# Sparse plus Low rank Network Identification: A Nonparametric Approach * 

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#### Abstract

Modeling and identification of high-dimensional stochastic processes is ubiquitous in many fields. In particular, there is a growing interest in modeling stochastic processes with simple and interpretable structures. In many applications, such as econometrics and biomedical sciences, it seems natural to describe each component of that stochastic process in terms of few factor variables, which are not accessible for observation, and possibly of few other components of the stochastic process. These relations can be encoded in graphical way via a structured dynamic network, referred to as "sparse plus low-rank ( $\mathrm{S}+\mathrm{L}$ ) network" hereafter. The problem of finding the $\mathrm{S}+\mathrm{L}$ network as well as the dynamic model can be posed as a system identification problem. In this paper, we introduce two new nonparametric methods to identify dynamic models for stochastic processes described by a $\mathrm{S}+\mathrm{L}$ network. These methods take inspiration from regularized estimators based on recently introduced kernels (e.g. "stable spline", "tuned-correlated" etc.). Numerical examples show the benefit to introduce the $\mathrm{S}+\mathrm{L}$ structure in the identification procedure.


Key words: Linear system identification, Sparsity and low-rank inducing priors, Kernel-based methods, Gaussian processes.

## 1 Introduction

In many applications, high-dimensional data are measured to describe the underlying phenomena. These data are typically measured over time and can thus be modeled as a stochastic process whose components are referred to as manifest variables, i.e. accessible for observation. Very often, in high dimensional time series modeling it is necessary to control the model complexity to be able to obtain sensible results from a finite set of measured data. In addition, models should be interpretable in the sense of providing an insight into the data generation mechanism.

One possible way to limit the complexity is to postulate, often very reasonably, that these observations share some common behaviour (comovements) which, in turn, can be described by a small set of (unmeasurable) variables, called factor variables.

These ideas have been exploited in so called dynamic factor models, see e.g. Heij et al. (1997), Deistler \& Zinner

[^0](2007), Forni et al. (2004), Zorzi \& Sepulchre (2015b) and references therein.

Another possible avenue to control complexity is to build sparse dynamic models, e.g. exploring Granger's causality structure (Granger, 1969) as done in Chiuso \& Pillonetto (2012).

In this paper we shall extend and merge these ideas, building so called sparse plus low-rank ( $S+L$ ) models, Zorzi \& Chiuso (2015); our aim is to model processes $y$ which Granger's causality structure is not necessarily sparse, but it may become so, in an appropriate manner to be defined later on, after some latent variables (called factors in analogy with factor models) are added.

In this way, the relations among manifest variables and factor variables will be described through a two-layer graph (or network) where the (few) nodes in the top (and hidden) layer denote the factor variables, whereas the ones in the bottom (and visible) layer denote the manifest variables. The direct relation, in a sense to be precisely defined later on, between two nodes is encoded by the presence of a connecting directed edge. If the relations among manifest variables are mostly encoded through the factor variables and the number of the latter is small (as compared to the number of manifest variables), this graph will be referred to as sparse plus low-
rank $(S+L)$ network; in fact, as we shall see, its structure translates into a $S+L$ structure for the dynamic model of the manifest process. In particular, the rank of the lowrank component will coincide with the number of factor variables whereas the sparse component depends on the number of edges among the nodes representing the manifest variables. This modeling framework seems natural in many applications, such as biomedical sciences (Liegegois et al. 2015), econometrics (Zorzi \& Sepulchre, 2015a; Chandrasekaran et al. 2010 Choi et al. 2010), and so on. Finally, this S+L structure have been considered also to perform robust principal component (PCA) analysis Candès et al. 2011).

In the first part of this paper we propose a new $S+L$ network defined as a directed graph wherein edges encode conditional Granger causality dependences (Granger, 1969, Ding et al. 2006) among variables. In the special case where there is no factor variable our model coincides with the sparse model considered in Chiuso \& Pillonetto (2012). Moreover, the $\mathrm{S}+\mathrm{L}$ model we propose is strictly connected to so called quasi-static factor models (Heij et al., 1997, Deistler \& Zinner, 2007; Deistler et al., 2015). Therefore, our S+L model can be understood as blend of those two models.

Within this framework, we shall formulate an identification problem to select, using a finite set of measured data, the most appropriate $\mathrm{S}+\mathrm{L}$ model (and thus the corresponding network).

A consolidated paradigm in system identification is the so-called prediction error method (PEM), see Ljung (1999); Söderström \& Stoica (1989). In the traditional setting, candidate models are described in fixed parametric model structures, e.g. ARMAX. However, there are two main difficulties in this setting. First a model selection problem (i.e. order selection), usually performed by AIC and BIC criteria (Akaike, 1974, Schwarz, 1978), needs to be solved. Second, the parameterization of the predictor is non-linear, so that minimizing the squared prediction error leads to a non-convex optimization problem. Regularization has been recently introduced in the PEM framework, see Pillonetto et al. (2011); Pillonetto \& De Nicolao (2010); Chen et al. (2012); Chiuso \& Pillonetto (2012); Pillonetto et al. (2014); Chiuso (2016), as an alternative approach to control complexity of the estimated models. With this latter setting we search the candidate model, described via the predictor impulse responses, in an infinite dimensional nonparametric model class; the inverse (ill-posed) problem of determining a specific model using a finite set of measured data can be made into a well posed one using a penalty term, whose duty is to favor models with specific features. In the Bayesian view, this is equivalent to the introduction of an a priori probability (prior) on the unknown model. In the nonparametric Gaussian regression approach proposed in Pillonetto et al. (2011), this prior distribution is completely characterized by
the covariance function, known also as kernel in the machine learning literature. The kernel encodes the $a$ priori knowledge about the predictor impulse responses. In our case, the a priori knowledge is that the predictor impulse responses must be Bounded Input Bounded Output (BIBO) stable and respect the $\mathrm{S}+\mathrm{L}$ structure. Starting from these a priori assumptions, we derive the corresponding kernel by using the maximum entropy principle. In particular, we consider two possible alternative formulations to endow this a priori properties in the kernel, and thus two different identification algorithms.

These kernels are characterized by the decay rate of the predictor impulse responses, by the number of conditional Granger causality relations among the manifest variables and by the number of factor variables. This ensemble of features is not known and is characterized by the so called hyperparameters vector. The latter is usually estimated by minimizing the negative log-marginal likelihood of the measured data (Rasmussen \& Williams, 2006). In our case, the challenge is to perform the joint estimation of the hyperparameters tuning the sparse part and the low-rank one. Indeed, it should be observed that the $\mathrm{S}+\mathrm{L}$ decomposition of a given model might not be unique. On the other hand, once the hyperparameters vector is fixed the uniqueness of the estimated model will be guaranteed through regularization. To estimate the hyperparameters minimizing the negative log-marginal likelihood, we propose an algorithm imposing an "hyper regularizer" on the low-rank hyperparameter to partially handle the non-uniqueness of the $\mathrm{S}+\mathrm{L}$ decomposition.

Numerical experiments involving both $\mathrm{S}+\mathrm{L}$ and generic models show the effectiveness of our identification procedure both in terms of complexity, predictive capability and impulse response fit.

The outline of the paper follows. In Section 2, we introduce the $\mathrm{S}+\mathrm{L}$ models. In Section 3, we introduce the $\mathrm{S}+\mathrm{L}$ identification problem. In Section 4, we derive the maximum entropy kernels inducing BIBO stability, sparsity and low-rank. Section 5 deals with the estimation of the hyperparameters vector. In Section 6, we provide some numerical examples to show the effectiveness of our method. Finally, conclusions are drawn in Section 7. In order to streamline the presentation all proofs are deferred to the Appendix.

## Notation

Throughout the paper, we will use the following notation. $\mathbb{N}$ is the set of natural numbers. Given a finite set $\mathcal{I}$, $|\mathcal{I}|$ denotes its cardinality. $\mathbb{E}[\cdot]$ denotes the expectation, while $\mathbb{E}[\cdot \mid \cdot]$ denotes the conditional mean. Given three (possibly infinite dimensional) random vectors $a, b$ and c we say that a is conditionally independent of $b$ given $c$ if

$$
\mathbb{E}[\mathrm{a} \mid \mathrm{b}, \mathrm{c}]=\mathbb{E}[\mathrm{a} \mid \mathrm{c}]
$$

Given $G \in \mathbb{R}^{n \times p},[G]_{i j}$ denotes the entry of $G$ in position $(i, j) . \mathcal{M}^{p}$ denotes the vectors space of symmetric matrices of dimension $p . \mathcal{M}_{+}^{p}$ is the cone of symmetric positive definite matrices of dimension $p$, and $\overline{\mathcal{M}}_{+}^{p}$ denotes its closure. $\ell_{2}(\mathbb{N})$ denotes the space of $\mathbb{R}$-valued infinite length sequences, which we think as infinite dimensional column vectors $g:=\left[\begin{array}{llll}g_{1} & g_{2} & \ldots & g_{j}\end{array}\right]^{\top}, g_{k} \in \mathbb{R}, k \in \mathbb{N}$, such that $\|g\|_{2}:=\sqrt{\sum_{k=1}^{\infty}\left|g_{k}\right|^{2}}<\infty . \ell_{2}^{p \times n}(\mathbb{N})$ is the space of matrices of sequences in $\ell_{2}(\mathbb{N})$

$$
\Phi=\left[\begin{array}{ccc}
\left(\phi^{[11]}\right)^{\top} & \ldots & \left(\phi^{[1 n]}\right)^{\top}  \tag{1}\\
\vdots & \ddots & \vdots \\
\left(\phi^{[p 1]}\right)^{\top} & \ldots & \left(\phi^{[p n]}\right)^{\top}
\end{array}\right]
$$

where $\phi^{[i j]} \in \ell_{2}(\mathbb{N}), i=1 \ldots p$ and $j=1 \ldots n . \ell_{1}(\mathbb{N})$ denotes the space of $\mathbb{R}$-valued infinite length sequences $g$ such that $\|g\|_{1}:=\sum_{k=1}^{\infty}\left|g_{k}\right|<\infty . \ell_{1}^{p \times n}(\mathbb{N})$ is defined in similar way. $\mathcal{S}_{2}(\mathbb{N})$ denotes the space of symmetric infinite dimensional matrices $K$ such that $\|K\|_{2}:=$ $\sqrt{\sum_{i, j=1}^{\infty}\left|[K]_{i j}\right|^{2}}<\infty . \mathcal{S}_{2}^{p}(\mathbb{N})$ is the space of symmetric infinite dimensional matrices

$$
K=\left[\begin{array}{cccc}
K^{[11]} & K^{[12]} & \ldots & K^{[1 p]}  \tag{2}\\
K^{[12]} & K^{[22]} & & K^{[2 p]} \\
\vdots & & \ddots & \vdots \\
K^{[1 p]} & K^{[2 p]} & \ldots & K^{[p p]}
\end{array}\right]
$$

