Increasing-Gain Observers for Nonlinear Systems: Stability and Design

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Abstract

For the purpose to estimate the state of nonlinear continuous-time systems, we focus on the increasing-gain observer regarded as generalization of the well-known high-gain observer. As compared with the previous results on increasing-gain observers, the assumptions under which the global asymptotic stability of the estimation error is proved are relaxed. The stability analysis is drawn by using a more general type of Lyapunov functional, where design parameters can be tuned to set the time-varying gain. A new design method based on such a Lyapunov functional is proposed to construct the observer. Simulation results affirm the effectiveness of the increasing-gain approach as compared with the high-gain observer.

Key words: Nonlinear observer, High-gain observer, Lyapunov stability.

1 Introduction

Among the various approaches to state estimation for nonlinear continuous-time systems, the high-gain observer is by far the most popular [8]. The high-gain observer is based on the idea of selecting a sufficiently large gain in such a way as to dominate the nonlinear contribution to the dynamics of the estimation error. Unfortunately, such a large gain is the source of the well-known peaking phenomenon, which may cause destabilization in the loop when the high-gain observer is used in cascade with a feedback regulator. In order to reduce peaking, a different approach was explored in [3], i.e., a time-varying estimatorwith the structure of the standard high-gain observer but with the possibility to assign a small gain in the first time instants and let it increase over time up to a point of maximum, thus motivating the name of increasing-gain observer. In this paper, new stability conditions on the estimation error as well as a novel design method are presented for such an observer.

The high-gain observer is the result of numerous research efforts devoted to the goal of state reconstruction for nonlinear systems with guarantees on the stability of the estimation error. First attempts to construct such observers are reported in [20,11]. Later on, other methods were proposed by relying on the idea of performing state transformation with a straightforward evaluation of the stability (see e.g., [12,13,7]). Nevertheless, the proof of global stability for the estimation error under general nonlinear assumptions on the system is still a challenge.

Recent researches are focused on enhancing the high-gain observer by using some adaptation machinery. For example, the adoption of a switching-gain strategy is analyzed in [1]. The combination of the tuning capabilities of the extended Kalman filter approach with the high-gain global stability properties is considered in [5]. Adaptive high-gain observers are investigated to achieve a tradeoff between transient response in a noise-free setting and sensitivity to disturbances in the presence of noise [9,14,10,4,16,15]. With respect to the previous works, here we do not focus on a specific adaptation law but deal with a generic time-varying gain likewise in [6], where the first ideas about time-varying high-gain observers are presented. Under the assumption of triangular structure and Lipschitz nonlinearities, in [3] the stability of the estimation error given by the increasing-gain observer is proved by means of a Lyapunov functional instead of the quadratic Lyapunov function, which is usually adopted to this scope (see also [18,17] about observers with the property of finite-time stability). As compared with [3], in this paper the stability analysis of the estimation error is drawn by using a novel Lyapunov functional because of some additional parameters that may ensure stability under more favorable conditions. Moreover, instead of the usual Lipschitz hypothesis we rely on a more general sublinear assumption composed of a Lipschitz bound with an additional term that

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converges to zero. Under such assumptions, the analysis is conducted by providing new conditions that ensure global asymptotic stability. Finally, a novel design method is presented that is based on a Taylor expansion and turns out to be much simpler and more effective as compared with the approach proposed in [3].

The paper is organized as follows. In Section 2, the system and observer assumptions are briefly described. The main results concerning stability are presented in Section 3. Section 4 deals with the observer design. Simulation results are shown in Section 5. Conclusions are drawn in Section 6.

Before concluding this section, let us introduce the notation adopted and remind some useful results. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the set of the nonnegative and strictly positive real numbers, respectively. The components of any column vector $x \in \mathbb{R}^n$ are denoted by $x_1, x_2, \ldots, x_n$, i.e., $x = (x_1, x_2, \ldots, x_n)$; moreover, $\|x\| := \sqrt{x^\top x}$. The norm of a matrix M is denoted by $\|M\| := \sqrt{\lambda_{\max}(M^\top M)} = \sqrt{\lambda_{\max}(M M^\top)}$. For a symmetric matrix P, $P > 0$ (< 0) denotes that P is positive (negative) definite; the minimum and maximum eigenvalues of P are denoted by $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$, respectively. If $P > 0$, $\|P\| = \lambda_{\max}(P)$.

2 System Assumptions and Observer Structure

Let us consider the class of nonlinear continuous-time system described by

$$\begin{cases} \dot{x} = A\,x + f(x,t) \\ y = C\,x \end{cases}, \; t \geq 0 \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $y(t) \in \mathbb{R}$ is a scalar measurement; $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$, and the function $f : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are defined as follows:

$$A := \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & 1 & & & \vdots \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{bmatrix}, \quad C := [1\ 0\ \ldots\ 0],$$

$$f(x,t) := \begin{bmatrix} f_1(x_1,t) \\ f_2(x_1,x_2,t) \\ \vdots \\ f_{n-1}(x_1,x_2,\ldots,x_{n-1},t) \\ f_n(x_1,x_2,\ldots,x_n,t) \end{bmatrix}.$$

To estimate $x(t)$, we consider the full-order observer

$$\dot{\hat{x}} = A\hat{x} + \hat{f}(\hat{x},t) + G(\gamma)\,(y - C\hat{x})\;,\; t \geq 0 \tag{2}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ at time t and

$$\hat{f}(\hat{x},t) := \begin{bmatrix} \hat{f}_1(\hat{x}_1,t) \\ \hat{f}_2(\hat{x}_1,\hat{x}_2,t) \\ \vdots \\ \hat{f}_{n-1}(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_{n-1},t) \\ \hat{f}_n(\hat{x}_1,\hat{x}_2,\ldots,\hat{x}_n,t) \end{bmatrix}, G(\gamma) := \begin{bmatrix} \gamma k_1 \\ \gamma^2 k_2 \\ \vdots \\ \gamma^n k_n \end{bmatrix}$$

with $\gamma \geq 1$ and $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$ to be suitably chosen.

Assumption 1 *The functions f and $\hat{f}$ are continuous and there exist δ and $\bar{\delta} > 0$, M and $\bar{M} \in \mathbb{R}^n_{\geq 0}$, L and $\bar{L} \in \mathbb{R}^n_{\geq 0}$ such that, for all $x, w \in \mathbb{R}^n$ and $t \geq 0$,*

$$\begin{aligned} &\big|f_i(x_1+w_1, x_2+w_2, \ldots, x_i+w_i, t) \\ &- f_i(x_1,x_2,\ldots,x_i,t)\big| \leq \bar{M}_i \exp(-\bar{\delta}t) + \bar{L}_i \sum_{j=1}^{i} |w_j| \end{aligned} \tag{3}$$

$$\begin{aligned} &\big|f_i(x_1+w_1, x_2+w_2, \ldots, x_i+w_i, t) \\ &- \hat{f}_i(x_1,x_2,\ldots,x_i,t)\big| \leq M_i \exp(-\delta t) + L_i \sum_{j=1}^{i} |w_j|\,. \end{aligned} \tag{4}$$

Instead of studying the stability of the estimation error $\hat{e} := x - \hat{x}$ that genuinely descends from (1) and (2), we perform a change of variables $\hat{e} = T(\gamma)\,e$, $e \in \mathbb{R}^n$ with

$$T(\gamma) = \mathrm{diag}\left(\gamma, \gamma^2, \ldots, \gamma^n\right)$$

and study the stability in the new coordinates under suitable conditions beginning with the existence of an inverse for $T(\gamma)$. Before proving our main results, we need to assume the following.

