR** method works well for real poles, but when complex conjugate poles are involved, it does not produce the real Schur form of A_c and can be problematic. In this paper, we put forward a modified Schur method, which improves the efficiency of **SCHUR** when complex conjugate poles are to be assigned. Besides producing the real Schur form of A_c , our approach also leads to a relatively small departure from normality of A_c . Numerical examples show that our modified method produces better or at least comparable results than both **place** and **robpole** algorithms, with much less computational costs.

Key words. pole assignment, state feedback control, robustness, departure from normality, real Schur form

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1 Introduction

Let the matrix pair (A, B) denotes the dynamic state equation

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1.1}$$

of the time invariant linear system, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the open-loop system matrix and the input matrix, respectively. The dynamic behavior of (1.1) is governed by the eigen-structure of A, especially the poles (eigenvalues). And in order to change the dynamic behavior of the open-loop system (1.1) in some desirable way (to achieve stability or to speed up response), one needs to modify the poles of (1.1). Typically, this may be actualized by the state-feedback control

$$u(t) = Fx(t), \tag{1.2}$$

where the feedback matrix $F \in \mathbb{R}^{m \times n}$ is to be chosen such that the closed-loop system

$$\dot{x}(t) = (A + BF)x(t) \equiv A_c x(t) \tag{1.3}$$

has specified poles.

Mathematically, the *state-feedback pole assignment problem* can be stated as:

State-Feedback Pole Assignment Problem (SFPA) Given $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and a set of *n* complex numbers $\mathfrak{L} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, closed under complex conjugation, find an $F \in \mathbb{R}^{m \times n}$ such that $\lambda(A + BF) = \mathfrak{L}$, where $\lambda(A + BF)$ is the eigenvalue set of A + BF.

A necessary and sufficient condition for the solvability of the **SFPA** for any set \mathfrak{L} of n selfconjugate complex numbers is that (A, B) is controllable, or equivalently, the controllability matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is of full row rank [25–27]. Many algorithms have been put forward to solve the **SFPA**, such as the invariant subspace method [18], the QR-like method [15,16], etc.. We refer readers to [3,4,7,10,12,17,20,24] for some other approaches.

When m > 1, the solution to the **SFPA** is generally not unique. We may then utilize the freedom of F to achieve some other desirable properties of the closed-loop system. In applications, one sympathetic character for system design is that the eigenvalues of the closed-loop system matrix A_c are insensitive to perturbations, which leads to the following *state-feedback robust pole* assignment problem:

State-Feedback Robust Pole Assignment Problem (SFRPA) Find a solution $F \in \mathbb{R}^{m \times n}$ to the SFPA, such that the closed-loop system is robust, that is, the eigenvalues of A_c are as insensitive to perturbations on A_c as possible.

The key to solve the **SFRPA** is to choose an appropriate measure of robustness formulated in quantitative form. Some measures can be found in [5, 8, 9, 13, 27], such as the condition number measurement $\kappa_F(X) = ||X||_F ||X^{-1}||_F$, where X is the eigenvector matrix of A_c , the departure from normality $\Delta_F(A_c) = \sqrt{\|A_c\|_F^2 - \sum_{j=1}^{j=n} |\lambda_j|^2}$ and so on. Ramar and Gourishankar [19] made an early contribution to the SFRPA and since then various optimization methods have been proposed based on different measures [5,6,8,9,13,14,23]. The most classic methods should be those proposed by Kautsky, Nichols and Van Dooren in [13], where $\kappa_F(X)$ is used as the measure of robustness of the closed-loop system matrix. However, Method 0 in [13] may fail to converge, Method 1 may suffer from slow convergence, and Method 2/3 may not perform well on ill-conditioned problems. Based on Method 0 in [13], Tits and Yang [23] proposed a method for solving the **SFRPA** by trying to maximize the absolute value of the determinant of the eigenvector matrix X. The optimization processes are iterative, and hence generally expensive. Recently, Chu [8] put forward a Schur-type method for the SFRPA by tending to minimize the departure from normality of the closed-loop system matrix A_c via the Schur decomposition of A_c . It computes the matrices X and T column by column, where $A_c = XTX^{-1}$, X, T are real and T is upper quasi-triangular, such that the strictly block upper triangular elements of matrix T are minimized in each step. If $\lambda_1, \ldots, \lambda_n$ are all real, **SCHUR** [8] will generate an orthogonal matrix X, that is, $A_c = XTX^{-1}$ is the Schur decomposition of A_c . This implies that the departures from normality of A_c and T are the same. Hence the strategy aiming to minimize the departure from normality of T is also pliable to A_c . However, in case of complex conjugate poles, it cannot produce an orthogonal X, suggesting that the departure from normality of A_c is generally not identical to that of T. Hence, although it attempts to optimize the departure from normality of T, that of A_c may still be large.

In this paper, we propose a modified Schur method upon **SCHUR** [8], where poles are assigned via the real Schur decomposition of $A_c = XTX^{\top}$, with X being real orthogonal and T being real upper quasi-triangular. In each step (assigning a real pole or a pair of conjugate poles), one optimization problem arises for purpose of minimizing the departure from normality of T. When assigning a real pole, we improve the efficiency of **SCHUR** by computing the SVD of a matrix, instead of computing the GSVD of a matrix pencil. When assigning a pair of conjugate poles, by exploring the properties of the posed optimization problem, we provide a polished way to obtain its suboptimal solution. Numerical examples show that our method outperforms **SCHUR** when complex conjugate poles are involved. We also compare our method with the MATLAB functions **place** (an implementation of Method 1 in [13]), **robpole** (an implementation of the method in [23]) and the **O-SCHUR** algorithm (an implementation of an optimization method in [8]) on some benchmark examples and randomly generated examples, where numerical results show that our method is comparable in accuracy and robustness, while with lower computational costs.

The paper is organized as follows. In Section 2, we give some preliminaries which will be used in subsequent sections. Our method is developed in Section 3, including both the real case and the complex conjugate case. Numerical results are presented in Section 4. Some concluding remarks are finally drawn in Section 5.

2 Preliminaries

In this section, we briefly review the parametric solutions to the **SFPA**, and the departure from normality.

2.1 Solutions to the SFPA

The parametric solutions to the **SFPA** can be expressed in several ways. In this paper, as in [8], we formulate it by using the real Schur decomposition of $A_c = A + BF$. Assume that the real Schur decomposition of A + BF is

$$A + BF = XTX^{\top}, \tag{2.1}$$

where $X \in \mathbb{R}^{n \times n}$ is orthogonal, $T \in \mathbb{R}^{n \times n}$ is upper quasi-triangular with only 1×1 and 2×2 diagonal blocks.

Without loss of generality, we may assume that B is of full column rank. Let

$$B = Q \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R \\ 0 \end{bmatrix} = Q_1 R$$
(2.2)

be the QR decomposition of B, where $Q \in \mathbb{R}^{n \times n}$ is orthogonal, $Q_1 \in \mathbb{R}^{n \times m}$, and $R \in \mathbb{R}^{m \times m}$ is nonsingular and upper triangular.

It follows from (2.1) that

$$AX + BFX - XT = 0. (2.3)$$

Pre-multiplying (2.3) by diag $(R^{-1}, I_{n-m}) \begin{bmatrix} Q_1 & Q_2 \end{bmatrix}^{\top}$ on both sides gives

$$\begin{cases} R^{-1}Q_1^{\top}AX + FX - R^{-1}Q_1^{\top}XT = 0, \\ Q_2^{\top}(AX - XT) = 0. \end{cases}$$
(2.4)

Consequently, if we get an orthogonal matrix X and an upper quasi-triangular matrix T from the second equation of (2.4), then a solution F to the **SFPA** will be obtained immediately from the first equation of (2.4) as

$$F = R^{-1}Q_1^{\top}(XTX^{\top} - A).$$
(2.5)

2.2 Departure from normality

In this paper, we adopt the departure from normality of $A_c = A + BF$ as a measure of robustness of the closed-loop system matrix as in [8], which is defined as ([11,22])

$$\Delta_F(A_c) = \sqrt{\|A_c\|_F^2 - \sum_{j=1}^n |\lambda_j|^2},$$

where $\lambda_1, \ldots, \lambda_n$ are the poles to be assigned, and hence eigenvalues of A_c . Now let D be the block diagonal part of T with only 1×1 and 2×2 blocks on its diagonal. Each 1×1 block of D admits a real eigenvalue λ_j of T, while each 2×2 block of D admits a pair of conjugate eigenvalues $\lambda_j = \alpha_j + i\beta_j, \lambda_{j+1} = \bar{\lambda}_j$ and is of the form $D_j = \begin{bmatrix} \alpha_j & \delta_j \beta_j \\ -\frac{\beta_j}{\delta_j} & \alpha_j \end{bmatrix} \in \mathbb{R}^{2 \times 2}$ with $\delta_j \beta_j \neq 0$, where δ_j is some real number. Let $N = T - D = [\breve{v}_1 \quad \breve{v}_2 \quad \cdots \quad \breve{v}_n]$ be the strictly upper quasi-triangular part of T with $\breve{v}_k = \begin{bmatrix} v_k^\top & 0 \end{bmatrix}^\top, v_k \in \mathbb{R}^{k-1}$ or \mathbb{R}^{k-2} . Direct calculations give rise to

$$\Delta_F^2(A_c) = \Delta_F^2(T) = \|N\|_F^2 + \sum_j (\delta_j - \frac{1}{\delta_j})^2 \beta_j^2, \qquad (2.6)$$

where the summation is over all 2×2 blocks of D.

When all poles $\lambda_1, \ldots, \lambda_n$ are real, the second part of $\Delta_F^2(A_c)$ in (2.6) will vanish. However, when some poles are non-real, not only the strictly block upper triangular part N contributes to the departure from normality, but also the block diagonal part D. When some $|\delta_j|$ is large or close to zero, the second term can be pretty large, which means that it is not negligible.

3 Solving the SFRPA via the real Schur form

In this section, we solve the **SFRPA** by finding an orthogonal matrix $X = [x_1 \ x_2 \ \cdots \ x_n]$ and an upper quasi-triangular matrix T = D + N satisfying the second equation of (2.4), such that $\Delta_F^2(A_c)$ in (2.6) is minimized. Obtaining a global optimization solution to the problem $\min\{\Delta_F^2(A_c)\}$ is rather difficult. In this paper, we propose an efficient method to get a suboptimal solution, which balances the contributions of N and D to the departure from normality. As in [8], we compute the matrices X and T column by column.