where $K^{[i j]} \in \mathcal{S}_{2}(\mathbb{N}), i, j=1 \ldots p$. Given $\Phi \in \ell_{2}^{p \times n}(\mathbb{N})$, $\Psi \in \ell_{2}^{m \times n}(\mathbb{N})$ and $K \in \mathcal{S}_{2}^{n}(\mathbb{N})$, the products $\Phi \Psi^{\top}$ and $\Phi K \Psi^{\top}$ are understood as $p \times m$ matrices whose entries are limits of infinite sequences (Jorgesen et al., 2011). Given $g \in \ell_{2}^{n \times 1}(\mathbb{N})$ and $K \in \mathcal{S}_{2}^{n}(\mathbb{N}),\|g\|_{K}^{2}:=g^{\top} K^{-1} g$. The definition is similar in the case that $g$ and $K$ have finite dimension. With some abuse of notation the symbol $z$ will denote both the complex variable as well as the shift operator $z^{-1} y(t):=y(t-1)$. Given a transfer matrix $L(z), z \in \mathbb{C}$, of dimension $p \times p$, with some abuse of terminology, we say that $L(z)$ has rank $n$, with $n \leq p$, if it admits the decomposition $L(z)=F H(z)$ where $F$ is a $p \times n$ matrix and $H(z)$ is a $n \times p$ transfer matrix. Given a stochastic process $y=\{y(t)\}_{t \in \mathbb{Z}}$, its $i$ th component is denoted by $y_{i}=\left\{y_{i}(t)\right\}_{t \in \mathbb{Z}}$. With some abuse of notation, $y(t)$ will both denote a random vector and its sample value. From now on the time $t$ will denote present and we shall talk about past and future with respect to time $t$. With this convention in mind,

$$
\begin{equation*}
\mathrm{y}^{-}=\left[y(t-1)^{\top} y(t-2)^{\top} \ldots\right]^{\top} \tag{3}
\end{equation*}
$$

denotes the (infinite length) past data vector of $y$ at time $t$. In similar way, $\mathrm{y}_{i}^{-}$denotes the past data vector of $y_{i}$ at time $t$.

## 2 Sparse plus Low rank Models

Consider a zero-mean stationary and Gaussian stochastic process $y$ taking values in $\mathbb{R}^{p}$. Let $y$ be manifest, i.e. it can be measured, and described by the innovation model

$$
\begin{equation*}
y(t)=G(z) y(t)+e(t) \tag{4}
\end{equation*}
$$

where $G(z)=\sum_{k=1}^{\infty} G_{k} z^{-k}$ and $(I-G(z))^{-1}$ are BIBO stable transfer matrices of dimension $p \times p$ and $e$ is a zero mean white Gaussian noise (WGN) with covariance matrix $\Sigma$. The minimum variance one-step ahead predictor of $y(t)$ based on the past data $\mathrm{y}^{-}$, denoted by $\hat{y}(t \mid t-1)$ is given by

$$
\begin{equation*}
\hat{y}(t \mid t-1)=G(z) y(t) \tag{5}
\end{equation*}
$$

so that $e$ is the one-step-ahead prediction error

$$
e(t)=y(t)-G(z) y(t)=y(t)-\hat{y}(t \mid t-1)
$$

Often $G(z)$ is approximated by a "simple structure" in order to simplify the analysis of the underlying system. One possible simple structure is $G(z)$ sparse, i.e. many of its entries are null transfer functions (Chiuso \& Pillonetto, 2012). Sparsity of the predictor transfer function $G(z)$ encodes the Granger's causality (Granger, 1969) of $y$ and can be graphically represented with a Bayesian network (see below). However, in many interesting applications (Zorzi \& Sepulchre, 2015a; Kolaczyk, 2009, Werhli et al., 2006; Forni et al. 2004) the components in $y$ are strongly related through common factors, not accessible for observation. In such situations approximating $G(z)$ with a sparse matrix will likely yield to poor results.

To overcome this limitation we consider a more general, but still "simple", structure by introducing an $n$ dimensional process $x(n \ll p)$, called factor. We assume that $x$ is zero-mean, jointly stationary and Gaussian with $y$. The underlying idea is that the (few) components of the factor $x$ are able to capture the common movement of $y$ so that, conditionally on $x$, the predictor of $y$ can now be well approximated by a sparse model. This can be formalized assuming that $y$ can be modeled as follows:

$$
\begin{equation*}
y(t)=F x(t)+S(z) y(t)+w(t) \tag{6}
\end{equation*}
$$

where $F \in \mathbb{R}^{n \times p}$ is the so-called factor loading matrix and, in analogy with dynamic factor models, $F x(t)$ is called the "common component" or "latent variable" of $y$ (Deistler \& Zinner, 2007); $S(z)=\sum_{k=1}^{\infty} S_{k} z^{-k}$ is a BIBO stable and sparse transfer matrix; $w$ is a white noise process uncorrelated with the past histories of $y$ and $x$. As a result, the one-step ahead predictor of $y$ is given by

$$
\begin{equation*}
\hat{y}(t \mid t-1)=F \hat{x}(t \mid t-1)+S(z) y(t) \tag{7}
\end{equation*}
$$

where $\hat{x}(t \mid t-1)$ is the minimum variance estimator of $x(t)$ based on $\mathrm{y}^{-}$, that is

$$
\begin{equation*}
\hat{x}(t \mid t-1):=\mathbb{E}\left[x(t) \mid \mathrm{y}^{-}\right]=H(z) y(t) \tag{8}
\end{equation*}
$$

where $H(z)=\sum_{k=1}^{\infty} H_{k} z^{-k}$ is a BIBO stable transfer matrix of dimension $n \times p$. We conclude that $\hat{y}(t \mid t-1)$ takes the form in (5) with $G(z)=S(z)+L(z)$ where $L(z):=F H(z)$. Hence, $G(z)$ has a sparse plus low-rank $(\mathrm{S}+\mathrm{L})$ structure.

It is possible to describe the structure of model (6) using a Bayesian network (Lauritzen, 1996) having two layers. The nodes in the top layer correspond to the scalar processes $x_{i} i=1 \ldots n$, i.e. the components of $x$, whereas the ones in the bottom layer correspond to the scalar processes $y_{i} i=1 \ldots p$, i.e. the components of $y$, see Figure 1. Then, the connections among the nodes obey the


Fig. 1. Example of a network describing a $\mathrm{S}+\mathrm{L}$ model with $p=6$ manifest variables and $n=1$ factor variable.
following rules:

- there is a directed link from node $y_{j}$ to node $y_{i}$ if

$$
\mathbb{E}\left[y_{i}(t) \mid x(t), \mathrm{y}^{-}\right] \neq \mathbb{E}\left[y_{i}(t) \mid x(t), \mathrm{y}_{k}^{-}, k=1, . ., p, k \neq j\right]
$$

i.e. if $\mathrm{y}_{j}^{-}$is needed to predict $y_{i}(t)$ given $\mathrm{y}_{k}^{-}$with $k \neq j$ and $x(t)$. In this case, we shall say $y_{j}$ conditionally Granger causes $y_{i}$

- there is a directed link from node $y_{j}$ to node $x_{i}$ if

$$
\mathbb{E}\left[x_{i}(t) \mid \mathrm{y}^{-}\right] \neq \mathbb{E}\left[x_{i}(t) \mid \mathrm{y}_{k}^{-}, k=1, . ., p, k \neq j\right]
$$

i.e. if $\mathrm{y}_{j}^{-}$is needed to predict $x_{i}(t)$ given $\mathrm{y}_{k}^{-}$with $k \neq j$. In this case, we shall say $y_{j}$ conditionally Granger causes $x_{i}$

- there is a direct link from node $x_{j}$ to node $y_{i}$ if

$$
\mathbb{E}\left[y_{i}(t) \mid \mathrm{y}^{-}, x(t)\right] \neq \mathbb{E}\left[y_{i}(t) \mid \mathrm{y}^{-}, x_{k}(t), k=1, . ., n, k \neq j\right]
$$

i.e. if $x_{j}(t)$ is needed to predict $y_{i}(t)$ given $\mathrm{y}^{-}$and $x_{k}(t)$ with $k \neq j$. In this case, we shall say $x_{j}$ conditionally Granger causes $y_{i}$.

Therefore, the $\mathrm{S}+\mathrm{L}$ model in (6) represents a network wherein manifest variables Granger cause each other mostly through few factor variables. In Figure 1 we provide an example of a network describing a $\mathrm{S}+\mathrm{L}$ model with $p=6$ manifest variables and $n=1$ factor variable. In particular, $y_{5}$ is conditionally Granger caused by $x$ and $y_{1}, y_{i}$ with $i \neq 5$ is conditionally Granger caused only by $x$, and $x$ is conditionally Granger caused by $y$. Therefore, the corresponding $\mathrm{S}+\mathrm{L}$ model in (6) has $S(z)$ with only one nonnull entry in position $(5,1)$ and $L(z)$ has rank equal to one.

The decomposition of a transfer matrix $G(z)$ into sparse plus low-rank, i.e. $G(z)=S(z)+L(z)$ may not be unique. As noticed in (Chandrasekaran et al., 2010), this degeneracy may occur when $L(z)$ is sparse, i.e. the factor variables are not sufficiently "diffuse" across the manifest variables, or the degree of sparsity of $S(z)$ is low, i.e. there are manifest variables conditionally Granger caused by too many manifest variables. On the other hand, it is possible to derive conditions under which such a decomposition is locally identifiable. We do not tackle this nonidentifiability issue in this paper, although it is important. Indeed, our aim is to find one $S+L$ decomposition (see the next Section) which is not necessarily unique.

### 2.1 Connections: feedback models and quasi-static factor models

It is also interesting to consider a "feedback" representation of the joint process $v:=\left[y^{\top} x^{\top}\right]^{\top}$ and see how this connects with $\mathrm{S}+\mathrm{L}$ model (6) and other models already considered in the literature, such as quasi-static factor models (Deistler \& Zinner, 2007).

Recall that (Gevers \& Anderson, 1982, Caines \& Chan, 1976), given the jointly stationary process $v$, there is an essentially unique feedback model of the form

$$
\begin{align*}
& y(t)=F(z) x(t)+d(t)  \tag{9}\\
& x(t)=C(z) y(t)+r(t)
\end{align*}
$$

where the driving noises $r(t)$ and $d(t)$ are uncorrelated and jointly stationary processes (possibly non-white) and the feedback interconnection (9) is internally stable, namely

$$
\left[\begin{array}{cc}
(I-F(z) C(z))^{-1} & (I-F(z) C(z))^{-1} F(z) \\
(I-C(z) F(z))^{-1} C(z) & (I-C(z) F(z))^{-1}
\end{array}\right]
$$

is analytic inside the closed unit disc and $C(\infty)=0$. In particular, given an analytic and minimum phase spec-
tral factor

$$
W(z)=\left[\begin{array}{ll}
W_{11}(z) & W_{12}(z) \\
W_{21}(z) & W_{22}(z)
\end{array}\right]
$$

of the joint spectrum $\Phi_{v}(z)=W(z) Q W^{\top}\left(z^{-1}\right)$ of $v$, lower block triangular and normalized at infinity (i.e. $\left.W_{11}(\infty)=I, W_{22}(\infty)=I, W_{21}(\infty)=0\right)$, the transfer function $F(z)$ is given by (see e.g. eq. (3.4) in Gevers \& Anderson (1982)) $F(z)=W_{12}(z) W_{22}^{-1}(z)$.