Assumption 2 *The time-varying parameter $\gamma(t)$ is such that $t \mapsto \gamma(t)$ is continuous on $\mathbb{R}_{\geq 0}$ and there exists $c_0 \geq 1$ such that $|\gamma(t)| \leq c_0$, $\forall t \geq 0$.*

Assumptions 1 and 2 are strictly connected to the necessity to ensure the existence of the solutions of the various differential equations we have to deal with. From the continuity of f, $\hat{f}$, and γ we deduce the local existence of a solution for (1) and (2). Moreover, conditions (3), (4), and the boundedness of γ ensure the existence of complete solutions [1] for both (1) and (2).

[1] A solution is complete if its domain of definition is $\mathbb{R}_{\geq 0}$.

The error dynamics is derived from (1) and (2) as follows:

$$\dot{\hat{e}}(t) = (A - G(\gamma(t))C)\,\hat{e}(t) + f(x(t),t) - \hat{f}(x(t) - \hat{e}(t), t)\,. \tag{5}$$

Lemma 1 *Let $t \mapsto \gamma(t)$ be such that $\gamma(t) \geq 1$ for all $t \geq 0$. Then, the following facts hold:*

(i) the origin is a Lyapunov asymptotically stable equilibrium point of the error dynamics in $\hat{e}(t)$ if and only if so it is also for the error dynamics in $e(t)$;

(ii) there exist $N > 0, k_f > 0$ such that

$$\left\| T(\gamma(t))^{-1}\left(f(x(t),t) - \hat{f}(x(t) - T(\gamma(t))\,e(t),t)\right)\right\| \leq N\exp(-\delta t) + k_f\,\|e(t)\| \tag{6}$$

for all $t \geq 0$, and k_f does not depend on $\gamma(t)$ and t.

Proof. It is omitted as quite similar to the proof of Lemma 1 in [3] (see p. 2846). □

To state the differential problem associated with the error dynamics, from now on we need to assume the following.

Assumption 3 *The function $t \mapsto \gamma(t)$ is a.e. differentiable with $\dot{\gamma}(t)$ a.e. bounded on $\mathbb{R}_{\geq 0}$.*

Based on the aforesaid, from (5) we obtain:

$$\dot{e}(t) = T(\gamma(t))^{-1}\,(A - G(\gamma(t))\,C)\,T(\gamma(t))\,e(t) - T(\gamma(t))^{-1}\,T'(\gamma(t))\,\dot{\gamma}(t)\,e(t) + T(\gamma(t))^{-1}\Big(f(x(t),t) - \hat{f}(x(t) - T(\gamma(t))\,e(t),t)\Big)\,.$$

Because of the particular observer structure, the previous equation can be written as follows:

$$\dot{e}(t) = \gamma(t)\,(A - KC)\,e(t) - \frac{\dot{\gamma}(t)}{\gamma(t)}\,D\,e(t) + T(\gamma(t))^{-1}\Big(f(x(t),t) - \hat{f}(x(t) - T(\gamma(t))\,e(t),t)\Big) \tag{7}$$

where $D := \mathrm{diag}\,(1,2,\ldots,n)$.

3 Stability Analysis

First of all, let us consider a basic condition to accomplish the stability analysis of the estimation error presented in the following. Since (A,C) is observable, there exist $\lambda > 0$, $K \in \mathbb{R}^n$, and a symmetric, definite positive matrix $P \in \mathbb{R}^{n\times n}$ such that

$$(A - KC)^\top P + P(A - KC) + \lambda I < 0 \tag{8}$$

with $K := [k_1\ k_2\ \cdots k_n]^\top$.

To avoid cumbersome notations, from now on we will drop the dependence on time unless ambiguity arises. For example, we will write $f(x)$ instead of $f(x,t)$ for the sake of brevity. Given $P > 0$ as in (8) and any $\alpha > 0$, for $\beta \in (1,2]$, let $\mathcal{V} : (z,t) \in \mathbb{R}^n \times [0,\infty) \to \mathbb{R}$ the functional

$$\mathcal{V}(z,t) := z^\top Pz + 2\exp\left(\alpha t^\beta\right)\int_t^\infty \Big|\big(T(\gamma(s))^{-1}\big(f(x(s)) - \hat{f}\big(x(s) - T(\gamma(s))e(s)\big)\big)\big)^\top Pe(s)\Big|\exp\left(-\alpha s^\beta\right)ds \tag{9}$$

where $t \mapsto x(t)$ and $t \mapsto e(t)$ are solutions of (1) and (7), respectively. Note that $\mathcal{V}(z,t) \geq 0$; moreover, $\mathcal{V}(z,t) = 0$ if and only if $e(t) = 0$ for all $t \geq 0$ and $z = 0$. For what is detailed later on, we need to state the following lemmas, which can be regarded as generalization of results stated in [3].

Lemma 2 *For $\tau \geq t_0$, let*

$$\begin{cases} \dot{e}(\tau) = \gamma(\tau)\,(A - KC)e(\tau) - \dfrac{\dot{\gamma}(\tau)}{\gamma(\tau)}\,D\,e(\tau) \\ \qquad + T(\gamma(\tau))^{-1}\big(f(x(\tau)) - \hat{f}\big(x(\tau) - T(\gamma(\tau))\,e(\tau)\big)\big) \\ e(t_0) = e_0 \in \mathbb{R}^n \end{cases}$$

the initial value problem associated with (7). If there exists $\eta > 0$ such that

$$\eta \geq \sup_{t\geq 0}\left\|\gamma(t)\,(A - K\,C) - \frac{\dot{\gamma}(t)}{\gamma(t)}\,D\right\|, \tag{10}$$

then

$$\|e(\tau)\| \leq \left(\|e_0\| + \frac{N}{\delta}\,\exp(-\delta\,t_0)\right)\exp((\eta + k_f)(\tau - t_0)), \qquad \tau \geq t_0\,. \tag{11}$$

Proof. It is omitted as quite similar to the proof of Lemma 2 in [3] (p. 2847). □

Lemma 3 *Let $\gamma(t) \geq 1$. If there exists $\eta > 0$ such that (10) is satisfied, then*

$$\exp\left(\alpha t^\beta\right)\int_t^\infty \left(N\exp(-\delta s) + k_f\|e(s)\|\right)\|e(s)\| \times \exp(-\alpha s^\beta)\,ds \leq \Big(\left(N/2 + 2k_f\right)\|e(t)\|^2 + N\big(1/2 + N/\delta + 2k_f N/\delta^2\big)\exp(-2\delta t)\Big)a_{\alpha,\beta,\eta}(t) \tag{12}$$

and

$$t^{\beta-1}\exp\left(\alpha t^{\beta}\right)\int_t^{\infty}\left(N\exp(-\delta s)+k_f\|e(s)\|\right)\|e(s)\|$$
$$\times\exp(-\alpha s^{\beta})\mathrm{d}s\leq\Big(2k_f\|e(t)\|^2+N\Big(\|e(t)\|+N\big(1/\delta$$
$$+2k_f/\delta^2\big)\Big)\exp(-\delta t)\Big)b_{\alpha,\beta,\eta}(t) \tag{13}$$

where $a_{\alpha,\beta,\eta}:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ *defined as*

$$a_{\alpha,\beta,\eta}(t):=\exp\left(\alpha t^{\beta}-2\left(\eta+k_f\right)t\right)\int_t^{\infty}\exp\big(-\alpha s^{\beta}$$
$$+2\left(\eta+k_f\right)s\big)\,ds \tag{14}$$

is differentiable and strictly decreasing and $b_{\alpha,\beta,\eta}$: $\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ *defined as*

$$b_{\alpha,\beta,\eta}(t):=t^{\beta-1}\exp(\alpha t^{\beta}-2(\eta$$
$$+k_f)t)\int_t^{\infty}\exp\big(-\alpha s^{\beta}+2\left(\eta+k_f\right)s\big)\,ds \tag{15}$$

is differentiable with

$$b'_{\alpha,\beta,\eta}(t)=t^{\beta-2}((\beta-1+\alpha\,\beta\,t^{\beta}-2(\eta$$
$$+k_f)t)\,a_{\alpha,\beta,\eta}(t)-t)\,. \tag{16}$$

Moreover,

(i) there exists $t_{\max}>0$ *such that* $b_{\alpha,\beta,\eta}$ *is strictly increasing on* $[0,t_{\max}]$ *and strictly decreasing on* $[t_{\max},+\infty)$ *(i.e.,* $b_{\alpha,\beta,\eta}$ *admits a global maximum in* $t_{\max}$ *on* $\mathbb{R}_{\geq 0}$*);*
(ii) $b_{\alpha,\beta,\eta}$ *is concave in* $[0,t_{\max}]$.

