# A Modified Schur Method for Robust Pole Assignment in State Feedback Control * 

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#### Abstract

Recently, a SCHUR method was proposed in [8 to solve the robust pole assignment problem in state feedback control. It takes the departure from normality of the closed-loop system matrix $A_{c}$ as the measure of robustness, and intends to minimize it via the real Schur form of $A_{c}$. The SCHUR method works well for real poles, but when complex conjugate poles are involved, it does not produce the real Schur form of $A_{c}$ and can be problematic. In this paper, we put forward a modified Schur method, which improves the efficiency of SCHUR when complex conjugate poles are to be assigned. Besides producing the real Schur form of $A_{c}$, our approach also leads to a relatively small departure from normality of $A_{c}$. Numerical examples show that our modified method produces better or at least comparable results than both place and robpole algorithms, with much less computational costs.


Key words. pole assignment, state feedback control, robustness, departure from normality, real Schur form

AMS subject classification. 15A18, 65F18, 93B55.

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## 1 Introduction

Let the matrix pair $(A, B)$ denotes the dynamic state equation

$$
\begin{equation*}
\dot{x}(t)=A x(t)+B u(t) \tag{1.1}
\end{equation*}
$$

of the time invariant linear system, where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ are the open-loop system matrix and the input matrix, respectively. The dynamic behavior of (1.1) is governed by the eigen-structure of $A$, especially the poles (eigenvalues). And in order to change the dynamic behavior of the open-loop system (1.1) in some desirable way (to achieve stability or to speed up response), one needs to modify the poles of (1.1). Typically, this may be actualized by the state-feedback control

$$
\begin{equation*}
u(t)=F x(t), \tag{1.2}
\end{equation*}
$$

where the feedback matrix $F \in \mathbb{R}^{m \times n}$ is to be chosen such that the closed-loop system

$$
\begin{equation*}
\dot{x}(t)=(A+B F) x(t) \equiv A_{c} x(t) \tag{1.3}
\end{equation*}
$$

has specified poles.
Mathematically, the state-feedback pole assignment problem can be stated as:
State-Feedback Pole Assignment Problem (SFPA) Given $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}$ and a set of $n$ complex numbers $\mathfrak{L}=\left\{\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right\}$, closed under complex conjugation, find an $F \in \mathbb{R}^{m \times n}$ such that $\lambda(A+B F)=\mathfrak{L}$, where $\lambda(A+B F)$ is the eigenvalue set of $A+B F$.

A necessary and sufficient condition for the solvability of the SFPA for any set $\mathfrak{L}$ of $n$ selfconjugate complex numbers is that $(A, B)$ is controllable, or equivalently, the controllability matrix $\left[\begin{array}{llll}B & A B & \cdots & A^{n-1} B\end{array}\right]$ is of full row rank [25-27]. Many algorithms have been put forward to solve the SFPA, such as the invariant subspace method [18, the QR-like method [15, 16], etc.. We refer readers to $3,4,7,10,12,17,20,24$ for some other approaches.

When $m>1$, the solution to the SFPA is generally not unique. We may then utilize the freedom of $F$ to achieve some other desirable properties of the closed-loop system. In applications, one sympathetic character for system design is that the eigenvalues of the closed-loop system matrix $A_{c}$ are insensitive to perturbations, which leads to the following state-feedback robust pole assignment problem:

State-Feedback Robust Pole Assignment Problem (SFRPA) Find a solution $F \in \mathbb{R}^{m \times n}$ to the SFPA, such that the closed-loop system is robust, that is, the eigenvalues of $A_{c}$ are as insensitive to perturbations on $A_{c}$ as possible.

The key to solve the SFRPA is to choose an appropriate measure of robustness formulated in quantitative form. Some measures can be found in [5, 8, 9, 13, 27, such as the condition number measurement $\kappa_{F}(X)=\|X\|_{F}\left\|X^{-1}\right\|_{F}$, where $X$ is the eigenvector matrix of $A_{c}$, the departure from normality $\Delta_{F}\left(A_{c}\right)=\sqrt{\left\|A_{c}\right\|_{F}^{2}-\sum_{j=1}^{j=n}\left|\lambda_{j}\right|^{2}}$ and so on. Ramar and Gourishankar [19] made an early contribution to the SFRPA and since then various optimization methods have been proposed based on different measures [5, 6, 8, $9,13,14,23$. The most classic methods should be those proposed by Kautsky, Nichols and Van Dooren in [13], where $\kappa_{F}(X)$ is used as the measure of robustness of the closed-loop system matrix. However, Method 0 in 13 may fail to converge, Method 1 may suffer from slow convergence, and Method $2 / 3$ may not perform well on ill-conditioned problems. Based on Method 0 in [13], Tits and Yang [23] proposed a method for solving the SFRPA by trying to maximize the absolute value of the determinant of the eigenvector matrix $X$. The optimization processes are iterative, and hence generally expensive. Recently, Chu [8] put forward a Schur-type method for the SFRPA by tending to minimize the departure from normality of the closed-loop system matrix $A_{c}$ via the Schur decomposition of $A_{c}$. It computes the matrices $X$ and $T$ column by column, where $A_{c}=X T X^{-1}, X, T$ are real and $T$ is upper quasi-triangular, such that the strictly block upper triangular elements of matrix $T$ are minimized in each step. If $\lambda_{1}, \ldots, \lambda_{n}$ are all real, SCHUR [8] will generate an orthogonal matrix $X$, that is, $A_{c}=X T X^{-1}$ is the Schur decomposition of $A_{c}$. This implies that the departures from normality of $A_{c}$ and $T$ are the same. Hence the strategy aiming to minimize the departure from normality of $T$ is also pliable to $A_{c}$. However, in case of complex conjugate poles, it cannot produce an orthogonal $X$, suggesting that the departure from normality of $A_{c}$ is generally not identical to that of $T$. Hence, although it attempts to optimize the departure from normality of $T$, that of $A_{c}$ may still be large.

In this paper, we propose a modified Schur method upon SCHUR [8], where poles are assigned via the real Schur decomposition of $A_{c}=X T X^{\top}$, with $X$ being real orthogonal and $T$ being real upper quasi-triangular. In each step (assigning a real pole or a pair of conjugate poles), one optimization problem arises for purpose of minimizing the departure from normality of $T$. When assigning a real pole, we improve the efficiency of SCHUR by computing the SVD of a matrix, instead of computing the GSVD of a matrix pencil. When assigning a pair of conjugate poles, by exploring the properties of the posed optimization problem, we provide a polished way to obtain its suboptimal solution. Numerical examples show that our method outperforms SCHUR when complex conjugate poles are involved. We also compare our method with the MATLAB functions place (an implementation of Method 1 in [13), robpole (an implementation of the method
in 23) and the O-SCHUR algorithm (an implementation of an optimization method in 8) on some benchmark examples and randomly generated examples, where numerical results show that our method is comparable in accuracy and robustness, while with lower computational costs.

The paper is organized as follows. In Section 2, we give some preliminaries which will be used in subsequent sections. Our method is developed in Section 3, including both the real case and the complex conjugate case. Numerical results are presented in Section 4. Some concluding remarks are finally drawn in Section 5.

## 2 Preliminaries

In this section, we briefly review the parametric solutions to the SFPA, and the departure from normality.

### 2.1 Solutions to the SFPA

The parametric solutions to the SFPA can be expressed in several ways. In this paper, as in [8], we formulate it by using the real Schur decomposition of $A_{c}=A+B F$. Assume that the real Schur decomposition of $A+B F$ is

$$
\begin{equation*}
A+B F=X T X^{\top} \tag{2.1}
\end{equation*}
$$

where $X \in \mathbb{R}^{n \times n}$ is orthogonal, $T \in \mathbb{R}^{n \times n}$ is upper quasi-triangular with only $1 \times 1$ and $2 \times 2$ diagonal blocks.

Without loss of generality, we may assume that $B$ is of full column rank. Let

$$
B=Q\left[\begin{array}{c}
R  \tag{2.2}\\
0
\end{array}\right]=\left[\begin{array}{ll}
Q_{1} & Q_{2}
\end{array}\right]\left[\begin{array}{c}
R \\
0
\end{array}\right]=Q_{1} R
$$

be the QR decomposition of $B$, where $Q \in \mathbb{R}^{n \times n}$ is orthogonal, $Q_{1} \in \mathbb{R}^{n \times m}$, and $R \in \mathbb{R}^{m \times m}$ is nonsingular and upper triangular.

It follows from (2.1) that

$$
\begin{equation*}
A X+B F X-X T=0 \tag{2.3}
\end{equation*}
$$

Pre-multiplying (2.3) by $\operatorname{diag}\left(R^{-1}, I_{n-m}\right)\left[\begin{array}{ll}Q_{1} & Q_{2}\end{array}\right]^{\top}$ on both sides gives

$$
\left\{\begin{array}{l}
R^{-1} Q_{1}^{\top} A X+F X-R^{-1} Q_{1}^{\top} X T=0  \tag{2.4}\\
Q_{2}^{\top}(A X-X T)=0
\end{array}\right.
$$

Consequently, if we get an orthogonal matrix $X$ and an upper quasi-triangular matrix $T$ from the second equation of (2.4), then a solution $F$ to the SFPA will be obtained immediately from
the first equation of (2.4) as

$$
\begin{equation*}
F=R^{-1} Q_{1}^{\top}\left(X T X^{\top}-A\right) . \tag{2.5}
\end{equation*}
$$

### 2.2 Departure from normality

In this paper, we adopt the departure from normality of $A_{c}=A+B F$ as a measure of robustness of the closed-loop system matrix as in [8], which is defined as ( [11, 22])

$$
\Delta_{F}\left(A_{c}\right)=\sqrt{\left\|A_{c}\right\|_{F}^{2}-\sum_{j=1}^{n}\left|\lambda_{j}\right|^{2}}
$$

where $\lambda_{1}, \ldots, \lambda_{n}$ are the poles to be assigned, and hence eigenvalues of $A_{c}$. Now let $D$ be the block diagonal part of $T$ with only $1 \times 1$ and $2 \times 2$ blocks on its diagonal. Each $1 \times 1$ block of $D$ admits a real eigenvalue $\lambda_{j}$ of $T$, while each $2 \times 2$ block of $D$ admits a pair of conjugate eigenvalues $\lambda_{j}=\alpha_{j}+i \beta_{j}, \lambda_{j+1}=\bar{\lambda}_{j}$ and is of the form $D_{j}=\left[\begin{array}{cc}\alpha_{j} & \delta_{j} \beta_{j} \\ -\frac{\beta_{j}}{\delta_{j}} & \alpha_{j}\end{array}\right] \in \mathbb{R}^{2 \times 2}$ with $\delta_{j} \beta_{j} \neq 0$, where $\delta_{j}$ is some real number. Let $N=T-D=\left[\begin{array}{llll}\breve{v}_{1} & \breve{v}_{2} & \cdots & \breve{v}_{n}\end{array}\right]$ be the strictly upper quasi-triangular part of $T$ with $\breve{v}_{k}=\left[\begin{array}{ll}v_{k}^{\top} & 0\end{array}\right]^{\top}, v_{k} \in \mathbb{R}^{k-1}$ or $\mathbb{R}^{k-2}$. Direct calculations give rise to

$$
\begin{equation*}
\Delta_{F}^{2}\left(A_{c}\right)=\Delta_{F}^{2}(T)=\|N\|_{F}^{2}+\sum_{j}\left(\delta_{j}-\frac{1}{\delta_{j}}\right)^{2} \beta_{j}^{2} \tag{2.6}
\end{equation*}
$$

where the summation is over all $2 \times 2$ blocks of $D$.
When all poles $\lambda_{1}, \ldots, \lambda_{n}$ are real, the second part of $\Delta_{F}^{2}\left(A_{c}\right)$ in (2.6) will vanish. However, when some poles are non-real, not only the strictly block upper triangular part $N$ contributes to the departure from normality, but also the block diagonal part $D$. When some $\left|\delta_{j}\right|$ is large or close to zero, the second term can be pretty large, which means that it is not negligible.

