Stability of Switched Nonlinear Systems with Delay and Disturbance ⋆

Yue-E Wang [a], Xi-Ming Sun [b], Frédéric Mazenc [c]

[a] *College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710119, PR China*

[b] *School of Control Science and Engineering, Dalian University of Technology, Dalian 116024, PR China*

[c] *EPI Inria-Saclay DISCO, Laboratoire des Signaux et Systèmes, CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192 Gif-sur-Yvette, France*

Abstract

We consider a class of nonlinear time-varying switched control systems for which stabilizing feedbacks are available. We study the effect of the presence of a delay in the input of switched nonlinear systems with an external disturbance. By contrast with most of the contributions available in the literature, we do not assume that all the subsystems of the switched system we consider are stable when the delay is present. Through a Lyapunov approach, we derive sufficient conditions in terms of size of the delay ensuring the global exponential stability of the switched system. Moreover, under appropriate conditions, the input-to-state stability of the system with respect to an external disturbance is established.

Key words: Switched nonlinear systems; Lyapunov-Krasovskii functionals; input delay; input-to-state stability.

1 Introduction

Switched systems in continuous-time are systems with discrete switching events. They rely on a switching signal, which indicates at each instant which subsystem operates. Switched systems have extensive applications in domains which pertain to complex dynamical networks Zhao et al. (2009), mobile robot Sankaranarayanan & Mahindrakar (2009), flight control Yang et al. (2009), and many others (see Liberzon (2003)). Control problems for these systems have been studied in many contributions Caliskan et al. (2013), Liberzon (2014), Zhao et al. (2012), Zhao & Hill (2008), Sun (2008). For instance, the work Zhao & Hill (2008) solved the L_2-gain analysis and an H_∞ control problem and gave necessary and sufficient conditions of stability of switched systems derived from multiple generalized Lyapunov-like functions. In Sun (2008), a robust switching signal was designed to render the switched system exponentially stable and robust against switching perturbations, based on the notion of relative distance.

It is well-known that in many applications, delays are present due in particular to measurement and transport phenomena. For instance, switched systems with delay model drilling systems and cellular mobile communication systems. In these systems, each user is allocated a channel on a per call basis, and upon termination of the call, the previously occupied channel is immediately switched to the available channel, which generally includes time delays and disturbances Rappaport (1996). Delays can degrade the performances of the controllers or even destabilize the systems whose stability in the absence of delay is ensured by an appropriate choice of feedback. In fact, the presence of delays in the input and of additive disturbances may cause the system breakdown or the system crash if the switching signal is designed inappropriately. For these reasons, the last ten years have witnessed significant developments in the domain of switched systems when a delay is present, as illustrated for instance by the contributions Ma & Zhao (2015), Wang et al. (2012), Vu & Kristi (2010)

⋆ This work was supported by National Natural Science Foundation of China under Grants 61403241, 61325014, 11371233, the China Postdoctoral Science Foundation under Grant 2014M560748, and the Fundamental Research Funds for the Central Universities under Grant GK201503011. This paper was not presented at any IFAC meeting. Corresponding author: Xi-Ming Sun. Tel. +86 29 85310232 Fax +86 29 89255877

Email addresses: baihewye@snnu.edu.cn (Yue-E Wang), sunxm@dlut.edu.cn (Xi-Ming Sun), frederic.mazenc@l2s.centralesupelec.fr (Frédéric Mazenc).

and Colaneri et al. (2008). Interestingly, the presence of a delay may destabilize some of the subsystems of a switched system, without destabilizing it, which implies that stability conditions based on the assumption that the delay does not destabilize any subsystem are conservative. Since establishing stability results with the largest possible delay is desirable, the research subject of the stability of switched systems with unstable subsystems is very appealing from an applied point of view. The first contribution devoted to it is Zhai et al. (2001). It presents results for linear systems only. From then on, the stability and the stabilization for switched linear time delay systems with both stable and unstable subsystems have been studied in many works. On the other hand, switched nonlinear time-varying time delay systems received less attention (with Muller & Liberzon (2012), Wang et al. (2013), Liu et al. (2011), Sun & Wang (2012) as notable exceptions), probably because of the difficulty of analyzing general families of these systems. But the study of these systems is fundamental because many models are nonlinear and tracking problems lead to time-varying systems. In fact, this subject is strongly motivated by recent applications, notably in the domains of mechanical systems Moon & Kalmar-Nagy (2001), wireless communications Rappaport (1996), mobile robots Malisoff et al. (2012).

These remarks motivate the present work. Before describing more precisely its purpose, a few preliminary comments are needed. Lyapunov-Krasovskii Functionals (LKFs) are very powerful tools when it comes to establishing stability conditions for time delay systems. For constructions of LKFs for linear time delay systems, the research monographs Fridman (2014) and Gu et al. (2003) have provided systematic methods, which have been already used to solve problems for switched linear time delay systems. But, in a general nonlinear context, even for stable systems with delay without switch, only a few constructions of Lyapunov-Krasovskii Functionals (LKFs) are available. Fortunately, recently, in Mazenc et al. (2013) (see also Mazenc et al. (2012)), a new family of Lyapunov-Krasovskii Functionals (LKFs) has been proposed to analyze the stability of systems belonging to a broad family of nonlinear time-varying systems with delays without switch. The delay is not supposed to be known, and it is established that a system in closed-loop with a feedback which globally asymptotically stabilize its origin is not destabilize by the delay provided it is smaller than an upper bound (given by an explicit formula). Our aim is to extend the preceding study Wang et al. (2014), which considered the case of a switched systems with delay whose subsystems are all stable, to show how the functionals constructed in Mazenc et al. (2013) can be used to establish a similar result for switched systems, when a constant point wise delay in the input destabilize some of the subsystems of the system. Besides, we aim at determining estimates of the norm of solutions when additive disturbances in the input are present. As in Wang et al. (2014), we shall prove the input-to-state stability (ISS) for switched nonlinear time delay systems with respect to the disturbance. It is worth pointing out that the assumptions imposed in Wang et al. (2014) guarantee the stability of the system by implying that all its subsystems are stable, which leads to conservative conditions in terms of the size of the delay and of the minimum allowable dwell time of the switching signal. Our aim is to relax these hypotheses.

The analysis we shall propose decomposes in two steps. First, by constructing a new LKF, we will determine sufficient conditions, involving the size of the delay, the growth properties of the vector fields and minimum dwell time, guaranteeing the stability of the switched nonlinear system and of all its subsystems. These conditions are less restrictive than those imposed in Wang et al. (2014) because, the LKF given in Wang et al. (2014) involves a common Lyapunov function, which results in a conservative constraint on the input delay upper bound. The new LKF we propose allows us to derive less conservative input delay upper bound and average dwell time. Second, we focus on the case where the input delay is larger than the upper bound given in the first step, so that some subsystems may be unstable. By modifying the LKF and considering a special switching signal, the ISS of the system is guaranteed for delays larger than the admissible delays of the first step.

Finally, we wish to observe that, although the main results of the present paper assume the existence of stabilizing control laws for the system without delay, they can be used in the context of the design of control laws because they make it possible to select, amongst the possible control laws, those that lead to good robustness properties with respect to the presence of a delay and of additive disturbances.

The paper is organized as follows. The problem is stated in Section 2. Section 3 gives two preliminary result, which are instrumental in establishing the main results, which are given in Section 4, where are given, results for switched systems whose subsystems are all stable and Section 5 which presents a solution in the case where some of the subsystems of the studied switched system are unstable. Section 6 is devoted to an illustrative example. Concluding remarks in Section 7 end paper.