The predictor for $y(t)$ given the past $\mathrm{y}^{-}$can be written, using (9), as

$$
\hat{y}(t \mid t-1)=F(z) \hat{x}(t \mid t-1)+\underbrace{\hat{\mathbb{E}}\left[d(t) \mid \mathrm{y}^{-}\right]}_{\hat{d}(t \mid t-1)} .
$$

Hence, model (6) is recaptured when
(a) $F(z):=W_{12}(z) W_{22}^{-1}(z)=F$ (i.e. $F(z)$ is constant) (b) $\hat{d}(t \mid t-1):=S(z) y(t)$ is sparse.

Note that condition (a) above is equivalent to $W_{12}(z)=$ $F W_{22}(z)$, while complete freedom is left to the other entries of $W(z)$.

To summarize, our model (6) is equivalent to assuming that there is a process $x$, jointly stationary with $y$, so that the pair $(y, x)$ admits the internally stable feedback representation:

$$
\begin{align*}
& y(t)=F x(t)+d(t) \\
& x(t)=C(z) y(t)+r(t) \tag{10}
\end{align*}
$$

with uncorrelated (and possibly non-white) noises $r(t)$ and $d(t)$.

It is worth noting that when $C(z)=0$, model (10) can be understood as a quasi-static factor model (Deistler \& Zinner, 2007) where $F$ is the factor loading matrix, $x$ is the $n$-dimensional factor and $d$ is colored noise uncorrelated with $x$, such that its minimum variance onestep ahead predictor based on $\mathrm{y}^{-}$is sparse: $\hat{d}(t \mid t-1)=$ $S(z) y(t)$.

In general, however, the feedback model (10) is a generalization of quasi-static feedback models as $d(t)$ might be correlated with $x(t)$ (which happens when $C(z) \neq 0$ in $(10)$ ). The predictor model (7) can be obtained from
(10) simply computing the predictor $\hat{x}(t \mid t-1)$, see (8):

$$
\begin{aligned}
\hat{x}(t \mid t-1) & =C(z) y(t)+\hat{r}(t \mid t-1) \\
& =C(z) y(t)+G_{r}(z) y(t) \\
& =\underbrace{H(z)}_{C(z)+G_{r}(z)} y(t)
\end{aligned}
$$

where $G_{r}(z) y(t)=\hat{r}(t \mid t-1)$.

## 3 Problem Formulation

Assume measured data $\{y(t)\}_{t=1, . ., N}$ are available from the manifest process $y$ generated by (6). The factor process $x$ cannot be measured nor its dimension $n$ is known. In this section, we address the problem of estimating $S(z)$ and $L(z)$ from $\{y(t)\}_{t=1, \ldots, N}$.

The transfer matrix $S(z)$ is parameterized in terms of its impulse response coefficients $S_{k}$. In particular, defining $s^{[i j]} \in \ell_{1}(\mathbb{N})$ to be the impulse response from input $j$ to output $i$ of the transfer matrix $S(z)$, we have:

$$
s^{[i j]}:=\left[\begin{array}{lll}
{\left[S_{1}\right]_{i j}} & {\left[S_{2}\right]_{i j} \ldots} & \left.\left.\ldots S_{k}\right]_{i j} \ldots\right]^{\top} . \tag{11}
\end{array}\right.
$$

The coefficient vector $\theta_{s}^{\top} \in \ell_{1}^{1 \times p^{2}}(\mathbb{N})$ is defined as follows:

$$
\begin{align*}
\theta_{s}^{\top}=[ & \left(s^{[11]}\right)^{\top} \ldots\left(s^{[1 p]}\right)^{\top} \mid \ldots \\
& \left.\ldots \mid\left(s^{[p 1]}\right)^{\top} \ldots\left(s^{[p p]}\right)^{\top}\right] . \tag{12}
\end{align*}
$$

Similarly, the impulse response coefficients $L_{k}$ parameterizing the transfer matrix $L(z)=\sum_{k=1}^{\infty} L_{k} z^{-k}$ are stacked in $\theta_{l}^{\top} \in \ell_{1}^{1 \times p^{2}}(\mathbb{N})$ as done above for $S(z)$. We first introduce $l^{[i j]} \in \ell_{1}(\mathbb{N})$ as

$$
\begin{equation*}
l^{[i j]}:=\left[\left[L_{1}\right]_{i j}\left[L_{2}\right]_{i j} \ldots\left[L_{k}\right]_{i j} \ldots\right]^{\top} \tag{13}
\end{equation*}
$$

and define

$$
\begin{align*}
\theta_{l}^{\top}=[ & \left(l^{[11]}\right)^{\top} \ldots\left(l^{[1 p]}\right)^{\top} \mid \ldots \\
& \left.\ldots \mid\left(l^{[p 1]}\right)^{\top} \ldots\left(l^{[p p]}\right)^{\top}\right] . \tag{14}
\end{align*}
$$

The measured data $y(1) \ldots y(N)$ are stacked in the vector $\mathrm{y}^{+}$as follows

$$
\begin{align*}
\mathrm{y}^{+}=[ & y_{1}(t)^{\top} \ldots y_{1}(t+N-1)^{\top} \mid \ldots \\
& \left.\ldots \mid y_{p}(t)^{\top} \ldots y_{p}(t+N-1)^{\top}\right]^{\top} . \tag{15}
\end{align*}
$$

The vector $\mathrm{e}^{+}$is defined analogously. Let us also introduce the Toeplitz matrix $\phi_{j} \in \ell_{2}^{N \times 1}(\mathbb{N})$ :

$$
\begin{equation*}
\left[\phi_{j}\right]_{k h}:=y_{j}(t+k-h-1) \tag{16}
\end{equation*}
$$

with $k=1 \ldots N$ and $h \in \mathbb{N}$.
Then, we define the regression matrix $\Phi \in \ell_{2}^{p N \times p^{2}}(\mathbb{N})$ as:

$$
\Phi=I_{p} \otimes\left[\begin{array}{lll}
\phi_{1} & \ldots & \phi_{p} \tag{17}
\end{array}\right]
$$

so that, from (6) the vector $\mathrm{y}^{+}$containing the measured data satisfy the linear mode ${ }^{1}$

$$
\begin{equation*}
\mathrm{y}^{+}=\underbrace{\Phi\left(\theta_{l}+\theta_{s}\right)}_{\hat{\mathrm{y}}^{+}}+\mathrm{e}^{+} \tag{18}
\end{equation*}
$$

where $\hat{\mathrm{y}}^{+}:=\Phi\left(\theta_{s}+\theta_{l}\right)$ is the one-step ahead predictor of $y^{+}$.

Therefore, our $\mathrm{S}+\mathrm{L}$ identification problem can be formulated in terms of PEM as follows.

Problem 1 Find $\theta_{s}^{\top}, \theta_{l}^{\top} \in \ell_{1}^{1 \times p^{2}}(\mathbb{N})$ corresponding to $a$ $S+L$ model minimizing the prediction error norm $\| \mathrm{y}^{+}-$ $\Phi\left(\theta_{s}+\theta_{l}\right) \|_{\Sigma^{-1} \otimes I_{N}}^{2}$.

Following the nonparametric Gaussian regression approach in Pillonetto et al. (2011), we model $\theta_{s}$ and $\theta_{l}$ as two zero-mean processes with kernels $K_{S} \in \mathcal{S}_{2}^{p^{2}}(\mathbb{N})$ and $K_{L} \in \mathcal{S}_{2}^{p^{2}}(\mathbb{N})$, respectively. These kernels may depend upon some tuning parameters, usually called hyperparameters and denoted with $\bar{\xi}$ hereafter. As illustrated in Section 4, according to the maximum entropy principle $\theta_{s}$ and $\theta_{l}$ will be modeled as Gaussian and independent. In the following $\mathcal{H}_{S}$ and $\mathcal{H}_{L}$ denote the reproducing Hilbert spaces Aronszajn, 1950) of deterministic functions on $\mathbb{N}$, associated with $K_{S}$ and $K_{L}$, with norm denoted by $\|\cdot\|_{K_{S}^{-1}}$ and $\|\cdot\|_{K_{L}^{-1}}$, respectively. We assume that the past data $\mathrm{y}^{-}$neither affects the a priori probability on $\theta_{s}$ and $\theta_{l}$ nor carries information on $\bar{\xi}$ and $\Sigma$ (Poggio \& Girosi, 1990), that is

$$
\begin{align*}
& \mathbf{p}\left(\mathrm{y}^{+}, \theta_{l}, \theta_{s}, \mathrm{y}^{-} \mid \bar{\xi}, \Sigma\right) \\
& \quad=\mathbf{p}\left(\mathrm{y}^{+} \mid \theta_{l}, \theta_{s}, \mathrm{y}^{-}, \bar{\xi}, \Sigma\right) \mathbf{p}\left(\theta_{l}, \theta_{s} \mid \mathrm{y}^{-}, \bar{\xi}, \Sigma\right) \mathbf{p}\left(\mathrm{y}^{-} \mid \bar{\xi}, \Sigma\right) \\
& \quad \approx \mathbf{p}\left(\mathrm{y}^{+} \mid \theta_{l}, \theta_{s}, \mathrm{y}^{-}, \bar{\xi}, \Sigma\right) \mathbf{p}\left(\theta_{l}, \theta_{s} \mid \bar{\xi}, \Sigma\right) \mathbf{p}\left(\mathrm{y}^{-}\right) \tag{19}
\end{align*}
$$

1 The matrix $\Phi$ contains in principle data in the remote past which are not available. In practice model (18) needs to be approximated truncating $\Phi$ (and thus $\theta_{l}$ and $\theta_{s}$ ). This corresponds to assuming zero initial conditions, and it is a reasonable approximation given the decay, as a function of $k$, of the coefficients $L_{k}$ and $S_{k}$ (which is a necessary condition for BIBO stability). We shall not enter into these details in the paper. See also Remark 3.

Let

$$
\begin{equation*}
\hat{\theta}_{s}=\mathbb{E}\left[\theta_{s} \mid \mathrm{y}^{+}, \bar{\xi}, \Sigma\right], \quad \hat{\theta}_{l}=\mathbb{E}\left[\theta_{l} \mid \mathrm{y}^{+}, \bar{\xi}, \Sigma\right] \tag{20}
\end{equation*}
$$

be, respectively, the minimum variance estimators of $\theta_{s}$ and $\theta_{l}$ given $\mathrm{y}^{+}, \bar{\xi}$ and $\Sigma$. In what follows, $\hat{S}(z)$ and $\hat{L}(z)$ denote the transfer matrices corresponding to $\hat{\theta}_{s}$ and $\hat{\theta}_{l}$, respectively. The next Proposition shows that $\hat{\theta}_{s}$ and $\hat{\theta}_{l}$ are, almost surely, solution to a Tikhonov-type variational problem and belong to the spaces $\mathcal{H}_{S}$ and $\mathcal{H}_{L}$, respectively.