## 3 Solving the SFRPA via the real Schur form

In this section, we solve the SFRPA by finding an orthogonal matrix $X=\left[\begin{array}{llll}x_{1} & x_{2} & \cdots & x_{n}\end{array}\right]$ and an upper quasi-triangular matrix $T=D+N$ satisfying the second equation of (2.4), such that $\Delta_{F}^{2}\left(A_{c}\right)$ in (2.6) is minimized. Obtaining a global optimization solution to the problem $\min \left\{\Delta_{F}^{2}\left(A_{c}\right)\right\}$ is rather difficult. In this paper, we propose an efficient method to get a suboptimal solution, which balances the contributions of $N$ and $D$ to the departure from normality. As in [8], we compute the matrices $X$ and $T$ column by column.

For any matrix $S$, we denote its range space and null space by $\mathcal{R}(S)$ and $\mathcal{N}(S)$, respectively.

Assume that we have already obtained $X_{j}=\left[\begin{array}{llll}x_{1} & x_{2} & \cdots & x_{j}\end{array}\right] \in \mathbb{R}^{n \times j}$ and $T_{j} \in \mathbb{R}^{j \times j}$ satisfying

$$
\begin{equation*}
Q_{2}^{\top}\left(A X_{j}-X_{j} T_{j}\right)=0, \quad X_{j}^{\top} X_{j}=I_{j}, \tag{3.1}
\end{equation*}
$$

where $T_{j}$ is upper quasi-triangular and $\lambda\left(T_{j}\right)=\left\{\lambda_{k}\right\}_{k=1}^{k=j}$. We then are to assign the pole $\lambda_{j+1}$ (if $\lambda_{j+1}$ is real) or poles $\lambda_{j+1}, \bar{\lambda}_{j+1}$ (if $\lambda_{j+1}$ is non-real) to get $x_{j+1}, \breve{v}_{j+1}$ or $x_{j+1}, x_{j+2}, \breve{v}_{j+1}, \breve{v}_{j+2}$, such that the departure from normality of $A_{c}$ is optimized in some sense. This procedure is repeated until all columns of $X$ and $T$ are acquired, and eventually a solution $F$ to the SFRPA would be computed from (2.5). In the following subsections we will distinguish two different cases when $\lambda_{j+1}$ is real or non-real.

Before this, we shall show how to get the first one (two) column(s) of $X$ and $T$. If $\lambda_{1}$ is real, the first column of $T$ is then $\lambda_{1} e_{1}$, or $T_{1}=\lambda_{1}$, and the first column $x_{1}$ of $X$ must satisfy

$$
\begin{equation*}
Q_{2}^{\top}\left(A-\lambda_{1} I_{n}\right) x_{1}=0, \tag{3.2}
\end{equation*}
$$

and $\left\|x_{1}\right\|_{2}=1$. Let the columns of $S \in \mathbb{R}^{n \times r}$ be an orthonormal basis of $\mathcal{N}\left(Q_{2}^{\top}\left(A-\lambda_{1} I_{n}\right)\right)$, then $x_{1}$ can be chosen to be any unit vector in $\mathcal{R}(S)$. We take

$$
x_{1}=\left(\begin{array}{lll}
\left.S\left[\begin{array}{lll}
1 & \ldots & 1
\end{array}\right]^{\top}\right) /\left\|S\left[\begin{array}{lll}
1 & \ldots & 1
\end{array}\right]^{\top}\right\|_{2} \tag{3.3}
\end{array}\right.
$$

in our algorithm as in [8], and then initially set $X_{1}=x_{1}, T_{1}=\lambda_{1}$.
If $\lambda_{1}=\alpha_{1}+i \beta_{1}$ is non-real, to get the real Schur form, we should place $\bar{\lambda}_{1}=\alpha_{1}-i \beta_{1}$ together with $\lambda_{1}$. Notice that $T_{2}$ is of the form $T_{2}=\left[\begin{array}{cc}\alpha_{1} & \delta_{1} \beta_{1} \\ -\beta_{1} / \delta_{1} & \alpha_{1}\end{array}\right]$ with $0 \neq \delta_{1} \in \mathbb{R}$, then the first two columns $x_{1}, x_{2} \in \mathbb{R}^{n}$ of $X$ should be chosen to satisfy

$$
Q_{2}^{\top}\left(\begin{array}{ll}
\left.A\left[\begin{array}{ll}
x_{1} & x_{2}
\end{array}\right]-\left[\begin{array}{ll}
x_{1} & x_{2}
\end{array}\right] T_{2}\right)=0, \quad x_{1}^{\top} x_{2}=0, \quad\left\|x_{1}\right\|_{2}=\left\|x_{2}\right\|_{2}=1, \text {, }, ~ \tag{3.4}
\end{array}\right.
$$

so that $\left(\delta_{1}-\frac{1}{\delta_{1}}\right)^{2} \beta_{1}^{2}$ is minimized, which obviously achieves its minimum when $\delta_{1}=1$. Let the columns of $S \in \mathbb{C}^{n \times r}$ be an orthonormal basis of $\mathcal{N}\left(Q_{2}^{\top}\left(A-\lambda_{1} I_{n}\right)\right)$, and $S_{R}=\operatorname{Re}(S), S_{I}=\operatorname{Im}(S)$. Direct calculations show that such $x_{1}, x_{2}$ satisfying (3.4) with $\delta_{1}=1$ can be obtained by $x_{1}=\left[\begin{array}{ll}S_{R} & -S_{I}\end{array}\right]\left[\begin{array}{llllll}\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots & \zeta_{r}\end{array}\right]^{\top}, \quad x_{2}=\left[\begin{array}{lllll}S_{I} & S_{R}\end{array}\right]\left[\begin{array}{lllll}\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots\end{array} \zeta_{r}\right]^{\top}$,
with $x_{1}^{\top} x_{2}=0$ and $\left\|x_{1}\right\|_{2}=\left\|x_{2}\right\|_{2}=1$. Clearly,

$$
\begin{align*}
& x_{1}^{\top} x_{2}+x_{2}^{\top} x_{1} \\
& =\left[\begin{array}{llllll}
\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots & \zeta_{r}
\end{array}\right]\left[\begin{array}{ccc}
S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R} & S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} \\
S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} & -\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right)
\end{array}\right]\left[\begin{array}{ccccc}
\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots \\
\zeta_{r}
\end{array}\right]^{\top} \text {, } \\
& x_{1}^{\top} x_{1}-x_{2}^{\top} x_{2} \\
& =\left[\begin{array}{llllll}
\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots & \zeta_{r}
\end{array}\right]\left[\begin{array}{ccccc}
S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} & -\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right) \\
-\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right) & S_{I}^{\top} S_{I}-S_{R}^{\top} S_{R}
\end{array}\right]\left[\begin{array}{lllll}
\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots \\
S_{r}
\end{array}\right]^{\top} . \tag{3.6}
\end{align*}
$$

Note that the two matrices in the above two equations are symmetric Hamiltonian systems owning special properties. So we exhibit some simple results about symmetric Hamiltonian system which will be used here and when assigning the complex conjugate poles. Both results can be verified directly, and we omit the proof.
Lemma 3.1. Let $A, B \in \mathbb{R}^{n \times n}$ satisfying $A^{\top}=A, B^{\top}=B$. If $\lambda$ is an eigenvalue of $\left[\begin{array}{cc}A & B \\ B & -A\end{array}\right]$ and $\left[\begin{array}{ll}x^{\top} & y^{\top}\end{array}\right]^{\top}$ is the corresponding eigenvector, then

$$
\left[\begin{array}{cc}
A & B \\
B & -A
\end{array}\right]\left[\begin{array}{cc}
x & -y \\
y & x
\end{array}\right]=\left[\begin{array}{cc}
x & -y \\
y & x
\end{array}\right]\left[\begin{array}{ll}
\lambda & \\
& -\lambda
\end{array}\right],
$$

and

$$
\left[\begin{array}{cc}
B & -A \\
-A & -B
\end{array}\right]\left[\begin{array}{cc}
x & -y \\
y & x
\end{array}\right]\left[\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{array}\right]=\left[\begin{array}{cc}
x & -y \\
y & x
\end{array}\right]\left[\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{array}\right]\left[\begin{array}{ll}
\lambda & \\
& -\lambda
\end{array}\right]
$$

Lemma 3.2. (Property of Two Hamiltonian Systems) Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric, and let $\left[\begin{array}{cc}A & B \\ B & -A\end{array}\right]=U \operatorname{diag}(\Theta,-\Theta) U^{\top}$ be the spectral decomposition, where $\Theta=\operatorname{diag}\left(\theta_{1}, \theta_{2}, \ldots, \theta_{n}\right)$ with $\theta_{1} \geq \theta_{2} \geq \ldots \geq \theta_{n} \geq 0$. If the $j$-th column $u_{j}$ and the $(n+j)$-th column $u_{n+j}$ of $U$ satisfy $u_{n+j}=\left[\begin{array}{ll}I_{n} & -\bar{I}_{n}\end{array}\right] u_{j}$, then $\left[\begin{array}{cc}B & -A \\ -A & -B\end{array}\right]=U\left[\begin{array}{cc}0 & -\Theta \\ -\Theta & 0\end{array}\right] U^{\top}$.

Applying Lemma 3.2 to the two symmetric Hamiltonian systems which appeared in (3.6), that is

$$
\begin{array}{r}
{\left[\begin{array}{cc}
S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R} & S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} \\
S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} & -\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right)
\end{array}\right]=U \operatorname{diag}(\Theta,-\Theta) U^{\top},} \\
{\left[\begin{array}{cc}
S_{R}^{\top} S_{R}-S_{I}^{\top} S_{I} & -\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right) \\
-\left(S_{R}^{\top} S_{I}+S_{I}^{\top} S_{R}\right) & S_{I}^{\top} S_{I}-S_{R}^{\top} S_{R}
\end{array}\right]=U\left[\begin{array}{cc}
0 & -\Theta \\
-\Theta & 0
\end{array}\right] U^{\top},}
\end{array}
$$

then if we let

$$
\left[\begin{array}{llllll}
\gamma_{1} & \ldots & \gamma_{r} & \zeta_{1} & \ldots & \zeta_{r}
\end{array}\right]^{\top}=U\left[\begin{array}{llllll}
\mu_{1} & \ldots & \mu_{r} & \nu_{1} & \ldots & \nu_{r} \tag{3.7}
\end{array}\right]^{\top},
$$

$x_{1}^{\top} x_{2}+x_{2}^{\top} x_{1}=\sum_{j=1}^{r} \theta_{j}\left(\mu_{j}^{2}-\nu_{j}^{2}\right)$ and $x_{1}^{\top} x_{1}-x_{2}^{\top} x_{2}=-2 \sum_{j=1}^{r} \theta_{j} \mu_{j} \nu_{j}$ follow. Without loss of generality, we may assume that $\theta_{1} \geq \theta_{2} \geq \ldots \geq \theta_{r} \geq 0$, then by taking

$$
\begin{align*}
& \mu_{3}=\nu_{3}=\ldots=\mu_{r}=\nu_{r}=0, \quad \mu_{1}=-\nu_{1}=\sqrt{\frac{\theta_{2}}{\theta_{1}} \mu_{2}^{2}}  \tag{3.8a}\\
& \left.\mu_{2}=\nu_{2}=\frac{1}{\|\left[\begin{array}{lll}
S_{R} & -S_{I}
\end{array}\right] U\left[\begin{array}{lllllllll}
\sqrt{\frac{\theta_{2}}{\theta_{1}}} & 1 & 0 & \cdots & 0 & -\sqrt{\frac{\theta_{2}}{\theta_{1}}} & 1 & 0 & \cdots
\end{array}\right.} 0\right]^{\top} \|_{2} \tag{3.8b}
\end{align*}
$$

it is easy to verify that (3.4) holds with $x_{1}$ and $x_{2}$ computed by (3.5) and (3.7). Hence, we can still choose initial vectors $x_{1}$ and $x_{2}$, so that $\left(\delta_{1}-\frac{1}{\delta_{1}}\right)^{2} \beta_{1}^{2}=0$. We then initially set

$$
X_{2}=\left[\begin{array}{ll}
x_{1} & x_{2}
\end{array}\right], \quad T_{2}=\left[\begin{array}{cc}
\alpha_{1} & \beta_{1}  \tag{3.9}\\
-\beta_{1} & \alpha_{1}
\end{array}\right] .
$$

Now assume that (3.1) has been satisfied with $j \geq 1$, we shall then assign the next pole $\lambda_{j+1}$.