Proof. See Appendix. ◻

The proof of Lemma 3 for $\beta=2$ with stronger assumptions on f is reported in [3], as the Lyapunov functional (9) with the choice $\beta=2$ is just the same adopted therein. As a matter of fact, Lemma 3 is more general for both the larger range in the choice of β and the weaker hypothesis on f. In the case $\beta=2$, using the bound in Assumption 1 of [3] instead of Assumption 1 of this paper, Lemma 3 provides the same result of Lemma 2 in [3].

Now, we are able to prove that the Lyapunov functional $\mathcal{V}$ is well defined and satisfies significant inequalities.

Theorem 1 *Let* e *a solution of the error dynamics (7);* $\lambda>0$, $P>0$, *and* K *such that (8) holds;* $\gamma(t)\geq 1$ *for all* $t\geq 0$ *and* $\dot\gamma(t)\geq 0$ *for a.e.* $t\geq 0$. *Then, the Lyapunov functional* $\mathcal{V}(z,t)$ *is well defined and satisfies the following inequalities for a.e.* $t\geq 0$*:*

$$\lambda_{\min}(P)\,\|e(t)\|^2\leq\mathcal{V}(e(t),t)\leq\lambda_{\max}(P)\,\Big(\big(1+(N$$
$$+4\,k_f)\,a_{\alpha,\beta,\eta}(0)\big)\,\|e(t)\|^2+N\Big(1+2N/\delta$$
$$+4k_fN/\delta^2\Big)\,a_{\alpha,\beta,\eta}(0)\exp(-2\delta t)\Big) \tag{17}$$
$$\frac{\partial\mathcal{V}(e(t),t)}{\partial z}\,\dot e+\frac{\partial\mathcal{V}(e(t),t)}{\partial t}\leq-\big(\gamma(t)\,\lambda-4\,\alpha\,\beta\,\lambda_{\max}(P)$$
$$\times\;k_f\,b_{\alpha,\beta,\eta}(t)\big)\,\|e(t)\|^2+2\,\alpha\,\beta\,\lambda_{\max}(P)\,N\Big(\|e(t)\|$$
$$+N\big(1/\delta+2k_f/\delta^2\big)\Big)\,b_{\alpha,\beta,\eta}(t)\exp(-\delta t)\,. \tag{18}$$

Proof. Obviously, $V_e(t):=\mathcal{V}(e(t),t)\geq\lambda_{\min}(P)\,\|e(t)\|^2$. The upper bound in the r.h.s. of (17) follows from (12) and the strict decreasing of the function $a_{\alpha,\beta,\eta}$. Using (6), (8), and (13) as well as the assumptions on $\gamma(t)$ and $\dot\gamma(t)$, we get (18) since $PD+DP>0$, which is straightforward to verify as D and P are symmetric and positive definite. ◻

To proceed with the sequel, we need a result that is related to Lemma 1 in [19] (p. 674) but more general since here asymptotic stability is proved without conditions on the various parameters of the Lyapunov inequalities.

Theorem 2 *Let* $F:[0,\infty)\times W\to\mathbb{R}^n$ *be a piecewise continuous in* t *and locally Lipschitz in* ξ *on* $[0,\infty)\times W$ *and* $W\subset\mathbb{R}^n$ *a domain that contains the origin* $\xi=0$. *Let* $\varphi=0$ *be an equilibrium point for*

$$\dot\varphi=F(t,\varphi)\,. \tag{19}$$

Let $V:[0,\infty)\times W\to\mathbb{R}_{\geq 0}$ *be a continuously differentiable function with Lyapunov parameters* c_1, c_2, c_3, c_4, ν, $\mu>0$ *such that for* $\xi\in W$ *and* $t\geq 0$*:*

$$c_2\|\xi\|^2-c_4\exp(-\nu t)\leq V(t,\xi)\leq c_3\left(\|\xi\|^2+\exp(-c_1t)\right) \tag{20}$$

$$\frac{\partial V(t,\xi)}{\partial t}+\frac{\partial V(t,\xi)}{\partial\xi}\cdot F(t,\xi)\leq-c_5\|\xi\|^2$$
$$+c_6\exp(-\mu t)\,(1+\|\xi\|)\quad\textit{for a.e. }\ t\geq 0\,. \tag{21}$$

Then there exist $C_0>0$ *and* $\theta>0$ *such that, for every trajectory of (19), we have*

$$\|\varphi(t)\|\leq C_0\left(\|\varphi(t_0)\|\,\exp\left(-\theta\left(t-t_0\right)\right)+\exp\left(-\theta\,t\right)\right)$$

for all $t_0\geq 0$ *and hence* $\varphi=0$ *is asymptotically stable.*

Proof. It is omitted as quite similar to the proof of Theorem 2 in [2] (see p. 3960). ◻

The stability of the estimation error is ensured by a suitable choice of $\gamma(t)$ and σ, as follows.

Theorem 3 *Given $\lambda > 0$, $P > 0$, K such that (8) holds; moreover, let $\gamma(t)$ such that $\dot{\gamma}(t) \geq 0$ for a.e. $t \geq 0$ and there exist $\alpha > 0$, $\beta \in (1, 2]$, and $\eta > 0$ such that (10) and*

$$\gamma(t) \geq \max\left(1, \frac{\sigma + 4\,\alpha\,\beta\,k_f\,\lambda_{\max}(P)\;b_{\alpha,\beta,\eta}(t)}{\lambda}\right) \tag{22}$$

hold for all $t \geq 0$ with

$$\sigma > \kappa\,\lambda_{\max}(P)\,\frac{1 + (N + 4k_f)\,a_{\alpha,\beta,\eta}(0)}{\sqrt{\lambda_{\min}(P)}} \tag{23}$$

where $\kappa := 2\,\alpha\,\beta\,\lambda_{\max}(P)\;N\,b_{\alpha,\beta,\eta}(t_{\max})\;\max(N(1 + 2k_f/\delta)/\delta, 1)$. Then the observer (2) admits an estimation error that is globally asymptotically stable at the origin and there exist $C_1, C_2, C_3 > 0$ such that

$$\begin{aligned}\|e(t)\| \leq C_2 \exp\left(-\frac{C_1}{2}(t - t_0)\right)\|e(t_0)\| \\ + C_3 \exp\left(-\frac{1}{2}\min(\delta, C_1)\,(t - t_0)\right)\end{aligned} \tag{24}$$

for all $t \geq t_0$.