Proposition 2 Under the assumption that y is a second order stationary process and under approximation (19), almost surely we have

$$
\begin{align*}
&\left(\hat{\theta}_{s}, \hat{\theta}_{l}\right)=\underset{\substack{\theta_{s} \in \mathcal{H}_{K_{S}} \\
\theta_{l} \in \mathcal{H}_{K_{L}}}}{\arg \min }\left\|\mathrm{y}^{+}-\Phi\left(\theta_{s}+\theta_{l}\right)\right\|_{\Sigma^{-1} \otimes I_{N}}^{2} \\
&+\left\|\theta_{s}\right\|_{K_{S}^{-1}}^{2}+\left\|\theta_{l}\right\|_{K_{L}^{-1}}^{2} \tag{21}
\end{align*}
$$

Moreover, almost surely:

$$
\begin{equation*}
\hat{\theta}_{s}=K_{S} \Phi^{\top} c, \quad \hat{\theta}_{l}=K_{L} \Phi^{\top} c \tag{22}
\end{equation*}
$$

where

$$
\begin{equation*}
c=\left(\Phi\left(K_{S}+K_{L}\right) \Phi^{\top}+\Sigma \otimes I_{N}\right)^{-1} \mathrm{y}^{+} . \tag{23}
\end{equation*}
$$

Remark 3 The semi-infinite regression matrix $\Phi$ depends on both $\mathrm{y}^{+}$and $\mathrm{y}^{-}$. Since $\mathrm{y}^{-}$is never completely known, a solution to handle the initial conditions consists of setting its unknown components to zero. In this way, the introduced error goes to zero as $N$ increases Ljung, 1999, Section 3.2). Alternatively it would be possible to incorporate initial conditions in the estimation problem, e.g. modeling also the free response of the system. This is however outside the scope of the paper and is only practically relevant when very slow dynamics are present.

The main task now is to design the kernels $K_{S}$ and $K_{L}$ in such a way that $\hat{S}(z)$ and $\hat{L}(z)$ are almost surely BIBO stable while favouring $\hat{S}(z)$ to be sparse and $\hat{L}(z)$ to be of low-rank.

## 4 Maximum entropy priors

One way to derive a probability law for the joint process $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ under desired constraints rests on the maximum entropy principle. The most common justification of maximum entropy solutions relies on "information" arguments essentially stating that the maximum entropy distribution is the one which entails the maximum "uncertainty" under the given constraints. There is another and very important motivation for adopting the maximum entropy solution: Shore \& Johnson (1980)
have shown that maximum entropy is the unique correct method satisfying some minimal consistency axioms; basically, these axioms state that the solution should be consistent when "there are different ways of taking the same information into account".

We shall make the rather mild assumption that the process $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{L}^{\top}\end{array}\right]^{\top}$ is zero-mean and absolutely continuous with respect to the Lebesgue measure. We will see that, under suitable constraints, the optimal solution (i.e. maximizing the differential entropy) is a Gaussian process where $\theta_{s}$ and $\theta_{l}$ are independent. Then, we will also characterize the corresponding kernels. In what follows, we propose two different ways to enforce BIBO stability and low-rank on $\hat{L}(z)$. This leads to two different types of kernel for $\theta_{l}$.

### 4.1 First type of kernel

We start with the constraints on $\theta_{s}$ inducing BIBO stability and sparsity on $\hat{S}(z)$. To do so, we exploit the following proposition:

Proposition 4 Let $P \in \mathcal{S}_{2}(\mathbb{N})$ be a strictly positive definite kernel (in the sense of Moore) such that $[P]_{t t} \leq$ $\kappa t^{\alpha} e^{-\beta t}, t \in \mathbb{N}$, with $\kappa, \beta>0$ and $\alpha \in \mathbb{R}$. Let also $\phi$ be a zero-mean process which satisfies the moment constraint

$$
\begin{equation*}
\mathbb{E}\left[\|\phi\|_{P^{-1}}^{2}\right] \leq c \tag{24}
\end{equation*}
$$

where $c \geq 0$. Then, for any $\varepsilon>1$ there exists $\bar{\kappa}_{\varepsilon}>0$ such that the covariance function (kernel) $K$ of $\phi$ satisfies:

$$
\begin{equation*}
[K]_{t t} \leq \bar{\kappa}_{\varepsilon} t^{\alpha+\varepsilon} e^{-\beta t}, \quad t \in \mathbb{N} \tag{25}
\end{equation*}
$$

Thus, we consider the constraint on $\theta_{s}$

$$
\begin{equation*}
\mathbb{E}\left[\left\|s^{[i j]}\right\|_{P-1}^{2}\right] \leq c_{i j} \tag{26}
\end{equation*}
$$

where $c_{i j} \geq 0, i, j=1 \ldots p$, and $P \in \mathcal{S}_{2}(\mathbb{N})$. If $c_{i j}=0$, then $s^{[i j]}$ is the null sequence in mean square and so is its posterior mean.

It is not difficult to see that the assumption on $P$ in Proposition 4 is satisfied by the kernels usually employed in the identification of dynamical models (e.g. stable spline, tuned/correlated and so on, see Pillonetto et al. (2014)). Thus, by (26) the covariance of the $k$-th element of $s^{[i j]}$ decays exponentially. We conclude that the posterior mean of the transfer function in position $(i, j)$ of $S(z)$, under constraint (26), is BIBO stable. Moreover, it is null if and only if $c_{i j}=0$.

Remark 5 Clearly, if $P$ in (26) is chosen as the covariance matrix of $D C(T C)$ prior, the resulting Maximum Entropy prior coincides with the DC (TC) prior.

Some recent literature has discussed the maximum entropy properties of some kernels, such as the TC or DC kernels, see Nicolao et al. (1998), Carli et al. (2014) and Chen et al. (2015), as well as extensions to more articulated kernels Prando et al. (2015). In these papers it was shown for TC and DC kernels that weaker constraints with respect to (26) lead to the same solution. We refer the reader to these papers form more details; here we stick to the simpler conditions (26) in order to streamline the derivation as well as to allow alternative choices of the matrix $P$ in (26).

In view of Proposition 4, to guarantee BIBO stability on $\hat{L}(z)$ we also impose the constraint

$$
\begin{equation*}
\sum_{i, j=1}^{p} \mathbb{E}\left[\left\|l^{[i j]}\right\|_{P^{-1}}^{2}\right] \leq r \tag{27}
\end{equation*}
$$

for some $r$ such that $0<r<\infty$.
We now switch our attention to the low rank property of the matrix $L(z)$. Let $A_{l}$ be the random semi-infinite matrix defined as

$$
A_{l}=\left[\begin{array}{lllll}
L_{1} & L_{2} & \ldots & L_{k} & \ldots \tag{28}
\end{array}\right]
$$

and consider the constraint on $\theta_{l}$

$$
\begin{equation*}
\mathbb{E}\left[A_{l} A_{l}^{\top}\right] \leq Q \tag{29}
\end{equation*}
$$

with $Q \in \overline{\mathcal{M}}_{+}^{p}$. If $Q$ has $p-n$ singular values equal to zero, so does the covariance $\mathbb{E}\left[A_{l} A_{l}^{\top}\right]$; therefore the posterior mean $\hat{A}_{l}$ of $A_{l}$ has rank less than or equal to $n$ thus admitting the decomposition

$$
\hat{A}_{l}=\left[\begin{array}{llll}
\hat{F} \hat{H}_{1} & \hat{F} \hat{H}_{2} \ldots \hat{F} \hat{H}_{k} \ldots \tag{30}
\end{array}\right]
$$

where $\hat{F} \in \mathbb{R}^{p \times n}$ and the $\hat{H}_{k} \in \mathbb{R}^{n \times p}, k \in \mathbb{N}$, as in Section 2. We conclude that, under constraints (27), $\hat{L}(z)$ is BIBO stable and under constraint (29), its rank is less than or equal to $n$ if and only if $Q$ has rank equal to $n$.

In order to build the desired prior distribution we make use of Kolmogorov extension Theorem, see Øksendal (1998), and work with finite vectors extracted from processes $\theta_{l}$ and $\theta_{s}$.

To do so, let us consider a finite index set $\mathcal{I}=\mathcal{I}_{s} \times \mathcal{I}_{l}$ in $\mathbb{N} \times \mathbb{N}$. Let $\check{\theta}_{s}$ and $\check{\theta}_{l}$ be the random vectors whose components are extracted, respectively, from the process $\theta_{s}$ and $\theta_{l}$ according to the index sets $\mathcal{I}_{s}$ and $\mathcal{I}_{l}$. We denote by $\mathbf{p}_{\mathcal{I}}\left(\check{\theta}_{s}, \check{\theta}_{l}\right)$ the joint probability density of $\check{\theta}_{s}$ and $\check{\theta}_{l}$. By Kolmogorov extension Theorem the joint process $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ can be characterized by specifying the joint probability density $\mathbf{p}_{\mathcal{I}}$ for all finite sets $\mathcal{I} \subset \mathbb{N} \times \mathbb{N}$. Thus,
the maximum entropy process $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ can be constructed building all marginals $\mathbf{p}_{\mathcal{I}}$ using the maximum entropy principle, which can thus be extended by Kolmogorov extension theorem. Such principle states that among all the probability densities $\mathbf{p}_{\mathcal{I}}$ satisfying the desired constraints, the optimal one should maximize the differential entropy (Cover \& Thomas, 1991)

$$
\begin{equation*}
\mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right)=-\mathbb{E}\left[\log \left(\mathbf{p}_{\mathcal{I}}\right)\right] . \tag{31}
\end{equation*}
$$

Constraints (26), (27) and (29) boil down, respectively, to

$$
\begin{align*}
& \mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right] \leq c_{i j}, \quad i, j=1 \ldots p  \tag{32}\\
& \sum_{i, j=1}^{p} \mathbb{E}\left[\left\|\check{l}^{[i j]}\right\|_{P_{\mathcal{I}_{l}}^{-1}}^{2}\right] \leq r  \tag{33}\\
& \mathbb{E}\left[\check{A}_{l} \check{A}_{l}^{\top}\right] \leq Q \tag{34}
\end{align*}
$$

where $\check{s}^{[i j]}$ and $\check{l}^{[i j]}$ are the vectors extracted from $s^{[i j]}$ and $l^{[i j]}$ according to the index set $\mathcal{I}_{s}$ and $\mathcal{I}_{l}$, respectively. $P_{\mathcal{I}_{s}}$ and $P_{\mathcal{I}_{l}}$ are the kernel matrices whose entries are extracted from $P$ according to $\mathcal{I}_{s}$ and $\mathcal{I}_{l}$, respectively. $\check{A}_{l}$ is the matrix whose blocks are extracted from $A_{l}$ according to $\mathcal{I}_{l}$. Therefore, we obtain the following maximum entropy problem

$$
\begin{align*}
& \max _{\mathbf{p}_{\mathcal{I}} \in \mathcal{P}} \mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right) \\
& \text { s.t. } \mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right] \leq c_{i j}, \quad i, j=1 \ldots p \\
& \quad \sum_{i, j=1}^{p} \mathbb{E}\left[\left\|\check{l}^{〔 i j]}\right\|_{P_{\mathcal{I}_{l}}^{-1}}^{2}\right] \leq r \\
& \quad \mathbb{E}\left[\check{A}_{l} \check{A}_{l}^{\top}\right] \leq Q \tag{35}
\end{align*}
$$

where $\mathcal{P}$ is the class of probability densities in $\mathbb{R}^{\left|\mathcal{I}_{s}\right|} \times$ $\mathbb{R}^{\left|\mathcal{I}_{l}\right|}$ which are bounded and taking positive values.