Notations. The notation will be simplified, e.g., by omitting the arguments of the functions, whenever no confusion can arise from the context. By τ, we denote the delay satisfying $0 \leq \tau \leq \bar{\tau}$, where $\bar{\tau}$ is a finite positive constant. The Banach Space of absolutely continuous functions $\phi : [-\bar{\tau}, 0] \to \mathbb{R}^{n_x}$ with $\dot{\phi} \in L_2([-\bar{\tau}, 0]; \mathbb{R}^{n_x})$, equipped with the norm $\|\phi\|_W = \max_{m\in[-\bar{\tau},0]} |\phi(m)| + \left(\int_{-\bar{\tau}}^{0} |\dot{\phi}(s)|^2 ds\right)^{\frac{1}{2}}$ is denoted by $W[-\bar{\tau}, 0]$. For $t \in \mathbb{R}^+$, set $x_t(s) := x(t + s), s \in [-\bar{\tau}, 0]$. For a measurable and essentially bounded function $u : \mathbb{R}^+ \to \mathbb{R}^p$, $||u||_\infty = ess\sup_{t\geq 0} |u(t)|$. If $||u||_\infty < \infty$, we write $u \in L^p_\infty$. For a continuous-time signal $w(t)$,

set $\|w[t_1,t_2]\| = \sup_{t_1\le s\le t_2}\{|w(s)|\}$. For any $t_2 > t_1 \ge 0$, let $N_\sigma(t_1,t_2)$ denote the number of switching of $\sigma(t)$ over (t_1,t_2). If $N_\sigma(t_1,t_2) \le N_0 + \frac{t_2-t_1}{\tau_a}$ holds for two constants $\tau_a > 0, N_0 \ge 0$, then τ_a is called average dwell time. A function $V : \mathbb{R}^{n_x}\times\mathbb{R}^+ \to \mathbb{R}^+$ is called uniformly proper and positive definite provided there are $\alpha_s, \alpha_l \in \mathcal{K}_\infty$ such that $\alpha_s(|x|) \le V(x,t) \le \alpha_l(|x|)$, for all $x \in \mathbb{R}^{n_x}$ and $t \ge 0$.

2 Problem formulation

Throughout the paper, we consider the switched nonlinear time-varying system:

$$\begin{aligned}\dot{x}(t) &= f_{\sigma(t)}(x,t) + g_{\sigma(t)}(x,t)[u_{\sigma(t)}(x(t-\tau),t) + w(t)],\\ x_{t_0}(m) &= \xi(m), m \in [-\bar{\tau}, 0],\end{aligned} \tag{1}$$

where $t_0 \ge 0$, $x \in \mathbb{R}^{n_x}$ is the state, $\dot{x}(t)$ denotes the right-hand derivative of $x(t)$, the delay τ satisfies $\tau \in [0,\bar{\tau}]$, $w \in L^p_\infty$ represents a disturbance and ξ is a differentiable initial function. The function $\sigma : [0,\infty) \to N = \{1,2,..,n\}$ is the switching signal. Associated with σ, we have the switching sequence $\{(i_0,t_0),...,(i_k,t_k),...,|i_k \in N, k \in \mathbb{N}\}$, which is such that the i_kth subsystem is active when $t \in [t_k, t_{k+1})$. For any $i \in N$, $u_i(x,t) \in \mathbb{R}^p$ is C^1 and is the predesigned stabilizing controller for the ith subsystem, f_i, g_i are locally Lipschitz with respect to x, and, for all $t \ge t_0$, $u_i(0,t) = 0$ and $f_i(0,t) = 0$. We assume that, for any finite time interval, there is only a finite number of switches and no jump occurs in the state at a switching instant.

For the sake of clarity, we give the definition of ISS for switched systems with delay, which is analogous to the one introduced in Pepe & Jiang (2006).

Definition 1. System (1) is said to be input-to-state stable (ISS) if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that for any initial condition $\xi \in W[-\bar{\tau},0]$ and for any $w \in L^p_\infty$, the solution of (1) exists over $[0,+\infty)$ and satisfies

$$|x(t)| \le \beta(||\xi||_W, t-t_0) + \gamma(||w[t_0,t]||), t \ge t_0 \ge 0. \tag{2}$$

We introduce an assumption.

Assumption 1. There are known C^1 uniformly proper and positive definite functions $V_i : \mathbb{R}^{n_x}\times\mathbb{R}^+ \to \mathbb{R}^+$, $i \in N$ such that
i) For each $i \in N$, there is a constant $\lambda_i > 0$ such that, for all $x \in \mathbb{R}^{n_x}$ and $t \ge 0$

$$\frac{\partial V_i}{\partial t}(x,t) + \frac{\partial V_i}{\partial x}(x,t)h_i(x,t) \le -\lambda_i V_i(x,t) \tag{3}$$

with

$$h_i(x,t) = f_i(x,t) + g_i(x,t)u_i(x,t). \tag{4}$$

ii) There is a constant $\mu \ge 1$ such that for all $i,j \in N$

$$V_i(x,t) \le \mu V_j(x,t) \tag{5}$$

for all $x \in \mathbb{R}^{n_x}$ and $t \ge 0$.

Next, we definite the piecewise continuous Lyapunov function $V(x,t) = V_{\sigma(t)}(x,t)$ for (1).

Let us recall that in Liberzon (2003) it is proved that if $\tau = 0$ and w is not present and if Assumption 1 is satisfied and the switching signal satisfies the average dwell time constraint $\tau_a > \frac{\ln\mu}{\min_{i\in N}\{\lambda_i\}}$, then the switched system (1) is globally exponentially stable.

3 Preliminary results

In this section, we establish some stability and growth properties for the solutions of the nonlinear system without switch

$$\begin{aligned}\dot{x}(t) &= f(x(t),t) + g(x(t),t)[u(x(t-\tau),t) + w(t)],\\ x_{t_0}(m) &= \xi(m), m \in [-\bar{\tau}, 0].\end{aligned} \tag{6}$$

with $\tau \in [0,\bar{\tau}]$, where $\bar{\tau}$ is a constant and w is an external disturbance. We suppose that a locally Lipschitz feedback $u(x,t)$ and a C^1 uniformly proper and positive definite Lyapunov function $V : \mathbb{R}^{n_x}\times\mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\frac{\partial V}{\partial t}(x,t) + \frac{\partial V}{\partial x}(x,t)h(x,t) \le -\lambda V(x,t), \tag{7}$$

where $\lambda > 0$ and $h(x,t) = f(x,t) + g(x,t)u(x,t)$, are known.

3.1 First result

In this section, we revisit the main construction of Mazenc et al. (2013) to adapt it to our main goal, i.e. the stability analysis of the system (1). We introduce an assumption.