### 3.1 Assigning a real pole

Assume that $\lambda_{j+1}$ is real, then the $(j+1)$-th diagonal element of $T$ must be $\lambda_{j+1}$. Comparing the $(j+1)$-th column of $Q_{2}^{\top} A X-Q_{2}^{\top} X T=0$ gives rise to

$$
\begin{equation*}
Q_{2}^{\top} A x_{j+1}-Q_{2}^{\top} X_{j} v_{j+1}-\lambda_{j+1} Q_{2}^{\top} x_{j+1}=0 \tag{3.10}
\end{equation*}
$$

Recall the definition of the departure from normality of $A_{c}$ in (2.6) and notice that we are now computing the $(j+1)$-th columns of $X$ and $T$, it is then natural to consider the following optimization problem:

$$
\begin{gather*}
\min _{\left\|x_{j+1}\right\|_{2}=1}\left\|v_{j+1}\right\|_{2}^{2}  \tag{3.11}\\
\text { s.t. } M_{j+1}\left[\begin{array}{l}
x_{j+1} \\
v_{j+1}
\end{array}\right]=0, \tag{3.12}
\end{gather*}
$$

where

$$
M_{j+1}=\left[\begin{array}{cc}
Q_{2}^{\top}\left(A-\lambda_{j+1} I_{n}\right) & -Q_{2}^{\top} X_{j}  \tag{3.13}\\
X_{j}^{\top} & 0
\end{array}\right]
$$

Let $r=\operatorname{dim} \mathcal{N}\left(M_{j+1}\right)$. Then it follows from the controllability of $(A, B)$ that $Q_{2}^{\top}\left(A-\lambda_{j+1} I_{n}\right)$ is of full row rank, indicating that $n-m \leq \operatorname{rank}\left(M_{j+1}\right) \leq n-m+j$ and $\mathcal{N}\left(M_{j+1}\right) \neq \emptyset$ ( [8]). Suppose that the columns of $S=\left[\begin{array}{ll}S_{1}^{\top} & S_{2}^{\top}\end{array}\right]^{\top}$ with $S_{1} \in \mathbb{R}^{n \times r}, S_{2} \in \mathbb{R}^{j \times r}$ form an orthonormal basis of $\mathcal{N}\left(M_{j+1}\right)$, then (3.12) shows that

$$
\begin{equation*}
x_{j+1}=S_{1} y, \quad v_{j+1}=S_{2} y, \quad \forall y \in \mathbb{R}^{r} \tag{3.14}
\end{equation*}
$$

Consequently, the optimization problem (3.11) subject to (3.12) is equivalent to the following problem:

$$
\begin{equation*}
\min _{y^{\top} S_{1}^{\top} S_{1} y=1} y^{\top} S_{2}^{\top} S_{2} y . \tag{3.15}
\end{equation*}
$$

Perceived that the discussions above can also be found in [8], and the constrained optimization problem (3.15) is solved via the GSVD of the matrix pencil $\left(S_{1}, S_{2}\right)$. We put forward a simpler approach here. Actually, since $S^{\top} S=I_{r}$, we have $S_{2}^{\top} S_{2}=I_{r}-S_{1}^{\top} S_{1}$. Thus the problem (3.15) is equivalent to

$$
\begin{equation*}
\min _{y^{\top} S_{1}^{\top} S_{1} y=1} y^{\top} y, \tag{3.16}
\end{equation*}
$$

whose minimum value is acquired when $y$ is an eigenvector of $S_{1}^{\top} S_{1}$ corresponding to its greatest eigenvalue and satisfies $y^{\top} S_{1}^{\top} S_{1} y=1$. Once such $y$ is obtained, $x_{j+1}$ and $v_{j+1}$ can be given by (3.14). We may then update $X_{j}$ and $T_{j}$ as

$$
X_{j+1}=\left[\begin{array}{ll}
X_{j} & x_{j+1}
\end{array}\right] \in \mathbb{R}^{n \times(j+1)}, \quad T_{j+1}=\left[\begin{array}{cc}
T_{j} & v_{j+1}  \tag{3.17}\\
0 & \lambda_{j+1}
\end{array}\right] \in \mathbb{R}^{(j+1) \times(j+1)}
$$

and continue with the next pole $\lambda_{j+2}$.

### 3.2 Assigning a pair of conjugate poles

In this subsection, we will consider the case that $\lambda_{j+1}$ is non-real. To obtain a real matrix $F$ from the real Schur form of $A_{c}=A+B F$, we would assign $\lambda_{j+1}$ and $\lambda_{j+2}=\bar{\lambda}_{j+1}$ simultaneously to get the $(j+1)$-th and $(j+2)$-th columns of $X$ and $T$.

### 3.2.1 Initial optimization problem

Assume that $\lambda_{j+1}=\alpha_{j+1}+i \beta_{j+1}\left(\beta_{j+1} \neq 0\right)$ and let $D_{\delta}=\left[\begin{array}{cc}\alpha_{j+1} & \delta \beta_{j+1} \\ -\beta_{j+1} / \delta & \alpha_{j+1}\end{array}\right]$ be the diagonal block in $T$ whose eigenvalues are $\lambda_{j+1}$ and $\bar{\lambda}_{j+1}$. By comparing the $(j+1)$-th and $(j+2)$-th columns of $Q_{2}^{\top} A X-Q_{2}^{\top} X T=0$, we have

$$
Q_{2}^{\top} A\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]-Q_{2}^{\top} X_{j}\left[\begin{array}{ll}
v_{j+1} & v_{j+2}
\end{array}\right]-Q_{2}^{\top}\left[\begin{array}{ll}
x_{j+1} & x_{j+2} \tag{3.18}
\end{array}\right] D_{\delta}=0
$$

Recalling the form of $\Delta_{F}^{2}\left(A_{c}\right)$ in (2.6), it is then natural to consider the following optimization problem:

$$
\begin{align*}
\min _{\delta, v_{j+1}, v_{j+2}} & \left\|v_{j+1}\right\|_{2}^{2}+\left\|v_{j+2}\right\|_{2}^{2}+\beta_{j+1}^{2}\left(\delta-\frac{1}{\delta}\right)^{2}  \tag{3.19a}\\
\text { s.t. } & Q_{2}^{\top}\left(\begin{array}{ll}
\left.A\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]-X_{j}\left[\begin{array}{ll}
v_{j+1} & v_{j+2}
\end{array}\right]-\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right] D_{\delta}\right)=0, \\
& X_{j}^{\top}\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=0, \\
& {\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]^{\top}\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=I_{2} .}
\end{array}\right. \tag{3.19b}
\end{align*}
$$

The constraints (3.19b) and (3.19d) are nonlinear. In [8, the author solves this optimization problem by taking $\delta=1$ and neglecting the orthogonal requirement $x_{j+1}^{\top} x_{j+2}=0$. These simplify the problem significantly. However, it cannot lead to the real Schur form of the closedloop system matrix $A_{c}$, since $x_{j+1}$ is generally not orthogonal to $x_{j+2}$. Moreover, the minimum value of the simplified optimization problem in [8 may be much greater than that of the original problem (3.19).

We may rewrite the optimization problem (3.19) into another equivalent form. If we write $\delta=\frac{\delta_{2}}{\delta_{1}}$ with $\delta_{1}, \delta_{2}>0$, and set $D_{0}=\left[\begin{array}{cc}\alpha_{j+1} & \beta_{j+1} \\ -\beta_{j+1} & \alpha_{j+1}\end{array}\right]$, then $D_{\delta}=\left[\begin{array}{cc}1 / \delta_{1} & \\ & 1 / \delta_{2}\end{array}\right] D_{0}\left[\begin{array}{ll}\delta_{1} & \\ & \delta_{2}\end{array}\right]$. Redefine $x_{j+1} \triangleq \frac{x_{j+1}}{\delta_{1}}, x_{j+2} \triangleq \frac{x_{j+2}}{\delta_{2}}, v_{j+1} \triangleq \frac{v_{j+1}}{\delta_{1}}, v_{j+2} \triangleq \frac{v_{j+2}}{\delta_{2}}$, then the optimization problem (3.19) is equivalent to

$$
\begin{align*}
\min _{\delta_{1}, \delta_{2}, v_{j+1}, v_{j+2}} & \left\|\delta_{1} v_{j+1}\right\|_{2}^{2}+\left\|\delta_{2} v_{j+2}\right\|_{2}^{2}+\beta_{j+1}^{2}\left(\frac{\delta_{1}}{\delta_{2}}-\frac{\delta_{2}}{\delta_{1}}\right)^{2}  \tag{3.20a}\\
\text { s.t. } & \left.Q_{2}^{\top}\left(\begin{array}{ll}
A\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]-X_{j}\left[v_{j+1}\right. & v_{j+2}
\end{array}\right]-\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right] D_{0}\right)=0,  \tag{3.20b}\\
& X_{j}^{\top}\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=0  \tag{3.20c}\\
& {\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]^{\top}\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=\left[\begin{array}{ll}
1 / \delta_{1}^{2} & \\
& 1 / \delta_{2}^{2}
\end{array}\right] } \tag{3.20d}
\end{align*}
$$

Here the constraint (3.20b) becomes linear. Once a solution to the optimization problem (3.20) is obtained, we need to redefine

$$
v_{j+1} \triangleq \frac{v_{j+1}}{\left\|x_{j+1}\right\|_{2}}, \quad v_{j+2} \triangleq \frac{v_{j+2}}{\left\|x_{j+2}\right\|_{2}}, \quad x_{j+1} \triangleq \frac{x_{j+1}}{\left\|x_{j+1}\right\|_{2}}, \quad x_{j+2} \triangleq \frac{x_{j+2}}{\left\|x_{j+2}\right\|_{2}}
$$

as the corresponding columns of $T$ and $X$.
The constraints (3.20b and 3.20c are linear. Actually, all vectors $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying these two constraints can be found via the null space of the matrix

$$
M_{j+1}=\left[\begin{array}{cc}
Q_{2}^{\top}\left(A-\left(\alpha_{j+1}+i \beta_{j+1}\right) I_{n}\right) & -Q_{2}^{\top} X_{j}  \tag{3.21}\\
X_{j}^{\top} & 0
\end{array}\right] .
$$

Specifically, for any $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying (3.20b) and (3.20c), direct calculations show that $M_{j+1}\left[\begin{array}{l}x_{j+1}+i x_{j+2} \\ v_{j+1}+i v_{j+2}\end{array}\right]=0$. Conversely, for any vector $\left[\begin{array}{ll}z^{\top} & w^{\top}\end{array}\right]^{\top} \in \mathcal{N}\left(M_{j+1}\right)$, the vectors $x_{j+1}=\operatorname{Re}(z), x_{j+2}=\operatorname{Im}(z), v_{j+1}=\operatorname{Re}(w), v_{j+2}=\operatorname{Im}(w)$ satisfy 3.20b and 3.20c). The constraint (3.20d) shows that $x_{j+1}^{\top} x_{j+2}=0$. For any vector $\left[\begin{array}{ll}z^{\top} & w^{\top}\end{array}\right]^{\top} \in \mathcal{N}\left(M_{j+1}\right)$ with $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ being linearly independent, we may then orthogonalize $\operatorname{Re}(z)$ and $\operatorname{Im}(z)$ by the Jacobi transformation as follows to get $x_{j+1}$ and $x_{j+2}$ satisfying $x_{j+1}^{\top} x_{j+2}=0$. Let $\varrho_{1}=\|\operatorname{Re}(z)\|_{2}^{2}, \varrho_{2}=$ $\|\operatorname{Im}(z)\|_{2}^{2}, \gamma=\operatorname{Re}(z)^{\top} \operatorname{Im}(z)$ and $\tau=\frac{\varrho_{2}-\varrho_{1}}{2 \gamma}$, and define $t$ as

$$
t= \begin{cases}1 /\left(\tau+\sqrt{1+\tau^{2}}\right), & \text { if } \quad \tau \geq 0, \\ -1 /\left(-\tau+\sqrt{1+\tau^{2}}\right), & \text { if } \quad \tau<0\end{cases}
$$