For any matrix S, we denote its range space and null space by $\mathcal{R}(S)$ and $\mathcal{N}(S)$, respectively.

Assume that we have already obtained $X_j = \begin{bmatrix} x_1 & x_2 & \cdots & x_j \end{bmatrix} \in \mathbb{R}^{n \times j}$ and $T_j \in \mathbb{R}^{j \times j}$ satisfying

$$Q_2^{\top}(AX_j - X_jT_j) = 0, \qquad X_j^{\top}X_j = I_j,$$
(3.1)

where T_j is upper quasi-triangular and $\lambda(T_j) = \{\lambda_k\}_{k=1}^{k=j}$. We then are to assign the pole λ_{j+1} (if λ_{j+1} is real) or poles $\lambda_{j+1}, \bar{\lambda}_{j+1}$ (if λ_{j+1} is non-real) to get x_{j+1}, \bar{v}_{j+1} or $x_{j+1}, x_{j+2}, \bar{v}_{j+1}, \bar{v}_{j+2}$, such that the departure from normality of A_c is optimized in some sense. This procedure is repeated until all columns of X and T are acquired, and eventually a solution F to the **SFRPA** would be computed from (2.5). In the following subsections we will distinguish two different cases when λ_{j+1} is real or non-real.

Before this, we shall show how to get the first one (two) column(s) of X and T. If λ_1 is real, the first column of T is then $\lambda_1 e_1$, or $T_1 = \lambda_1$, and the first column x_1 of X must satisfy

$$Q_2^{\top}(A - \lambda_1 I_n) x_1 = 0, (3.2)$$

and $||x_1||_2 = 1$. Let the columns of $S \in \mathbb{R}^{n \times r}$ be an orthonormal basis of $\mathcal{N}(Q_2^{\top}(A - \lambda_1 I_n))$, then x_1 can be chosen to be any unit vector in $\mathcal{R}(S)$. We take

$$x_1 = (S \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^{\top}) / \|S \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^{\top} \|_2$$
 (3.3)

in our algorithm as in [8], and then initially set $X_1 = x_1, T_1 = \lambda_1$.

If $\lambda_1 = \alpha_1 + i\beta_1$ is non-real, to get the real Schur form, we should place $\bar{\lambda}_1 = \alpha_1 - i\beta_1$ together with λ_1 . Notice that T_2 is of the form $T_2 = \begin{bmatrix} \alpha_1 & \delta_1\beta_1 \\ -\beta_1/\delta_1 & \alpha_1 \end{bmatrix}$ with $0 \neq \delta_1 \in \mathbb{R}$, then the first two columns $x_1, x_2 \in \mathbb{R}^n$ of X should be chosen to satisfy

$$Q_2^{\top}(A \begin{bmatrix} x_1 & x_2 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \end{bmatrix} T_2) = 0, \quad x_1^{\top} x_2 = 0, \quad \|x_1\|_2 = \|x_2\|_2 = 1,$$
(3.4)

so that $(\delta_1 - \frac{1}{\delta_1})^2 \beta_1^2$ is minimized, which obviously achieves its minimum when $\delta_1 = 1$. Let the columns of $S \in \mathbb{C}^{n \times r}$ be an orthonormal basis of $\mathcal{N}(Q_2^\top(A - \lambda_1 I_n))$, and $S_R = \operatorname{Re}(S)$, $S_I = \operatorname{Im}(S)$. Direct calculations show that such x_1, x_2 satisfying (3.4) with $\delta_1 = 1$ can be obtained by

$$x_1 = \begin{bmatrix} S_R & -S_I \end{bmatrix} \begin{bmatrix} \gamma_1 & \dots & \gamma_r & \zeta_1 & \dots & \zeta_r \end{bmatrix}^\top, \qquad x_2 = \begin{bmatrix} S_I & S_R \end{bmatrix} \begin{bmatrix} \gamma_1 & \dots & \gamma_r & \zeta_1 & \dots & \zeta_r \end{bmatrix}^\top,$$
(3.5)

with $x_1^{\top} x_2 = 0$ and $||x_1||_2 = ||x_2||_2 = 1$. Clearly,

$$\begin{aligned} x_{1}^{\top}x_{2} + x_{2}^{\top}x_{1} \\ &= \left[\gamma_{1} \quad \dots \quad \gamma_{r} \quad \zeta_{1} \quad \dots \quad \zeta_{r}\right] \begin{bmatrix} S_{R}^{\top}S_{I} + S_{I}^{\top}S_{R} & S_{R}^{\top}S_{R} - S_{I}^{\top}S_{I} \\ S_{R}^{\top}S_{R} - S_{I}^{\top}S_{I} & -(S_{R}^{\top}S_{I} + S_{I}^{\top}S_{R}) \end{bmatrix} \begin{bmatrix} \gamma_{1} \quad \dots \quad \gamma_{r} \quad \zeta_{1} \quad \dots \quad \zeta_{r} \end{bmatrix}^{\top}, \\ &x_{1}^{\top}x_{1} - x_{2}^{\top}x_{2} \\ &= \left[\gamma_{1} \quad \dots \quad \gamma_{r} \quad \zeta_{1} \quad \dots \quad \zeta_{r}\right] \begin{bmatrix} S_{R}^{\top}S_{R} - S_{I}^{\top}S_{I} & -(S_{R}^{\top}S_{I} + S_{I}^{\top}S_{R}) \\ -(S_{R}^{\top}S_{I} + S_{I}^{\top}S_{R}) & S_{I}^{\top}S_{I} - S_{R}^{\top}S_{R} \end{bmatrix} \begin{bmatrix} \gamma_{1} \quad \dots \quad \gamma_{r} \quad \zeta_{1} \quad \dots \quad \zeta_{r} \end{bmatrix}^{\top}. \end{aligned}$$

$$(3.6)$$

Note that the two matrices in the above two equations are symmetric Hamiltonian systems owning special properties. So we exhibit some simple results about symmetric Hamiltonian system which will be used here and when assigning the complex conjugate poles. Both results can be verified directly, and we omit the proof.

Lemma 3.1. Let $A, B \in \mathbb{R}^{n \times n}$ satisfying $A^{\top} = A, B^{\top} = B$. If λ is an eigenvalue of $\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$ and $\begin{bmatrix} x^{\top} & y^{\top} \end{bmatrix}^{\top}$ is the corresponding eigenvector, then

$$\begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} \lambda \\ & -\lambda \end{bmatrix},$$

and

$$\begin{bmatrix} B & -A \\ -A & -B \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \lambda \\ & -\lambda \end{bmatrix}.$$

Lemma 3.2. (Property of Two Hamiltonian Systems) Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric, and let $\begin{bmatrix} A & B \\ B & -A \end{bmatrix} = U \operatorname{diag}(\Theta, -\Theta) U^{\top}$ be the spectral decomposition, where $\Theta = \operatorname{diag}(\theta_1, \theta_2, \dots, \theta_n)$ with $\theta_1 \ge \theta_2 \ge \dots \ge \theta_n \ge 0$. If the j-th column u_j and the (n + j)-th column u_{n+j} of U satisfy $u_{n+j} = \begin{bmatrix} -I_n \\ I_n \end{bmatrix} u_j$, then $\begin{bmatrix} B & -A \\ -A & -B \end{bmatrix} = U \begin{bmatrix} 0 & -\Theta \\ -\Theta & 0 \end{bmatrix} U^{\top}$.

Applying Lemma 3.2 to the two symmetric Hamiltonian systems which appeared in (3.6), that is

$$\begin{bmatrix} S_R^{\top} S_I + S_I^{\top} S_R & S_R^{\top} S_R - S_I^{\top} S_I \\ S_R^{\top} S_R - S_I^{\top} S_I & -(S_R^{\top} S_I + S_I^{\top} S_R) \end{bmatrix} = U \operatorname{diag}(\Theta, -\Theta) U^{\top},$$

$$\begin{bmatrix} S_R^{\top} S_R - S_I^{\top} S_I & -(S_R^{\top} S_I + S_I^{\top} S_R) \\ -(S_R^{\top} S_I + S_I^{\top} S_R) & S_I^{\top} S_I - S_R^{\top} S_R \end{bmatrix} = U \begin{bmatrix} 0 & -\Theta \\ -\Theta & 0 \end{bmatrix} U^{\top},$$

then if we let

$$\begin{bmatrix} \gamma_1 & \dots & \gamma_r & \zeta_1 & \dots & \zeta_r \end{bmatrix}^\top = U \begin{bmatrix} \mu_1 & \dots & \mu_r & \nu_1 & \dots & \nu_r \end{bmatrix}^\top, \tag{3.7}$$

 $x_1^{\top}x_2 + x_2^{\top}x_1 = \sum_{j=1}^r \theta_j(\mu_j^2 - \nu_j^2)$ and $x_1^{\top}x_1 - x_2^{\top}x_2 = -2\sum_{j=1}^r \theta_j \mu_j \nu_j$ follow. Without loss of generality, we may assume that $\theta_1 \ge \theta_2 \ge \ldots \ge \theta_r \ge 0$, then by taking

$$\mu_3 = \nu_3 = \dots = \mu_r = \nu_r = 0, \quad \mu_1 = -\nu_1 = \sqrt{\frac{\theta_2}{\theta_1}\mu_2^2},$$
(3.8a)

$$\mu_2 = \nu_2 = \frac{1}{\| [S_R - S_I] U \left[\sqrt{\frac{\theta_2}{\theta_1}} \ 1 \ 0 \ \cdots \ 0 \ -\sqrt{\frac{\theta_2}{\theta_1}} \ 1 \ 0 \ \cdots \ 0 \right]^\top \|_2},$$
(3.8b)

it is easy to verify that (3.4) holds with x_1 and x_2 computed by (3.5) and (3.7). Hence, we can still choose initial vectors x_1 and x_2 , so that $(\delta_1 - \frac{1}{\delta_1})^2 \beta_1^2 = 0$. We then initially set

$$X_2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}.$$
(3.9)

Now assume that (3.1) has been satisfied with $j \ge 1$, we shall then assign the next pole λ_{j+1} .