Assumption 2. There exist constants $c > 0$, $k_j \ge 0$ $(j = 2,...,5)$ and a function $\alpha \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_x}$ and $t \ge 0$, the following inequalities hold:
i) $\left|\frac{\partial V}{\partial x}(x,t)g(x,t)\right| \le \alpha(|x|), \left|\frac{\partial u}{\partial x}(x,t)\right| \le c$,
ii) $|f(x,t)|^2 \le k_2\alpha^2(|x|), |g(x,t)|^2 \le k_3\alpha(|x|) + k_4$,
iii) $(|g(x,t)||u(z,t)|)^2 \le k_5\alpha^2(|x|) + \alpha^2(|z|)$,
iv) $\alpha^2(|x|) \le V(x,t)$,
v) $\bar{\tau} < \mathfrak{T}$ with

$$\mathfrak{T} = \frac{\lambda}{1 + \sqrt{2}c\sqrt{k_2 + 2k_5 + \frac{17}{8}}}. \tag{8}$$

Remark 1. To reduce the number of parameters in Assumption 2, we do not impose in i) and iii) the more general inequalities $\left|\frac{\partial V}{\partial x}(x,t)g(x,t)\right| \leq k_1\alpha(|x|)$ and $[|g(x,t)||u(z,t)|]^2 \leq k_5\alpha^2(|x|)+k_6\alpha^2(|z|)$ where $k_1>0$ and $k_6>0$ are arbitrary constants. Indeed, through a scalar rescaling of V, we can always replace the constant k_6 by 1 and in a second step, we can remove k_1 by observing that if the inequality $\left|\frac{\partial V}{\partial x}(x,t)g(x,t)\right| \leq k_1\alpha(|x|)$ is satisfied, then by replacing g by $\tilde{g}=\frac{g}{k_1}$ and u by $\tilde{u}=k_1u$ the requirement i) is satisfied, with $\tilde{g}$ and $\tilde{u}$ instead of g and u.

Let us introduce a LKF defined, along the trajectories of (6), by:

$$U(x_t,t)=V(x,t)+\Gamma_1(x_t)+\Gamma_2(x_t), \tag{9}$$

where

$$\Gamma_1(x_t)=d_1\int_{t-\tau}^{t}e^{-a(t-s)}\alpha^2(|x(s)|)ds, \tag{10}$$

$$\Gamma_2(x_t)=d_2\int_{t-\bar{\tau}}^{t}\int_{m}^{t}e^{-a(t-s)}|\dot{x}(s)|^2dsdm, \tag{11}$$

where a, d_1, d_2 are positive constants appropriately chosen. We are ready to state and prove the following result:

Lemma 1 *Let the system (6) satisfy Assumption 2. Then there exist positive constants a, d_1, d_2 and q such that*

$$\theta<\lambda, \tag{12}$$

$$4d_2\bar{\tau}-d_1e^{-a\bar{\tau}}\leq 0, \tag{13}$$

$$\frac{\bar{\tau}}{2q}-d_2e^{-a\bar{\tau}}\leq 0, \tag{14}$$

with $\theta=\frac{q}{2}c^2+d_1+\left(2d_2k_2+4d_2k_5+1+\frac{d_2}{4}\right)\bar{\tau}$ and, along the trajectory of the system (6), for all $t\geq t_0$, the inequality

$$U(x_t,t)\leq e^{-a(t-t_0)}U(x_{t_0},t_0)+\int_{t_0}^{t}e^{-a(t-s)}\delta(|w(s)|)ds \tag{15}$$

holds, where U is the LKF defined in (9), $\delta(m)=16d_2\bar{\tau}k_3^2m^4+\left(4d_2\bar{\tau}k_4+\frac{1}{4\bar{\tau}}\right)m^2$. Moreover, the system (6) is ISS with respect to w.

Remark 2. In the linear case, a positive definite quadratic function V can be selected and then i), ii), iii), iv) in Assumption 2 is satisfied. However, these requirements are satisfied for a much larger family of positive definite functions V.

Proof. Let us prove that there are constants a, d_1, d_2 and q such that (12), (13) and (14) are satisfied.

From v) in Assumption 2 and Young's inequality, it follows that there is a constant $q>0$ such that

$$\frac{q}{2}c^2+\left(k_2+2k_5+\frac{17}{8}\right)\frac{\bar{\tau}^2}{q}+\bar{\tau}<\lambda. \tag{16}$$

It follows that one can choose $d_2>0$ so that $\frac{\bar{\tau}}{2q}<d_2$ and

$$\frac{q}{2}c^2+4\bar{\tau}d_2+\left(2k_2+4k_5+\frac{1}{4}\right)\bar{\tau}d_2+\bar{\tau}<\lambda. \tag{17}$$

Consequently, one can find $d_1>0$ such that (12) is satisfied and $4\bar{\tau}d_2<d_1$. To summarize, we observe that we have (12) and

$$4d_2\bar{\tau}-d_1<0 \quad \text{and} \quad \frac{\bar{\tau}}{2q}-d_2<0. \tag{18}$$

We deduce that there is $a\in(0,\lambda-\theta]$ such that (13) and (14) are satisfied; this is the case for instance of $a=\inf\left\{\lambda-\theta,\frac{1}{\bar{\tau}}\ln\left(\frac{d_1}{4d_2\bar{\tau}}\right),\frac{1}{\bar{\tau}}\ln\left(\frac{2qd_2}{\bar{\tau}}\right)\right\}$.

Now, let us establish (15). Elementary calculations give

$$\dot{\Gamma}_1=-a\Gamma_1+d_1\alpha^2(|x(t)|)-d_1e^{-a\bar{\tau}}\alpha^2(|x(t-\tau)|), \tag{19}$$

$$\dot{\Gamma}_2=-a\Gamma_2+d_2\bar{\tau}|\dot{x}(t)|^2-d_2e^{-a\bar{\tau}}\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2ds \tag{20}$$

and that the derivative of V along the trajectories of system (6) satisfies

$$\begin{aligned}\dot{V}&=\frac{\partial V}{\partial t}(x,t)+\frac{\partial V}{\partial x}h(x,t)+\frac{\partial V}{\partial x}gw(t)\\&\quad+\frac{\partial V}{\partial x}gu(x(t-\tau),t)-\frac{\partial V}{\partial x}gu(x,t)\\&\leq-\lambda V+\left|\frac{\partial V}{\partial x}g\right||w|+\left|\frac{\partial V}{\partial x}g\right|c|x(t)-x(t-\tau)|,\end{aligned}$$

where the last inequality is a consequence of (7). Using the equality $x(t)-x(t-\tau)=\int_{t-\tau}^{t}\dot{x}(s)ds$, Young's and Cauchy-Schwarz's inequalities, we deduce that

$$\begin{aligned}\dot{V}&\leq-\lambda V+\frac{q}{2}\left|\frac{\partial V}{\partial x}g\right|^2c^2+\frac{1}{2q}\left|\int_{t-\tau}^{t}\dot{x}(s)ds\right|^2\\&\quad+\bar{\tau}\left|\frac{\partial V}{\partial x}g\right|^2+\frac{1}{4\bar{\tau}}|w|^2\\&\leq-\lambda V+\frac{qc^2}{2}\alpha^2(|x(t)|)+\frac{\bar{\tau}}{2q}\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2ds\\&\quad+\bar{\tau}\alpha^2(|x(t)|)+\frac{1}{4\bar{\tau}}|w|^2,\end{aligned} \tag{21}$$

where the last inequality is consequence of the condition i) in Assumption 2.