Let $c=1 / \sqrt{1+t^{2}}, s=t c$. Then $x_{j+1}$ and $x_{j+2}$ obtained by

$$
\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=\left[\begin{array}{ll}
\operatorname{Re}(z) & \operatorname{Im}(z)
\end{array}\right]\left[\begin{array}{cc}
c & s  \tag{3.22}\\
-s & c
\end{array}\right]
$$

satisfy $x_{j+1}^{\top} x_{j+2}=0$. Moreover, if we let

$$
\left[\begin{array}{ll}
v_{j+1} & v_{j+2}
\end{array}\right]=\left[\begin{array}{ll}
\operatorname{Re}(w) & \operatorname{Im}(w)
\end{array}\right]\left[\begin{array}{cc}
c & s  \tag{3.23}\\
-s & c
\end{array}\right],
$$

then $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfy (3.20b) and (3.20c). Hence, we can get $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying the constrains (3.20b)-(3.20d) in this way. Furthermore,

$$
\begin{equation*}
1 / \delta_{1}^{2}=\left\|x_{j+1}\right\|_{2}^{2}=\|x\|_{2}^{2}-\omega, \quad 1 / \delta_{2}^{2}=\left\|x_{j+2}\right\|_{2}^{2}=\|y\|_{2}^{2}+\omega \tag{3.24}
\end{equation*}
$$

where $x=\operatorname{Re}(z), y=\operatorname{Im}(z), \omega=\frac{2\left(x^{\top} y\right)^{2}}{\|y\|_{2}^{2}-\|x\|_{2}^{2}+\sqrt{4\left(x^{\top} y\right)^{2}+\left(\|y\|_{2}^{2}-\|x\|_{2}^{2}\right)^{2}}}$ if $\|x\|_{2}<\|y\|_{2}$; and $\omega=$ $\frac{2\left(x^{\top} y\right)^{2}}{\|y\|_{2}^{2}-\|x\|_{2}^{2}-\sqrt{4\left(x^{\top} y\right)^{2}+\left(\|y\|_{2}^{2}-\|x\|_{2}^{2}\right)^{2}}}$ if $\|x\|_{2} \geq\|y\|_{2}$.

### 3.2.2 The suboptimal strategy

It is hard to get an optimal solution to (3.20) since it is a nonlinear optimization problem with quadratic constraints. Even if such an optimal solution can be found, the cost will be expensive. So instead of finding an optimal solution, we prefer to get a suboptimal one with less computational cost.

Let the columns of $S=\left[\begin{array}{ll}S_{1}^{\top} & S_{2}^{\top}\end{array}\right]^{\top} \in \mathbb{C}^{(n+j) \times r}$ with $S_{1} \in \mathbb{C}^{n \times r}$ and $S_{2} \in \mathbb{C}^{j \times r}$ form an orthonormal basis of $\mathcal{N}\left(M_{j+1}\right)$, and let $S_{1}=U \Sigma V^{*}$ be the SVD of $S_{1}$. Since $S_{1}^{*} S_{1}+S_{2}^{*} S_{2}=I_{r}$, it follows that $S_{2}^{*} S_{2}=V\left(I_{r}-\Sigma^{*} \Sigma\right) V^{*}$. For any vector $\left[z^{\top} \quad w^{\top}\right]^{\top} \in \mathcal{N}\left(M_{j+1}\right)$ with $z \in \mathbb{C}^{n}$ and $w \in \mathbb{C}^{j}$, there exists $b \in \mathbb{C}^{r}$ such that $z=S_{1} b=U\left(\Sigma V^{*} b\right)$ and $w=S_{2} b$. Hence

$$
\|z\|_{2} \leq \sigma_{1}\|b\|_{2} \quad \text { and } \quad\|w\|_{2}^{2} \geq\left(1-\sigma_{1}^{2}\right)\|b\|_{2}^{2}
$$

where $\sigma_{1}$ is the largest singular value of $S_{1}$. Now suppose that the real part and the imaginary part of $z$ are linearly independent satisfying $\|\operatorname{Re}(z)\|_{2} \leq\|\operatorname{Im}(z)\|_{2}$, and $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ are obtained from the the Jacobi orthogonal process (3.22), (3.23). Define $C=\frac{\|z\|_{2}}{\left\|x_{j+1}\right\|_{2}}$, then $C \geq \sqrt{2}$ and the objective function in 3.20a becomes

$$
\begin{align*}
& \left\|\delta_{1} v_{j+1}\right\|_{2}^{2}+\left\|\delta_{2} v_{j+2}\right\|_{2}^{2}+\beta_{j+1}^{2}\left(\frac{\delta_{1}}{\delta_{2}}-\frac{\delta_{2}}{\delta_{1}}\right)^{2} \\
= & \frac{C^{2}}{C^{2}-1} \frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}}+\frac{C^{4}-2 C^{2}}{C^{2}-1} \frac{\left\|v_{j+1}\right\|_{2}^{2}}{\|z\|_{2}^{2}}+\beta_{j+1}^{2}\left(C^{2}-3+\frac{1}{C^{2}-1}\right) . \tag{3.25}
\end{align*}
$$

Obviously,

$$
\begin{equation*}
\frac{C^{2}}{C^{2}-1} \frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}} \leq \frac{C^{2}}{C^{2}-1} \frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}}+\frac{C^{4}-2 C^{2}}{C^{2}-1} \frac{\left\|v_{j+1}\right\|_{2}^{2}}{\|z\|_{2}^{2}} \leq C^{2} \frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}} \tag{3.26}
\end{equation*}
$$

So the objective function in (3.20a) depends on $\frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}}$ and $C$ with $\min \frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}}=\frac{1-\sigma_{1}^{2}}{\sigma_{1}^{2}}$. In our suboptimal strategy, we will first take $b$ from $\operatorname{span}\left\{V e_{1}\right\}$, where $e_{i}$ is the $i$-th column of the identity matrix. With this choice, $\frac{\|w\|_{2}^{2}}{\|z\|_{2}^{2}}$ achieves its minimum value. And the following theorem shows the relevant results.

Theorem 3.1. With the notations above, let $u_{1}$ be the first column of $U$ and assume that $\operatorname{Re}\left(u_{1}\right)$ and $\operatorname{Im}\left(u_{1}\right)$ are linearly independent. Let $x_{j+1}$ and $x_{j+2}$ be the vectors obtained from $\operatorname{Re}\left(u_{1}\right)$ and $\operatorname{Im}\left(u_{1}\right)$ via the Jacobi orthogonal process

$$
\left[\begin{array}{ll}
x_{j+1} & x_{j+2}
\end{array}\right]=\left[\begin{array}{ll}
\operatorname{Re}\left(u_{1}\right) & \operatorname{Im}\left(u_{1}\right)
\end{array}\right]\left[\begin{array}{cc}
c & s \\
-s & c
\end{array}\right]
$$

and let

$$
\left[\begin{array}{ll}
v_{j+1} & v_{j+2}
\end{array}\right]=\left[\begin{array}{ll}
\operatorname{Re}(w) & \operatorname{Im}(w)
\end{array}\right]\left[\begin{array}{cc}
c & s \\
-s & c
\end{array}\right]
$$

where $w=S_{2} V e_{1} / \sigma_{1}$. Then $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfy the constrains (3.20b) - 3.20d), and the value of the corresponding objective function specified by (3.20a) will be no larger than

$$
\frac{1}{\min \left\{\left\|x_{j+1}\right\|_{2}^{2},\left\|x_{j+2}\right\|_{2}^{2}\right\}}\left(\frac{1-\sigma_{1}^{2}}{\sigma_{1}^{2}}+\beta_{j+1}^{2}\right)
$$

Proof. The first part of the theorem is obvious. To prove the second part, note that here $b=\frac{V e_{1}}{\sigma_{1}}$, $\|z\|_{2}=\left\|u_{1}\right\|_{2}=1,\|w\|_{2}^{2}=\frac{1-\sigma_{1}^{2}}{\sigma_{1}^{2}}$. If $\left\|\operatorname{Re}\left(u_{1}\right)\right\|_{2} \leq\left\|\operatorname{Im}\left(u_{1}\right)\right\|_{2}$, it then follows directly from (3.25), (3.26) and $C^{2}-3+\frac{1}{C^{2}-1} \leq C^{2}$ with $C=\frac{1}{\left\|x_{j+1}\right\|_{2}}$. The case when $\left\|\operatorname{Re}\left(u_{1}\right)\right\|_{2} \geq\left\|\operatorname{Im}\left(u_{1}\right)\right\|_{2}$ can be proved similarly.

Theorem 3.1shows that if $\operatorname{Re}\left(u_{1}\right)$ and $\operatorname{Im}\left(u_{1}\right)$ are linearly independent, and $\min \left\{\left\|x_{j+1}\right\|_{2},\left\|x_{j+2}\right\|_{2}\right\}$ is not pathologically small, the above procedure will generate $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ satisfying the constrains (3.20b)-3.20d), and the value of the corresponding objective function in (3.20a) is not too large. We then take these $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ as the suboptimal solution. However, if $\operatorname{Re}\left(u_{1}\right)$ and $\operatorname{Im}\left(u_{1}\right)$ are linearly dependent, we cannot get orthogonal $x_{j+1}$ and $x_{j+2}$ via the Jacobi orthogonal process. Even if $\operatorname{Re}\left(u_{1}\right)$ and $\operatorname{Im}\left(u_{1}\right)$ are linearly independent, the resulted $\min \left\{\left\|x_{j+1}\right\|_{2},\left\|x_{j+2}\right\|_{2}\right\}$ might be fairly small, which means that the corresponding value of the objective function might be large. In this case, we would choose $b$ from $\operatorname{span}\left\{V e_{1}, V e_{2}\right\}$.