3.1 Assigning a real pole

Assume that λ_{j+1} is real, then the (j+1)-th diagonal element of T must be λ_{j+1} . Comparing the (j+1)-th column of $Q_2^{\top}AX - Q_2^{\top}XT = 0$ gives rise to

$$Q_2^{\top} A x_{j+1} - Q_2^{\top} X_j v_{j+1} - \lambda_{j+1} Q_2^{\top} x_{j+1} = 0.$$
(3.10)

Recall the definition of the departure from normality of A_c in (2.6) and notice that we are now computing the (j + 1)-th columns of X and T, it is then natural to consider the following optimization problem:

$$\min_{\|x_{j+1}\|_2=1} \|v_{j+1}\|_2^2 \tag{3.11}$$

s.t.
$$M_{j+1} \begin{bmatrix} x_{j+1} \\ v_{j+1} \end{bmatrix} = 0,$$
 (3.12)

where

$$M_{j+1} = \begin{bmatrix} Q_2^{\top} (A - \lambda_{j+1} I_n) & -Q_2^{\top} X_j \\ X_j^{\top} & 0 \end{bmatrix}.$$
 (3.13)

Let $r = \dim \mathcal{N}(M_{j+1})$. Then it follows from the controllability of (A, B) that $Q_2^{\top}(A - \lambda_{j+1}I_n)$ is of full row rank, indicating that $n - m \leq \operatorname{rank}(M_{j+1}) \leq n - m + j$ and $\mathcal{N}(M_{j+1}) \neq \emptyset$ ([8]). Suppose that the columns of $S = \begin{bmatrix} S_1^{\top} & S_2^{\top} \end{bmatrix}^{\top}$ with $S_1 \in \mathbb{R}^{n \times r}, S_2 \in \mathbb{R}^{j \times r}$ form an orthonormal basis of $\mathcal{N}(M_{j+1})$, then (3.12) shows that

$$x_{j+1} = S_1 y, \quad v_{j+1} = S_2 y, \qquad \forall y \in \mathbb{R}^r.$$

$$(3.14)$$

Consequently, the optimization problem (3.11) subject to (3.12) is equivalent to the following problem:

$$\min_{y^{\top}S_{1}^{\top}S_{1}y=1} y^{\top}S_{2}^{\top}S_{2}y.$$
(3.15)

Perceived that the discussions above can also be found in [8], and the constrained optimization problem (3.15) is solved via the GSVD of the matrix pencil (S_1, S_2) . We put forward a simpler approach here. Actually, since $S^{\top}S = I_r$, we have $S_2^{\top}S_2 = I_r - S_1^{\top}S_1$. Thus the problem (3.15) is equivalent to

$$\min_{y^{\top}S_{1}^{\top}S_{1}y=1} y^{\top}y,$$
(3.16)

whose minimum value is acquired when y is an eigenvector of $S_1^{\top}S_1$ corresponding to its greatest eigenvalue and satisfies $y^{\top}S_1^{\top}S_1y = 1$. Once such y is obtained, x_{j+1} and v_{j+1} can be given by (3.14). We may then update X_j and T_j as

$$X_{j+1} = \begin{bmatrix} X_j & x_{j+1} \end{bmatrix} \in \mathbb{R}^{n \times (j+1)}, \qquad T_{j+1} = \begin{bmatrix} T_j & v_{j+1} \\ 0 & \lambda_{j+1} \end{bmatrix} \in \mathbb{R}^{(j+1) \times (j+1)}, \tag{3.17}$$

and continue with the next pole λ_{j+2} .

3.2 Assigning a pair of conjugate poles

In this subsection, we will consider the case that λ_{j+1} is non-real. To obtain a real matrix F from the real Schur form of $A_c = A + BF$, we would assign λ_{j+1} and $\lambda_{j+2} = \overline{\lambda}_{j+1}$ simultaneously to get the (j + 1)-th and (j + 2)-th columns of X and T.

3.2.1 Initial optimization problem

Assume that $\lambda_{j+1} = \alpha_{j+1} + i\beta_{j+1} \ (\beta_{j+1} \neq 0)$ and let $D_{\delta} = \begin{bmatrix} \alpha_{j+1} & \delta\beta_{j+1} \\ -\beta_{j+1}/\delta & \alpha_{j+1} \end{bmatrix}$ be the diagonal block in T whose eigenvalues are λ_{j+1} and $\bar{\lambda}_{j+1}$. By comparing the (j+1)-th and (j+2)-th columns of $Q_2^{\top}AX - Q_2^{\top}XT = 0$, we have

$$Q_2^{\top} A [x_{j+1} \quad x_{j+2}] - Q_2^{\top} X_j [v_{j+1} \quad v_{j+2}] - Q_2^{\top} [x_{j+1} \quad x_{j+2}] D_{\delta} = 0.$$
(3.18)

Recalling the form of $\Delta_F^2(A_c)$ in (2.6), it is then natural to consider the following optimization problem:

$$\min_{\delta, v_{j+1}, v_{j+2}} \|v_{j+1}\|_2^2 + \|v_{j+2}\|_2^2 + \beta_{j+1}^2 (\delta - \frac{1}{\delta})^2$$
(3.19a)

s.t.
$$Q_2^{\top}(A[x_{j+1} \ x_{j+2}] - X_j[v_{j+1} \ v_{j+2}] - [x_{j+1} \ x_{j+2}]D_{\delta}) = 0,$$
 (3.19b)

$$X_j^{\top} \begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix} = 0, \tag{3.19c}$$

$$\begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix}^{\top} \begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix} = I_2.$$
(3.19d)

The constraints (3.19b) and (3.19d) are nonlinear. In [8], the author solves this optimization problem by taking $\delta = 1$ and neglecting the orthogonal requirement $x_{j+1}^{\top}x_{j+2} = 0$. These simplify the problem significantly. However, it cannot lead to the real Schur form of the closedloop system matrix A_c , since x_{j+1} is generally not orthogonal to x_{j+2} . Moreover, the minimum value of the simplified optimization problem in [8] may be much greater than that of the original problem (3.19).

We may rewrite the optimization problem (3.19) into another equivalent form. If we write $\delta = \frac{\delta_2}{\delta_1}$ with $\delta_1, \delta_2 > 0$, and set $D_0 = \begin{bmatrix} \alpha_{j+1} & \beta_{j+1} \\ -\beta_{j+1} & \alpha_{j+1} \end{bmatrix}$, then $D_{\delta} = \begin{bmatrix} 1/\delta_1 \\ 1/\delta_2 \end{bmatrix} D_0 \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$. Redefine $x_{j+1} \triangleq \frac{x_{j+1}}{\delta_1}, x_{j+2} \triangleq \frac{x_{j+2}}{\delta_2}, v_{j+1} \triangleq \frac{v_{j+1}}{\delta_1}, v_{j+2} \triangleq \frac{v_{j+2}}{\delta_2}$, then the optimization problem (3.19) is equivalent to

$$\min_{\delta_1, \delta_2, v_{j+1}, v_{j+2}} \|\delta_1 v_{j+1}\|_2^2 + \|\delta_2 v_{j+2}\|_2^2 + \beta_{j+1}^2 (\frac{\delta_1}{\delta_2} - \frac{\delta_2}{\delta_1})^2$$
(3.20a)

s.t.
$$Q_2^{\top}(A[x_{j+1} \quad x_{j+2}] - X_j[v_{j+1} \quad v_{j+2}] - [x_{j+1} \quad x_{j+2}]D_0) = 0,$$
 (3.20b)

$$X_j^{\top} [x_{j+1} \quad x_{j+2}] = 0, \tag{3.20c}$$

$$\begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix}^{\top} \begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix} = \begin{bmatrix} 1/\delta_1^2 \\ & 1/\delta_2^2 \end{bmatrix}.$$
 (3.20d)

Here the constraint (3.20b) becomes linear. Once a solution to the optimization problem (3.20) is obtained, we need to redefine

$$v_{j+1} \triangleq \frac{v_{j+1}}{\|x_{j+1}\|_2}, \quad v_{j+2} \triangleq \frac{v_{j+2}}{\|x_{j+2}\|_2}, \quad x_{j+1} \triangleq \frac{x_{j+1}}{\|x_{j+1}\|_2}, \quad x_{j+2} \triangleq \frac{x_{j+2}}{\|x_{j+2}\|_2}$$

as the corresponding columns of T and X.

The constraints (3.20b) and (3.20c) are linear. Actually, all vectors $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying these two constraints can be found via the null space of the matrix

$$M_{j+1} = \begin{bmatrix} Q_2^\top (A - (\alpha_{j+1} + i\beta_{j+1})I_n) & -Q_2^\top X_j \\ X_j^\top & 0 \end{bmatrix}.$$
 (3.21)

Specifically, for any $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying (3.20b) and (3.20c), direct calculations show that $M_{j+1} \begin{bmatrix} x_{j+1} + ix_{j+2} \\ v_{j+1} + iv_{j+2} \end{bmatrix} = 0$. Conversely, for any vector $\begin{bmatrix} z^{\top} & w^{\top} \end{bmatrix}^{\top} \in \mathcal{N}(M_{j+1})$, the vectors $x_{j+1} = \operatorname{Re}(z), x_{j+2} = \operatorname{Im}(z), v_{j+1} = \operatorname{Re}(w), v_{j+2} = \operatorname{Im}(w)$ satisfy (3.20b) and (3.20c). The constraint (3.20d) shows that $x_{j+1}^{\top}x_{j+2} = 0$. For any vector $\begin{bmatrix} z^{\top} & w^{\top} \end{bmatrix}^{\top} \in \mathcal{N}(M_{j+1})$ with $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ being linearly independent, we may then orthogonalize $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ by the Jacobi transformation as follows to get x_{j+1} and x_{j+2} satisfying $x_{j+1}^{\top}x_{j+2} = 0$. Let $\varrho_1 = \|\operatorname{Re}(z)\|_2^2$, $\varrho_2 = \|\operatorname{Im}(z)\|_2^2$, $\gamma = \operatorname{Re}(z)^{\top}\operatorname{Im}(z)$ and $\tau = \frac{\rho_2 - \rho_1}{2\gamma}$, and define t as

$$t = \begin{cases} 1/(\tau + \sqrt{1 + \tau^2}), & \text{if } \tau \ge 0, \\ -1/(-\tau + \sqrt{1 + \tau^2}), & \text{if } \tau < 0. \end{cases}$$

Let $c = 1/\sqrt{1+t^2}$, s = tc. Then x_{j+1} and x_{j+2} obtained by

$$\begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix} = \begin{bmatrix} \operatorname{Re}(z) & \operatorname{Im}(z) \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$
(3.22)

satisfy $x_{j+1}^{\top} x_{j+2} = 0$. Moreover, if we let

$$\begin{bmatrix} v_{j+1} & v_{j+2} \end{bmatrix} = \begin{bmatrix} \operatorname{Re}(w) & \operatorname{Im}(w) \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$
(3.23)

then $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfy (3.20b) and (3.20c). Hence, we can get $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying the constraints (3.20b)-(3.20d) in this way. Furthermore,

$$1/\delta_1^2 = \|x_{j+1}\|_2^2 = \|x\|_2^2 - \omega, \quad 1/\delta_2^2 = \|x_{j+2}\|_2^2 = \|y\|_2^2 + \omega, \tag{3.24}$$

where $x = \operatorname{Re}(z), y = \operatorname{Im}(z), \omega = \frac{2(x^{\top}y)^2}{\|y\|_2^2 - \|x\|_2^2 + \sqrt{4(x^{\top}y)^2 + (\|y\|_2^2 - \|x\|_2^2)^2}}$ if $\|x\|_2 < \|y\|_2$; and $\omega = \frac{2(x^{\top}y)^2}{\|y\|_2^2 - \|x\|_2^2 - \sqrt{4(x^{\top}y)^2 + (\|y\|_2^2 - \|x\|_2^2)^2}}$ if $\|x\|_2 \ge \|y\|_2$.