Now, considering (14) and applying (19)-(21) lead to

$$\begin{aligned}\dot{U} \leq & -\lambda V - a\Gamma_1 - a\Gamma_2 + \frac{1}{4\bar{\tau}}|w|^2 \\ & -d_1 e^{-a\bar{\tau}}\alpha^2(|x(t-\tau)|) \\ & + \left[\frac{qc^2}{2} + \bar{\tau} + d_1\right]\alpha^2(|x(t)|) + d_2\bar{\tau}|\dot{x}(t)|^2. \end{aligned} \quad (22)$$

Using

$$|\dot{x}(t)|^2 \leq 2|f|^2 + 4(|gu(x(t-\tau),t)|^2 + |gw(t)|^2) \quad (23)$$

and $ii)$ and $iii)$ in Assumption 2, we obtain

$$\begin{aligned}|\dot{x}(t)|^2 \leq & 2k_2\alpha^2(|x|) + 4k_5\alpha^2(|x|) + 4\alpha^2(|x(t-\tau)|) \\ & +4[k_3\alpha(|x|) + k_4]|w|^2 \\ \leq & 2k_2\alpha^2(|x|) + 4k_5\alpha^2(|x|) + 4\alpha^2(|x(t-\tau)|) \\ & +16k_3^2|w|^4 + \frac{1}{4}\alpha^2(|x|) + 4k_4|w|^2. \end{aligned} \quad (24)$$

Then, combining (22) and (24), we deduce that

$$\begin{aligned}\dot{U} \leq & -\lambda V - a\Gamma_1 - a\Gamma_2 + \left[\frac{qc^2}{2} + \bar{\tau} + d_1 + 2d_2\bar{\tau}k_2\right. \\ & \left. +4d_2\bar{\tau}k_5 + \frac{1}{4}d_2\bar{\tau}\right]\alpha^2(|x(t)|) \\ & + \left[4d_2\bar{\tau} - d_1 e^{-a\bar{\tau}}\right]\alpha^2(|x(t-\tau)|) + \delta(|w|). \end{aligned} \quad (25)$$

From $iv)$ in Assumption 2, (12), (13) and $a \in (0, \lambda - \theta]$, we deduce that

$$\dot{U} \leq -aU + \delta(|w|). \quad (26)$$

By integrating this inequality, we obtain (15).

From the definitions of V, Γ_1 and Γ_2, we deduce easily that there are functions $\gamma_a \in \mathcal{K}_\infty$ and $\gamma_b \in \mathcal{K}_\infty$ such that $\gamma_a(|\phi(0)|) \leq U(\phi,t) \leq \gamma_b(||\phi||_W)$ for all $t \geq 0$, $\phi \in W$. Therefore, from (15), we deduce that the system (6) is ISS (for more details, see Mazenc et al. (2008)). This completes the proof. □

3.2 Second result

Now, we consider the system (6) in the case where $\bar{\tau}$ can be larger than $\mathfrak{T}$ defined in (8).

We provide an estimate of the evolution along the trajectories of LKF candidate:

$$U(x_t,t) = V(x,t) + \Gamma_1(x_t) + \Gamma_2(x_t) \quad (27)$$

with

$$\Gamma_1(x_t) = d_1\int_{t-\tau}^{t}\alpha^2(|x(s)|)ds, \quad (28)$$

$$\Gamma_2(x_t) = d_2\int_{t-\bar{\tau}}^{t}\int_{m}^{t}|\dot{x}(s)|^2 ds dm \quad (29)$$

where d_1 and d_2 are positive constants.

Lemma 2 *Under the conditions $i), ii), iii), iv)$ of Assumption 2, along the trajectory of the system (6), the LKF (27) with d_1 and d_2 such that*

$$4d_2\bar{\tau} - d_1 \leq 0 \quad (30)$$

satisfies, for all $t \geq t_0$:

$$U(x_t,t) \leq e^{b(t-t_0)}U(x_{t_0},t_0) + \int_{t_0}^{t} e^{b(t-s)}\delta(|w(s)|)ds, \quad (31)$$

where $b = \theta - \lambda$, $\theta = \frac{q}{2}c^2 + \bar{\tau} + d_1 + 2d_2\bar{\tau}k_2 + 4d_2\bar{\tau}k_5 + \frac{1}{4}d_2\bar{\tau}$, $q = \frac{\bar{\tau}}{2d_2}$ and $\delta(m) = 16d_2\bar{\tau}k_3^2 m^4 + \left(4d_2\bar{\tau}k_4 + \frac{1}{4\bar{\tau}}\right)m^2$.

Proof. Arguing as we did in the proof of Lemma 1, one can prove that, along the trajectories of system (6),

$$\begin{aligned}\dot{V} \leq & -\lambda V + \frac{qc^2}{2}\alpha^2(|x|) + \frac{\bar{\tau}}{2q}\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2 ds \\ & +\bar{\tau}\alpha^2(|x|) + \frac{1}{4\bar{\tau}}|w|^2, \end{aligned} \quad (32)$$

$$\dot{\Gamma}_1 = d_1\alpha^2(|x(t)|) - d_1\alpha^2(|x(t-\tau)|), \quad (33)$$

$$\dot{\Gamma}_2 = d_2\bar{\tau}|\dot{x}(t)|^2 - d_2\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2 ds. \quad (34)$$

Considering (24), (30) and applying (32)-(34) lead to

$$\begin{aligned}\dot{U} \leq & -\lambda V + \left(\frac{qc^2}{2} + \bar{\tau} + d_1\right)\alpha^2(|x|) \\ & +\frac{1}{4\bar{\tau}}|w|^2 - d_1\alpha^2(|x(t-\tau)|) + d_2\bar{\tau}|\dot{x}(t)|^2 \\ & +\frac{\bar{\tau}}{2q}\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2 ds - d_2\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2 ds \\ \leq & -\lambda V + \theta\alpha^2(|x|) + \left(\frac{\bar{\tau}}{2q} - d_2\right)\int_{t-\bar{\tau}}^{t}|\dot{x}(s)|^2 ds \\ & +\delta(|w|) + (4d_2\bar{\tau} - d_1)\,\alpha^2(|x(t-\tau)|) \\ \leq & bU + \delta(|w|). \end{aligned} \quad (35)$$

By integrating this inequality, we obtain (31). This concludes the proof. □

4 Switched system with stable subsystems

In this section, we prove the stability for the switched nonlinear system (1) under the following assumption:

Assumption 3. For any $i \in N$, there exist constants $c_i > 0$, $k_{ji} \geq 0$, $(j = 2, ..., 5)$ such that for all $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_x}$ and $t \geq 0$,
$i)'$ $\left|\frac{\partial V_i}{\partial x}(x,t)g_i(x,t)\right| \leq \alpha(|x|), \left|\frac{\partial u_i}{\partial x}(x,t)\right| \leq c_i$,
$ii)'$ $|f_i(x,t)|^2 \leq k_{2i}\alpha^2(|x|)$, $|g_i(x,t)|^2 \leq k_{3i}\alpha(|x|) + k_{4i}$,
$iii)'$ $(|g_i(x,t)||u_i(z,t)|)^2 \leq k_{5i}\alpha^2(|x|) + \alpha^2(|z|)$,
$iv)'$ $\alpha^2(|x|) \leq V_i(x,t)$,
$v)'$ For all $i \in N$, $\bar{\tau} < \mathfrak{T}_i$ with

$$\mathfrak{T}_i = \frac{\lambda_i}{1 + \sqrt{2}c_i\sqrt{k_{2i} + 2k_{5i} + \frac{17}{8}}}. \tag{36}$$

For given constants $a_i > 0, i \in N$, we define, along the trajectories of (1) the LKF candidate

$$\begin{aligned} U(x_t,t) &= U_\sigma(x_t,t) \\ &= V_\sigma(x,t) + \Gamma_{1\sigma}(x_t) + \Gamma_{2\sigma}(x_t), \end{aligned} \tag{37}$$

where, for any $i \in N$,

$$\Gamma_{1i}(x_t) = d_{1i}\int_{t-\tau}^{t} e^{-a_i(t-s)}\alpha^2(|x(s)|)ds, \tag{38}$$

$$\Gamma_{2i}(x_t) = d_{2i}\int_{t-\bar{\tau}}^{t}\int_{m}^{t} e^{-a_i(t-s)}|\dot{x}(s)|^2 dsdm, \tag{39}$$

and d_{1i} and d_{2i} are positive constants.