Define

$$
\begin{array}{ll}
\tilde{x}_{1}+i \tilde{y}_{1}=z_{1}=u_{1}=\frac{S_{1} V e_{1}}{\sigma_{1}}, & w_{1}=\frac{S_{2} V e_{1}}{\sigma_{1}} \\
\tilde{x}_{2}+i \tilde{y}_{2}=z_{2}=u_{2}=\frac{S_{1} V e_{2}}{\sigma_{2}}, & w_{2}=\frac{S_{2} V e_{2}}{\sigma_{2}} \tag{3.27}
\end{array}
$$

where $\sigma_{1}, \sigma_{2}$ are the first two greatest singular values of $S_{1}$. Let $b=\left[\begin{array}{ll}\frac{V e_{1}}{\sigma_{1}} & \frac{V e_{2}}{\sigma_{2}}\end{array}\right]\left[\begin{array}{c}\gamma_{1}+i \zeta_{1} \\ \gamma_{2}+i \zeta_{2}\end{array}\right]$ with $\gamma_{1}^{2}+\gamma_{2}^{2}+\zeta_{1}^{2}+\zeta_{2}^{2}=1$, then

$$
x+i y=z=S_{1} b=\left[\begin{array}{ll}
z_{1} & z_{2}
\end{array}\right]\left[\begin{array}{l}
\gamma_{1}+i \zeta_{1}  \tag{3.28}\\
\gamma_{2}+i \zeta_{2}
\end{array}\right], \quad w=S_{2} b=\left[\begin{array}{ll}
w_{1} & w_{2}
\end{array}\right]\left[\begin{array}{l}
\gamma_{1}+i \zeta_{1} \\
\gamma_{2}+i \zeta_{2}
\end{array}\right] .
$$

Denoting $\tilde{X}=\left[\begin{array}{cc}\tilde{x}_{1} & \tilde{x}_{2}\end{array}\right], \tilde{Y}=\left[\begin{array}{ll}\tilde{y}_{1} & \tilde{y}_{2}\end{array}\right]$, it can be easily verified that

$$
x=\left[\begin{array}{ll}
\tilde{X} & -\tilde{Y}
\end{array}\right]\left[\begin{array}{llll}
\gamma_{1} & \gamma_{2} & \zeta_{1} & \zeta_{2}
\end{array}\right]^{\top}, \quad y=\left[\begin{array}{cc}
\tilde{Y} & \tilde{X}
\end{array}\right]\left[\begin{array}{llll}
\gamma_{1} & \gamma_{2} & \zeta_{1} & \zeta_{2} \tag{3.29}
\end{array}\right]^{\top},
$$

and

$$
\left.\begin{array}{l}
x^{\top} y+y^{\top} x=\left[\begin{array}{llll}
\gamma_{1} & \gamma_{2} & \zeta_{1} & \zeta_{2}
\end{array}\right]\left[\begin{array}{cc}
\tilde{X}^{\top} \tilde{Y}^{2}+\tilde{Y}^{\top} \tilde{X} & \tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} \\
\tilde{X}^{\top}-\tilde{Y}^{\top} \tilde{Y} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right)
\end{array}\right]\left[\begin{array}{lll}
\gamma_{1} & \gamma_{2} & \zeta_{1}
\end{array} \zeta_{2}\right.
\end{array}\right]^{\top}, ~(3.3 \text {. }
$$

Obviously, the two matrices in (3.30) and (3.31) are symmetric Hamiltonian systems and they satisfy the property in Lemma 3.2. Hence we can get the following lemma.

Lemma 3.3. Let $\phi_{m}, \phi_{M}$ be the two smallest singular values of $\left[\begin{array}{cc}\tilde{Y} & \tilde{X}\end{array}\right]$ and $\left[\begin{array}{l}p_{1} \\ q_{1}\end{array}\right],\left[\begin{array}{l}p_{2} \\ q_{2}\end{array}\right]$ be the corresponding right singular vectors respectively. Define

$$
\Omega=\left[\begin{array}{cccc}
p_{1} & p_{2} & -q_{1} & -q_{2}  \tag{3.32}\\
q_{1} & q_{2} & p_{1} & p_{2}
\end{array}\right],
$$

$\Phi=\operatorname{diag}\left(\phi_{1}, \phi_{2},-\phi_{1},-\phi_{2}\right)$ with $\phi_{1}=1-2 \phi_{m}^{2}, \phi_{2}=1-2 \phi_{M}^{2}$, then

$$
\left[\begin{array}{cc}
\tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right)  \tag{3.33}\\
-\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) & \tilde{Y}^{\top} \tilde{Y}-\tilde{X}^{\top} \tilde{X}
\end{array}\right]=\Omega \Phi \Omega^{\top}
$$

and

$$
\left[\begin{array}{cc}
\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X} & \tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} \\
\tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right)
\end{array}\right]=\Omega\left(\begin{array}{cc|ll} 
& & \phi_{1} & \\
& & & \phi_{2} \\
\hline \phi_{1} & & & \\
& \phi_{2} & &
\end{array}\right) \Omega^{\top} .
$$

Proof. Since $\left(\tilde{X}^{\top}-i \tilde{Y}^{\top}\right)(\tilde{X}+i \tilde{Y})=\left[\begin{array}{ll}z_{1} & z_{2}\end{array}\right]^{*}\left[\begin{array}{ll}z_{1} & z_{2}\end{array}\right]=I_{2}$, so $\tilde{X}^{\top} \tilde{X}+\tilde{Y}^{\top} \tilde{Y}=I_{2}$ and $\tilde{X}^{\top} \tilde{Y}=$ $\tilde{Y}^{\top} \tilde{X}$. Thus

$$
\left[\begin{array}{cc}
\tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) \\
-\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) & \tilde{Y}^{\top} \tilde{Y}-\tilde{X}^{\top} \tilde{X}^{2}
\end{array}\right]=\left[\begin{array}{cc}
I_{2}-2 \tilde{Y}^{\top} \tilde{Y} & -2 \tilde{Y}^{\top} \tilde{X} \\
-2 \tilde{X}^{\top} \tilde{Y}^{2} & I_{2}-2 \tilde{X}^{\top} \tilde{X}
\end{array}\right]=I_{4}-2\left[\begin{array}{c}
\tilde{Y}^{\top} \\
\tilde{X}^{\top}
\end{array}\right]\left[\begin{array}{cc}
\tilde{Y} & \tilde{X}
\end{array}\right]
$$

From the above equation, it obviously holds that $\phi_{1}, \phi_{2}$ are the two nonnegative eigenvalues of $\left[\begin{array}{cc}\tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y}^{2} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) \\ -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) & \tilde{Y}^{\top} \tilde{Y}-\tilde{X}^{\top} \tilde{X}\end{array}\right]$ with $\left[\begin{array}{c}p_{1} \\ q_{1}\end{array}\right],\left[\begin{array}{c}p_{2} \\ q_{2}\end{array}\right]$ being the corresponding eigenvectors. Note that $\left[\begin{array}{cc}\tilde{X}^{\top} \tilde{X}-\tilde{Y}^{\top} \tilde{Y} & -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) \\ -\left(\tilde{X}^{\top} \tilde{Y}+\tilde{Y}^{\top} \tilde{X}\right) & \tilde{Y}^{\top} \tilde{Y}-\tilde{X}^{\top} \tilde{X}\end{array}\right]$ is a Hamiltonian matrix, thus the results follow immediately from Lemma 3.1 and Lemma 3.2.

Now by defining

$$
\left[\begin{array}{llll}
\mu_{1} & \mu_{2} & \nu_{1} & \nu_{2}
\end{array}\right]^{\top}=\Omega^{\top}\left[\begin{array}{llll}
\gamma_{1} & \gamma_{2} & \zeta_{1} & \zeta_{2} \tag{3.34}
\end{array}\right]^{\top}
$$

we have

$$
\begin{equation*}
x^{\top} y+y^{\top} x=2 \phi_{1} \mu_{1} \nu_{1}+2 \phi_{2} \mu_{2} \nu_{2}, \quad x^{\top} x-y^{\top} y=\phi_{1}\left(\mu_{1}^{2}-\nu_{1}^{2}\right)+\phi_{2}\left(\mu_{2}^{2}-\nu_{2}^{2}\right) \tag{3.35}
\end{equation*}
$$

Theorem 3.2. With the notations above, there exist $\mu_{1}, \mu_{2}, \nu_{1}, \nu_{2} \in \mathbb{R}$ such that $x^{\top} y=0$ and $\|x\|_{2}=\|y\|_{2}=\frac{\sqrt{2}}{2}$. For these $\mu_{1}, \mu_{2}, \nu_{1}, \nu_{2}$, let $\gamma_{1}, \gamma_{2}, \zeta_{1}, \zeta_{2}$ be computed from (3.34), where $\Omega$ is as in (3.32). Then $x_{j+1}=x, x_{j+2}=y, v_{j+1}=\operatorname{Re}(w)$ and $v_{j+2}=\operatorname{Im}(w)$, where $w$ is computed by (3.28), satisfy the constrains (3.20b) - (3.20d), and the value of the corresponding objective function in (3.20a) will be no larger than $\frac{2\left(1-\sigma_{2}^{2}\right)}{\sigma_{2}^{2}}$.

Proof. It is easy to check that all solutions of the following system of equations

$$
\begin{cases}\phi_{1} \mu_{1} \nu_{1}+\phi_{2} \mu_{2} \nu_{2} & =0  \tag{3.36}\\ \phi_{1}\left(\mu_{1}^{2}-\nu_{1}^{2}\right)+\phi_{2}\left(\mu_{2}^{2}-\nu_{2}^{2}\right) & =0 \\ \mu_{1}^{2}+\mu_{2}^{2}+\nu_{1}^{2}+\nu_{2}^{2} & =1\end{cases}
$$

are

$$
\left\{\begin{array} { l l } 
{ \mu _ { 2 } = \pm \sqrt { \frac { \phi _ { 1 } } { \phi _ { 1 } + \phi _ { 2 } } - \nu _ { 2 } ^ { 2 } } }  \tag{3.37}\\
{ \mu _ { 1 } = - \sqrt { \frac { \phi _ { 2 } } { \phi _ { 1 } } \nu _ { 2 } } } \\
{ \nu _ { 1 } } & { = \pm \sqrt { \frac { \phi _ { 2 } } { \phi _ { 1 } + \phi _ { 2 } } - \frac { \phi _ { 2 } } { \phi _ { 1 } } \nu _ { 2 } ^ { 2 } } }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
\mu_{2}= \pm \sqrt{\frac{\phi_{1}}{\phi_{1}+\phi_{2}}-\nu_{2}^{2}} \\
\mu_{1}=\sqrt{\frac{\phi_{2}}{\phi_{1}} \nu_{2}} \\
\nu_{1}=\mp \sqrt{\frac{\phi_{2}}{\phi_{1}+\phi_{2}}-\frac{\phi_{2}}{\phi_{1}} \nu_{2}^{2}}
\end{array}\right.\right.
$$

with $\nu_{2}^{2} \leq \frac{\phi_{1}}{\phi_{1}+\phi_{2}}$. Note (3.35) and $\|x\|_{2}^{2}+\|y\|_{2}^{2}=1$, so with the values in (3.37), it holds that $x^{\top} y=$ 0 and $\|x\|_{2}=\|y\|_{2}=\frac{\sqrt{2}}{2}$. Since $\left[\begin{array}{ll}z^{\top} & w^{\top}\end{array}\right]^{\top} \in \mathcal{N}\left(M_{j+1}\right)$, so $\left[\begin{array}{cc}x_{j+1} & x_{j+2} \\ v_{j+1} & v_{j+2}\end{array}\right]=\left[\begin{array}{cc}x & y \\ \operatorname{Re}(w) & \operatorname{Im}(w)\end{array}\right]$ satisfy the constrains (3.20b)- 3.20 d$)$ with $\delta_{1}=\delta_{2}=\frac{\sqrt{2}}{2}$. Hence

$$
\begin{aligned}
& \left\|\delta_{1} v_{j+1}\right\|_{2}^{2}+\left\|\delta_{2} v_{j+2}\right\|_{2}^{2}+\beta_{j+1}^{2}\left(\frac{\delta_{1}}{\delta_{2}}-\frac{\delta_{2}}{\delta_{1}}\right)^{2} \\
= & 2\|w\|_{2}^{2}=2\left(\gamma_{1}^{2}+\zeta_{1}^{2}\right) \frac{1-\sigma_{1}^{2}}{\sigma_{1}^{2}}+2\left(\gamma_{2}^{2}+\zeta_{2}^{2}\right) \frac{1-\sigma_{2}^{2}}{\sigma_{2}^{2}} \leq \frac{2\left(1-\sigma_{2}^{2}\right)}{\sigma_{2}^{2}},
\end{aligned}
$$

which completes the proof of the theorem.
From the proof of Theorem 3.2 we can see that with such choice of $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$, the value of the corresponding objective function is just $2\|w\|_{2}^{2}$. Define $\xi_{1}=p_{1}^{\top} \Xi p_{1}, \xi_{2}=p_{2}^{\top} \Xi p_{2}, \eta_{1}=$ $q_{1}^{\top} \Xi q_{1}, \eta_{2}=q_{2}^{\top} \Xi q_{2}, \zeta_{12}=q_{1}^{\top} \Xi p_{2}, \zeta_{21}=q_{2}^{\top} \Xi p_{1}$, with $\Xi=\operatorname{diag}\left\{\left(1-\sigma_{1}^{2}\right) / \sigma_{1}^{2},\left(1-\sigma_{2}^{2}\right) / \sigma_{2}^{2}\right\}$, it then
follows

$$
\|w\|_{2}^{2}= \begin{cases}\frac{\phi_{2}}{\phi_{1}+\phi_{2}}\left(\xi_{1}+\eta_{1}\right)+\frac{\phi_{1}}{\phi_{1}+\phi_{2}}\left(\xi_{2}+\eta_{2}\right)+2 \sqrt{\frac{\phi_{2}}{\phi_{1}}} \frac{\phi_{1}}{\phi_{1}+\phi_{2}}\left(\zeta_{21}-\zeta_{12}\right) \quad \text { if }\left(\mu_{1} \nu_{2}\right) \leq 0  \tag{3.38}\\ \frac{\phi_{2}}{\phi_{1}+\phi_{2}}\left(\xi_{1}+\eta_{1}\right)+\frac{\phi_{1}}{\phi_{1}+\phi_{2}}\left(\xi_{2}+\eta_{2}\right)+2 \sqrt{\frac{\phi_{2}}{\phi_{1}} \frac{\phi_{1}}{\phi_{1}+\phi_{2}}\left(\zeta_{12}-\zeta_{21}\right)} \quad \text { if }\left(\mu_{1} \nu_{2}\right)>0\end{cases}
$$

So in order to get a smaller $\|w\|_{2}$, we can take $\mu_{1}, \mu_{2}, \nu_{1}, \nu_{2}$ satisfying $\mu_{1} \nu_{2} \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_{1} \nu_{2}>0$ if $\zeta_{21}>\zeta_{12}$.