Proof. The thesis is proved by means of the same arguments of Theorem 3 in [2] (see p. 3961). □

It is worth noting that, as compared with [3], the stability proved in Theorem 3 is not uniform w.r.t. time just because of the more general hypothesis given by (3) in Assumption 1.

4 Observer Design

Based on the results presented so far, a novel method derived from [3] is described to select the design parameters of the proposed observer in such a way to fully exploit the potential of the Lyapunov functional (9). As a primary goal, we aim to design observers with a peaking as much reduced as possible by searching for the smallest gain that is consistent with the stability conditions.

First, we need to account for (8), which can be treated by solving the equivalent LMI

$$A^\top P - C^\top Y^\top + PA - YC + \lambda I < 0 \tag{25}$$

where the unknowns are $\lambda > 0$, $Y = PK \in \mathbb{R}^n$, and $P > 0$. Having solved such an LMI, we obtain the gain $K = P^{-1}Y$. If (10) and (23) hold, Theorem 3 ensures the global stability of the estimation error by using

$$\gamma(t) = \begin{cases} \max\left(1, \dfrac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\,b_{\alpha,\beta,\eta}(t)}{\lambda}\right) & t \in [0, t_{\max}] \\ \max\left(1, \dfrac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\,b_{\alpha,\beta,\eta}(t_{\max})}{\lambda}\right) & t > t_{\max}\,. \end{cases} \tag{26}$$

Unfortunately, in general the selection of the design parameters in such a way to satisfy (10) is not easy. To overcome this difficulty, a simple remedy is proposed in [3] that consists in exploiting the concavity of $b_{\alpha,\beta,\eta}$ by replacing the curve with its tangent line at a given time instant $t_* \in [0, t_{\max}]$ to be chosen in such a way as to ensure

$$\sup_{t\geq 0} \dot{\gamma}(t) \leq \dot{\gamma}(t_*) \tag{27}$$

through a convenient choice of $\gamma(t)$. Toward this end, instead of (26) we consider

$$\gamma(t) = \begin{cases} \max\Big(1, (\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\,\big(b_{\alpha,\beta,\eta}(t_*) \\ \quad + b'_{\alpha,\beta,\eta}(t_*)(t - t_*)\big))/\lambda_*\Big) \quad t \in [0, t_*] \\ \max\left(1, \dfrac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\,b_{\alpha,\beta,\eta}(t)}{\lambda_*}\right) \\ \quad t \in (t_*, t_{\max}] \\ \max\left(1, \dfrac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\;b_{\alpha,\beta,\eta}(t_{\max})}{\lambda_*}\right) \\ \quad t > t_{\max} \end{cases} \tag{28}$$

where the design parameters t_*, $t_{\max}$, α, β, σ, λ_*, and η have to be carefully chosen. In [3,2] such parameters are obtained by solving a nonlinear programming problem of pure feasibility. For the purpose of achieving a reduced peaking, here we propose a new formulation of the design problem aimed to find the lowest value of $\gamma(t)$ that is admissible with the stability conditions. More specifically, we consider the following problem:

$$\min\max\left(1, \frac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\;k_f\,b_{\alpha,\beta,\eta}(t_{\max})}{\lambda_*}\right) \tag{29a}$$

$$\text{w.r.t.}\quad t_*,\, t_{\max},\, \alpha,\, \beta,\, \sigma,\, \lambda_*,\, \eta \quad \text{s.t.}\quad t_* \geq 0, t_{\max} \geq 0, \alpha > 0, \beta \in (1, 2], \sigma > 0, \lambda_* \in (0, \lambda], \eta > 0 \tag{29b}$$

and

$$t_* \leq t_{\max} \tag{29c}$$

$$b'_{\alpha,\eta}(t_{\max}) = 0 \tag{29d}$$

$$\eta \geq \max\left(1, \frac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_{\max})}{\lambda_*}\right) \|A - KC\| + \frac{4\,n\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b'_{\alpha,\beta,\eta}(t_*)}{\max\left(\lambda_*, \sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_*)\right)} \tag{29e}$$

$$\sigma > k_0\,\alpha\,\beta\, b_{\alpha,\beta,\eta}(t_{\max})\; \left(1 + (N + 4k_f)\, a_{\alpha,\beta,\eta}(0)\right) \tag{29f}$$

where

$$k_0 := 2\,\lambda_{\max}(P)^2\, N \max\left(N(1+2k_f/\delta)/\delta, 1\right)/\sqrt{\lambda_{\min}(P)}$$

is a constant with a given value after solving (25). The statement of such a problem can be motivated as follows. Since (28) satisfies (27) by construction, the design parameters that result from the solution of (29) comply with (22). Moreover, (10) is taken into account since (29e) follows from the inequalities

$$\begin{aligned}
&\left\|\gamma(t)\,(A - K\,C) - \frac{\dot\gamma(t)}{\gamma(t)} D\right\| \leq \|\gamma(t)\,(A - K\,C)\| \\
&+ \left\|\frac{\dot\gamma(t)}{\gamma(t)} D\right\| \leq \gamma(t)\; \|A - K\,C\| + \frac{\dot\gamma(t)}{\gamma(t)}\, \|D\| \\
&\leq \max\left(1, \frac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_{\max})}{\lambda_*}\right) \|A - KC\| + \frac{4\,n\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b'_{\alpha,\beta,\eta}(t_*)}{\max\left(\lambda_*, \sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_*)\right)}
\end{aligned}$$

where $\|D\| = n$. Finally, the constraint (29f) accounts for (23).

The main difficulty in searching the solution of (29) is due to the need of analytic expressions of $t \mapsto b_{\alpha,\beta,\eta}(t)$ and $t \mapsto b'_{\alpha,\beta,\eta}(t)$ as well as the value of $a_{\alpha,\beta,\eta}(0)$. Here, a new approach to the design of the increasing-gain observer is presented as compared with that described in [3], where an expansion series of $a_{\alpha,\beta,\eta}(t)$ based on the Dawson function is used in the various constraints of the optimization problem. Instead of (29) we formulate a problem with (29e) and (29f) replaced by stronger conditions that are not expressed by means of $a_{\alpha,\beta,\eta}(t)$, $b_{\alpha,\beta,\eta}(t)$, and $b'_{\alpha,\beta,\eta}(t)$. After solving such a problem, we fix a Taylor approximation of both $b_{\alpha,\beta,\eta}(t)$ and $b'_{\alpha,\beta,\eta}(t)$, and reoptimize the selection of the design parameters as far as possible.

The stronger conditions that enforce the satisfaction of (29e) and (29f) without using $a_{\alpha,\beta,\eta}(t)$, $b_{\alpha,\beta,\eta}(t)$, and $b'_{\alpha,\beta,\eta}(t)$ are given in the following proposition, where from now on

$$g_{\alpha,\beta,\eta}(t) := (\beta - 1)/t + \alpha\,\beta\, t^{\beta-1} - 2\,(\eta + k_f)\,.$$

Proposition 1 *Let $t_* \geq 0, t_{\max} \geq 0, \alpha > 0, \beta \in (1, 2], \sigma > 0, \lambda_* > 0, \eta > 0$; we have*

$$b_{\alpha,\beta,\eta}(t_{\max}) = \frac{t_{\max}^{\beta}}{g_{\alpha,\beta,\eta}(t_{\max})}\,. \tag{30}$$

Moreover,

(i) if

$$\eta \geq \max\left(\frac{\|A - KC\|}{\lambda_*}\left(\sigma + 4\alpha\,\beta\,\lambda_{\max}(P)\; k_f \times b_{\alpha,\beta,\eta}(t_{\max})\right) + n\, g_{\alpha,\beta,\eta}(t_*)\,, \frac{\|A - KC\|}{\lambda_*} \max\left(\sigma + 4\alpha\beta\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_{\max}), \lambda_*\right) + \frac{4n\alpha\beta\lambda_{\max}(P)\, k_f}{\lambda_*}\left(g_{\alpha,\beta,\eta}(t_*)\, b_{\alpha,\beta,\eta}(t_{\max}) - t_*^{\beta-1}\right)\right) \tag{31}$$

then (29e) holds;

(ii) if

$$\sigma > k_0\,\alpha\beta\, b_{\alpha,\beta,\eta}(t_{\max})\Big(1 + (N + 4k_f)\Big((2(\eta + k_f)/\alpha)^{1/(\beta-1)} \exp\big(2(\eta + k_f)\,(1 - 1/\beta)\,(2(\eta + k_f)/(\alpha\beta))^{1/(\beta-1)}\big) + 1\Big)\Big) \tag{32}$$

then (29f) holds.