From the definition of U, it follows that there are two functions $\gamma_a, \gamma_b \in \mathcal{K}_\infty$ such that

$$\gamma_a(|\phi(0)|) \leq U(\phi,t) \leq \gamma_b(\|\phi\|_W) \tag{40}$$

for all $t \geq 0$, $\phi \in W$.

We are ready to state and prove the following result.

Theorem 1 *Under Assumptions 1 and 3 and the LKF (37), for any $i \in N$, there exist positive constants a_i, d_{1i}, d_{2i} and q_i such that*

$$\theta_i < \lambda_i, \tag{41}$$

$$4d_{2i}\bar{\tau} - d_{1i}e^{-a_i\bar{\tau}} \leq 0, \tag{42}$$

$$\frac{\bar{\tau}}{2q_i} - d_{2i}e^{-a_i\bar{\tau}} \leq 0 \tag{43}$$

with $\theta_i = \frac{q_i}{2}c_i^2 + d_{1i} + \left(1 + 2d_{2i}k_{2i} + 4d_{2i}k_{5i} + \frac{1}{4}d_{2i}\right)\bar{\tau}$. Moreover, the switched system (1) is ISS for any switching signal satisfying

$$\tau_a > \tau_a^* = \frac{\ln\mu}{\underline{a}}, \tag{44}$$

where $\underline{a} = \min_{i\in N}\{a_i\}$ and $\mu > 1$ is such that

$$e^{a_j\bar{\tau}}d_{1i} \leq \mu d_{1j}, e^{a_j\bar{\tau}}d_{2i} \leq \mu d_{2j}, \forall i,j \in N. \tag{45}$$

Proof. Arguing as in the first part of the proof of Lemma 1, one can prove the existence of positive constants a_i, d_{1i}, d_{2i} and q_i such that (41), (42), and (43) are satisfied and $a_i \in (0, \lambda_i - \theta_i]$. Moreover, one can prove easily that the assumptions imply that the system (1) is forward complete. Then, it follows from Lemma 1 that when $t \in [t_k, t_{k+1})$, for any $i \in N$,

$$U(t) \leq e^{-a_i(t-t_k)}U_i(t_k) + \int_{t_k}^{t} e^{-a_i(t-s)}\delta_i(|w(s)|)ds, \tag{46}$$

where $\delta_i(l) = 16d_{2i}\bar{\tau}k_{3i}^2l^4 + \left(4d_{2i}\bar{\tau}k_{4i} + \frac{1}{4\bar{\tau}}\right)l^2$. Consequently

$$U(t) \leq e^{-\underline{a}(t-t_k)}U_i(t_k) + \int_{t_k}^{t} e^{-\underline{a}(t-s)}\bar{\delta}(|w(s)|)ds \tag{47}$$

with $\bar{\delta}(l) = \max_{i\in N}\{\delta_i(l)\}$. On the other hand, the inequalities (5) and (45) give, for all $t \geq 0$,

$$U_i(x_t,t) \leq \mu U_j(x_t,t), \forall i,j \in N. \tag{48}$$

Let $t_1, ..., t_{N_\sigma}$ denote the switching times of σ in $(t_0, t]$. Using (48), we deduce from (47) and (44) that

$$\begin{aligned} U(t) \leq{}& \mu^{N_0}e^{(\frac{\ln\mu}{\tau_a} - \underline{a})(T-t_0)}U(t_0) \\ &+ \mu^{N_0}\frac{\tau_a}{\tau_a\underline{a} - \ln\mu}\bar{\delta}(\|w[t_0,t]\|). \end{aligned} \tag{49}$$

The inequality (40) implies that $|x(t)| \leq \gamma_a^{-1}(U(t))$. This inequality, the fact that for all $s \geq 0$, ℓ, the inequality $\gamma_a^{-1}(s+\ell) \leq \gamma_a^{-1}(2s) + \gamma_a^{-1}(2\ell)$ holds and (49) give

$$\begin{aligned} |x(t)| \leq{}& \gamma_a^{-1}\left(2\mu^{N_0}e^{(\frac{\ln\mu}{\tau_a} - \underline{a})(t-t_0)}U(t_0)\right) \\ &+ \gamma_a^{-1}\left(2\mu^{N_0}\frac{\tau_a}{\tau_a\underline{a} - \ln\mu}\bar{\delta}(\|w[t_0,t]\|)\right). \end{aligned} \tag{50}$$

Then, we can conclude with the functions $\beta(s,\ell) = \gamma_a^{-1}\left(2\mu^{N_0}e^{(\frac{\ln\mu}{\tau_a} - \underline{a})\ell}\gamma_b(s)\right)$ and $\gamma(s) = \gamma_a^{-1}\left(2\frac{\tau_a\mu^{N_0}}{\tau_a\underline{a} - \ln\mu}\bar{\delta}(s)\right)$.

Remark 3. Assumption 3 implies less stringent requirements on the norms of $g_i(x,t)$ and $u_i(z,t)$ than the assumption in Wang et al. (2014). Indeed, in Wang et al. (2014), the terms $|g_i(x,t)u_i(z,t)|^2$ are required to satisfy the common constraint of being smaller than $k_4(\alpha^2(|x|) + \alpha^2(|z|))$ and in the present paper only the inequality in $iii)'$ is imposed. Moreover, in Wang et al.

(2014), the functional

$$d_i \int_{t-\tau}^{t} e^{-\frac{\ln(4\bar{\tau}k_4)}{\bar{\tau}}(s-t)} \alpha^2(|x(s)|)ds$$

is used instead of the functional Γ_{1i}. The presence of the common function $e^{-\frac{\ln(4\bar{\tau}k_4)}{\bar{\tau}}(s-t)}$ in the functionals Γ_{1i} leads to the strict constraint $4\bar{\tau}k_4 < 1$. In the present paper, the new LKF and the assumptions we propose lead to a less conservative delay upper bound, as illustrated by an example in Section 6.

5 Switched system with some unstable subsystems

In the previous section, ISS property has been established under the assumption that all the subsystems with input delay are stable. The consequence of this assumption is that Theorem 1 applies only when $\bar{\tau} \leq \min_{i\in N}\{\mathfrak{T}_i\}$, with $\mathfrak{T}_i$ defined in (36). However, when the input delay is larger than the upper bound $\bar{\tau}$, the stability of some subsystems can still be guaranteed while others may be unstable and then, in some cases, the switched system is stable. This motivates the present section. We consider the system (1) without assuming that all its subsystems are stable. Without loss of generality, we suppose that the subsystems $i(i \in N^- = \{1, 2, ..., r\}, 1 \leq r < n, r \in \mathbb{N})$ are stable while the other subsystems $j(j \in N^+ = \{r+1, r+2, ..., n\})$ may be unstable. For any switching signal and any $0 \leq T_1 \leq T_2$, we let $T^+(T_1, T_2)$ (resp., $T^-(T_1, T_2)$) denote the total activation time of unstable subsystems (resp., stable subsystems) during (T_1, T_2).