Theorem 6 Under the assumption that $c_{i j}>0, i, j=$ $1 \ldots p$, and $Q \in \mathcal{M}_{+}^{p}$, the unique optimal solution to the maximum entropy problem (35) is such that $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are independent, Gaussian with zero mean and kernel matrix, respectively,

$$
\begin{align*}
\check{K}_{S} & =\operatorname{diag}\left(\gamma_{1} \ldots \gamma_{p^{2}}\right) \otimes P_{\mathcal{I}_{s}}  \tag{36}\\
\check{K}_{L} & =\left(\lambda I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}+\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right)^{-1} \tag{37}
\end{align*}
$$

where $\gamma_{i}>0, i=1 \ldots p^{2}, \tilde{\Lambda} \in \overline{\mathcal{M}}_{+}^{p}$ and $\lambda \geq 0$.
In what follows we assume that constraints (33) and (34) are totally binding in problem (35). Since $\lambda$ and $\tilde{\Lambda}$ are the Lagrange multipliers associated to those constraints, it is not difficult to see that in this situation $\lambda>0$ and $\tilde{\Lambda} \in \mathcal{M}_{+}^{p}$. Moreover, we define $\Lambda=\tilde{\Lambda}^{-1}$. As noticed before, we are interested in the limiting cases where $c_{i j}$
might be equal to zero and $Q$ might be of low-rank in order to have sparse and low-rank estimators. To include these scenarios, we consider the limits as $c_{i j} \rightarrow 0$ and $Q$ tends to a low rank matrix and extend the maximum entropy solution by continuity.

Proposition 7 Let $\mathbf{C}=\left\{(i, j)\right.$ s.t. $\left.c_{i j}=0\right\}$ and $\mathbf{Q}=$ $\{v$ s.t. $Q v=0\}$. Then, the maximum entropy solution extended by continuity is the probability density such that $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are independent, Gaussian, zero-mean, and with kernel matrices

$$
\begin{align*}
\check{K}_{S}= & \operatorname{diag}\left(\gamma_{1} \ldots \gamma_{p^{2}}\right) \otimes P_{\mathcal{I}_{s}}  \tag{38}\\
\check{K}_{L}= & \lambda^{-1} I_{p^{2}} \otimes P_{\mathcal{I}_{l}}-\lambda^{-2} I_{p^{2}} \otimes P_{\mathcal{I}_{l}} \\
& \times\left(\lambda^{-1} I_{p^{2}} \otimes P_{\mathcal{I}_{l}}+\Lambda \otimes I_{p\left|\mathcal{I}_{l}\right|}\right)^{-1} I_{p^{2}} \otimes P_{\mathcal{I}_{l}} \tag{39}
\end{align*}
$$

where $\gamma_{(i-1) p+j}=0$ if and only if $(i, j) \in \mathbf{C}$ and $\Lambda v=0$ if and only if $v \in \mathbf{Q}$.

Finally, in view of Kolmogorov extension Theorem, from the probability density of $\left[\check{\theta}_{s}^{\top} \check{\theta}_{l}^{\top}\right]^{\top}$ we can characterize the probability law of $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ maximizing the differential entropy.

Corollary 8 Consider the joint process $\left[\theta_{s}^{\top} \theta_{l}^{\top}\right]^{\top}$ where $\theta_{s}$ and $\theta_{l}$ are Gaussian independent processes with kernels, respectively,

$$
\begin{align*}
K_{S}= & \operatorname{diag}\left(\gamma_{1} \ldots \gamma_{p^{2}}\right) \otimes P \\
K_{L}= & \lambda^{-1} I_{p^{2}} \otimes P-\lambda^{-2} I_{p^{2}} \otimes P \\
& \times\left(\lambda^{-1} I_{p^{2}} \otimes P+\Lambda \otimes I_{\infty}\right)^{-1} I_{p^{2}} \otimes P \tag{40}
\end{align*}
$$

where $I_{\infty}$ such that $\left[I_{\infty}\right]_{t t}=1, t \in \mathbb{N}$, and zero otherwise. For all finite sets $\mathcal{I} \subset \mathbb{N} \times \mathbb{N}$, its joint probability density is the extended solution to the maximum entropy problem (35).

It is worth noting that the maximum entropy kernels are characterized by the hyperparameters $\gamma_{i}, i=1 \ldots p^{2}$, which control sparsity on $\hat{S}(z), \Lambda$ tuning the rank (and column space) of $\hat{L}(z)$, while $\beta$ (see Proposition 4) controls the decay rate (as a function of the time index $k \in \mathbb{N}$ ) of the estimators $\hat{L}_{k}$ and $\hat{S}_{k}$ and thus BIBO stability of $\hat{S}(z)$ and $\hat{L}(z)$, see Pillonetto \& De Nicolao (2010). Finally, $\lambda$ represents a trade-off between BIBO stability and low-rank on $\hat{L}(z)$. The structure of $\Lambda$ is very general. We suggest that $\Lambda$ can be reparameterized by introducing few hyperparameters to reduce its degrees of freedom. In Section 5 we will propose one possible reparameterization.

Remark 9 It is possible to derive the same structure for $K_{L}$ by adopting the regularization point of view. In Wipf (2012), it has been shown that the penalty term $\log \operatorname{det}(R)$, with $R \in \mathcal{M}_{+}^{p}$, induces low-rank on $R$. Moreover, the term $\log \operatorname{det}(R)$ admits the variational upper
bound

$$
\begin{equation*}
\log \operatorname{det}(R) \leq \operatorname{tr}\left(\Lambda^{-1} R\right)-\log \operatorname{det}\left(\Lambda^{-1}\right)-p \tag{41}
\end{equation*}
$$

where $\Lambda \in \mathcal{M}_{+}^{p}$ and equality holds if and only if $R=\Lambda$. Consider the random vector $\check{\theta}_{l}$ extracted from $\theta_{l}$ according to $\mathcal{I}_{l}$. Thus, we can induce low-rank on $R=\check{A}_{l} \check{A}_{l}^{\top}$ by considering the penalty

$$
\begin{equation*}
\operatorname{tr}\left(\Lambda^{-1} \check{A}_{l} \check{A}_{l}^{\top}\right)-\log \operatorname{det}(\Lambda)-p \tag{42}
\end{equation*}
$$

where $\Lambda$ represents a rough estimate of $\check{A}_{l} \check{A}_{l}^{\top}$. The unique term depending on $\check{\theta}_{l}$ in (42) is $\operatorname{tr}\left(\Lambda^{-1} \check{A}_{l} \check{A}_{l}^{\top}\right)=$ $\left\|\check{\theta}_{l}\right\|_{\Lambda^{-1} \otimes I_{p\left|I_{l}\right|}}$ which is one part of the norm $\left\|\check{\theta}_{l}\right\|_{\tilde{K}_{L}^{-1}}^{2}$ with $\check{K}_{L}$ kernel matrix defined in (37). Thus, this penalty terms induces low-rank on $\check{A}_{l}$.

### 4.2 Second type of kernel

The BIBO stability and low-rank constraints on $\hat{L}(z)$ can be imposed by using only one constraint. Consider the random semi-infinite matrix $A_{l}$ defined in (28) and the constraint

$$
\begin{equation*}
\mathbb{E}\left[A_{l}\left(P^{-1} \otimes I_{p}\right) A_{l}^{\top}\right] \leq Q \tag{43}
\end{equation*}
$$

with $Q \in \overline{\mathcal{M}}_{+}^{p}$. Similarly to the previous case, if the null space of $Q$ has dimension $m-n$, then the posterior mean $\hat{A}_{l}$ of $A_{l}$ has rank less than or equal to $n$; therefore also $\hat{L}(z)$ has rank less than or equal to $n$. This statement is formalized in the following proposition:

Proposition 10 Assume that $P \in \mathcal{S}_{2}(\mathbb{N})$ is strictly positive definite and such that $[P]_{t t} \leq \kappa t^{\alpha} e^{-\beta t}, t \in \mathbb{N}$, with $\kappa, \beta>0$ and $\alpha \in \mathbb{R}$. Then, under constraint (43), for any $\varepsilon>1$ there exists $\bar{\kappa}_{\varepsilon}>0$ such that $\mathbb{E}\left[\left|\left[L_{t}\right]_{i j}\right|^{2}\right] \leq$ $\bar{\kappa}_{\varepsilon} t^{\alpha+\varepsilon} e^{-\beta t}, i, j=1 \ldots p, t \in \mathbb{N}$. In addition, if the vector $v$ belongs to the nullspace of $Q$, then $v^{\top} A_{l}$ is zero in mean square; therefore $A_{l}$ has low rank (in mean square) and its null space contains the one of $Q$.

Similarly to the previous case, the joint process $\left[\theta_{s}^{\top} \theta_{l}^{\top}\right]^{\top}$ is characterized through $\mathbf{p}_{\mathcal{I}}$ using the maximum entropy principle. In particular, constraint (43) becomes

$$
\begin{equation*}
\mathbb{E}\left[\check{A}_{l}\left(P_{\mathcal{I}_{l}}^{-1} \otimes I_{p}\right) \check{A}_{l}^{\top}\right] \leq Q \tag{44}
\end{equation*}
$$

The corresponding maximum entropy problem is

$$
\begin{array}{rl}
\max _{\mathbf{p}_{\mathcal{I}} \in \mathcal{P}} & \mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right) \\
\text { s.t. } & \mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right] \leq c_{i j}, \quad i, j=1 \ldots p \\
& \mathbb{E}\left[\check{A}_{l}\left(P_{\mathcal{I}_{l}}^{-1} \otimes I_{p}\right) \check{A}_{l}^{\top}\right] \leq Q \tag{45}
\end{array}
$$

Theorem 11 Under the assumption that $c_{i j}>0, i, j=$ $1 \ldots p$, and $Q \in \mathcal{M}_{+}^{p}$, the unique optimal solution to the maximum entropy problem (45) is such that $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are independent, Gaussian with zero mean and kernel matrix, respectively,

$$
\begin{align*}
\check{K}_{S} & =\operatorname{diag}\left(\gamma_{1} \ldots \gamma_{p^{2}}\right) \otimes P_{\mathcal{I}_{s}}  \tag{46}\\
\check{K}_{L} & =\Lambda \otimes I_{p} \otimes P_{\mathcal{I}_{l}} \tag{47}
\end{align*}
$$

where $\gamma_{i}>0, i=1 \ldots p^{2}$, and $\Lambda \in \mathcal{M}_{+}^{p}$.
Similarly to the previous case, we extend the maximum entropy solution by continuity to the case of interest.

Proposition 12 Let $\mathbf{C}=\left\{(i, j)\right.$ s.t. $\left.c_{i j}=0\right\}$ and $\mathbf{Q}=$ $\{v$ s.t. $Q v=0\}$. Then, the maximum entropy solution extended by continuity is the probability density such that $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are independent, Gaussian, zero-mean, and with kernel matrices (46) and (47), respectively, where $\gamma_{(i-1) p+j}=0$ if and only if $(i, j) \in \mathbf{C}$ and $\Lambda v=0$ if and only if $v \in \mathbf{Q}$.

Finally, from the probability density of $\left[\begin{array}{cc}\check{\theta}_{s}^{\top} & \check{\theta}_{l}^{\top}\end{array}\right]^{\top}$ we characterize the probability law of the joint process $\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ using the Kolmogorov extension Theorem.