Proof. See Appendix. □

Proposition 1 enables us to consider the following optimization problem instead of (29):

$$\min \max\left(1, \frac{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t_{\max})}{\lambda_*}\right) \tag{33a}$$

$$\text{w.r.t.}\quad t_*,\, t_{\max},\, \alpha,\, \beta,\, \sigma,\, \lambda_*,\, \eta \quad \text{s.t.}\quad t_* > 0, t_{\max} > 0, \alpha > 0, \beta \in (1, 2], \sigma > 0, \lambda_* \in (0, \lambda], \eta > 0,\ (29c),\ (30), (31), \text{and } (32) \text{ hold.} \tag{33b}$$

It is worth noting that (32) reduces to $\sigma > 0$ for $N = 0$ since k_0 turns to be equal to zero, namely, if Assumption 1 stands as a pure Lipschitz condition. In such a case, the optimization w.r.t. σ is meaningless and hence σ should be removed by the statement of the problem (33) and fixed to any arbitrary strictly positive value.

To construct the increasing-gain observer, we have to find a Taylor polynomial centered in $t_{\max}$ that approximates $b_{\alpha,\beta,\eta}$ to compute (28) with the design parameters that result from the solution of (33).

Proposition 2 *Let $b_{\alpha,\beta,\eta}$ as in (15) with $t_{\max}$ as its point of global maximum on $\mathbb{R}_{\geq 0}$. Then there exists $\bar{t} \geq 0$ with $|\bar{t} - t_{\max}| < |t - t_{\max}|$ such that*

$$b_{\alpha,\beta,\eta}(t) = \sum_{i=0}^{n} \frac{b^{(i)}_{\alpha,\beta,\eta}(t_{\max})\,(t-t_{\max})^i}{i!} + \frac{b^{(n+1)}_{\alpha,\beta,\eta}(\bar{t})\,(t-t_{\max})^{n+1}}{(n+1)!} \tag{34}$$

where

$$b'_{\alpha,\beta,\eta}(t_{\max}) = 0 \tag{35a}$$

$$b^{(n)}_{\alpha,\beta,\eta}(t) = \sum_{k=0}^{n-1} \binom{n-1}{k} g^{(n-1-k)}_{\alpha,\beta,\eta}(t)\, b^{(k)}_{\alpha,\beta,\eta}(t) - (\beta - 1)\ldots(\beta-n+1)\, t^{\beta-n}\,,\ n = 2,3,\ldots \tag{35b}$$

with the n-th derivative of $g_{\alpha,\beta,\eta}(t)$ for $t > 0$ given by

$$g^{(n)}_{\alpha,\beta,\eta}(t) = (-1)^n\,(\beta-1)\,n!\,t^{-n-1} + \alpha\,\beta\,(\beta-1)\ldots(\beta-n)\,t^{\beta-n-1}\,,\ n = 1,2,\ldots\,.$$

Proof. Clearly, (35a) follows from the definition and it is straightforward to obtain (35b) from (16) by using Leibniz's rule and (30). □

The choice of the approximation order may be done by bounding the rest in the r.h.s. of (34) in the interval $[t_*, t_{\max}]$. However, a simpler and less conservative technique to choose the order of the approximating polynomial consists in repeating the evaluation of the maximum error on $[t_*, t_{\max}]$ after increasing the polynomial order as far as such an error is less than or equal to the computational precision tolerance.

5 Simulation Results

Let us consider

$$\begin{cases} \dot{x}_1 = x_2 - c\sin(x_1) + \exp(-t) \\ \dot{x}_2 = x_3 - c\sin(x_2) + \exp(-t) \\ \dot{x}_3 = -c\sin(x_3) + \exp(-t) + u \\ y = x_1 \end{cases} \tag{36}$$

where $x(t) = (x_1(t), x_2(t), x_3(t))$ is the vector to be estimated by using only the measurements of the first variable and u is an external input. Thus, we have

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = [1\ 0\ 0]\,,$$

$$f(x,u,t) = \begin{bmatrix} -c\sin(x_1) + \exp(-t) \\ -c\sin(x_2) + \exp(-t) \\ -c\sin(x_3) + \exp(-t) + u \end{bmatrix},$$

where the positive constant c is scaled in such a way to yield different values of the Lipschitz constant of $f(\cdot,u,t)$, which will be denoted by k_f. Moreover, we chose $u(t) = \sin t$. First, we solved (25), thus obtaining

$$\lambda = 0.2195\,, \quad K = [8.5818 \quad 15.0909 \quad 7.7818]^\top\,,$$

$$P = \begin{bmatrix} 1.0976 & -0.4390 & -0.2744 \\ -0.4390 & 0.5488 & -0.4390 \\ -0.2744 & -0.4390 & 1.0976 \end{bmatrix}.$$

Second, we considered such a system with different values of c so as to obtain k_f equal to 0.1, 1, 10, and 15, respectively; using the routine *fmincon* of the Matlab Optimization Toolbox, we solved (33) and Taylor approximations of order 20 to compute $b_{\alpha,\beta,\eta}$ and $b'_{\alpha,\beta,\eta}$ (see the final minimizers shown in Table 1). Fig. 1 shows the corresponding mapping $t \mapsto \gamma(t)$ given by (28). For all the observers, we chose

$$\hat{f}(\hat{x},u,t) = [-c\sin(\hat{x}_1) \ \ -c\sin(\hat{x}_2) \ \ -c\sin(\hat{x}_3) + u]^\top\,.$$

Likewise for the increasing-gain observers, the high-gain observers were designed by minimizing the corresponding gain under constraints that ensure the asymptotic stability of the corresponding estimation errors. Such a minimization was accomplished by using the same routine *fmincon* adopted for the solution of (33).

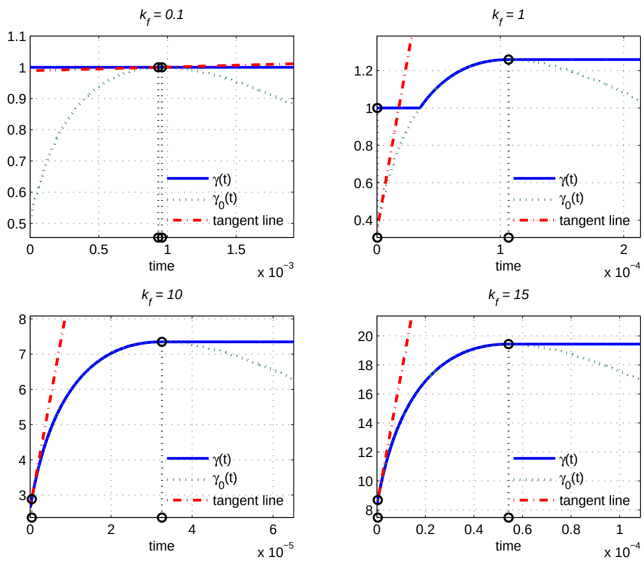

Fig. 1. Plots of $\gamma(t)$, its tangent line in t_*, and $\gamma_0(t) := (\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\; k_f\, b_{\alpha,\beta,\eta}(t))/\lambda_*$.