For given positive constants $a_i, i \in N^-$, we introduce the following LKF candidate:

$$\begin{aligned} U(x_t, t) &= U_\sigma(x_t, t) \\ &= V_\sigma(x, t) + \Gamma_{1\sigma}(x_t) + \Gamma_{2\sigma}(x_t) \end{aligned} \tag{51}$$

with, for any $i \in N^-$,

$$\Gamma_{1i}(x_t) = d_{1i} \int_{t-\tau}^{t} e^{-a_i(t-s)} \alpha^2(|x(s)|)ds, \tag{52}$$

$$\Gamma_{2i}(x_t) = d_{2i} \int_{t-\bar{\tau}}^{t} \int_{m}^{t} e^{-a_i(t-s)} |\dot{x}(s)|^2 dsdm \tag{53}$$

and, for any $i \in N^+$,

$$\Gamma_{1i}(x_t) = d_{1i} \int_{t-\tau}^{t} \alpha^2(|x(s)|)ds, \tag{54}$$

$$\Gamma_{2i}(x_t) = d_{2i} \int_{t-\bar{\tau}}^{t} \int_{m}^{t} |\dot{x}(s)|^2 dsdm, \tag{55}$$

where for all $i \in N$, d_{1i}, d_{2i} are positive constants. We define for later use the constants:

$$\overline{a} = \max_{i\in N^-}\{a_i\}, \quad \underline{a} = \min_{i\in N^-}\{a_i\}. \tag{56}$$

Before stating and proving the main result of the section, we introduce 2 new assumptions:

Assumption 4. There exists a time sequence $p_0 = t_0 < p_1 < p_2 < ... < p_l ...$ such that, for all $l \in \mathbb{N}$,

$$0 < p_{l+1} - p_l \leq \eta, \tag{57}$$

where η is a positive constant and is a subsequence of switching sequence $t_0 < t_1 < ... < t_j < ...$ associated with $\sigma(t)$. Moreover, there is a constant $\kappa > 0$ such that for all $l \in \mathbb{N}$,

$$\frac{T^-(p_l, p_{l+1})}{T^+(p_l, p_{l+1})} \geq \kappa. \tag{58}$$

Assumption 5. The conditions $i)'$, $ii)'$, $iii)'$, $iv)'$ of Assumption 3 are satisfied and, for all $i \in N^-$, $\bar{\tau} < \mathfrak{T}_i$ with

$$\mathfrak{T}_i = \frac{\lambda_i}{1 + \sqrt{2}c_i\sqrt{k_{2i} + 2k_{5i} + \frac{17}{8}}}. \tag{59}$$

Theorem 2 *Under Assumptions 1, 4 and 5 and LKF (51), there exist positive constants a_i, d_{1i}, d_{2i} and q_i such that for any $i \in N^-$,*

$$\chi_i - \lambda_i < 0, \tag{60}$$

$$4d_{2i}\bar{\tau} - d_{1i}e^{-a_i\bar{\tau}} \leq 0, \tag{61}$$

$$\frac{\bar{\tau}}{2q_i} - d_{2i}e^{-a_i\bar{\tau}} \leq 0 \tag{62}$$

with $\chi_i = \frac{q_i}{2}c_i^2 + d_{1i} + \left(1 + 2d_{2i}k_{2i} + 4d_{2i}k_{5i} + \frac{d_{2i}}{4}\right)\bar{\tau}$. Moreover, the switched system (1) is ISS with respect to w for the switching signal such that

$$\kappa = \frac{\overline{b} + \rho}{\underline{a} - \rho}, \tag{63}$$

where κ is the constant in (58) and ρ is a constant satisfying $\rho \in (0, \underline{a})$, $\overline{b} = \max_{i\in N^+}\{b_i\}$ with $b_i = \chi_i - \lambda_i, q_i = \frac{\bar{\tau}}{2d_{2i}}, i \in N^+$,

$$\frac{\ln \mu}{\tau_a} \in (0, \rho) \tag{64}$$

where $\mu > 1$ satisfies, for any $i, j \in N, i \neq j$,

$$e^{\overline{a}\bar{\tau}}d_{1i} \leq \mu d_{1j}, e^{\overline{a}\bar{\tau}}d_{2i} \leq \mu d_{2j}. \tag{65}$$

Remark 4. For the sake of simplicity, we consider the

case where all the subsystems share a common average dwell time. The results can be extended to the mode-dependent average dwell time framework.

Proof. Arguing as in the first part of the proof of Lemma 1, one can prove the existence of positive constants $a_i \in (0, \lambda_i - \chi_i]$, d_{1i}, d_{2i} and q_i such that (60), (61) and (62) are satisfied. Moreover, one can prove easily that the assumptions imply that the system (1) is forward complete.

It follows from Lemma 1 and Lemma 2 that, when $t \in [t_k, t_{k+1})$, for $i \in N^-$,

$$U(t) \leq e^{-a_i(t-t_k)} U_i(t_k) + \int_{t_k}^{t} e^{-a_i(t-s)} \delta_i(|w(s)|) ds, \tag{66}$$

and for $i \in N^+$,

$$U(t) \leq e^{b_i(t-t_k)} U_i(t_k) + \int_{t_k}^{t} e^{b_i(t-s)} \delta_i(|w(s)|) ds, \tag{67}$$

where $\delta_i(s) = 16 d_{2i} \bar{\tau} k_{3i}^2 s^4 + \left(4 d_{2i} \bar{\tau} k_{4i} + \frac{1}{4\bar{\tau}}\right) s^2$. Let $\bar{\delta}(s) = \max_{i \in N} \{\delta_i(s)\}$. Then, for $i \in N^-$,

$$U(t) \leq e^{-\underline{a}(t-t_k)} U_i(t_k) + \int_{t_k}^{t} e^{-\underline{a}(t-s)} \bar{\delta}(|w(s)|) ds, \tag{68}$$

and for $i \in N^+$,

$$U(t) \leq e^{\bar{b}(t-t_k)} U_i(t_k) + \int_{t_k}^{t} e^{\bar{b}(t-s)} \bar{\delta}(|w(s)|) ds. \tag{69}$$

From (5) and (65), it is easy to show that

$$U_i(x_t, t) \leq \mu U_j(x_t, t), \forall i, j \in N. \tag{70}$$

Suppose that $t \in [t_k, t_{k+1}) \subseteq [p_m, p_{m+1})$, where $k \geq 0$ and $m \geq 0$. From (68-70), we deduce that, along the trajectories of the system (1),

$$U(t) \leq e^{\psi(t_k,t)} U_{\sigma(t_k)}(t_k) + \int_{t_k}^{t} e^{\psi(s,t)} \bar{\delta}(|w|) ds \tag{71}$$

with

$$\psi(s, \ell) = \bar{b} T^+(s, \ell) - \underline{a} T^-(s, \ell). \tag{72}$$

From (70), we deduce that

$$U(t) \leq \mu e^{\psi(t_k,t)} U_{\sigma(t_k^-)}(t_k^-) + \int_{t_k}^{t} e^{\psi(s,t)} \bar{\delta}(|w|) ds. \tag{73}$$