Corollary 13 Consider the joint process $\left[\begin{array}{cc}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$ where $\theta_{s}$ and $\theta_{l}$ are Gaussian independent processes with kernel, respectively,

$$
\begin{align*}
& K_{S}=\operatorname{diag}\left(\gamma_{1} \ldots \gamma_{p^{2}}\right) \otimes P \\
& K_{L}=\Lambda \otimes I_{p} \otimes P \tag{48}
\end{align*}
$$

For all finite sets $\mathcal{I} \subset \mathbb{N} \times \mathbb{N}$, its joint probability density is the extended solution to the maximum entropy problem (45).

In respect to the first type of kernel, we do not need the extra hyperparameter $\lambda$. Furthermore, in order to compute $K_{L}$ in (48) we do not need to invert an infinite dimensional matrix ${ }_{2}$ as in (40). We conclude that the computation of $K_{L}$ in the second type of kernel is more efficient.

Remark 14 The derivation of the maximum entropy kernels given above requires that a matrix $P$, satisfying the assumptions of Proposition 4 and Proposition 10, be fixed. In what follows, we consider the filtered kernel proposed in Pillonetto et al. (2011)

$$
\begin{equation*}
P=F P_{S S} F^{\top} \tag{49}
\end{equation*}
$$

where $P_{S S} \in \mathcal{S}_{2}(\mathbb{N})$ is the stable spline (SS) kernel and $F$ is a Toeplitz matrix which can be used to shape the kernel

[^1]emphasising certain frequencies. For instance, when the focus is prediction, the predictor impulse might exhibit an oscillatory behavior due to "high" frequency zeros of the noise spectrum. In this paper, following Pillonetto et al. (2011), we have built $F$ from the impulse response of a second order oscillatory system; the poles of this system are estimated as hyperparameters. Note that, instead of SS one might choose other types of kernel, such as diagonal, tuned/correlated, diagonal/correlated and so on.

## 5 Hyperparameters Estimation

In order to compute $\hat{\theta}_{s}$ and $\hat{\theta}_{l}$, estimates of the noise variance $\Sigma$ and of the hyperparameters vector $\bar{\xi}$ are needed. Let $\bar{\xi}:=\{\tau, \xi\}$ where $\tau$ denotes the hyperparameters of $P$ defined in (49), $\xi:=\left\{\gamma_{1} \ldots \gamma_{p^{2}}, \Lambda, \lambda\right\}$ for the first type of kernel, and $\xi:=\left\{\gamma_{1} \ldots \gamma_{p^{2}}, \Lambda\right\}$ for the second one. In this paper $\Sigma$ is estimated using a low-bias ARX-model as suggested in (Goodwin et al. 1992). To estimate the hyperparameters of $P$ we consider the unstructured model for $y$

$$
\begin{equation*}
y(t)=G(t) y(t)+e(t) \tag{50}
\end{equation*}
$$

where $G(z)=\sum_{k=1}^{\infty} G_{k} z^{-k}$ is BIBO stable and $e$ WGN with covariance matrix $\Sigma$. Equation (50) can be written as a linear regression model

$$
\begin{equation*}
\mathrm{y}^{+}=\Phi \theta+\mathrm{e}^{+} \tag{51}
\end{equation*}
$$

where $\theta^{\top} \in \ell^{1 \times p^{2}}(\mathbb{N})$ contains the coefficients of $G(z)$. Following the Gaussian regression approach, $\theta$ in (51) is modelled as a zero-mean Gaussian process with kernel $K=I_{p^{2}} \otimes P$. Doing so, the hyperparameters of $P$ can be estimated by minimizing the negative log-marginal likelihood of $\mathrm{y}^{+}$computed using model (51), see Pillonetto et al. (2011).

Then the hyperparameters vector $\xi$ can be estimated minimizing the negative log-marginal likelihood $\ell$ of $\mathrm{y}^{+}$ under model (18) with $P$ fixed as above. Under the assumptions of Proposition 2, we have

$$
\begin{align*}
\ell\left(\mathrm{y}^{+}, \xi\right) & =\frac{1}{2} \log \operatorname{det} V+\frac{1}{2}\left(\mathrm{y}^{+}\right)^{\top} V^{-1} \mathrm{y}^{+}+\text {const. term } \\
V & =\Phi\left(K_{S}+K_{L}\right) \Phi^{\top}+\Sigma \otimes I_{N} \tag{52}
\end{align*}
$$

where $K_{L}$ and $K_{S}$ are defined in (40) or in (48). Notice that, the minimization of (52) with respect to $\xi$ is a nonconvex constrained optimization problem. Accordingly, only local minima can be computed. However, the real challenge is to perform the joint estimation of the $\gamma_{i}$ 's and $\Lambda$ because the sparse and the low-rank part might be not identifiable from the measured data. As a partial
remedy it is useful to reduce the degrees of freedom of $\Lambda$ (see also Prando et al. (2015)), constraining its structure as follows:

$$
\begin{equation*}
\Lambda=\alpha\left(I-U U^{\top}\right)+U \operatorname{diag}\left(\beta_{1} \ldots \beta_{n}\right) U^{\top} \tag{53}
\end{equation*}
$$

The matrix $U \in \mathbb{R}^{p \times n}$ is built, in view also of Remark 9 , using the first $n$ singular vectors of an estimate $\hat{A}_{l} \hat{A}_{l}^{\top}$ of $A_{l} A_{l}^{\top}$, that is $U=\left[\begin{array}{lll}u_{1} \ldots u_{n}\end{array}\right]^{\top}$ where $u_{1} \ldots u_{n}$ are the first $n$ singular vectors of $\hat{A}_{l} \hat{A}_{l}^{\top}$. In this way, the constraints in $\Lambda$ are decoupled along the "most relevant" $n$ singular vectors of $\hat{A}_{l} \hat{A}_{l}^{\top}$ and their orthogonal complement. This has the effect of steering the factor loading matrix $F$ toward the columns space of $U$. Regarding the hyperparameters $\gamma_{i}$ 's, it has been shown in the literature (MacKay, 1994, Tipping, 2001, Aravkin et al. 2014) that the minimization of (52) automatically leads to sparsity in the $\hat{\gamma}_{i}$ 's and therefore in $\hat{S}(z)$. Therefore, the minimization of (52) is performed with respect to $\tilde{\xi}:=\left\{\gamma_{1} \ldots \gamma_{p^{2}}, \alpha, \beta_{1} \ldots \beta_{n}, \lambda\right\}$ for the first type of kernel and with respect to $\tilde{\xi}:=\left\{\gamma_{1} \ldots \gamma_{p^{2}}, \alpha, \beta_{1} \ldots \beta_{n}\right\}$ for the second one while $n$ and $U$ are fixed. The complete procedure is outlined in Algorithm 1. $n^{(k)}, U^{(k)}, \tilde{\xi}^{(k)}, \hat{A}_{l}^{(k)}$ and $\hat{L}^{(k)}(z)$ denote, respectively, $n, U, \tilde{\xi}, \hat{A}_{l}$ and $\hat{L}(z)$ at the $k$-th iteration of the algorithm. The marginal likelihood function (52), when $\Lambda$ is constrained to be of the form (53), is a function of $U$. This dependence is made explicit in the notation

$$
\begin{aligned}
\ell\left(\mathrm{y}^{+}, \tilde{\xi}, U\right) & =\frac{1}{2} \log \operatorname{det} V+\frac{1}{2}\left(\mathrm{y}^{+}\right)^{\top} V^{-1} \mathrm{y}^{+}+\text {const. term } \\
V & =\Phi\left(K_{S}+K_{L}\right) \Phi^{\top}+\Sigma \otimes I_{N} \\
\Lambda & =\alpha\left(I-U U^{\top}\right)+U \operatorname{diag}\left(\beta_{1} \ldots \beta_{n}\right) U^{\top}
\end{aligned}
$$

where $K_{S}$ and $K_{L}$ are defined in (40) for the first type of kernel and in (48) for the second one. Finally, to efficiently compute a local minimum of (52) in the algorithm we used the scaled gradient projection algorithm developed in Bonettini et al. (2015).

It is worth noting that our algorithm is similar to the non-separable reweighting scheme proposed in Wipf \& Nagarajan (2010) for solving a sparse Bayesian learning problem. That algorithm iteratively alternates the computation of the optimal estimate (in our case $\hat{L}(z)$ ) and the closed form update of the hyperparameters (in our case it is given by (53)).

## 6 Numerical experiments

We consider two Monte Carlo studies of 100 runs where at any run a manifest process $y$ of dimension $m=6$ is considered. For each run in the Monte Carlo experiments an identification data set of size 500 and a test set of size 1000 are generated. We compare the true model, denoted by TRUE, with the following estimators

```
Algorithm 1 Computation of \(n, U\) and \(\xi\)
    \(k=0\)
    \(n^{(0)} \leftarrow 0\)
    \(U_{O P T}^{(0)} \leftarrow\) empty matrix
    \(\tilde{\xi}^{(0)} \leftarrow \underset{\tilde{c}}{\operatorname{argmin} \ell\left(\mathrm{y}^{+}, \tilde{\xi}, U_{O P T}^{(0)}\right)}\)
    \(\tilde{\xi}_{O P T}^{(0)} \leftarrow \stackrel{\tilde{\xi}^{(0)}}{ }\)
    repeat
        \(k \leftarrow k+1\)
        \(n^{(k)} \leftarrow n^{(k-1)}+1\)
        if \(n^{(k)}=1\) then
            \(\hat{A}_{l}^{(k)} \leftarrow\left[\begin{array}{lll}\hat{G}_{1} & \hat{G}_{2} & \ldots\end{array}\right]\) where \(\hat{G}_{1}, \hat{G}_{2}, \ldots\) are the
        coefficients of \(\hat{G}(z)\) estimated from (50) with
        \(K=I_{p^{2}} \otimes P\)
        \({ }_{\text {else }}^{\hat{A}_{l}^{(k)}} \leftarrow\left[\begin{array}{lll}\hat{L}_{1}^{(k)} & \hat{L}_{2}^{(k)} & \ldots\end{array}\right]\) where \(\hat{L}_{1}^{(k)}, \hat{L}_{2}^{(k)}, \ldots\) are
        the coefficients of \(\hat{L}^{(k)}(z)\) estimated from (21) with
        \(K_{S}\) and \(K_{L}\) having hyperparameters given by
        \(U_{O P T}^{(n(k)-1)}\) and \(\tilde{\xi}_{O P T}^{\left(n^{(k)}-1\right)}\)
        end if
        \(\tilde{\tilde{\xi}}^{(k)} \leftarrow\) first \(n^{(k)}\) singular vectors of \(\hat{A}_{l}^{(k)} \hat{A}_{l}^{(k)^{\top}}\)
        \(\tilde{\xi}^{(k)} \leftarrow \operatorname{argmin} \ell\left(\mathrm{y}^{+}, \tilde{\xi}, U^{(k)}\right)\)
        repeat
            \(\tilde{\xi}_{O P T}^{\left(n^{(k)}\right)} \leftarrow \tilde{\xi}^{(k)}\)
            \(U_{O P T}^{\left(n^{(k)}\right)} \leftarrow U^{(k)}\)
            \(k \leftarrow k+1\)
            \(n^{(k)} \leftarrow n^{(k-1)}\)
        \(\hat{A}_{l}^{(k)} \leftarrow\left[\begin{array}{lll}\hat{L}_{1}^{(k)} & \hat{L}_{2}^{(k)} & \ldots] \text { where } \hat{L}_{1}^{(k)}, \hat{L}_{2}^{(k)}, \ldots \text { are }\end{array}\right.\)
        the coefficients of \(\hat{L}^{(k)}(z)\) estimated from (21)
        with \(K_{S}\) and \(K_{L}\) having hyperparameters given by
        \(U^{(k-1)}\) and \(\tilde{\xi}^{(k-1)}\)
        \(U^{(k)} \leftarrow\) first \(n^{(k)}\) singular vectors of \(\hat{A}_{l}^{(k)} \hat{A}_{l}^{(k)}{ }^{\top}\)
        \(\tilde{\xi}^{(k)} \leftarrow \operatorname{argmin} \ell\left(\mathrm{y}^{+}, \tilde{\xi}, U^{(k)}\right)\)
            until \(\ell\left(\mathrm{y}^{+}, \tilde{\xi}^{\tilde{\xi}}, U^{(k)}\right)<\ell\left(\mathrm{y}^{+}, \tilde{\xi}^{(k-1)}, U^{(k-1)}\right)\)
    until
        \(\ell\left(y^{+}, \tilde{\xi}_{O P T}^{\left(n^{(k)}\right)}, U_{O P T}^{\left(n^{(k)}\right)}\right)<\ell\left(\mathrm{y}^{+}, \tilde{\xi}_{O P T}^{\left(n^{(k)}-1\right)}, U_{O P T}^{\left(n^{(k)}-1\right)}\right)\)
    \(: n \leftarrow n^{(k)}-1\)
    \(U \leftarrow U_{O P T}^{\left(n^{(k)}-1\right)}\)
    \(\tilde{\xi} \leftarrow \tilde{\xi}_{O P T}^{\left(n^{(k)}-1\right)}\)
```