An example of a single run simulation is shown in Fig. 2. Table 2 reports the medians of the maximum abso-

Table 1
Optimal design parameters.

	k_f			
	0.1	1	10	15
t_*	$9.322681 \cdot 10^{-4}$	$3.9848386 \cdot 10^{-7}$	$3.9505594 \cdot 10^{-7}$	$3.9698746 \cdot 10^{-7}$
$t_{\max}$	$9.590693 \cdot 10^{-4}$	$1.0672627 \cdot 10^{-4}$	$3.2540019 \cdot 10^{-5}$	$5.4250563 \cdot 10^{-5}$
α	819.7327	4054.3032	7849.0366	5844.1067
β	1.615669	1.577717	1.5271657	1.5365227
σ	0.09365816	0.044591381	0.36900222	1.3555074
λ_*	0.210754	0.21905934	0.21940956	0.21938864
η	287.0292	2336.391	6976.7783	4299.2249

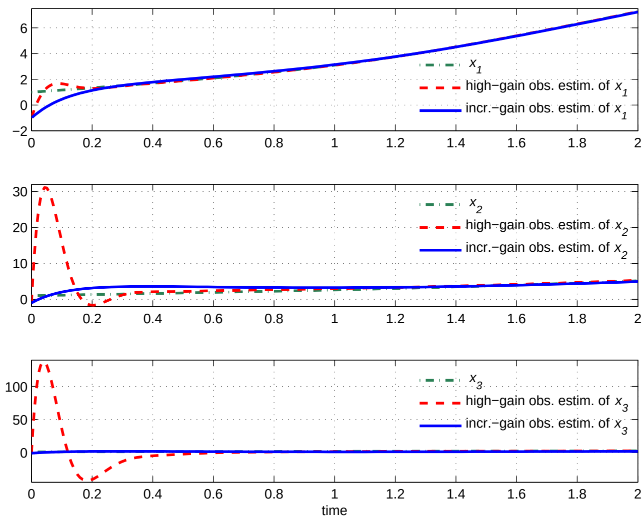

Fig. 2. Comparison in the case $k_f = 1$: simulation run with $x(0) = (1\ 1\ 1)$ and $\hat{x}(0) = (-1\ \ -1\ \ -1)$ for both observers.

lute errors with initial values of all the state variables randomly chosen according to a uniform distribution in $[-1, 1]$. Such results clearly show the increasing-gain observer outperforms the high-gain observer as to the peaking of the estimation error of inaccessible state variables (i.e., x_2 and x_3), especially for large values of k_f.

Table 2
Medians of the maximum absolute errors for each state variable over 1000 simulation runs on the time interval [0,5].

		k_f			
		0.1	1	10	15
incr.-gain obs.	x_1	0.60819	0.62997	0.6273	0.62252
	x_2	1.2561	1.3326	6.3256	17.0862
	x_3	0.9401	1.0216	22.5384	157.9092
high-gain obs.	x_1	0.60819	0.62997	0.6273	0.62252
	x_2	1.34	9.342	93.5494	17.0862
	x_3	0.96176	42.0036	4065.2572	9040.3839

6 Conclusions

The increasing-gain observer originally proposed in [3] has been enhanced by using a more general Lyapunov functional to prove global asymptotic stability and derive novel design conditions. The motivation of the approach with the goal of peaking reduction has been confirmed by simulations. Such an advantage may be ascribed to the richer structure of the Lyapunov functional used to construct the increasing-gain observer as compared with the quadratic Lyapunov function adopted for the high-gain observer. As a future work, we will focus on improving the optimization especially as to the local minima trapping, which may undermine the effective selection of the design parameters. Another subject of interest is the observer design in the presence of system and measurement disturbances.

Appendix

For the sake of notational convenience, let us omit the indices of the functions $a_{\alpha,\beta,\eta}$, $b_{\alpha,\beta,\eta}$, and $g_{\alpha,\beta,\eta}$, which will be denoted by a, b, and g, respectively.

Proof of Lemma 3. First, inequalities (12) and (13) follow by applying (11) to each l.h.s., respectively. Let us prove only (12), as the verification of (13) is quite similar. Using the well-known Young's inequality and since $s \geq t$, we obtain

$$\begin{aligned}
&\exp\left(\alpha t^{\beta}\right) \int_t^{\infty} \left(N \exp(-\delta s) + k_f \, \|e(s)\|\,\right) \|e(s)\| \\
&\times \exp(-\alpha s^{\beta})\, ds \leq \exp\left(\alpha t^{\beta} - 2(\eta + k_f)t\right) \int_t^{\infty} \Big(N \\
&\times \exp(-\delta t) + k_f \left(\|e(t)\| + (N/\delta)\exp(-\delta t)\right)\Big)\Big(\|e(t)\| \\
&+ (N/\delta)\exp(-\delta t)\Big) \exp(-\alpha s^{\beta} + 2(\eta + k_f)s)\, ds \\
&\leq \Big((N/2 + 2k_f)\|e(t)\|^2 + N(1/2 + N/\delta \\
&+ 2k_f N/\delta^2)\exp(-2\delta t)\Big) a(t)\,.
\end{aligned}$$

Note that the function a in (14) is differentiable and strictly positive for all $t \geq 0$. Using the De l'Hôpital theorem, we obtain

$$\lim_{t\to\infty} a(t) = \lim_{t\to\infty} \frac{\int_t^{\infty} \exp\left(-\alpha s^{\beta} + 2\,(\eta + k_f)\, s\right) ds}{\exp\left(-\alpha t^{\beta} + 2\,(\eta + k_f)\, t\right)} = 0\,. \tag{37}$$

To prove that a is strictly decreasing, let us consider the derivative of a given by

$$a'(t) = \exp\left(\alpha t^{\beta} - 2\,(\eta + k_f)\,t\right)\,\omega(t)$$

where the function $\omega : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as

$$\omega(t) := \left(\alpha\,\beta\,t^{\beta-1} - 2\,(\eta + k_f)\right)\int_t^{\infty}\exp\left(-\alpha s^{\beta} + 2(\eta + k_f)s\right)ds - \exp\left(-\alpha t^{\beta} + 2(\eta + k_f)t\right)$$

is differentiable. Note that $\lim_{t\to\infty}\omega(t) = 0$ since

$$\lim_{t\to\infty}\int_t^{\infty}\exp\left(-\alpha s^{\beta} + 2\,(\eta + k_f)\,s\right)ds = 0$$

of order higher than 1 because of (37). Moreover, ω is strictly increasing on $\mathbb{R}_{\geq 0}$ and $\omega(0) < 0$. Hence $\omega(t) < 0$ for all $t \geq 0$ and this proves the monotonicity of a.