Next, we deduce easily that

$$\begin{aligned}
U(t) \leq & \mu e^{\psi(t_{k-1},t)} U_{\sigma(t_{k-1})}(t_{k-1}) \\
& + \mu \int_{t_{k-1}}^{t_k} e^{\psi(s,t)} \bar{\delta}(|w|) ds \\
& + \int_{t_k}^{t} e^{\psi(s,t)} \bar{\delta}(|w|) ds \\
\leq & \mu^{N_\sigma(t_0,t)} e^{\psi(t_0,t)} U_{\sigma(t_0)}(t_0) \\
& + \int_{t_0}^{t_1} \mu^{N_\sigma(s,t)} e^{\psi(s,t)} \bar{\delta}(|w|) ds \\
& + \dots \\
& + \int_{t_k}^{t} \mu^{N_\sigma(s,t)} e^{\psi(s,t)} \bar{\delta}(|w|) ds \\
\leq & \mu^{N_\sigma(t_0,t)} e^{\psi(t_0,t)} U_{\sigma(t_0)}(t_0) \\
& + \int_{t_0}^{t} \mu^{N_\sigma(s,t)} e^{\psi(s,t)} \bar{\delta}(|w|) ds.
\end{aligned} \tag{74}$$

Now, let us show that, for any $s \in [t_0, t)$,

$$\psi(s, t) \leq -\rho(t-s) + 2\nu \tag{75}$$

with $\nu = \frac{(\underline{a}-\rho)(\bar{b}+\rho)}{\underline{a}+\bar{b}} \eta$.

To prove (75), distinguish between two cases: *i)* $s \in [p_l, p_{l+1})$, $0 \leq l \leq m-1$ and *ii)* $s \in [p_m, t)$.

In the case *i)*, we have:

$$\begin{aligned}
\psi(s,t) = & (\bar{b}+\rho) T^+(s,t) - (\underline{a}-\rho) T^-(s,t) - \rho(t-s) \\
= & -\rho(t-s) + (\bar{b}+\rho)[T^+(s, p_{l+1}) \\
& + T^+(p_{l+1}, p_m) + T^+(p_m, t)] \\
& - (\underline{a}-\rho)[T^-(s, p_{l+1}) + T^-(p_{l+1}, p_m) \\
& + T^-(p_m, t)].
\end{aligned} \tag{76}$$

We deduce that

$$\begin{aligned}
\psi(s,t) \leq & -\rho(t-s) + [-(\underline{a}-\rho) T^-(p_{l+1}, p_m) \\
& + (\bar{b}+\rho) T^+(p_{l+1}, p_m)] \\
& + (\bar{b}+\rho) T^+(s, p_{l+1}) + (\bar{b}+\rho) T^+(p_m, t).
\end{aligned} \tag{77}$$

From Assumption 4 and (63), we deduce that

$$\begin{aligned}
& -(\underline{a}-\rho) T^-(p_{l+1}, p_m) + (\bar{b}+\rho) T^+(p_{l+1}, p_m) \\
= & \sum_{i=l+1}^{m-1} [-(\underline{a}-\rho) T^-(p_i, p_{i+1}) \\
& + (\bar{b}+\rho) T^+(p_i, p_{i+1})] \leq 0,
\end{aligned} \tag{78}$$

and, for all $j \in \mathbb{N}$,

$$T^+(p_j, p_{j+1}) \leq \frac{\underline{a} - \rho}{\bar{b} + \underline{a}}\eta. \tag{79}$$

Combining (77), (78), and (79) gives (75). Similarly, in the case ii) the following inequality

$$\psi(t_0, t) \leq -\rho(t - t_0) + \nu \tag{80}$$

can be derived.

Now, applying (75) and (80) into (74) and considering (64) lead to

$$\begin{aligned}
U(t) \leq & \mu^{N_\sigma(t_0,t)} e^{-\rho(t-t_0)+\nu} U(t_0) \\
& + \int_{t_0}^{t} \mu^{N_\sigma(s,t)} e^{-\rho(t-s)+2\nu} \bar{\delta}(|w|) ds \\
\leq & \mu^{N_0} e^{\nu} e^{(\rho^*-\rho)(t-t_0)} U(t_0) \\
& + \mu^{N_0} e^{2\nu} \int_{t_0}^{t} e^{(\rho^*-\rho)(t-s)} \bar{\delta}(|w|) ds \\
\leq & \mu^{N_0} e^{\nu+(\rho^*-\rho)(t-t_0)} U(t_0) + \frac{\mu^{N_0} e^{2\nu}}{\rho - \rho^*} \bar{\delta}(\|w[t_0, t]\|)
\end{aligned}$$

with $\rho^* = \frac{\ln \mu}{\tau_a}$.

Finally arguing as in the end of the proof of Theorem 1, we can conclude. □

6 Numerical example

In this section, we present an example to illustrate the effectiveness of the proposed method. We consider the following switched nonlinear system:

$$\dot{x}(t) = f_\sigma(x, t) + g_\sigma(x, t)[u_\sigma(x(t - \tau), t) + w(t)]. \tag{81}$$

with $f_1 = \begin{bmatrix} -x_2 \sin x_1 \\ \frac{1}{2} x_1 \sin x_1 - x_2 \end{bmatrix}$, $f_2 = \begin{bmatrix} -\frac{1}{2} x_1 \\ -3x_1 - \frac{1}{2} x_2 \end{bmatrix}$, $g_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$, $g_2 = \begin{bmatrix} \frac{1}{6} \\ 0 \end{bmatrix}$, $u_1 = -2x_1$ and $u_2 = 6x_2$.

Choosing the Lyapunov functions $V_1(x) = x_1^2 + 2x_2^2$ and $V_2(x) = 3x_1^2 + x_2^2$, we obtain $\mu = 3$, $\lambda_1 = 2$ and $\lambda_2 = 1$ and we deduce from the main result of Liberzon (2003) (see Section 2), the system (81) with $\tau = 0$ and $w = 0$ is globally exponentially stable when the switching signal is such that $\tau_a > 1.0986$.

Now, let $\alpha(m) = m$ and observe that we have

$$\begin{aligned}
& |\tfrac{\partial V_1}{\partial x} g_1| \leq |x|, |\tfrac{\partial u_1}{\partial x}| = 2, |\tfrac{\partial u_2}{\partial x}| = 6, |f_1|^2 \leq \tfrac{9}{4}|x|^2, \\
& |\tfrac{\partial V_2}{\partial x} g_2| \leq |x|, |g_1|^2 = \tfrac{1}{4}, |g_2|^2 = \tfrac{1}{36}, |f_2|^2 \leq \tfrac{19}{2}|x|^2, \\
& |g_1(-2z_1)|^2 \leq |z|^2, |g_2(6z_2)|^2 \leq |z|^2,
\end{aligned} \tag{82}$$

which implies $k_{21} = \frac{9}{4}$, $k_{22} = \frac{19}{2}$, $k_{31} = k_{32} = 0$, $k_{41} = \frac{1}{4}, k_{42} = \frac{1}{36}$, $c_1 = 2$, $c_2 = 6$, $k_{51} = k_{52} = 0$.

Giving $d_{11} = \frac{1}{5}$, $d_{12} = \frac{1}{12}$, $d_{21} = \frac{1}{4}$, $d_{22} = \frac{1}{6}$, $q_1 = 0.1$, $q_2 = 0.04$, $a_1 = 1$, $a_2 = 0.3221$ and solving the conditions in Theorem 1, we obtain $\bar{\tau} = 0.0132$ and the minimum average dwell time $\tau_a^* = 3.4108$. Letting $N_{01} = N_{02} = 0$, the state response of the resulting system with $w(t) = 0$ and $w(t) = 0.3 \sin t$ are plotted in Fig. 1. and Fig. 2, respectively.