3.2.2 The suboptimal strategy

It is hard to get an optimal solution to (3.20) since it is a nonlinear optimization problem with quadratic constraints. Even if such an optimal solution can be found, the cost will be expensive. So instead of finding an optimal solution, we prefer to get a suboptimal one with less computational cost.

Let the columns of $S = \begin{bmatrix} S_1^\top & S_2^\top \end{bmatrix}^\top \in \mathbb{C}^{(n+j)\times r}$ with $S_1 \in \mathbb{C}^{n\times r}$ and $S_2 \in \mathbb{C}^{j\times r}$ form an orthonormal basis of $\mathcal{N}(M_{j+1})$, and let $S_1 = U\Sigma V^*$ be the SVD of S_1 . Since $S_1^*S_1 + S_2^*S_2 = I_r$, it follows that $S_2^*S_2 = V(I_r - \Sigma^*\Sigma)V^*$. For any vector $\begin{bmatrix} z^\top & w^\top \end{bmatrix}^\top \in \mathcal{N}(M_{j+1})$ with $z \in \mathbb{C}^n$ and $w \in \mathbb{C}^j$, there exists $b \in \mathbb{C}^r$ such that $z = S_1 b = U(\Sigma V^* b)$ and $w = S_2 b$. Hence

$$||z||_2 \le \sigma_1 ||b||_2$$
 and $||w||_2^2 \ge (1 - \sigma_1^2) ||b||_2^2$

where σ_1 is the largest singular value of S_1 . Now suppose that the real part and the imaginary part of z are linearly independent satisfying $\|\operatorname{Re}(z)\|_2 \leq \|\operatorname{Im}(z)\|_2$, and $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ are obtained from the the Jacobi orthogonal process (3.22), (3.23). Define $C = \frac{\|z\|_2}{\|x_{j+1}\|_2}$, then $C \geq \sqrt{2}$ and the objective function in (3.20a) becomes

$$\begin{aligned} \|\delta_1 v_{j+1}\|_2^2 + \|\delta_2 v_{j+2}\|_2^2 + \beta_{j+1}^2 (\frac{\delta_1}{\delta_2} - \frac{\delta_2}{\delta_1})^2 \\ = & \frac{C^2}{C^2 - 1} \frac{\|w\|_2^2}{\|z\|_2^2} + \frac{C^4 - 2C^2}{C^2 - 1} \frac{\|v_{j+1}\|_2^2}{\|z\|_2^2} + \beta_{j+1}^2 (C^2 - 3 + \frac{1}{C^2 - 1}). \end{aligned}$$
(3.25)

Obviously,

$$\frac{C^2}{C^2 - 1} \frac{\|w\|_2^2}{\|z\|_2^2} \le \frac{C^2}{C^2 - 1} \frac{\|w\|_2^2}{\|z\|_2^2} + \frac{C^4 - 2C^2}{C^2 - 1} \frac{\|v_{j+1}\|_2^2}{\|z\|_2^2} \le C^2 \frac{\|w\|_2^2}{\|z\|_2^2}.$$
(3.26)

So the objective function in (3.20a) depends on $\frac{\|w\|_2^2}{\|z\|_2^2}$ and C with $\min \frac{\|w\|_2^2}{\|z\|_2^2} = \frac{1-\sigma_1^2}{\sigma_1^2}$. In our suboptimal strategy, we will first take b from span{ Ve_1 }, where e_i is the *i*-th column of the identity matrix. With this choice, $\frac{\|w\|_2^2}{\|z\|_2^2}$ achieves its minimum value. And the following theorem shows the relevant results.

Theorem 3.1. With the notations above, let u_1 be the first column of U and assume that $Re(u_1)$ and $Im(u_1)$ are linearly independent. Let x_{j+1} and x_{j+2} be the vectors obtained from $Re(u_1)$ and $Im(u_1)$ via the Jacobi orthogonal process

$$\begin{bmatrix} x_{j+1} & x_{j+2} \end{bmatrix} = \begin{bmatrix} Re(u_1) & Im(u_1) \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix},$$

and let

$$\begin{bmatrix} v_{j+1} & v_{j+2} \end{bmatrix} = \begin{bmatrix} Re(w) & Im(w) \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

where $w = S_2 V e_1/\sigma_1$. Then $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfy the constraints (3.20b)-(3.20d), and the value of the corresponding objective function specified by (3.20a) will be no larger than

$$\frac{1}{\min\{\|x_{j+1}\|_2^2, \|x_{j+2}\|_2^2\}} (\frac{1-\sigma_1^2}{\sigma_1^2} + \beta_{j+1}^2).$$

Proof. The first part of the theorem is obvious. To prove the second part, note that here $b = \frac{Ve_1}{\sigma_1}$, $\|z\|_2 = \|u_1\|_2 = 1, \|w\|_2^2 = \frac{1-\sigma_1^2}{\sigma_1^2}$. If $\|\operatorname{Re}(u_1)\|_2 \leq \|\operatorname{Im}(u_1)\|_2$, it then follows directly from (3.25), (3.26) and $C^2 - 3 + \frac{1}{C^2 - 1} \leq C^2$ with $C = \frac{1}{\|x_{j+1}\|_2}$. The case when $\|\operatorname{Re}(u_1)\|_2 \geq \|\operatorname{Im}(u_1)\|_2$ can be proved similarly.

Theorem 3.1 shows that if $\operatorname{Re}(u_1)$ and $\operatorname{Im}(u_1)$ are linearly independent, and $\min\{||x_{j+1}||_2, ||x_{j+2}||_2\}$ is not pathologically small, the above procedure will generate $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying the constrains (3.20b)-(3.20d), and the value of the corresponding objective function in (3.20a) is not too large. We then take these $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ as the suboptimal solution. However, if $\operatorname{Re}(u_1)$ and $\operatorname{Im}(u_1)$ are linearly dependent, we cannot get orthogonal x_{j+1} and x_{j+2} via the Jacobi orthogonal process. Even if $\operatorname{Re}(u_1)$ and $\operatorname{Im}(u_1)$ are linearly independent, the resulted $\min\{||x_{j+1}||_2, ||x_{j+2}||_2\}$ might be fairly small, which means that the corresponding value of the objective function might be large. In this case, we would choose b from $\operatorname{span}\{Ve_1, Ve_2\}$.

Define

$$\tilde{x}_{1} + i\tilde{y}_{1} = z_{1} = u_{1} = \frac{S_{1}Ve_{1}}{\sigma_{1}}, \qquad \qquad w_{1} = \frac{S_{2}Ve_{1}}{\sigma_{1}}, \\ \tilde{x}_{2} + i\tilde{y}_{2} = z_{2} = u_{2} = \frac{S_{1}Ve_{2}}{\sigma_{2}}, \qquad \qquad w_{2} = \frac{S_{2}Ve_{2}}{\sigma_{2}}, \qquad (3.27)$$

where σ_1, σ_2 are the first two greatest singular values of S_1 . Let $b = \begin{bmatrix} \frac{Ve_1}{\sigma_1} & \frac{Ve_2}{\sigma_2} \end{bmatrix} \begin{bmatrix} \gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2 \end{bmatrix}$ with $\gamma_1^2 + \gamma_2^2 + \zeta_1^2 + \zeta_2^2 = 1$, then

$$x + iy = z = S_1 b = \begin{bmatrix} z_1 & z_2 \end{bmatrix} \begin{bmatrix} \gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2 \end{bmatrix}, \quad w = S_2 b = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \gamma_1 + i\zeta_1 \\ \gamma_2 + i\zeta_2 \end{bmatrix}.$$
 (3.28)

Denoting $\tilde{X} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \end{bmatrix}$, $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 & \tilde{y}_2 \end{bmatrix}$, it can be easily verified that

$$x = \begin{bmatrix} \tilde{X} & -\tilde{Y} \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \qquad y = \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^\top, \tag{3.29}$$

and

$$x^{\top}y + y^{\top}x = \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix} \begin{bmatrix} \tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X} & \tilde{X}^{\top}\tilde{X} - \tilde{Y}^{\top}\tilde{Y} \\ \tilde{X}^{\top}\tilde{X} - \tilde{Y}^{\top}\tilde{Y} & -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^{\top}, \quad (3.30)$$
$$x^{\top}x - y^{\top}y = \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix} \begin{bmatrix} \tilde{X}^{\top}\tilde{X} - \tilde{Y}^{\top}\tilde{Y} & -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) \\ -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) & \tilde{Y}^{\top}\tilde{Y} - \tilde{X}^{\top}\tilde{X} \end{bmatrix} \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^{\top}. \quad (3.31)$$

Obviously, the two matrices in (3.30) and (3.31) are symmetric Hamiltonian systems and they satisfy the property in Lemma 3.2. Hence we can get the following lemma.