- SL-I: this is the sparse plus low-rank estimator (22) with $K_{S}$ and $K_{L}$ as in (40)
- SL-II: this is the sparse plus low-rank estimator (22) with $K_{S}$ and $K_{L}$ as in (48)
- L-I: this is the low-rank estimator (22) with $K_{S}$ set equal to zero and $K_{L}$ as in (40)
- L-II: this is the low-rank estimator (22) with $K_{S}$ set equal to zero and $K_{L}$ as in (48)
- S: this is the sparse estimator (22) with $K_{L}$ set equal to zero
- SS: this is the estimator based on model (50) where $G(z)$ is modeled as a zero-mean Gaussian process with kernel $K=I_{p^{2}} \otimes P$ and $P$ defined in (49).

For implementation purposes, the impulse responses are truncated to a certain length $T$. The latter represents
the "practical" length of those impulse responses (Pillonetto et al., 2011). Note that, the truncation does not introduce bias-variance tradeoff because the filtered kernel $P$ forces the estimated impulse responses to decay exponentially. In these experiments we used $T=50$.

The following performance indexes are considered:

- Relative complexity of the estimated model. It is quantified with

$$
\begin{equation*}
C=\lim _{T \rightarrow \infty} \frac{\# \mathcal{E}_{T}}{m^{2} T} \tag{54}
\end{equation*}
$$

where $\# \mathcal{E}_{T}$ is the minimum number of parameters needed to characterize the model with impulse responses truncated at $T$. For instance, for the $\mathrm{S}+\mathrm{L}$ model (6) with $S(z)$ having $s$ nonnull transfer functions and $L(z)=F H(z)$ having rank $n$, we have $\# \mathcal{E}_{T}=s T+m n(T+1)$. The denominator $m^{2} T$ is the number of coefficients needed in unstructured predictor model $G(z)=\sum_{k=1}^{T} G_{k} z^{-k}$ in (50)

If we quantify the complexity of the S+L network as the number of edges among the manifest nodes and from the factor ones to the manifest ones, then it is not difficult to see that its complexity is equal to $m^{2} C$. Similar conclusions can be found for networks having only the sparse or the low-rank part. Therefore, the smaller $C$ is, the simpler the network is.

- One-step-ahead Coefficient of Determination, denoted by $\mathrm{COD}_{1}$. Such index quantifies how much of the test set variance is explained by the forecast, and is defined as:

$$
\begin{equation*}
\operatorname{COD}_{1}=1-\frac{\frac{1}{1000} \sum_{t=1}^{1000}\left\|y^{\text {test }}(t)-\hat{y}^{\text {test }}(t \mid t-1)\right\|^{2}}{\frac{1}{1000} \sum_{t=1}^{1000}\left\|y^{\text {test }}(t)-\bar{y}^{\text {test }}\right\|^{2}} \tag{55}
\end{equation*}
$$

where $\bar{y}^{\text {test }}$ denotes the sample mean of the test set data $y(1)^{\text {test }} \ldots y^{\text {test }}(1000)$ and $\hat{y}^{\text {test }}(t \mid t-1)$ is the onestep ahead prediction computed using the estimated model. Notice that, the larger $\mathrm{COD}_{1}$ is, the better predictive performance of the estimator is;

- Average impulse response fit

$$
\begin{equation*}
\operatorname{AIRF}=100\left(1-\sqrt{\frac{\frac{1}{T} \sum_{k=1}^{T}\left\|G_{k}-\hat{G}_{k}\right\|^{2}}{\frac{1}{T} \sum_{k=1}^{T}\left\|G_{k}-\bar{G}\right\|^{2}}}\right) \tag{56}
\end{equation*}
$$

where $G_{k}$ and $\hat{G}_{k}$ are the impulse response coefficients of the true and estimated model, respectively, and $\bar{G}=$ $\frac{1}{T} \sum_{k=1}^{T} G_{k}$. In the case the true (estimated) model is $\mathrm{S}+\mathrm{L}$ we have $G_{k}=S_{k}+L_{k}\left(\hat{G}_{k}=\hat{S}_{k}+\hat{L}_{k}\right)$.

In the first Monte Carlo experiment, $y$ is generated through model (6). In particular, $L(z)$ is randomly generated with rank $n=1 . S(z)$ is randomly generated


Fig. 2. Relative complexity of the models obtained by using the estimators SL-I, SL-II, L-I, L-II, and S.
with 7 nonnull transfer functions. The position of those transfer functions is randomly chosen. The top panel of Figure 2 shows the relative complexity of the models obtained with the estimators SL-I, SL-II, L-I, L-II and S reporting the boxplots of the values of $C$ after the 100 runs. One can see that L-II is the best estimator, then SL-I and S-II, then L-I and finally S. The top panel of Figure 3, reporting the boxplots of $\mathrm{COD}_{1}$, shows the one-step predictive performance of the estimators is similar. The top panel of Figure 4 shows the boxplots of AIRF. It is clear that SL-I and SL-II outperform the others.

In the second experiment, $y$ is generated through a generic (i.e. unstructured) model. The bottom panels of Figure 2, Figure 3 and Figure 4 show the performance



Fig. 3. One step ahead coefficient of determination $\left(\mathrm{COD}_{1}\right)$ obtained by the 6 estimators described in Section 6. Moreover, TRUE provides an upper bound on the performance of those estimators, being the true model for the manifest process $y$.
of the employed estimators as before. One can see that the best estimators in terms of $\mathrm{COD}_{1}$ are SL-I, SL-II and L-I. SL-I and SL-II provide simpler models than those obtained by L-I. On the other hand in terms of AIRF, SL-I and SL-II are slightly worse than L-I. These two Monte Carlo experiments suggest that the proposed $\mathrm{S}+\mathrm{L}$ models and estimators (SL-I and SL-II) provide an effective way of estimating complex model, yielding a good tradeoff among model complexity, prediction accuracy and average impulse response fit.



Fig. 4. Average impulse response fit (AIRF) obtained by the 6 estimators described in Section 6.

## 7 Conclusions

In this paper, we proposed two procedures, based on a nonparametric Gaussian regression approach, to identify stochastic processes having a sparse plus low-rank $(\mathrm{S}+\mathrm{L})$ network. The kernels inducing the $\mathrm{S}+\mathrm{L}$ structure have been derived using the maximum entropy principle. Simulations show that the proposed $\mathrm{S}+\mathrm{L}$ estimators have good predictive capability as well as low complexity compared with the sparse estimator and the low-rank estimators.

## Appendix

## Proof of Proposition 2

It is sufficient to observe that Problem (21) can be rewritten as

$$
\begin{equation*}
\hat{\theta}=\underset{\tilde{\theta} \in \mathcal{H}_{\tilde{K}}}{\arg \min }\left\|\mathrm{y}^{+}-\tilde{\Phi} \tilde{\theta}\right\|_{\Sigma^{-1} \otimes I_{N}}^{2}+\|\tilde{\theta}\|_{\tilde{K}^{-1}}^{2} \tag{57}
\end{equation*}
$$

where $\tilde{\theta}=\left[\begin{array}{ll}\theta_{s}^{\top} & \theta_{l}^{\top}\end{array}\right]^{\top}$, the regression matrix is $\tilde{\Phi}=$ $[\Phi \Phi]$, and $\mathcal{H}_{\tilde{K}}$ is the reproducing Hilbert space of deterministic functions on $\mathbb{N}$ with kernel

$$
\tilde{K}=\left[\begin{array}{cc}
K_{S} & 0  \tag{58}\\
0 & K_{L}
\end{array}\right]
$$

Then, the statement follows from Proposition 3 in Pillonetto et al. (2011). More precisely, it is not difficult to see that Proposition 3 still holds when the covariance matrix of the noise vector is not diagonal.

## Proof of Proposition 4

Define $\breve{P} \in \mathcal{S}_{2}(\mathbb{N})$ such that

$$
[\breve{P}]_{t s}=\left\{\begin{array}{c}
\kappa t^{\alpha} e^{-\beta t} \quad t=s  \tag{59}\\
{[P]_{t s} \quad \text { otherwise }}
\end{array}\right.
$$

hence $P \leq \breve{P}$. Let $\varepsilon>1$ and consider the decomposition

$$
\begin{equation*}
\breve{P}=D \bar{P} D \tag{60}
\end{equation*}
$$

where $D \in \mathcal{S}_{2}(\mathbb{N})$ is an infinite diagonal matrix with $\sqrt{t^{\alpha+\varepsilon}} e^{-\frac{\beta}{2} t}$ as $t$-th element in the main diagonal, and

$$
[\bar{P}]_{t s}=\left\{\begin{array}{cc}
\kappa t^{-\varepsilon} & t=s  \tag{61}\\
\star & \text { otherwise }
\end{array}\right.
$$

Note that, $\bar{P}$ is strictly positive definite by construction. Moreover,

$$
\begin{equation*}
\operatorname{tr}(\bar{P})=\kappa \sum_{t=1}^{\infty} t^{-\varepsilon}<\infty \tag{62}
\end{equation*}
$$

because $\varepsilon>1$. This means the sum of all the (nonnegative) eigenvalues of $\bar{P}$ is bounded, therefore its maximum eigenvalue, say $\lambda$, is bounded. Therefore, by (24)
we have

$$
\begin{align*}
c & \geq \mathbb{E}\left[\phi^{\top} P^{-1} \phi\right] \geq \mathbb{E}\left[\phi^{\top} \breve{P}^{-1} \phi\right] \\
& \geq \mathbb{E}\left[\phi^{\top} D^{-1} \bar{P}^{-1} D^{-1} \phi\right] \\
& \geq \lambda^{-1} \mathbb{E}\left[\phi^{\top} D^{-2} \phi\right]=\lambda^{-1} \sum_{t=1}^{\infty} t^{-(\alpha+\varepsilon)} e^{\beta t}[K]_{t t} \tag{63}
\end{align*}
$$

accordingly $\sum_{t=1}^{\infty} t^{-(\alpha+\varepsilon)} e^{\beta t}[K]_{t t}$ is bounded and thus $[K]_{t t} \leq \bar{\kappa}_{\varepsilon} t^{\alpha+\varepsilon} e^{-\beta t}, t \in \mathbb{N}$, for some $\bar{\kappa}_{\varepsilon}>0$.