The function b in (15) is differentiable for $t > 0$, $b(0) = 0$, $b(t) > 0$ for all $t > 0$, and

$$\lim_{t\to\infty} b(t) = 1/(\alpha\beta) \tag{38}$$

since

$$\lim_{t\to\infty} t^{\beta-1}\int_t^{\infty}\exp\left(-\alpha s^{\beta} + 2\,(\eta + k_f)\,s\right)ds = 0\,.$$

Deriving $b(t)$, we obtain (16) and write

$$b'(t) = t^{\beta-2}\exp\left(\alpha t^{\beta} - 2\,(\eta + k_f)\,t\right)\,\psi(t) \tag{39}$$

where the function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ defined as

$$\psi(t) := (\beta - 1)\int_t^{\infty}\exp\left(-\alpha s^{\beta} + 2(\eta + k_f)s\right)ds + \left(\alpha\,\beta\,t^{\beta} - 2(\eta + k_f)t\right)\int_t^{\infty}\exp\left(-\alpha s^{\beta} + 2(\eta + k_f)s\right)ds - t\,\exp\left(-\alpha t^{\beta} + 2(\eta + k_f)t\right)$$

is differentiable. Note that $\psi(0) > 0$, $\lim_{t\to\infty}\psi(t) = 0$, $\psi'(0) < 0$, and $\lim_{t\to\infty}\psi'(t) = 0$. Moreover,

$$\lim_{t\to 0^+}\psi''(t) = \begin{cases} +\infty & \text{if } 1 < \beta < 2 \\ 4\alpha\displaystyle\int_0^{\infty}\exp\left(-\alpha s^2 + 2(\eta + k_f)s\right)ds & \\ -2(\eta + k_f) & \text{if } \beta = 2 \end{cases}$$

and $\lim_{t\to\infty}\psi''(t) = 0$. If $1 < \beta < 2$, for $t > 0$, we have

$$\psi''(t) = (\beta - 1)\exp((-\alpha t^{\beta} + 2(\eta + k_f)t)\left(-2(\eta + k_f) + \alpha\,\beta^2\,a(t)/t^{2-\beta}\right). \tag{40}$$

The function $t \mapsto \nu(t) := \alpha\,\beta^2\,a(t)/t^{2-\beta}$ is continuous and differentiable in $\mathbb{R}_{>0}$, $\lim_{t\to 0^+}\nu(t) = +\infty$, $\lim_{t\to+\infty}\nu(t) = 0$ and it is strictly decreasing. Then $\nu\left(\mathbb{R}_{>0}\right) = \mathbb{R}_{>0}$, so there exists a unique point $t_2 > 0$ for which $\nu(t_2) = 2(\eta + k_f)$. If $\beta = 2$, since

$$\psi''(0) > 2\left((\eta + k_f) + \sqrt{\alpha\,\pi}\,\exp\left(\frac{(\eta + k_f)^2}{\alpha}\right)\right) > 0$$

and $\psi'''(t) > 0$ if and only if

$$t > t_1 := \left((\eta + k_f)^2 + \alpha\right)/\left(\alpha\,(\eta + k_f)\right),$$

it follows that ψ'' is decreasing in $[0, t_1]$ and increasing in $[t_1, \infty)$. In virtue of $\psi''(0) > 0$ and $\lim_{t\to\infty}\psi''(t) = 0$, there exists a point denoted by $t_2 \in (0, t_1)$ such that $\psi''(t_2) = 0$. Now we can even out the proof for every $1 < \beta \leq 2$. Since $\psi''(t) > 0$ for $t \in (0, t_2)$, $\psi''(t) < 0$ for $t \in (t_2, \infty)$, we obtain that ψ' is increasing in $[0, t_2]$ and decreasing in $[t_2, \infty)$. As $\psi'(0) < 0$ and $\lim_{t\to\infty}\psi'(t) = 0$, there exists $t_3 \in (0, t_2)$ such that $\psi'(t) < 0$ for $t \in [0, t_3)$ and $\psi'(t) > 0$ for $t \in (t_3, \infty)$. Thus, ψ is decreasing in $[0, t_3]$ and increasing in $[t_3, \infty)$. Since $\psi(0) > 0$ and $\lim_{t\to\infty}\psi(t) = 0$, there exists $t_{\max} \in (0, t_3)$ such that $\psi(t) > 0$ for $t \in [0, t_{\max})$ and $\psi(t) < 0$ for $t \in (t_{\max}, \infty)$. Therefore, from (39) we obtain that b is increasing in $[0, t_{\max}]$ and decreasing in $[t_{\max}, \infty)$. Moreover, since $b(0) = 0$ and $\lim_{t\to\infty} b(t) = 1/(\alpha\beta)$ (see (38)), b admits a global maximum in $t_{\max}$ and $b(t_{\max}) > 1/(\alpha\beta)$. To prove the concavity of b in $[0, t_{\max}]$, let us study the function b' in more detail. Such a function is differentiable,

$$\lim_{t\to 0^+} b'(t) = \begin{cases} +\infty & \text{if } 1 < \beta < 2 \\ \displaystyle\int_0^{\infty}\exp\left(-\alpha s^2 + 2(\eta + k_f)s\right)ds > 0 & \\ & \text{if } \beta = 2 \end{cases}$$

and, obviously, $b'(t_{\max}) = 0$. Moreover, from (39) and for $t > 0$, we obtain

$$b''(t) = t^{\beta-2}\exp\left(\alpha t^{\beta} - 2(\eta + k_f)t\right)\zeta(t)$$

where the function $\zeta : \mathbb{R}_{>0} \to \mathbb{R}$ defined as

$$\zeta(t) := \psi'(t) + \left(\alpha\,\beta\,t^{\beta-1} - 2(\eta + k_f) + (\beta - 2)/t\right)\psi(t)$$

is differentiable, $\lim_{t\to 0^+}\zeta(t) = -\infty$, $\zeta(t_{\max}) = \psi'(t_{\max}) < 0$.

Before proceeding with the derivative of ζ, we recall two inequalities that we will use to evaluate the sign of ζ' in $[0, t_{\max}]$. First, for every $t \in [0, t_{\max})$, we have

$$\left(\beta - 1 + \alpha\,\beta\,t^{\beta} - 2(\eta + k_f)t\right)\int_t^{\infty}\exp(-\alpha s^{\beta} + 2(\eta + k_f)s)\,ds > t\exp\left(-\alpha t^{\beta} + 2(\eta + k_f)t\right) \tag{41}$$

as $\psi(t) > 0$ for all $t \in [0, t_{\max})$. Second, since $\psi'(t) < 0$ for all $t \in [0, t_{\max}]$, then

$$\big(\alpha\beta^2 t^{\beta-1} - 2(\eta + k_f)\big) \int_t^\infty \exp(-\alpha s^\beta + 2(\eta + k_f)s)\, ds \\ < \beta \exp\big(-\alpha t^\beta + 2(\eta + k_f)t\big)\,. \tag{42}$$

Using the expressions of $\psi'(t)$ and $\psi''(t)$ and taking into account (41) and (42), with a little algebra we obtain

$$\zeta'(t) > h(t) \int_t^\infty \exp(-\alpha s^2 + 2(\eta + k_f)s)\, ds$$

where

$$h(t) := -(\beta - 2)(\beta - 1)\ \big(1/t^2 - 2(\eta + k_f)/(\beta t)\big) \\ + 4(\eta + k_f)^2\,(1 - 1/\beta)\,.$$

The function $h : \mathbb{R}_{>0} \to \mathbb{R}$ is differentiable and

$$h'(t) = (\beta - 2)(\beta - 1)\ \big(2/t^3 - 2(\eta + k_f)/(\beta t^2)\big)\ .$$

Moreover, $\lim_{t\to 0^+} h(t) = +\infty$ and $\lim_{t\to+\infty} h(t) = 4(\eta + k_f)^2\,(1 - 1/\beta) > 0$ for $\beta > 1$. Then h admits an absolute minimum in $\tilde{t} = \beta/(\eta + k_f)$ and, if $1 < \beta \le 2$, it follows $h(\tilde{t}) = (\eta + k_f)^2\,(5\beta^2 - 7\beta + 2)/\beta^2 > 0$. Therefore, we obtain that $h(t) > 0$ on $\mathbb{R}_{>0}$; hence $\zeta'(t) > 0$ and ζ strictly increasing for all $t \in (0, t_{\max}]$. This allows one to conclude that $\zeta(t) < 0$, i.e., $b''(t) < 0$ for all $t \in [0, t_{\max}]$, thus ensuring the strict decrease of b' and the concavity of b in such an interval. □

Proof of Proposition 1. First of all, let us find a convenient expression for $b(t_{\max})$, which will be used later on. Since $b'(t_{\max}) = 0$ and $a(t_{\max}) = b(t_{\max})/t_{\max}^{\beta-1}$, from (16) we obtain (30) with a denominator that has to be necessarily strictly positive.