Lemma 3.3. Let ϕ_m, ϕ_M be the two smallest singular values of $\begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix}$ and $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}, \begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ be the corresponding right singular vectors respectively. Define

$$\Omega = \begin{bmatrix} p_1 & p_2 & -q_1 & -q_2 \\ q_1 & q_2 & p_1 & p_2 \end{bmatrix},$$
(3.32)

 $\Phi = \text{diag}(\phi_1, \phi_2, -\phi_1, -\phi_2)$ with $\phi_1 = 1 - 2\phi_m^2$, $\phi_2 = 1 - 2\phi_M^2$, then

$$\begin{bmatrix} \tilde{X}^{\top}\tilde{X} - \tilde{Y}^{\top}\tilde{Y} & -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) \\ -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) & \tilde{Y}^{\top}\tilde{Y} - \tilde{X}^{\top}\tilde{X} \end{bmatrix} = \Omega\Phi\Omega^{\top},$$
(3.33)

and

$$\begin{bmatrix} \tilde{X}^{\top} \tilde{Y} + \tilde{Y}^{\top} \tilde{X} & \tilde{X}^{\top} \tilde{X} - \tilde{Y}^{\top} \tilde{Y} \\ \tilde{X}^{\top} \tilde{X} - \tilde{Y}^{\top} \tilde{Y} & -(\tilde{X}^{\top} \tilde{Y} + \tilde{Y}^{\top} \tilde{X}) \end{bmatrix} = \Omega \begin{pmatrix} \phi_1 & \\ \phi_2 & \\ \phi_1 & \\ \phi_2 & \\ \phi$$

Proof. Since $(\tilde{X}^{\top} - i\tilde{Y}^{\top})(\tilde{X} + i\tilde{Y}) = \begin{bmatrix} z_1 & z_2 \end{bmatrix}^* \begin{bmatrix} z_1 & z_2 \end{bmatrix} = I_2$, so $\tilde{X}^{\top}\tilde{X} + \tilde{Y}^{\top}\tilde{Y} = I_2$ and $\tilde{X}^{\top}\tilde{Y} = \tilde{Y}^{\top}\tilde{X}$. Thus

$$\begin{bmatrix} \tilde{X}^{\top}\tilde{X} - \tilde{Y}^{\top}\tilde{Y} & -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) \\ -(\tilde{X}^{\top}\tilde{Y} + \tilde{Y}^{\top}\tilde{X}) & \tilde{Y}^{\top}\tilde{Y} - \tilde{X}^{\top}\tilde{X} \end{bmatrix} = \begin{bmatrix} I_2 - 2\tilde{Y}^{\top}\tilde{Y} & -2\tilde{Y}^{\top}\tilde{X} \\ -2\tilde{X}^{\top}\tilde{Y} & I_2 - 2\tilde{X}^{\top}\tilde{X} \end{bmatrix} = I_4 - 2\begin{bmatrix} \tilde{Y}^{\top} \\ \tilde{X}^{\top} \end{bmatrix} \begin{bmatrix} \tilde{Y} & \tilde{X} \end{bmatrix}.$$

From the above equation, it obviously holds that ϕ_1, ϕ_2 are the two nonnegative eigenvalues of $\begin{bmatrix} \tilde{X}^\top \tilde{X} - \tilde{Y}^\top \tilde{Y} & -(\tilde{X}^\top \tilde{Y} + \tilde{Y}^\top \tilde{X}) \\ -(\tilde{X}^\top \tilde{Y} + \tilde{Y}^\top \tilde{X}) & \tilde{Y}^\top \tilde{Y} - \tilde{X}^\top \tilde{X} \end{bmatrix}$ with $\begin{bmatrix} p_1 \\ q_1 \end{bmatrix}, \begin{bmatrix} p_2 \\ q_2 \end{bmatrix}$ being the corresponding eigenvectors. Note that $\begin{bmatrix} \tilde{X}^\top \tilde{X} - \tilde{Y}^\top \tilde{Y} & -(\tilde{X}^\top \tilde{Y} + \tilde{Y}^\top \tilde{X}) \\ -(\tilde{X}^\top \tilde{Y} + \tilde{Y}^\top \tilde{X}) & \tilde{Y}^\top \tilde{Y} - \tilde{X}^\top \tilde{X} \end{bmatrix}$ is a Hamiltonian matrix, thus the results follow immediately from Lemma 3.1 and Lemma 3.2. Now by defining

$$\begin{bmatrix} \mu_1 & \mu_2 & \nu_1 & \nu_2 \end{bmatrix}^{\top} = \Omega^{\top} \begin{bmatrix} \gamma_1 & \gamma_2 & \zeta_1 & \zeta_2 \end{bmatrix}^{\top}, \qquad (3.34)$$

we have

$$x^{\top}y + y^{\top}x = 2\phi_1\mu_1\nu_1 + 2\phi_2\mu_2\nu_2, \qquad x^{\top}x - y^{\top}y = \phi_1(\mu_1^2 - \nu_1^2) + \phi_2(\mu_2^2 - \nu_2^2).$$
(3.35)

Theorem 3.2. With the notations above, there exist $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{R}$ such that $x^{\top}y = 0$ and $\|x\|_2 = \|y\|_2 = \frac{\sqrt{2}}{2}$. For these $\mu_1, \mu_2, \nu_1, \nu_2$, let $\gamma_1, \gamma_2, \zeta_1, \zeta_2$ be computed from (3.34), where Ω is as in (3.32). Then $x_{j+1} = x, x_{j+2} = y, v_{j+1} = Re(w)$ and $v_{j+2} = Im(w)$, where w is computed by (3.28), satisfy the constraints (3.20b)-(3.20d), and the value of the corresponding objective function in (3.20a) will be no larger than $\frac{2(1-\sigma_2^2)}{\sigma_2^2}$.

Proof. It is easy to check that all solutions of the following system of equations

$$\begin{cases} \phi_1 \mu_1 \nu_1 + \phi_2 \mu_2 \nu_2 &= 0, \\ \phi_1 (\mu_1^2 - \nu_1^2) + \phi_2 (\mu_2^2 - \nu_2^2) &= 0, \\ \mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2 &= 1. \end{cases}$$
(3.36)

are

$$\begin{cases} \mu_{2} = \pm \sqrt{\frac{\phi_{1}}{\phi_{1} + \phi_{2}} - \nu_{2}^{2}} \\ \mu_{1} = -\sqrt{\frac{\phi_{2}}{\phi_{1}}}\nu_{2} \\ \nu_{1} = \pm \sqrt{\frac{\phi_{2}}{\phi_{1} + \phi_{2}} - \frac{\phi_{2}}{\phi_{1}}}\nu_{2}^{2} \end{cases} \text{ and } \begin{cases} \mu_{2} = \pm \sqrt{\frac{\phi_{1}}{\phi_{1} + \phi_{2}} - \nu_{2}^{2}} \\ \mu_{1} = \sqrt{\frac{\phi_{2}}{\phi_{1}}}\nu_{2} \\ \nu_{1} = \mp \sqrt{\frac{\phi_{2}}{\phi_{1} + \phi_{2}} - \frac{\phi_{2}}{\phi_{1}}}\nu_{2}^{2} \end{cases}$$
(3.37)

with $\nu_2^2 \leq \frac{\phi_1}{\phi_1 + \phi_2}$. Note (3.35) and $\|x\|_2^2 + \|y\|_2^2 = 1$, so with the values in (3.37), it holds that $x^\top y = 0$ and $\|x\|_2 = \|y\|_2 = \frac{\sqrt{2}}{2}$. Since $\begin{bmatrix} z^\top & w^\top \end{bmatrix}^\top \in \mathcal{N}(M_{j+1})$, so $\begin{bmatrix} x_{j+1} & x_{j+2} \\ v_{j+1} & v_{j+2} \end{bmatrix} = \begin{bmatrix} x & y \\ \operatorname{Re}(w) & \operatorname{Im}(w) \end{bmatrix}$ satisfy the constraints (3.20b)-(3.20d) with $\delta_1 = \delta_2 = \frac{\sqrt{2}}{2}$. Hence

$$\|\delta_1 v_{j+1}\|_2^2 + \|\delta_2 v_{j+2}\|_2^2 + \beta_{j+1}^2 (\frac{\delta_1}{\delta_2} - \frac{\delta_2}{\delta_1})^2$$

=2 $\|w\|_2^2 = 2(\gamma_1^2 + \zeta_1^2) \frac{1 - \sigma_1^2}{\sigma_1^2} + 2(\gamma_2^2 + \zeta_2^2) \frac{1 - \sigma_2^2}{\sigma_2^2} \le \frac{2(1 - \sigma_2^2)}{\sigma_2^2},$

which completes the proof of the theorem.

From the proof of Theorem 3.2 we can see that with such choice of $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$, the value of the corresponding objective function is just $2\|w\|_2^2$. Define $\xi_1 = p_1^\top \Xi p_1, \xi_2 = p_2^\top \Xi p_2, \eta_1 = q_1^\top \Xi q_1, \eta_2 = q_2^\top \Xi q_2, \zeta_{12} = q_1^\top \Xi p_2, \zeta_{21} = q_2^\top \Xi p_1$, with $\Xi = \text{diag}\{(1 - \sigma_1^2)/\sigma_1^2, (1 - \sigma_2^2)/\sigma_2^2\}$, it then

follows

$$\|w\|_{2}^{2} = \begin{cases} \frac{\phi_{2}}{\phi_{1}+\phi_{2}}(\xi_{1}+\eta_{1}) + \frac{\phi_{1}}{\phi_{1}+\phi_{2}}(\xi_{2}+\eta_{2}) + 2\sqrt{\frac{\phi_{2}}{\phi_{1}}}\frac{\phi_{1}}{\phi_{1}+\phi_{2}}(\zeta_{21}-\zeta_{12}) & \text{if } (\mu_{1}\nu_{2}) \leq 0, \\ \frac{\phi_{2}}{\phi_{1}+\phi_{2}}(\xi_{1}+\eta_{1}) + \frac{\phi_{1}}{\phi_{1}+\phi_{2}}(\xi_{2}+\eta_{2}) + 2\sqrt{\frac{\phi_{2}}{\phi_{1}}}\frac{\phi_{1}}{\phi_{1}+\phi_{2}}(\zeta_{12}-\zeta_{21}) & \text{if } (\mu_{1}\nu_{2}) > 0. \end{cases}$$
(3.38)

So in order to get a smaller $||w||_2$, we can take $\mu_1, \mu_2, \nu_1, \nu_2$ satisfying $\mu_1\nu_2 \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_1\nu_2 > 0$ if $\zeta_{21} > \zeta_{12}$.