Till now we have proposed two strategies for computing $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$. The first strategy computes $x_{j+1}, x_{j+2}, v_{j+1}$ and $v_{j+2}$ by using the Jacobi orthogonal process (3.22) and (3.23) with $z=u_{1}$ and $w=\frac{S_{2} V e_{1}}{\sigma_{1}}$. While the second one first computes $\mu_{1}, \mu_{2}, \nu_{1}, \nu_{2}$ by (3.37) satisfying $\mu_{1} \nu_{2} \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_{1} \nu_{2}>0$ if $\zeta_{21}>\zeta_{12}$, and then compute $\gamma_{1}, \gamma_{2}, \zeta_{1}, \zeta_{2}$ from (3.34), where $\Omega$ is as in (3.32), and finally set $x_{j+1}=x, x_{j+2}=y, v_{j+1}=\operatorname{Re}(w)$ and $v_{j+2}=\operatorname{Im}(w)$, where $x, y, w$ are computed by (3.28). We cannot tell which strategy is better. So we suggest to apply both strategies, compare the corresponding values of the objective function and adopt the one which gives better results. Specifically, if the value of the objective function corresponding to the first strategy is smaller, we would update $X_{j}$ and $T_{j}$ as

$$
X_{j+2}=\left[\begin{array}{lll}
X_{j} & \delta_{1} x_{j+1} & \delta_{2} x_{j+2}
\end{array}\right] \in \mathbb{R}^{n \times(j+2)}, \quad T_{j+2}=\left[\begin{array}{ccc}
T_{j} & \delta_{1} v_{j+1} & \delta_{2} v_{j+2}  \tag{3.39}\\
0 & \alpha_{j+1} & \delta \beta_{j+1} \\
0 & -\frac{1}{\delta} \beta_{j+1} & \alpha_{j+1}
\end{array}\right] \in \mathbb{R}^{(j+2) \times(j+2)},
$$

where $\delta_{1}=\frac{1}{\left\|x_{j+1}\right\|_{2}}, \delta_{2}=\frac{1}{\left\|x_{j+2}\right\|_{2}}, \delta=\frac{\delta_{2}}{\delta_{1}}$. Otherwise, we update $X_{j}$ and $T_{j}$ as

$$
X_{j+2}=\left[\begin{array}{lll}
X_{j} & \sqrt{2} x & \sqrt{2} y
\end{array}\right] \in \mathbb{R}^{n \times(j+2)}, \quad T_{j+2}=\left[\begin{array}{ccc}
T_{j} & \sqrt{2} \operatorname{Re}(w) & \sqrt{2} \operatorname{Im}(w)  \tag{3.40}\\
0 & \alpha_{j+1} & \beta_{j+1} \\
0 & -\beta_{j+1} & \alpha_{j+1}
\end{array}\right] \in \mathbb{R}^{(j+2) \times(j+2)},
$$

with $x, y$ and $w$ defined as in (3.28). This completes the assignment of the complex conjugate poles $\lambda_{j+1}, \lambda_{j+2}=\bar{\lambda}_{j+1}$, and we can then continue with the next pole $\lambda_{j+3}$.

These two strategies essentially choose $z$ from $\mathcal{R}\left(u_{1}\right)$ and $\mathcal{R}\left(\left[\begin{array}{ll}u_{1} & u_{2}\end{array}\right]\right)$, respectively. If the results by these two strategies are not satisfactory, theoretically, we can choose $z$ from a higher dimensional space, i.e. $z \in \operatorname{span}\left\{u_{1}, u_{2}, \ldots, u_{k}\right\}, k \geq 3$, with $u_{l}$ being the $l$-th column of $U$. However the resulted optimization problem is much more complicated. More importantly, numerical examples show that these two strategies with $k=1,2$ can produce fairly satisfying results for most problems.

### 3.3 Algorithm

In this part, we give the framework of our algorithm.

## Algorithm 1 Framework of our Schur-rob algorithm. Input:

$A, B$ and $\mathfrak{L}=\left\{\lambda_{1}, \ldots, \lambda_{n}\right\}$ (complex conjugate poles appear in pairs).

## Output:

The feedback matrix $F$.
If $\lambda_{1}$ is real, compute $x_{1}$ by (3.3) and set $X_{1}=x_{1}, T_{1}=\lambda_{1}, j=1$. If $\lambda_{1}$ is non-real, compute $x_{1}, x_{2}$ by (3.5), (3.7), (3.8), and set $X_{2}, T_{2}$ as in (3.9), $j=2$.
while $j<n$ do
if $\lambda_{j+1}$ is real then
Find $S=\left[\begin{array}{ll}S_{1}^{\top} & S_{2}^{\top}\end{array}\right]^{\top}$, whose columns form an orthonormal basis of $\mathcal{N}\left(M_{j+1}\right)$ in (3.13);

Compute $y$ by (3.16);
Compute $x_{j+1}$ and $v_{j+1}$ by (3.14), update $X_{j}$ and $T_{j}$ as (3.17) and set $j=j+1$.
else
Find $S=\left[\begin{array}{ll}S_{1}^{\top} & S_{2}^{\top}\end{array}\right]^{\top}$, whose columns form an orthonormal basis of $\mathcal{N}\left(M_{j+1}\right)$ in (3.21);

Compute the SVD of $S_{1}$ as $S_{1}=U \Sigma V^{*}$;
if $\operatorname{Re}\left(U e_{1}\right)$ and $\operatorname{Im}\left(U e_{1}\right)$ are linearly independent then
Compute $x_{j+1}, x_{j+2}, v_{j+1}, v_{j+2}$ by (3.22) and (3.23) with $z=\frac{S_{1} V e_{1}}{\sigma_{1}}, w=\frac{S_{2} V e_{1}}{\sigma_{1}}$;
Set $\delta_{1}=\frac{1}{\left\|x_{j+1}\right\|_{2}}, \delta_{2}=\frac{1}{\left\|x_{j+2}\right\|_{2}}$ and $\delta=\frac{\delta_{2}}{\delta_{1}}$;
Compute $\operatorname{dep}_{1}=\left\|\delta_{1} v_{j+1}\right\|_{2}^{2}+\left\|\delta_{2} v_{j+2}\right\|_{2}^{2}+\beta_{j+1}^{2}\left(\delta-\frac{1}{\delta}\right)^{2} ;$
else
Set $d e p_{1}=\infty ;$
end if
Let $\tilde{X}=\left[\begin{array}{cc}\tilde{x}_{1} & \tilde{x}_{2}\end{array}\right], \tilde{Y}=\left[\begin{array}{cc}\tilde{y}_{1} & \tilde{y}_{2}\end{array}\right]$ with $\tilde{x}_{1}, \tilde{y}_{1}, \tilde{x}_{2}, \tilde{y}_{2}$ defined as in (3.27), and compute the spectral decomposition (3.33);

Compute $\mu_{1}, \mu_{2}, \nu_{1}, \nu_{2}$ by (3.37) satisfying $\mu_{1} \nu_{2} \leq 0$ if $\zeta_{21} \leq \zeta_{12}$, and $\mu_{1} \nu_{2}>0$ if $\zeta_{21}>\zeta_{12}$, and then compute $\gamma_{1}, \gamma_{2}, \zeta_{1}, \zeta_{2}$ from (3.34), where $\Omega$ is as in (3.32);

Compute $z, w$ by (3.28), set $x_{j+1}=\operatorname{Re}(z), x_{j+2}=\operatorname{Im}(z), v_{j+1}=\operatorname{Re}(w)$ and $v_{j+2}=$ $\operatorname{Im}(w)$. Compute dep ${ }_{2}=2\left[\left(\gamma_{1}^{2}+\zeta_{1}^{2}\right) \frac{1-\sigma_{1}^{2}}{\sigma_{1}^{2}}+\left(\gamma_{2}^{2}+\zeta_{2}^{2}\right) \frac{1-\sigma_{2}^{2}}{\sigma_{2}^{2}}\right] ;$
If $d e p_{1}<d e p_{2}$, update $X_{j}$ and $T_{j}$ as in (3.39); otherwise, update them as in (3.40). Set $j=j+2$.
end if
end while
Set $X=X_{n}, T=T_{n}$, and compute $F$ by (2.5).

## 4 Numerical Examples

In this section, we give some numerical examples to illustrate the performance of our Schur-rob algorithm, and compare it with some of the different versions of SCHUR in [8, the MATLAB functions robpole [23] and place [13]. Each algorithm computes a feedback matrix $F$ such that the eigenvalues of $A+B F$ are those given in $\mathfrak{L}$, and $A+B F$ is robust. When applying robpole to all test examples, we set the maximum number of sweep to be the default value 5 . All calculations are carried out on an Intel $\circledR$ Core ${ }^{\mathrm{TM}} \mathrm{i} 3$, dual core, 2.27 GHz machine, with 2.00 GB RAM. MATLAB R2012a is used with machine epsilon $\epsilon \approx 2.2 \times 10^{-16}$.

With $\lambda_{1} \in \mathbb{R}$ fixed, the choice of $x_{1}$ in Schur-rob ignores the freedom of $x_{1}$. Inspired by O-SCHUR [8], we may regard $x_{1}$ as a free parameter and manage to optimize the robustness. Specifically, we may run Schur-rob with several different choices of $x_{1}$, and keep the solution $F$ corresponding to the minimum departure from normality. We denote such method as "O-Schurrob".