As to (i), we consider two cases to account for (29e) by addressing separately (a) $\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_*) > \lambda_*$ and (b) $\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_*) \le \lambda_*$ and finally combining what results from (a) and (b). As to (a), we have to deal with

$$\eta \ge \frac{\|A - KC\|}{\lambda_*}\ (\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max})) \\ + \frac{4\,n\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b'(t_*)}{\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_*)} \tag{43}$$

since $\lambda_* < \sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_*) \le \sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max})$. As a matter of fact, (43) can be replaced by a stronger condition given by

$$\eta \ge \frac{\|A - KC\|}{\lambda_*}\ (\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max})) \\ + n\,b'(t_*)/b(t_*)\,. \tag{44}$$

Since the r.h.s. of (16) is upper bounded by $g(t)\,b(t)$, for all $t_* > 0$ we obtain

$$b'(t_*)/b(t_*) \le g(t_*)\,. \tag{45}$$

Using (30) and (45), it follows that the inequality

$$\eta \ge \frac{\|A - KC\|}{\lambda_*}\ (\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max})) \\ +n\,g(t_*) \tag{46}$$

implies (44) and hence (43). As to (b), (29e) takes on the form of

$$\eta \ge \frac{\|A - KC\|}{\lambda_*}\ \max(\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max}), \lambda_*) \\ + \frac{4\,n\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b'(t_*)}{\lambda_*} \tag{47}$$

because of (30). Since $b(t_*) \le b(t_{\max})$, (16) yields

$$b'(t_*) \le g(t_*)\,b(t_{\max}) - t_*^{\beta-1}$$

and hence it is straightforward to impose (47) by means of the following inequality:

$$\eta \ge \frac{\|A - KC\|}{\lambda_*}\ \max(\sigma + 4\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f\,b(t_{\max}), \lambda_*) \\ + \frac{4\,n\,\alpha\,\beta\,\lambda_{\max}(P)\ k_f}{\lambda_*}\ \Big(g(t_*)\,b(t_{\max}) - t_*^{\beta-1}\Big). \tag{48}$$

Combining the conclusions drawn in case (a) and (b) (i.e., (46) and (48)), we obtain that (29e) is implied by (31) and thus end the proof of (i).

As to (ii), we need to evaluate the integral

$$a(0) = \int_0^\infty \exp\big(-\alpha s^\beta + 2\,(\eta + k_f)\,s\big)\,\mathrm{d}s\,.$$

Note that

$$a(0) = \int_0^{l_0} \exp\big(-\alpha s^\beta + 2(\eta + k_f)s\big)\,\mathrm{d}s + a(l_0) \tag{49}$$

where $l_0 := (2(\eta + k_f)/\alpha)^{1/(\beta-1)}$. The first term in (49) is upper bounded with the value of the area of the rectangle having base length and height equal to l_0 and l_1, respectively, where $l_1 := \exp\Big(2(\eta + k_f)\ (1 - 1/\beta)\ (2(\eta + k_f)/(\alpha\beta))^{1/(\beta-1)}\Big)$. Thus, we have $a(0) \le l_0\,l_1 + a(l_0)$ and, after noting that $a(l_0)$ can be bounded from above by 1 since $-\alpha s^\beta + 2\,(\eta + k_f)\,s < -s + l_0$ for $s > l_0$ and using (30), it is straightforward to verify that (29f) is implied by (32). □

References

[1] J. Ahrens, H. Khalil. High-gain observers in the presence of measurement noise: a switched-gain approach. *Automatica*, 45(4):936–943, 2009.

[2] A. Alessandri, A. Rossi. On increasing-gain observers for nonlinear continuous-time systems. In *Proc. 52nd Conf. on Decision and Control*, 3957–3962, Firenze, Italy, 2013.

[3] A. Alessandri, A. Rossi. Time-varying increasing-gain observers for nonlinear systems. *Automatica*, 49(9):2845–2852, 2013.

[4] V. Andrieu, L. Praly, A. Astolfi. High gain observers with updated gain and homogeneous correction terms. *Automatica*, 45(2):422–428, 2009.

[5] N. Boizot, E. Busvelle, J. Gauthier. An adaptive high-gain observer for nonlinear systems. *Automatica*, 46(9):1483–1488, 2010.

[6] Y. Chitour. Time-varying high-gain observers for numerical differentiation. *IEEE Trans. on Automatic Control*, 47(9):1565–1569, 2002.

[7] G. Ciccarella, M. Dalla Mora, and A. Germani. A Luenberger-like observer for nonlinear systems. *Int. J. of Control*, 57(3):537–556, 1994.

[8] J.P. Gauthier, I. Kupka. *Deterministic Observation Theory and Applications*. Cambridge University Press, Cambridge, UK, 2001.

[9] H. Hammouri, B. Targui, F. Armanet. High gain observer based on a triangular structure. *International Journal of Robust and Nonlinear Control*, 12(6):497518, 2002.

[10] S. Ibrir. Adaptive observers for time-delay nonlinear systems in triangular form. *Automatica*, 45(10):2392–2399, 2009.

[11] S.R. Kou, D.L. Elliott, T.J. Tarn. Exponential observers for nonlinear dynamic systems. *Information and Control*, 29:204–216, 1975.

[12] A.J. Krener, A. Isidori. Linearization by output injection and nonlinear observers. *Systems & Control Letters*, 3:47–52, 1983.

[13] A.J. Krener, W. Respondek. Nonlinear observer with linearizable error dynamics. *SIAM J. on Control and Optimization*, 23(2):197–216, 1985.

[14] P. Krishnamurthy, F. Khorrami. A high-gain scaling technique for adaptive output feedback control of feedforward systems. *IEEE Trans. on Automatic Control*, 49(12):2286–2292, 2004.

[15] M. Oueder, M. Farza, R. Ben Abdennour, M. M'Saad. A high gain observer with updated gain for a class of MIMO non-triangular systems. *Systems & Control Letters*, 61(2):298–308, 2012.

[16] R. Sanfelice, L. Praly. On the performance of high-gain observers with gain adaptation under measurement noise. *Automatica*, 47(10):2165–2176, 2011.

[17] Y. Shen, Y. Huang. Uniformly observable and globally Lipschitzian nonlinear systems admit global finite-time observers. *IEEE Trans. on Automatic Control*, 54(11):2621–2625, 2009.

[18] Y. Shen, X. Xia. Semi-global finite-time observers for nonlinear systems. *Automatica*, 44(12):3152–3156, 2008.

[19] Y.J. Sun, C.H.H. Lien, J.G. Hsieh. Global exponential stabilization for a class of uncertain nonlinear systems with control constraint. *IEEE Trans. on Automatic Control*, 43(5):674–677, 1998.

[20] F.E. Thau. Observing the state of non-linear dynamic systems. *Int. J. of Control*, 17(3):471–479, 1973.