Till now we have proposed two strategies for computing $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$. The first strategy computes $x_{j+1}, x_{j+2}, v_{j+1}$ and v_{j+2} by using the Jacobi orthogonal process (3.22) and (3.23) with $z = u_1$ and $w = \frac{S_2 V e_1}{\sigma_1}$. While the second one first computes $\mu_1, \mu_2, \nu_1, \nu_2$ by (3.37) satisfying $\mu_1 \nu_2 \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_1 \nu_2 > 0$ if $\zeta_{21} > \zeta_{12}$, and then compute $\gamma_1, \gamma_2, \zeta_1, \zeta_2$ from (3.34), where Ω is as in (3.32), and finally set $x_{j+1} = x, x_{j+2} = y, v_{j+1} = \text{Re}(w)$ and $v_{j+2} = \text{Im}(w)$, where x, y, w are computed by (3.28). We cannot tell which strategy is better. So we suggest to apply both strategies, compare the corresponding values of the objective function and adopt the one which gives better results. Specifically, if the value of the objective function corresponding to the first strategy is smaller, we would update X_i and T_j as

$$X_{j+2} = \begin{bmatrix} X_j & \delta_1 x_{j+1} & \delta_2 x_{j+2} \end{bmatrix} \in \mathbb{R}^{n \times (j+2)}, \qquad T_{j+2} = \begin{bmatrix} T_j & \delta_1 v_{j+1} & \delta_2 v_{j+2} \\ 0 & \alpha_{j+1} & \delta\beta_{j+1} \\ 0 & -\frac{1}{\delta}\beta_{j+1} & \alpha_{j+1} \end{bmatrix} \in \mathbb{R}^{(j+2) \times (j+2)},$$
(3.39)

where $\delta_1 = \frac{1}{\|x_{j+1}\|_2}, \delta_2 = \frac{1}{\|x_{j+2}\|_2}, \delta = \frac{\delta_2}{\delta_1}$. Otherwise, we update X_j and T_j as

$$X_{j+2} = \begin{bmatrix} X_j & \sqrt{2}x & \sqrt{2}y \end{bmatrix} \in \mathbb{R}^{n \times (j+2)}, \quad T_{j+2} = \begin{bmatrix} T_j & \sqrt{2} \operatorname{Re}(w) & \sqrt{2} \operatorname{Im}(w) \\ 0 & \alpha_{j+1} & \beta_{j+1} \\ 0 & -\beta_{j+1} & \alpha_{j+1} \end{bmatrix} \in \mathbb{R}^{(j+2) \times (j+2)},$$
(3.40)

with x, y and w defined as in (3.28). This completes the assignment of the complex conjugate poles $\lambda_{j+1}, \lambda_{j+2} = \bar{\lambda}_{j+1}$, and we can then continue with the next pole λ_{j+3} .

These two strategies essentially choose z from $\mathcal{R}(u_1)$ and $\mathcal{R}([u_1 \quad u_2])$, respectively. If the results by these two strategies are not satisfactory, theoretically, we can choose z from a higher dimensional space, i.e. $z \in \text{span}\{u_1, u_2, \ldots, u_k\}, k \geq 3$, with u_l being the *l*-th column of U. However the resulted optimization problem is much more complicated. More importantly, numerical examples show that these two strategies with k = 1, 2 can produce fairly satisfying results for most problems.

3.3 Algorithm

In this part, we give the framework of our algorithm.

Algorithm 1 Framework of our Schur-rob algorithm.

Input:

A, B and $\mathfrak{L} = \{\lambda_1, \ldots, \lambda_n\}$ (complex conjugate poles appear in pairs).

Output:

The feedback matrix F.

- 1: If λ_1 is real, compute x_1 by (3.3) and set $X_1 = x_1, T_1 = \lambda_1, j = 1$. If λ_1 is non-real, compute x_1, x_2 by (3.5), (3.7), (3.8), and set X_2, T_2 as in (3.9), j = 2.
- 2: while j < n do
- 3: **if** λ_{j+1} is real **then**

4: Find $S = \begin{bmatrix} S_1^\top & S_2^\top \end{bmatrix}^\top$, whose columns form an orthonormal basis of $\mathcal{N}(M_{j+1})$ in (3.13);

- 5: Compute y by (3.16);
- 6: Compute x_{j+1} and v_{j+1} by (3.14), update X_j and T_j as (3.17) and set j = j + 1.
- 7: else
- 8: Find $S = \begin{bmatrix} S_1^\top & S_2^\top \end{bmatrix}^\top$, whose columns form an orthonormal basis of $\mathcal{N}(M_{j+1})$ in (3.21);
- Compute the SVD of S_1 as $S_1 = U\Sigma V^*$; 9: if $\operatorname{Re}(Ue_1)$ and $\operatorname{Im}(Ue_1)$ are linearly independent then 10: Compute $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ by (3.22) and (3.23) with $z = \frac{S_1 V e_1}{\sigma_1}, w = \frac{S_2 V e_1}{\sigma_1}$; 11: Set $\delta_1 = \frac{1}{\|x_{j+1}\|_2}, \delta_2 = \frac{1}{\|x_{j+2}\|_2}$ and $\delta = \frac{\delta_2}{\delta_1}$; 12:Compute $dep_1 = \|\delta_1 v_{j+1}\|_2^2 + \|\delta_2 v_{j+2}\|_2^2 + \beta_{j+1}^2 (\delta - \frac{1}{\delta})^2;$ 13:else 14:Set $dep_1 = \infty$; 15:end if 16:Let $\tilde{X} = \begin{bmatrix} \tilde{x}_1 & \tilde{x}_2 \end{bmatrix}$, $\tilde{Y} = \begin{bmatrix} \tilde{y}_1 & \tilde{y}_2 \end{bmatrix}$ with $\tilde{x}_1, \tilde{y}_1, \tilde{x}_2, \tilde{y}_2$ defined as in (3.27), and compute the 17:spectral decomposition (3.33); Compute $\mu_1, \mu_2, \nu_1, \nu_2$ by (3.37) satisfying $\mu_1\nu_2 \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_1\nu_2 > 0$ if $\zeta_{21} > \zeta_{12}$, 18: and then compute $\gamma_1, \gamma_2, \zeta_1, \zeta_2$ from (3.34), where Ω is as in (3.32); Compute z, w by (3.28), set $x_{j+1} = \text{Re}(z), x_{j+2} = \text{Im}(z), v_{j+1} = \text{Re}(w)$ and $v_{j+2} =$ 19: Im(w). Compute $dep_2 = 2[(\gamma_1^2 + \zeta_1^2)\frac{1-\sigma_1^2}{\sigma_1^2} + (\gamma_2^2 + \zeta_2^2)\frac{1-\sigma_2^2}{\sigma_2^2}];$ If $dep_1 < dep_2$, update X_j and T_j as in (3.39); otherwise, update them as in (3.40). Set 20: j = j + 2.end if 21: 22: end while 23: Set $X = X_n, T = T_n$, and compute F by (2.5).

4 Numerical Examples

In this section, we give some numerical examples to illustrate the performance of our **Schur-rob** algorithm, and compare it with some of the different versions of **SCHUR** in [8], the MATLAB functions **robpole** [23] and **place** [13]. Each algorithm computes a feedback matrix F such that the eigenvalues of A + BF are those given in \mathfrak{L} , and A + BF is robust. When applying **robpole** to all test examples, we set the maximum number of sweep to be the default value 5. All calculations are carried out on an Intel®CoreTMi3, dual core, 2.27 GHz machine, with 2.00 GB RAM. MATLAB R2012a is used with machine epsilon $\epsilon \approx 2.2 \times 10^{-16}$.

With $\lambda_1 \in \mathbb{R}$ fixed, the choice of x_1 in **Schur-rob** ignores the freedom of x_1 . Inspired by **O-SCHUR** [8], we may regard x_1 as a free parameter and manage to optimize the robustness. Specifically, we may run **Schur-rob** with several different choices of x_1 , and keep the solution F corresponding to the minimum departure from normality. We denote such method as "**O-Schur-rob**".

In this section, results on precision and robustness obtained by different algorithms are displayed. Here the precision refers to the accuracy of the eigenvalues of computed $A_c = A + BF$, compared with the prescribed poles in \mathfrak{L} . Precisely, we list

$$precs = \left\lfloor \min_{1 \le j \le n} \left(-\log(\left|\frac{\lambda_j - \hat{\lambda}_j}{\lambda_j}\right|) \right) \right\rfloor,$$

where $\hat{\lambda}_j, j = 1, ..., n$ are eigenvalues of computed $A_c = A + BF$. Larger values of precs indicate more accurate computed eigenvalues. The robustness is, however, more complicated, since different measures of robustness are used in these algorithms. Specifically, let the spectral decomposition and the real Schur decomposition of A + BF respectively be

$$A + BF = X\Lambda X^{-1}, \qquad A + BF = UTU^{\top},$$

where Λ is a diagonal matrix whose diagonal elements are those in \mathfrak{L} , U is orthogonal, and Tis the real Schur form. The MATLAB function **place** tends to minimize $||X^{-1}||_F$ and **robpole** aims to maximum $|\det(X)|$. Both measures are closely related to the condition number $\kappa_F(X) =$ $||X||_F ||X^{-1}||_F$. While different versions of **SCHUR** [8] and our **Schur-rob** try to minimize the departure from normality of $A_c = A + BF$. Hence, in the following tests, we adopt the following two measures of robustness: the departure from normality of A_c (denoted as "dep.") and the condition number of X (denoted as " $\kappa_F(X)$ ").