To compare the proposed method with the existing ones, we apply Theorem 1 in Wang et al. (2014) to the system (81). Let us choose the same functions $V_i, i = 1, 2$ and α. Then, one can check that all the conditions in Wang et al. (2014) are satisfied for $\bar{\tau} = 0.0114$, which is smaller than that given by the present paper. Moreover, even for the small input delay $\tau = 0.01$, the minimum average dwell time is $\tau_a^* = 10.1068$.

However, when $0.0114 < \tau \leq \bar{\tau} = 0.0476$, from Lemma 1, the subsystem 1 is ISS, but the stability of the subsystem 2 cannot be derived. Consequently, we resort to Theorem 2 to give the stability of the switched system. If $\tau = 0.02$, by computing, we obtain $a = 1$, $b = 0.2475$. Suppose $p_{i+1} - p_i = \eta$ for all $i \in \mathbb{N}$ and choose $\eta = 100$s, $\rho = 0.5$, $\rho^* = 0.25$, by conditions (58) and (63), we have $\frac{T^-}{T^+} \geq 1.4950$ and $\tau_a \geq \frac{\ln \mu}{0.25} = 4.3944$. In order to satisfy both the switching law and the average dwell time condition, we choose to activate the subsystem 1 and subsystem 2 with time periods 8s and 5s. Letting $N_{01} = N_{02} = 0$, the state response of the resulting system with $w = 0$ and $w = 0.3 \sin t$ are plotted in Fig. 3. and Fig. 4, respectively.

Fig. 1. The state response of the system (81) with $\tau = 0.01$ and $w = 0$.

Fig. 2. The state response of the system (81) with $\tau = 0.01$ and $w = 0.3 \sin t$.

Fig. 3. The state response of the system (81) with $\tau = 0.02$ and $w = 0$.

Fig. 4. The state response of the system (81) with $\tau = 0.02$ and $w = 0.3 \sin t$.

7 Conclusion

We have studied the stability properties of switched nonlinear time-varying systems with input pointwise delay and external disturbance by means of LKFs. First, we have proposed sufficient conditions ensuring the stability of the switched nonlinear system. In particular, these conditions include an upper bound on the delay. Then, we studied the case where the input delay is larger than the upper bound given in the first step. For this case, which includes cases where some subsystems are unstable, we have determined conditions ensuring the input-

to-state stability of the system with respect to an additive disturbance.

References

Caliskan, S. Y., Ozbay, H., Niculescu, S. I. (2013). Dwell-time computation for stability of switched systems with time delays. *IET Control Theory and Applications*, 7(10), 1422-1428.

Colaneri, P., Geromel J.C., Astolfi, A. (2008). Stabilization of continuous-time switched nonlinear systems. *Systems & Control Letters*, 57, (1), pp. 95-103.

Fridman, E. (2014). *Introduction to Time-Delay Systems*. Berlin: Birkhauser.

Fridman, E., Dambrine, M., Yeganefar, N. (2008). On input-to-state stability of systems with time-delay: A matrix inequalities approach. *Automatica*, 44(9), 2364-2369.

Gu, K., Kharitonov, V., Chen, J. (2003). *Stability of time-delay systems*. Berlin: Birkhauser.

Liberzon, D. (2003). *Switching in systems and control*. Berlin: Birkhauser.

Liberzon, D. (2014). Finite data-rate feedback stabilization of switched and hybrid linear systems. *Automatica*, 50(2), 409-420.

Liu, J., Liu, X. Z., Xie, W. C. (2011). Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, 47(5), 899-908.

Ma, D., Zhao, J. (2015). Stabilization of networked switched linear systems: An asynchronous switching delay system approach. *Systems & Control Letters*, 77, 46-54.

Malisoff, M., Mazenc, F., Zhang, F. M. (2012). Stability and robustness analysis for curve tracking control using input-to-state stability. *IEEE Transactions on Automatic Control*, 57(5), 1320-1326.

Mazenc, F., Malisoff, M., Dinh, T. N. (2013). Robustness of nonlinear systems with respect to delay and sampling of the controls. *Automatica*, 49(6), 1925-1931.

Mazenc, F., Malisoff, M., Lin, Z. L. (2008). Further results on input-to-state stability for nonlinear systems with delayed feedbacks. *Automatica*, 44(9), 2415-2421.

Mazenc, F., Niculescu, S. I., Krstic, M. (2012). Lyapunov-Krasovskii functionals and application to input delay compensation for linear time-invariant systems. *Automatica*, 48(7), 1317-1323.

Moon, F. C., Kalmar-Nagy, T. (2001). Nonlinear models for complex dynamics in cutting materials. *Philosophical transactions of the Royal Society of London*, 359(1781), 695-711.

M*ü*ller, M. A., Liberzon, D. (2012). Input/output-to-state stability and state-norm estimators for switched nonlinear systems. *Automatica*, 48(9), 2029-2039.

Pepe, P., and Jiang, Z. P. (2006). A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12), 1006–1014.

Rappaport, T. S. (1996). *Wireless communications: principles and practice*. New Jersey: prentice hall PTR.

Sankaranarayanan, V., Mahindrakar, A. D. (2009). Switched control of a nonholonomic mobile robot. *Communications in Nonlinear Science and Numerical Simulation*, 14(5), 2319-2327.

Sun, X. M., Wang, W. (2012). Integral input-to-state stability for hybrid delayed systems with unstable continuous dynamics. *Automatica*, 48(9), 2359-2364.

Sun, Z. D. (2008). Robust switching of discrete-time switched linear systems. *Automatica*, 48(1), 239-242.

Vu, L., Kristi, M. A. (2010). Stability of time-delay feedback switched linear systems. *IEEE Transactions on Automatic Control*, 55(10), 2385-2389.

Wang, D., Shi, P., Wang, J. L., Wang, W. (2012). Delay-dependent exponential H_∞ filtering for disxrete-time switched delay systems. *International Journal of Robust and Nonlinear Control*, 22(13), 1522-1536.

Wang, Y. E., Sun, X. M., Wang, Z., Zhao, J. (2014). Construction of Lyapunov-Krasovskii functionals for switched nonlinear systems with input delay. *Automatica*, 50(4), 1249-1253.

Wang, Y. E., Wang, R., Zhao, J. (2013). Input-to-State stability of non-linear impulsive and switched delay systems. *IET Control Theory and Applications*, 7(8), 1179-1185.

Yang, H., Cocquempot, V., Jiang, B. (2009). On stabilization of switched nonlinear systems with unstable modes. *Systems & Control Letters*, 58(10-11), 703-708.

Zhai, G. S., Hu, B., Yasuda, K., Michel, A. N. (2001). Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach. *International Journal of Systems Science*, 32(8), 1055-1061.

Zhao, J., Hill, D. J. (2008). On stability, L_2-gain and H_∞ control for switched systems. *Automatica*, 44(5), 1220-1232.

Zhao, J., David, J. H., Liu, T. (2009). Synchronization of complex dynamical networks with switching topology: A switched system point of view. *Automatica*, 45(11), 2502-2511.

Zhao, X. D., Zhang, L. X., Shi, P., Liu, M. (2012). Stability and stabilization of switched linear systems with mode-dependent average dwell time. *IEEE Transactions on Automatic Control*, 57(7), 1809-1815.