In this section, results on precision and robustness obtained by different algorithms are displayed. Here the precision refers to the accuracy of the eigenvalues of computed $A_{c}=A+B F$, compared with the prescribed poles in $\mathfrak{L}$. Precisely, we list

$$
\text { precs }=\left\lfloor\min _{1 \leq j \leq n}\left(-\log \left(\left|\frac{\lambda_{j}-\hat{\lambda}_{j}}{\lambda_{j}}\right|\right)\right)\right\rfloor,
$$

where $\hat{\lambda}_{j}, j=1, \ldots, n$ are eigenvalues of computed $A_{c}=A+B F$. Larger values of precs indicate more accurate computed eigenvalues. The robustness is, however, more complicated, since different measures of robustness are used in these algorithms. Specifically, let the spectral decomposition and the real Schur decomposition of $A+B F$ respectively be

$$
A+B F=X \Lambda X^{-1}, \quad A+B F=U T U^{\top}
$$

where $\Lambda$ is a diagonal matrix whose diagonal elements are those in $\mathfrak{L}, U$ is orthogonal, and $T$ is the real Schur form. The MATLAB function place tends to minimize $\left\|X^{-1}\right\|_{F}$ and robpole aims to maximum $|\operatorname{det}(X)|$. Both measures are closely related to the condition number $\kappa_{F}(X)=$ $\|X\|_{F}\left\|X^{-1}\right\|_{F}$. While different versions of SCHUR [8] and our Schur-rob try to minimize the departure from normality of $A_{c}=A+B F$. Hence, in the following tests, we adopt the following two measures of robustness: the departure from normality of $A_{c}$ (denoted as "dep.") and the condition number of $X$ (denoted as " $\kappa_{F}(X)$ ").

| $(n, k)$ | dep. |  |  |  | precs |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | SCHUR | $9.5 \mathrm{e}+1$ | $2.2 \mathrm{e}+1$ | $4.3 \mathrm{e}+1$ | $2.7 \mathrm{e}+0$ | 14 | 14 |  |
| $(4,1 \mathrm{e}+2)$ | $1.5 \mathrm{e}+4$ | $8.2 \mathrm{e}+2$ | $1.4 \mathrm{e}+4$ | $3.3 \mathrm{e}+2$ | 11 | 13 | 14 |  |
| $(4,1 \mathrm{e}+3)$ | $1.4 \mathrm{e}+6$ | $6.6 \mathrm{e}+4$ | $1.2 \mathrm{e}+6$ | $6.6 \mathrm{e}+2$ | 7 | 8 | 11 |  |
| $(4,1 \mathrm{e}+4)$ | $2.9 \mathrm{e}+8$ | $9.9 \mathrm{e}+5$ | $4.3 \mathrm{e}+7$ | $1.0 \mathrm{e}+4$ | 4 | 10 | 7 |  |
| $(4,1 \mathrm{e}+5)$ | $1.8 \mathrm{e}+10$ | $7.3 \mathrm{e}+6$ | $1.2 \mathrm{e}+10$ | $3.8 \mathrm{e}+5$ | 3 | 7 | 14 |  |
| $(20,1 \mathrm{e}+1)$ | $4.0 \mathrm{e}+1$ | $7.6 \mathrm{e}+0$ | $1.7 \mathrm{e}+1$ | $4.6 \mathrm{e}+0$ | 13 | 14 | 10 |  |
| $(20,1 \mathrm{e}+2)$ | $7.7 \mathrm{e}+4$ | $2.6 \mathrm{e}+2$ | $2.4 \mathrm{e}+2$ | $1.8 \mathrm{e}+1$ | 9 | 12 | 14 |  |
| $(20,1 \mathrm{e}+3)$ | $2.0 \mathrm{e}+5$ | $4.4 \mathrm{e}+3$ | $9.3 \mathrm{e}+4$ | $4.7 \mathrm{e}+2$ | 9 | 11 | 11 |  |
| $(20,1 \mathrm{e}+4)$ | $3.2 \mathrm{e}+7$ | $2.4 \mathrm{e}+4$ | $5.2 \mathrm{e}+6$ | $1.9 \mathrm{e}+3$ | 6 | 10 | 10 |  |
| $(20,1 \mathrm{e}+5)$ | $1.7 \mathrm{e}+9$ | $1.2 \mathrm{e}+6$ | $8.8 \mathrm{e}+8$ | $6.0 \mathrm{e}+4$ | 3 | 9 | 8 |  |
| $(50,1 \mathrm{e}+1)$ | $1.1 \mathrm{e}+1$ | $2.9 \mathrm{e}+0$ | $4.4 \mathrm{e}+0$ | $4.4 \mathrm{e}+0$ | 13 | 12 | 12 |  |
| $(50,1 \mathrm{e}+2)$ | $2.0 \mathrm{e}+4$ | $5.9 \mathrm{e}+2$ | $8.8 \mathrm{e}+2$ | $1.8 \mathrm{e}+1$ | 10 | 12 | 12 |  |
| $(50,1 \mathrm{e}+3)$ | $1.1 \mathrm{e}+6$ | $7.8 \mathrm{e}+2$ | $5.8 \mathrm{e}+4$ | $5.5 \mathrm{e}+2$ | 8 | 11 | 12 |  |
| $(50,1 \mathrm{e}+4)$ | $8.8 \mathrm{e}+7$ | $3.2 \mathrm{e}+4$ | $9.6 \mathrm{e}+6$ | $2.1 \mathrm{e}+3$ | 6 | 10 | 9 |  |
| $(50,1 \mathrm{e}+5)$ | $8.4 \mathrm{e}+9$ | $2.0 \mathrm{e}+5$ | $4.8 \mathrm{e}+8$ | $3.7 \mathrm{e}+4$ | 3 | 9 | 7 |  |

Table 4.1: Numerical results for Example 4.1

Example 4.1. Let

$$
\begin{array}{ll}
A & =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & I_{n-2} & 0 \\
0 & 0.5 \times e^{\top} & 0.5
\end{array}\right],
\end{array} \quad B=\left[\begin{array}{c}
I_{n-1} \\
0
\end{array}\right],
$$

where $e^{\top}$ is the row vector with its all entries being 1 , "randn $(1, n-2)$ " is a row vector of dimension $n-2$, generated by the MATLAB function randn. We set $k$ as $1 e+1,1 e+2,1 e+3,1 e+4,1 e+5$, and apply the four algorithms SCHUR, SCHUR-D, O-SCHUR and Schur-rob on these examples, where "SCHUR-D" denotes the algorithm combining the $D_{k}$ varying strategy in [8] with SCHUR. In [8], the author points out that minimizing the departure from normality via the $D_{k}$ varying technique can be achieved by optimizing the condition number of $X^{\top} X$ or $X$, which actually is hard to realize. So here, the numerical results associated with "SCHUR-D" are obtained by taking many different vectors from the null space of (6) in [8], which lead to orthogonal columns in $X$ when placing complex conjugate poles, and adopting the one owning the minimal departure from normality as the solution to the SFRPA. All numerical results are summarized in Table 4.1 which shows that our algorithm outperforms SCHUR and O-SCHUR on these examples with complex conjugate poles to be assigned.

We now compare our Schur-rob, O-Schur-rob algorithms with the MATLAB functions place, robpole and the SCHUR, O-SCHUR algorithms by applying them on some benchmark sets. The tested benchmark sets include eleven illustrated examples from 5, ten multi-

| num. |  | 5 | 7 | 8 | 9 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| dep. | place | $7.4 \mathrm{e}-1$ | $3.5 \mathrm{e}+0$ | $1.3 \mathrm{e}+1$ | $1.2 \mathrm{e}+1$ | $2.5 \mathrm{e}-3$ |
|  | robpole | $7.4 \mathrm{e}-1$ | $3.4 \mathrm{e}+0$ | $5.0 \mathrm{e}+0$ | $1.2 \mathrm{e}+1$ | $3.6 \mathrm{e}-1$ |
|  | SCHUR | $7.2 \mathrm{e}-1$ | $7.2 \mathrm{e}+0$ | $7.0 \mathrm{e}+0$ | $1.9 \mathrm{e}+1$ | $2.3 \mathrm{e}+0$ |
|  | O-SCHUR | $7.1 \mathrm{e}-1$ | $4.8 \mathrm{e}+0$ | $6.0 \mathrm{e}+0$ | $1.7 \mathrm{e}+1$ | $6.0 \mathrm{e}-1$ |
|  | Schur-rob | $7.2 \mathrm{e}-1$ | $3.7 \mathrm{e}+0$ | $7.5 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $2.4 \mathrm{e}-1$ |
|  | O-Schur-rob | $7.1 \mathrm{e}-1$ | $3.2 \mathrm{e}+0$ | $3.3 \mathrm{e}+0$ | $1.1 \mathrm{e}+1$ | $1.4 \mathrm{e}-1$ |
| $\kappa_{F}(X)$ | place | $1.5 \mathrm{e}+2$ | $1.2 \mathrm{e}+1$ | $3.7 \mathrm{e}+1$ | $2.4 \mathrm{e}+1$ | $4.0 \mathrm{e}+0$ |
|  | robpole | $1.5 \mathrm{e}+2$ | $1.2 \mathrm{e}+1$ | $6.2 \mathrm{e}+0$ | $2.4 \mathrm{e}+1$ | $4.1 \mathrm{e}+0$ |
|  | SCHUR | $2.7 \mathrm{e}+3$ | $1.3 \mathrm{e}+2$ | $1.1 \mathrm{e}+1$ | $5.6 \mathrm{e}+1$ | $6.0 \mathrm{e}+0$ |
|  | O-SCHUR | $1.1 \mathrm{e}+3$ | $4.5 \mathrm{e}+1$ | $7.5 \mathrm{e}+0$ | $5.5 \mathrm{e}+1$ | $4.1 \mathrm{e}+0$ |
|  | Schur-rob | $1.9 \mathrm{e}+3$ | $2.5 \mathrm{e}+1$ | $1.2 \mathrm{e}+1$ | $5.8 \mathrm{e}+1$ | $4.1 \mathrm{e}+0$ |
|  | O-Schur-rob | $1.2 \mathrm{e}+3$ | $2.2 \mathrm{e}+1$ | $9.6 \mathrm{e}+0$ | $3.3 \mathrm{e}+1$ | $4.0 \mathrm{e}+0$ |

Table 4.2: Robustness of the closed-loop system for the examples from [5]
input CARE examples and nine multi-input DARE examples in benchmark collections [1,2]. All examples are numbered in the order as they appear in the references.

Example 4.2. The first benchmark set includes eleven small examples from [5. Applying the six algorithms on these examples, all algorithms produce comparable precisions of the assigned poles, which are greater than 10, and we omit the results here. Table 4.2 lists two measures of robustness, i.e. dep. and $\kappa_{F}(X)$, for five examples. The results are generally comparable. The remaining six examples are not displayed in the table, as the results of the six algorithms applying on these examples are quite similar.

Now we apply the six algorithms on ten CARE and nine DARE examples from the SLICOT CARE/DARE benchmark collections [1,2]. Table 4.3 to Table 4.6 present the numerical results, respectively. The "-"s in the first columns in Table 4.4 and Table 4.6 corresponding to place, robpole, SCHUR and O-SCHUR mean that all four algorithms fail to output a solution, since the multiplicity of some pole is greater than $m$. Note that the "precs" in the last six columns associated with SCHUR and O-SCHUR in Table 4.3 and those in the third and eighth columns in Table 4.4 are also " -"s, which suggest that there exists at least one eigenvalue of $A+B F$, which owns no relative accuracy compared with the assigned poles. From Table 4.3, we know that the relative accuracy "precs" of the poles in example 4 and 5 corresponding to Schur-rob and O-Schur-rob are lower than those produced by place and robpole. And the reason is that there are semi-simple eigenvalues in both examples. So how to dispose the issue that semisimple eigenvalues can achieve higher relative accuracy deserves further exploration and we will treat it in a separate paper. For the sixth column in Table 4.3. "precs" from our algorithms are also smaller than those obtained from place and robpole for the existence of poles which are

| precs |  |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| place | 14 | 14 | 11 | 11 | 11 | 9 | 14 | 11 | 13 | 11 |
| robpole | 14 | 14 | 12 | 13 | 12 | 11 | 14 | 14 | 13 | 10 |
| SCHUR | 12 | 13 | 9 | 6 | - | - | - | - | - | - |
| O-SCHUR | 14 | 16 | 10 | 7 | - | - | - | - | - | - |
| Schur-rob | 14 | 14 | 12 | 8 | 9 | 6 | 14 | 14 | 12 | 9 |
| O-Schur-rob | 15 | 15 | 13 | 8 | 9 | 6 | 14 | 14 | 12 | 9 |

Table 4.3: Accuracy for CARE examples

| precs |  |
| :---: | :---: |
|  | $1 \begin{array}{lllllllll} \\ 1 & 2 & 3 & 4 & 6 & 7 & 9\end{array}$ |
| place | - $1514147115-13$ |
| robpole | - 1514147111 - 13 |
| SCHUR | $1-14781-12$ |
| O-SCHUR | - 1 -14892-15 |
| Schur-rob | $151515158104-12$ |
| O-Schur-rob | $151515158104-13$ |