(n, l_{i})			ep.		precs						
(n,k)	SCHUR	SCHUR-D	O-SCHUR	Schur-rob	SCHUR	SCHUR-D	O-SCHUR	Schur-rob			
(4, 1e+1)	$9.5e{+1}$	$2.2e{+1}$	$4.3e{+1}$	2.7e+0	14	14	14	15			
(4, 1e+2)	1.5e+4	8.2e + 2	$1.4e{+}4$	$3.3e{+}2$	11	13	11	14			
(4, 1e+3)	1.4e+6	6.6e + 4	1.2e + 6	6.6e + 2	7	8	7	10			
(4, 1e+4)	2.9e + 8	9.9e + 5	$4.3e{+7}$	$1.0e{+4}$	4	10	6	13			
(4, 1e+5)	$1.8e{+}10$	7.3e + 6	$1.2e{+}10$	3.8e + 5	3	7	3	10			
(20, 1e+1)	$4.0e{+1}$	7.6e + 0	$1.7e{+1}$	4.6e + 0	13	14	14	14			
(20, 1e+2)	7.7e+4	2.6e + 2	$2.4e{+}2$	$1.8e{+1}$	9	12	11	12			
(20, 1e+3)	2.0e+5	4.4e + 3	$9.3e{+}4$	4.7e + 2	9	11	10	12			
(20, 1e+4)	$3.2e{+7}$	$2.4e{+}4$	5.2e + 6	1.9e + 3	6	10	8	11			
(20, 1e+5)	1.7e + 9	1.2e+6	8.8e + 8	6.0e + 4	3	9	6	10			
(50, 1e+1)	$1.1e{+1}$	$2.9e{+}0$	$4.4e{+}0$	4.4e + 0	13	12	13	13			
(50, 1e+2)	2.0e+4	5.9e + 2	8.8e + 2	$1.8e{+1}$	10	12	11	12			
(50, 1e+3)	1.1e+6	7.8e + 2	5.8e + 4	5.5e + 2	8	11	9	12			
(50, 1e+4)	8.8e+7	$3.2e{+4}$	9.6e + 6	$2.1e{+}3$	6	10	7	11			
(50, 1e+5)	8.4e+9	2.0e+5	4.8e + 8	$3.7e{+4}$	3	9	5	10			

Table 4.1: Numerical results for Example 4.1

Example 4.1. Let

$$\begin{split} A &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0.5 \times e^{\top} & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} I_{n-1} \\ 0 \end{bmatrix} \\ \mathfrak{L} &= \{randn(1, n-2), \ 0.5 + ki, \ 0.5 - ki\}, \end{split}$$

where e^{\top} is the row vector with its all entries being 1, "randn(1, n-2)" is a row vector of dimension n-2, generated by the MATLAB function **randn**. We set k as 1e+1, 1e+2, 1e+3, 1e+4, 1e+5, and apply the four algorithms **SCHUR**, **SCHUR-D**, **O-SCHUR** and **Schur-rob** on these examples, where "**SCHUR-D**" denotes the algorithm combining the D_k varying strategy in [8] with **SCHUR**. In [8], the author points out that minimizing the departure from normality via the D_k varying technique can be achieved by optimizing the condition number of $X^{\top}X$ or X, which actually is hard to realize. So here, the numerical results associated with "**SCHUR-D**" are obtained by taking many different vectors from the null space of (6) in [8], which lead to orthogonal columns in X when placing complex conjugate poles, and adopting the one owning the minimal departure from normality as the solution to the **SFRPA**. All numerical results are summarized in Table 4.1, which shows that our algorithm outperforms **SCHUR** and **O-SCHUR** on these examples with complex conjugate poles to be assigned.

We now compare our Schur-rob, O-Schur-rob algorithms with the MATLAB functions place, robpole and the SCHUR, O-SCHUR algorithms by applying them on some benchmark sets. The tested benchmark sets include eleven illustrated examples from [5], ten multi-

		۲	7	0	0	11
	num.	5	1	8	9	11
	place	7.4e-1	3.5e+0	$1.3e{+1}$	$1.2e{+1}$	2.5e-3
	robpole	7.4e-1	$3.4e{+}0$	5.0e + 0	$1.2e{+1}$	$3.6e{-1}$
7	SCHUR	7.2e-1	7.2e + 0	7.0e + 0	$1.9e{+1}$	$2.3e{+}0$
dep.	O-SCHUR	7.1e-1	4.8e + 0	6.0e + 0	$1.7e{+1}$	6.0e-1
	Schur-rob	7.2e-1	3.7e + 0	7.5e + 0	$1.8e{+1}$	2.4e-1
	O-Schur-rob	7.1e-1	$3.2e{+}0$	$3.3e{+}0$	$1.1e{+1}$	1.4e-1
	place	1.5e+2	$1.2e{+1}$	$3.7e{+1}$	$2.4e{+1}$	4.0e+0
	robpole	$1.5e{+2}$	$1.2e{+1}$	6.2e + 0	$2.4e{+1}$	$4.1e{+}0$
$(\mathbf{T}_{\mathbf{T}})$	SCHUR	2.7e + 3	$1.3e{+}2$	$1.1e{+1}$	$5.6e{+1}$	6.0e + 0
$\kappa_F(X)$	O-SCHUR	1.1e + 3	$4.5e{+1}$	7.5e + 0	$5.5e{+1}$	$4.1e{+}0$
	Schur-rob	1.9e + 3	$2.5e{+1}$	$1.2e{+1}$	$5.8e{+1}$	$4.1e{+}0$
	O-Schur-rob	1.2e+3	$2.2e{+1}$	9.6e + 0	$3.3e{+1}$	$4.0e{+}0$

Table 4.2: Robustness of the closed-loop system for the examples from [5]

input CARE examples and nine multi-input DARE examples in benchmark collections [1,2]. All examples are numbered in the order as they appear in the references.

Example 4.2. The first benchmark set includes eleven small examples from [5]. Applying the six algorithms on these examples, all algorithms produce comparable precisions of the assigned poles, which are greater than 10, and we omit the results here. Table 4.2 lists two measures of robustness, i.e. *dep.* and $\kappa_F(X)$, for five examples. The results are generally comparable. The remaining six examples are not displayed in the table, as the results of the six algorithms applying on these examples are quite similar.

Now we apply the six algorithms on ten CARE and nine DARE examples from the SLICOT CARE/DARE benchmark collections [1,2]. Table 4.3 to Table 4.6 present the numerical results, respectively. The "-"s in the first columns in Table 4.4 and Table 4.6 corresponding to **place**, **robpole**, **SCHUR** and **O-SCHUR** mean that all four algorithms fail to output a solution, since the multiplicity of some pole is greater than m. Note that the "precs" in the last six columns associated with **SCHUR** and **O-SCHUR** in Table 4.3 and those in the third and eighth columns in Table 4.4 are also "-"s, which suggest that there exists at least one eigenvalue of A + BF, which owns no relative accuracy compared with the assigned poles. From Table 4.3, we know that the relative accuracy "precs" of the poles in example 4 and 5 corresponding to **Schur-rob** and **O-Schur-rob** are lower than those produced by **place** and **robpole**. And the reason is that there are semi-simple eigenvalues in both examples. So how to dispose the issue that semi-simple eigenvalues can achieve higher relative accuracy deserves further exploration and we will treat it in a separate paper. For the sixth column in Table 4.3, "precs" from our algorithms are also smaller than those obtained from **place** and **robpole** for the existence of poles which are

precs											
	1	2	3	4	5	6	7	8	9	10	
place	14	14	11	11	11	9	14	11	13	11	
robpole	14	14	12	13	12	11	14	14	13	10	
SCHUR	12	13	9	6	-	-	-	-	-	-	
O-SCHUR	14	16	10	7	-	-	-	-	-	-	
Schur-rob	14	14	12	8	9	6	14	14	12	9	
O-Schur-rob	15	15	13	8	9	6	14	14	12	9	

precs $2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9$ 1 place 15 14 14 7 11 5 - 13 robpole 15 14 14 7 11 1 - 13 SCHUR - 14781-12 1 **O-SCHUR** - 14892-15 1 Schur-rob 15 15 15 15 8 10 4 - 12 **O-Schur-rob** 15 15 15 15 8 10 4 - 13

Table 4.3: Accuracy for CARE examples

Table 4.4: Accuracy for DARE examples

	num.	1	2	3	4	5	6	7	8	9	10
	place	$5.2e{+}0$	3.0e-1	$7.3e{+}2$	1.5e+6	2.9e+6	2.3e+7	$7.6e{+}0$	$2.2e{+1}$	$6.1e{+}0$	4.9e + 9
dep.	robpole	5.2e + 0	2.9e-1	5.7e + 2	7.5e+5	2.9e+6	$2.3e{+7}$	8.1e + 0	$2.0e{+1}$	6.0e + 0	3.8e + 9
	SCHUR	8.4e + 1	7.2e + 0	5.0e + 2	1.7e+6	3.0e + 9	5.3e+7	6.2e + 1	$8.9e{+}2$	7.5e + 0	$4.4e{+}17$
1	O-SCHUR	4.7e + 1	2.6e + 0	$3.8e{+}2$	8.0e + 5	5.4e + 8	$2.6e{+7}$	7.3e + 0	1.7e + 2	6.8e + 0	$2.3e{+}17$
	Schur-rob	7.6e + 0	3.0e-1	$1.4e{+}2$	1.1e+5	7.3e+6	2.3e+7	7.5e + 0	$2.1e{+1}$	$8.4e{+}0$	$2.2e{+}10$
	O-Schur-rob	7.3e + 0	2.6e-1	$1.4e{+}2$	1.1e+5	2.5e+6	2.3e+7	6.8e + 0	$2.0e{+1}$	6.8e + 0	$2.2e{+}10$
	place	$7.4e{+}0$	$8.0e{+}0$	$4.3e{+1}$	$1.7e{+}15$	8.5e+4	4.8e+6	$1.6e{+}1$	$9.8e{+1}$	$1.5e{+2}$	2.3e+6
	robpole	7.3e + 0	8.0e + 0	$4.2e{+1}$	$2.2e{+7}$	$8.9e{+4}$	$3.2e{+}6$	1.6e + 1	$9.0e{+1}$	1.4e + 2	2.3e+6
$\kappa_F(X)$	SCHUR	2.2e + 2	$1.0e{+1}$	1.7e + 3	9.1e + 9	6.0e + 11	$4.0e{+}13$	3.5e + 8	6.1e + 9	1.3e + 9	4.6e + 13
		1.2e + 2	$5.1e{+1}$	2.1e + 3	1.0e+9	$2.4e{+}10$	1.2e + 8	1.0e + 8	3.7e + 9	4.1e + 9	5.7e + 13
	Schur-rob	$1.1e{+1}$	8.2e + 0	$9.2e{+}2$	$9.0e{+7}$	2.0e+6	3.2e + 8	$3.3e{+1}$	5.7e + 2	6.5e + 3	4.3e+6
	O-Schur-rob	1.0e + 1	8.0e + 0	$9.1e{+}2$	6.5e+7	1.3e+6	1.2e+8	$2.8e{+1}$	4.2e + 2	3.4e + 3	4.3e+6

Table 4.5: Robustness of the closed-loop system matrix for ten CARE examples

relatively badly separated from the imaginary axis. And this is a weakness of our algorithm.

We now test the five methods **place**, **robpole**, **SCHUR**, **O-SCHUR** and **Schur-rob** on some random examples generated by the MATLAB function **randn**.

Example 4.3. This test set includes 33 examples where *n* varies from 3 to 25 increased by 2, and *m* is set to be $2, \lfloor \frac{n}{2} \rfloor, n-1$ for each *n*. The examples are generated as following. We first randomly generate the matrices *A*, *B* and *F* by the MATLAB function **randn**, and then get \mathfrak{L} using the MATLAB function **eig**, that is, $\mathfrak{L} = eig(A + BF)$. We then apply the five algorithms on the *A*, *B* and \mathfrak{L} as input.

Fig. 4.1 to Fig. 4.4, respectively exhibit the departure from normality of the computed A_c , the condition number of the eigenvector matrix X, the relative accuracy of the poles and the CPU time of the five algorithms applied on these randomly generated examples. In these figures, the x-axis represents the number of the 33 different (n, m). For example, (3, 2), (5, 2) and (5, 4) correspond to 1, 2 and 3 in the x-axis, respectively. And the values along the y-axis are the mean values over 50 trials for a certain (n, m).

		-1	0	0	4	ž	0	-	0	0
	num.	1	2	3	4	5	6	7	8	9
	place	-	2.2e-1	3.9e-1	4.3e-1	1.7e+0	1.4e+0	$2.3e{+1}$	4.3e+7	$8.9e{+}0$
	robpole	-	2.2e-1	3.9e-1	3.6e-1	1.7e+0	$1.3e{+}0$	$1.8e{+1}$	$3.9e{+}12$	$8.0e{+}0$
dep.	SCHUR	-	4.1e-1	$1.1e{+}2$	5.9e-1	1.8e + 0	$1.1e{+1}$	$3.2e{+}2$	3.4e + 2	$1.1e{+1}$
	O-SCHUR	-	3.3e-1	$4.9e{+1}$	4.1e-1	1.7e + 0	$1.1e{+}0$	$1.7e{+}2$	$1.2e{+1}$	8.0e + 0
	Schur-rob	1.0e-1	2.5e-1	$1.3e{+}0$	3.4e-1	1.7e + 0	2.0e+0	$1.9e{+1}$	$9.8e{+}0$	$9.9e{+}0$
	O-Schur-rob	1.0e-1						$1.8e{+1}$		6.6e + 0
	place	-	5.2e + 0	4.9e+0	5.4e + 0	$1.8e{+1}$	$1.3e{+1}$	2.3e + 8	9.2e + 292	$3.4e{+}2$
	robpole	-	5.2e + 0	5.0e + 0	5.3e + 0	$1.8e{+1}$	$1.2e{+1}$	2.9e + 8	1.3e + 308	3.0e + 2
$\kappa_F(X)$	SCHUR	-	4.0e + 7	1.2e + 9	5.7e + 0	$1.8e{+1}$	5.8e + 3	$1.9e{+}11$	2.8e + 295	4.7e + 3
	O-SCHUR	-	$3.3e{+7}$	8.0e + 8	5.4e + 0	$1.8e{+1}$	1.7e + 3	$2.0e{+}11$	3.3e + 295	2.6e + 3
	Schur-rob	$7.1e{+}15$	5.5e + 0	5.6e + 0	7.2e + 0	$1.8e{+1}$	$3.8e{+1}$	1.7e + 9	5.6e + 292	$2.2e{+4}$
	O-Schur-rob	$2.5e{+}15$	5.5e+0	5.5e+0	7.2e+0	$1.8e{+1}$	$3.8e{+1}$	1.2e+9	5.6e + 292	4.7e + 3

Table 4.6: Robustness of the closed-loop system matrix for nine DARE examples

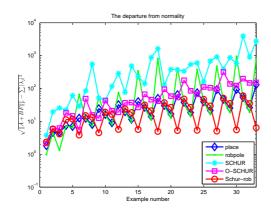


Fig. 4.1: dep. over 50 trials

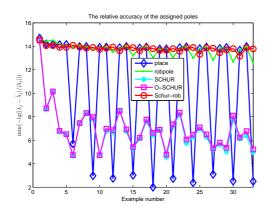


Fig. 4.3: precs over 50 trials

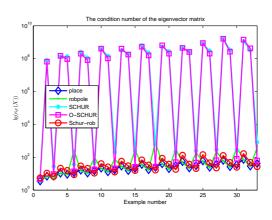


Fig. 4.2: $\kappa_F(X)$ over 50 trials

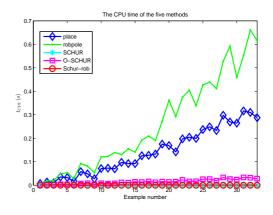


Fig. 4.4: CPU time over 50 trials

All these figures show that our **Schur-rob** algorithm can produce comparable or even better results as **place** and **robpole**, but with much less CPU time.

5 Conclusion

Pole assignment problem for multi-input control is generally under-determined. And utilizing this freedom to make the closed-loop system matrix to be insensitive to perturbations as far as possible evokes the state-feedback robust pole assignment problem (SFRPA) arising. Based on SCHUR [8], we propose a new direct method to solve the SFRPA, which obtains the real Schur form of the closed-loop system matrix and tends to minimize its departure from normality via solving some standard eigen-problems. Many numerical examples show that our algorithm can produce comparable or even better results than existing methods, but with much less computational costs than the two classic methods place and robpole.

References

- J. Abels and P. Benner, CAREX A collection of benchmark examples for continuous-time algebraic Riccati equations (Version 2.0), Katholieke Universiteit Leuven, ESAT/SISTA, Leuven, Belgium, SLICOT Working Note 1999-14, Nov. 1999. [Online]. Available: http://www.slicot.de/REPORTS/SLWN1999-14.ps.gz.
- [2] J. Abels and P. Benner, DAREX A collection of benchmark examples for discrete-time algebraic Riccati equations (Version 2.0), Katholieke Universiteit Leuven, ESAT/SISTA, Leuven, Belgium, SLICOT Working Note 1999-16, Dec. 1999. [Online]. Available: http://www.slicot.de/REPORTS/SLWN1999-16.ps.gz.
- [3] A.N. Andry, E.Y. Shapiro and J.C. Chung, Eigenstructure assignment for linear systems, IEEE Transactions on Aerospace and Electronic Systems, 19(1983), 711–729.
- [4] S.P. Bhattacharyya and E. De Souza, Pole assignment via Sylvester's equation, Systems & Control Letters, 1(1982), 261–263.
- [5] R. Byers and S.G. Nash, Approaches to robust pole assignment, International Journal of Control, 49(1989), 97–117.
- [6] R.K. Cavin and S.P. Bhattacharyya, Robust and well-conditioned eigenstructure assignment via sylvester's equation, Optimal Control Applications and Methods, 4(1983), 205–212.

- [7] E.K.W. Chu, A pole-assignment algorithm for linear state feedback, System & Control Letters, 7(1986), 289–299.
- [8] E.K.W. Chu, Pole assignment via the schur form, Systems & Control Letters, 56(2007), 303– 314.
- [9] A. Dickman, On the robustness of multivariable linear feedback systems in state-space representation, *IEEE Transactions on Automatic Control*, 32(1987), 407–410.
- [10] M. Fahmy and J. O'Reilly, On eigenstructure assignment in linear multivariable systems, *IEEE Transactions on Automatic Control*, 27(1982), 690–693.
- [11] P. Henrici, Bounds for iterates, inverses, spectral variation and fields of values of non-normal matrices, *Numerische Mathematik*, 4(1962), 24–40.
- [12] S.K. Katti, Pole placement in multi-input systems via elementary transformations, International Journal of Control, 37(1983), 315–347.
- [13] J. Kautsky, N.K. Nichols and P. Van Dooren, Robust pole assignment in linear state feedback, International Journal of Control, 41(1985), 1129–1155.
- [14] J. Lam and W.Y. Van, A gradient flow approach to the robust pole-placement problem, International Journal of Robust and Nonlinear Control, 5(1995), 175–185.
- [15] G.S. Miminis and C.C. Paige, A direct algorithm for pole assignment of time-invariant multiinput linear systems using state feedback, *Automatica*, 24(1988), 343–356.
- [16] G.S. Miminis and C.C. Paige, A QR-like approach for the eigenvalue assignment problem, in *Proceedings of the 2nd Hellenic Conference on Mathematics and Informatics*, Athens, Greece, Sept., 1994.
- [17] R.V. Patel and P. Misra, Numerical algorithms for eigenvalue assignment by state feedback, *Proceedings of the IEEE*, 72(1984), 1755–1764.
- [18] P.Hr. Petkov, N.D. Christov and M.M. Konstantinov, A computational algorithm for pole assignment of linear multiinput systems, *IEEE Transactions on Automatic Control*, 31(1986), 1044–1047.
- [19] K. Ramar and V. Gourishankar, Utilization of the design freedom of pole assignment feedback controllers of unrestricted rank, *International Journal of Control*, 24(1976), 423–430.

- [20] D.G. Retallack and A.G.J. MacFarlane, Pole-shifting techniques for multivariable feedback systems, *Proceedings of the Institution of Electrical Engineers*, 117(1970), 1037–1038.
- [21] V. Sima, A.L. Tits and Y. Yang, Computational experience with robust pole assignment algorithms, in *Proceedings of the 2006 IEEE Conference on Computer Aided Control Systems Design*, Munich, Germany, Oct. 4-6, 2006.
- [22] G.W. Stewart and J.G. Sun, Matrix Perturbation Theory, Academic Press, New York, 1990.
- [23] A.L. Tits and Y. Yang, Globally convergent algorithms for robust pole assignment by state feedback, *IEEE Transactions on Automatic Control* 41(1996), 1432–1452.
- [24] A. Varga, A Schur method for pole assignment, IEEE Transactions on Automatic Control, 26(1981), 517–519.
- [25] W.M. Wonham, On pole assignment in multi-input controllable linear systems, *IEEE Trans*actions on Automatic Control, 12(1967), 660–665.
- [26] W.M. Wonham, Linear Multivariable Control: A Geometric Approach, 2nd ed., Springer-Verlag, New York, 1979.
- [27] S.F. Xu, An Introduction to Inverse Algebraic Eigenvalue Problems, Peking University Press, Beijing, and Vieweg, Braunschweig, 1998.