Table 4.4: Accuracy for DARE examples

| num. |  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| dep. | place | $5.2 \mathrm{e}+0$ | 3.0e-1 | $7.3 \mathrm{e}+2$ | $1.5 \mathrm{e}+6$ | $2.9 \mathrm{e}+6$ | $2.3 \mathrm{e}+7$ | $7.6 \mathrm{e}+0$ | $2.2 \mathrm{e}+16$ | $1 \mathrm{e}+0$ | $4.9 \mathrm{e}+9$ |
|  | robpole 5 | $5.2 \mathrm{e}+0$ | 2.9e-1 | $5.7 \mathrm{e}+2$ | $7.5 \mathrm{e}+5$ | $2.9 \mathrm{e}+6$ | $2.3 \mathrm{e}+7$ | $8.1 \mathrm{e}+02$ | $2.0 \mathrm{e}+16$ | $6.0 \mathrm{e}+0$ | $3.8 \mathrm{e}+9$ |
|  | SCHUR 8 | $8.4 \mathrm{e}+17$ | $7.2 \mathrm{e}+0$ | 5.0e+2 | $1.7 \mathrm{e}+6$ | $3.0 \mathrm{e}+9$ | $5.3 \mathrm{e}+7$ | $6.2 \mathrm{e}+1$ | $8.9 \mathrm{e}+27$ | $7.5 \mathrm{e}+04$ | $4.4 \mathrm{e}+17$ |
|  | O-SCHUR | $4.7 \mathrm{e}+1$ | $2.6 \mathrm{e}+0$ | $3.8 \mathrm{e}+2$ | $8.0 \mathrm{e}+5$ | $5.4 \mathrm{e}+8$ | $2.6 \mathrm{e}+7$ | $7.3 \mathrm{e}+0$ | $1.7 \mathrm{e}+26$ | $6.8 \mathrm{e}+02$ | $2.3 \mathrm{e}+17$ |
|  | Schur-rob | $7.6 \mathrm{e}+0$ | 3.0e-1 | $1.4 \mathrm{e}+2$ | $1.1 \mathrm{e}+5$ | $7.3 \mathrm{e}+6$ | $2.3 \mathrm{e}+7$ | $7.5 \mathrm{e}+0$ | $2.1 \mathrm{e}+18$ | $8.4 \mathrm{e}+02$ | $2.2 \mathrm{e}+10$ |
|  | O-Schur-rob | $7.3 \mathrm{e}+0$ | 2.6e-1 | $1.4 \mathrm{e}+2$ | $1.1 \mathrm{e}+5$ | $2.5 \mathrm{e}+6$ | $2.3 \mathrm{e}+7$ | $6.8 \mathrm{e}+02$ | $2.0 \mathrm{e}+16$ | $6.8 \mathrm{e}+02$ | $2.2 \mathrm{e}+10$ |
| $\kappa_{F}(X)$ | place | $7.4 \mathrm{e}+08$ | $8.0 \mathrm{e}+0$ | $4.3 \mathrm{e}+1$ | $1.7 \mathrm{e}+15$ | $8.5 \mathrm{e}+4$ | $4.8 \mathrm{e}+6$ | $1.6 \mathrm{e}+1$ | $9.8 \mathrm{e}+11$ | $1.5 \mathrm{e}+2$ | $2.3 \mathrm{e}+6$ |
|  | robpole 7 | $7.3 \mathrm{e}+08$ | $8.0 \mathrm{e}+0$ | $4.2 \mathrm{e}+1$ | $2.2 \mathrm{e}+7$ | $8.9 \mathrm{e}+4$ | $3.2 \mathrm{e}+6$ | $1.6 \mathrm{e}+1$ | $9.0 \mathrm{e}+11$ | $1.4 \mathrm{e}+2$ | $2.3 \mathrm{e}+6$ |
|  | SCHUR | $2.2 \mathrm{e}+2$ | $1.0 \mathrm{e}+1$ | $11.7 \mathrm{e}+3$ | 9.1e+9 | $6.0 \mathrm{e}+11$ | $4.0 \mathrm{e}+13$ | $3.5 \mathrm{e}+8$ | 6.1e+91 | $1.3 \mathrm{e}+94$ | $4.6 \mathrm{e}+13$ |
|  | O-SCHUR | $1.2 \mathrm{e}+25$ | $5.1 \mathrm{e}+1$ | 2.1e+3 | $1.0 \mathrm{e}+9$ | $2.4 \mathrm{e}+10$ | $1.2 \mathrm{e}+8$ | $1.0 \mathrm{e}+8$ | $3.7 \mathrm{e}+94$ | $4.1 \mathrm{e}+95$ | $5.7 \mathrm{e}+13$ |
|  | Schur-rob | $1.1 \mathrm{e}+1$ | $8.2 \mathrm{e}+0$ | $9.2 \mathrm{e}+2$ | 9.0e+7 | $2.0 \mathrm{e}+6$ | $3.2 \mathrm{e}+8$ | $3.3 \mathrm{e}+1$ | $5.7 \mathrm{e}+26$ | $6.5 \mathrm{e}+3$ | $4.3 \mathrm{e}+6$ |
|  | O-Schur-rob | $1.0 \mathrm{e}+18$ | $8.0 \mathrm{e}+0$ | $9.1 \mathrm{e}+2$ | $6.5 \mathrm{e}+7$ | $1.3 \mathrm{e}+6$ | $1.2 \mathrm{e}+8$ | $2.8 \mathrm{e}+1$ | $4.2 \mathrm{e}+23$ | $3.4 \mathrm{e}+3$ | $4.3 \mathrm{e}+6$ |

Table 4.5: Robustness of the closed-loop system matrix for ten CARE examples
relatively badly separated from the imaginary axis. And this is a weakness of our algorithm.

We now test the five methods place, robpole, SCHUR, O-SCHUR and Schur-rob on some random examples generated by the MATLAB function randn.

Example 4.3. This test set includes 33 examples where $n$ varies from 3 to 25 increased by 2, and $m$ is set to be $2,\left\lfloor\frac{n}{2}\right\rfloor, n-1$ for each $n$. The examples are generated as following. We first randomly generate the matrices $A, B$ and $F$ by the MATLAB function randn, and then get $\mathfrak{L}$ using the MATLAB function eig, that is, $\mathfrak{L}=e i g(A+B F)$. We then apply the five algorithms on the $A, B$ and $\mathfrak{L}$ as input.

Fig. 4.1 to Fig. 4.4 respectively exhibit the departure from normality of the computed $A_{c}$, the condition number of the eigenvector matrix $X$, the relative accuracy of the poles and the CPU time of the five algorithms applied on these randomly generated examples. In these figures, the $x$-axis represents the number of the 33 different $(n, m)$. For example, $(3,2),(5,2)$ and $(5,4)$ correspond to 1,2 and 3 in the $x$-axis, respectively. And the values along the $y$-axis are the mean values over 50 trials for a certain $(n, m)$.

| num. |  | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| dep. | place |  | $2.2 \mathrm{e}-1$ | $3.9 \mathrm{e}-1$ | 4.3e-1 | 1.7 | 4e | $2.3 \mathrm{e}+1$ | $4.3 \mathrm{e}+7$ | $8.9 \mathrm{e}+0$ |
|  | robpole | - | $2.2 \mathrm{e}-1$ | $3.9 \mathrm{e}-1$ | $3.6 \mathrm{e}-1$ | $1.7 \mathrm{e}+0$ | $1.3 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $3.9 \mathrm{e}+12$ | $8.0 \mathrm{e}+0$ |
|  | SCHUR | - | $4.1 \mathrm{e}-1$ | $1.1 \mathrm{e}+2$ | $5.9 \mathrm{e}-1$ | $1.8 \mathrm{e}+0$ | $1.1 \mathrm{e}+1$ | $3.2 \mathrm{e}+2$ | $3.4 \mathrm{e}+2$ | $1.1 \mathrm{e}+1$ |
|  | O-SCHUR | - | $3.3 \mathrm{e}-1$ | $4.9 \mathrm{e}+1$ | $4.1 \mathrm{e}-1$ | $1.7 \mathrm{e}+0$ | $1.1 \mathrm{e}+0$ | $1.7 \mathrm{e}+2$ | $1.2 \mathrm{e}+1$ | $8.0 \mathrm{e}+0$ |
|  | Schur-rob | $1.0 \mathrm{e}-1$ | $2.5 \mathrm{e}-1$ | $1.3 \mathrm{e}+0$ | $3.4 \mathrm{e}-1$ | $1.7 \mathrm{e}+0$ | $2.0 \mathrm{e}+0$ | $1.9 \mathrm{e}+1$ | $9.8 \mathrm{e}+0$ | $9.9 \mathrm{e}+0$ |
|  | O-Schur-rob | $1.0 \mathrm{e}-1$ | $2.5 \mathrm{e}-1$ | $1.3 \mathrm{e}+0$ | $3.4 \mathrm{e}-1$ | $1.7 \mathrm{e}+0$ | $1.2 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $9.4 \mathrm{e}+0$ | 6.6e+0 |
| $\kappa_{F}(X)$ | place | - | $5.2 \mathrm{e}+0$ | $4.9 \mathrm{e}+0$ | $5.4 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $1.3 \mathrm{e}+1$ | $2.3 \mathrm{e}+8$ | $9.2 \mathrm{e}+292$ | $3.4 \mathrm{e}+2$ |
|  | robpole | - | $5.2 \mathrm{e}+0$ | $5.0 \mathrm{e}+0$ | $5.3 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $1.2 \mathrm{e}+1$ | $2.9 \mathrm{e}+8$ | $1.3 \mathrm{e}+308$ | $3.0 \mathrm{e}+2$ |
|  | SCHUR | - | $4.0 \mathrm{e}+7$ | $1.2 \mathrm{e}+9$ | $5.7 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $5.8 \mathrm{e}+3$ | $1.9 \mathrm{e}+11$ | $2.8 \mathrm{e}+295$ | $4.7 \mathrm{e}+3$ |
|  | O-SCHUR | - | $3.3 \mathrm{e}+7$ | $8.0 \mathrm{e}+8$ | $5.4 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $1.7 \mathrm{e}+3$ | $2.0 \mathrm{e}+11$ | $3.3 \mathrm{e}+295$ | $2.6 \mathrm{e}+3$ |
|  | Schur-rob | $7.1 \mathrm{e}+15$ | $5.5 \mathrm{e}+0$ | $5.6 \mathrm{e}+0$ | $7.2 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $3.8 \mathrm{e}+1$ | $1.7 \mathrm{e}+9$ | $5.6 \mathrm{e}+292$ | $2.2 \mathrm{e}+4$ |
|  | O-Schur-rob | $2.5 \mathrm{e}+15$ | $5.5 \mathrm{e}+0$ | $5.5 \mathrm{e}+0$ | $7.2 \mathrm{e}+0$ | $1.8 \mathrm{e}+1$ | $3.8 \mathrm{e}+1$ | $1.2 \mathrm{e}+9$ | $5.6 \mathrm{e}+292$ | $4.7 \mathrm{e}+3$ |

Table 4.6: Robustness of the closed-loop system matrix for nine DARE examples


Fig. 4.1: dep. over 50 trials


Fig. 4.3: precs over 50 trials


Fig. 4.2: $\kappa_{F}(X)$ over 50 trials


Fig. 4.4: CPU time over 50 trials

All these figures show that our Schur-rob algorithm can produce comparable or even better results as place and robpole, but with much less CPU time.

## 5 Conclusion

Pole assignment problem for multi-input control is generally under-determined. And utilizing this freedom to make the closed-loop system matrix to be insensitive to perturbations as far as possible evokes the state-feedback robust pole assignment problem (SFRPA) arising. Based on SCHUR [8], we propose a new direct method to solve the SFRPA, which obtains the real Schur form of the closed-loop system matrix and tends to minimize its departure from normality via solving some standard eigen-problems. Many numerical examples show that our algorithm can produce comparable or even better results than existing methods, but with much less computational costs than the two classic methods place and robpole.

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