## Proof of Theorem 6

We characterize the optimal solution by exploiting duality theory. We consider the Lagrange function

$$
\begin{align*}
\mathcal{L}\left(\mathbf{p}_{\mathcal{I}}, \tilde{\Gamma}, \lambda, \tilde{\Lambda}\right)= & \mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right) \\
& +\frac{1}{2} \sum_{i, j=1}^{p} \tilde{\gamma}_{(i-1) p+j}\left(c_{i j}-\mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right]\right) \\
& +\frac{1}{2} \lambda\left(r-\sum_{i, j=1}^{p} \mathbb{E}\left[\left\|\check{l}^{i j]}\right\|_{\left.P_{\mathcal{I}_{l}}^{-1}\right]}^{2}\right]\right. \\
& +\frac{1}{2} \operatorname{tr}\left(\tilde{\Lambda}\left(Q-\mathbb{E}\left[\check{A}_{l} \check{A}_{l}^{\top}\right]\right)\right) \tag{64}
\end{align*}
$$

where $\tilde{\Gamma}=\operatorname{diag}\left(\tilde{\gamma}_{1} \ldots \tilde{\gamma}_{p^{2}}\right)$ with $\tilde{\gamma}_{i} \geq 0, i=1 \ldots p^{2}$, $\lambda \geq 0$ and $\tilde{\Lambda} \in \overline{\mathcal{M}}_{+}^{p}$ are the Lagrange multipliers. It is not difficult to see that $\mathcal{L}$ is strictly concave over $\mathcal{P}$. Moreover, its unique maximum point is given by annihilating its first derivative. Therefore, we obtain

$$
\begin{align*}
& \mathbf{p}_{\mathcal{I}}\left(\check{\theta}_{s}, \check{\theta}_{l}\right)=\frac{1}{c} \exp \left(-\frac{1}{2} \sum_{i, j=1}^{p} \tilde{\gamma}_{(i-1) p+j}\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right. \\
& \left.-\frac{1}{2} \lambda \sum_{i, j=1}^{p}\left\|\check{l}^{[i j]}\right\|_{P_{\mathcal{I}_{l}}^{-1}}^{2}-\frac{1}{2} \operatorname{tr}\left(\tilde{\Lambda} \check{A}_{l} \check{A}_{l}^{\top}\right)\right) \tag{65}
\end{align*}
$$

where $c$ is the normalization constant. Let $e_{i}, i=1 \ldots p^{2}$, denote the $j$-th vector of the canonical basis of $\mathbb{R}^{p^{2}}$. Then, we have

$$
\begin{align*}
& \sum_{i, j=1}^{p} \tilde{\gamma}_{(i-1) p+j}\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2} \\
& \quad=\sum_{i, j=1}^{p} \tilde{\gamma}_{(i-1) p+j} \check{\theta}_{s}^{\top}\left(e_{(i-1) p+j} e_{(i-1) p+j}^{\top} \otimes P_{\mathcal{I}_{s}}^{-1}\right) \check{\theta}_{s} \\
& \quad=\check{\theta}_{s}^{\top}\left(\tilde{\Gamma} \otimes P_{\mathcal{I}_{s}}^{-1}\right) \check{\theta}_{s} \tag{66}
\end{align*}
$$

In similar way,

$$
\begin{equation*}
\sum_{i, j=1}^{p}\|\check{l}\lceil i j]\|_{P_{\mathcal{I}_{l}}^{-1}}^{2}=\check{\theta}_{l}^{\top}\left(I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}\right) \check{\theta}_{l} \tag{67}
\end{equation*}
$$

Moreover, it is not difficult to see that

$$
\begin{equation*}
\operatorname{tr}\left(\tilde{\Lambda} \check{A}_{l} \check{A}_{l}^{\top}\right)=\check{\theta}_{l}^{\top}\left(\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right) \check{\theta}_{l} \tag{68}
\end{equation*}
$$

Therefore, the optimal solution (if it does exist) is such that $\mathbf{p}_{\mathcal{I}}=\mathbf{p}_{\mathcal{I}_{s}} \mathbf{p}_{\mathcal{I}_{l}}$, where

$$
\begin{aligned}
& \mathbf{p}_{\mathcal{I}_{s}}\left(\check{\theta}_{s}\right)=\frac{1}{c_{s}} \exp \left(-\frac{1}{2} \check{\theta}_{s}^{\top}\left(\tilde{\Gamma} \otimes P_{\mathcal{I}_{s}}^{-1}\right) \check{\theta}_{s}\right) \\
& \mathbf{p}_{\mathcal{I}_{l}}\left(\check{\theta}_{l}\right)=\frac{1}{c_{l}} \exp \left(-\frac{1}{2} \check{\theta}_{l}^{\top}\left(\lambda I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}+\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right) \check{\theta}_{l}\right)
\end{aligned}
$$

with $c_{s}$ and $c_{l}$ normalization constants. Note that, $\mathbf{p}_{\mathcal{I}_{s}}$ and $\mathbf{p}_{\mathcal{I}_{l}}$ denote the marginal probability density of $\check{\theta}_{s}$ and $\check{\theta}_{l}$, respectively, Therefore, $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are independent, Gaussian with zero mean and covariance matrix

$$
\begin{align*}
\check{K}_{S} & =\tilde{\Gamma}^{-1} \otimes P_{\mathcal{I}_{s}} \\
\check{K}_{L} & =\left(\lambda I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}+\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right)^{-1} \tag{69}
\end{align*}
$$

Next, we prove the existence of such a solution showing that the dual problem does admits solution. Note that,
$\mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right)=\frac{1}{2} \log \operatorname{det}\left(\check{K}_{S}\right)+\frac{1}{2} \log \operatorname{det}\left(\check{K}_{L}\right)+$ const. term.
Therefore, the dual problem is equivalent to minimize the function

$$
\begin{align*}
& J(\tilde{\Gamma}, \lambda, \tilde{\Lambda})=-\left|\mathcal{I}_{s}\right| \log \operatorname{det}(\tilde{\Gamma})+\operatorname{tr}(\tilde{\Gamma} C)+\lambda r \\
& \quad+\operatorname{tr}(\tilde{\Lambda} Q)-\log \operatorname{det}\left(\lambda I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}+\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right) \tag{70}
\end{align*}
$$

where

$$
\begin{equation*}
C=\operatorname{diag}\left(c_{11}, \ldots, c_{1 p}, \ldots, c_{p 1}, \ldots, c_{p p}\right) \tag{71}
\end{equation*}
$$

Since $C \in \mathcal{M}_{+}^{p^{2}}, r>0$ and $Q \in \mathcal{M}_{+}^{p}$, it is not difficult to see that $J$ is lower bounded. It takes infinite value if and only if $\tilde{\Gamma}$ and/or $\lambda$ and/or $\tilde{\Lambda}$ are not bounded (the formal proof follows the one of Proposition 5.1 in Ferrante et al. (2012), see also Zorzi (2014a) and Zorzi (2014b)). Moreover, if $\tilde{\Gamma}$ and/or $\lambda I_{p^{2}} \otimes P_{\left|\mathcal{I}_{l}\right|}^{-1}+\tilde{\Lambda} \otimes I_{p\left|\mathcal{I}_{l}\right|}$ tend to be singular then $J$ approaches infinity. Accordingly, we can restrict the search of the Lagrange multipliers over the closed and bounded set

$$
\begin{align*}
\{(\tilde{\Gamma}, \lambda, \tilde{\Lambda}) \text { s.t. } & \varepsilon_{1} I_{p^{2}} \leq \tilde{\Gamma} \leq M_{1} I_{p^{2}} \\
& \left.\varepsilon_{2} \leq \lambda \leq M_{2}, \quad \varepsilon_{3} I_{p} \leq \tilde{\Lambda} \leq M_{3} I_{p}\right\} \tag{72}
\end{align*}
$$

for some and $\varepsilon_{1}, M_{1}>0$ and $\varepsilon_{2}, \varepsilon_{3}, M_{2}, M_{3} \geq 0$. The latter is a compact set because we are in a finite dimensional space. Since $J$ is continuous over that set, by the Weiestrass Theorem, the dual problem admits solution. Accordingly, the kernel matrices for $\check{\theta}_{s}$ and $\check{\theta}_{l}$ solution to (35) does exist and are unique. Finally, setting $\gamma_{i}=\tilde{\gamma}_{i}^{-1}$ we obtain matrix $K_{S}$ in the statement.

## Proof of Proposition 7

Consider the maximum entropy problem (35) where $c_{i j}$ and $Q$ have been replaced by $c_{i j}^{(k)}$ and $Q^{(k)}$ arbitrarily extracted from some sequences $\left\{c_{i j}^{(k)}\right\}_{k \geq 0}, c_{i j}^{(k)}>0$, and $\left\{Q^{(k)}\right\}_{k \geq 0}, Q^{(k)} \in \mathcal{M}_{+}^{p}$, respectively, and such that $c_{i j}^{(k)} \rightarrow 0 \forall(i, j) \in \mathbf{C}$ and $Q^{(k)} v \rightarrow 0 \forall v \in \mathbf{Q}$ as $k \rightarrow \infty$. Let $\left[\check{\theta}_{s}^{(k) \top} \check{\theta}_{l}^{(k) \top}\right]^{\top}$ be the random vector solution to this maximum entropy problem. The corresponding kernel matrices are (36) and (37) where $\gamma_{i}, \lambda$ and $\Lambda$ have been replaced with $\gamma_{i}^{(k)}>0, \lambda^{(k)}>0$ and $\Lambda^{(k)} \in \mathcal{M}_{+}^{p}$ with $k \geq 0$. Then, substituting this random vector into (32) we obtain

$$
\begin{align*}
c_{i j}^{(k)} & \geq \mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{l}}^{-1}}^{2}\right]=\operatorname{tr}\left(\mathbb{E}\left[\check{s}^{[i j]} \check{s}^{[i j]^{\top}}\right] P_{\mathcal{I}_{s}}^{-1}\right) \\
& =\left|\mathcal{I}_{s}\right| \gamma_{(i-1) p+j}^{(k)}>0 . \tag{73}
\end{align*}
$$

Accordingly, if $(i, j) \in \mathbf{C}$ then $\gamma_{(i-1) p+j}^{(k)} \rightarrow 0$ as $k \rightarrow \infty$. Let $\check{A}_{l}^{(k)}$ be the matrix built from the maximum entropy random vector. Then, substituting it in constraint (34) pre- and post-multiplied by an arbitrary $v \in \mathbb{R}^{p}$, we have

$$
\begin{align*}
v^{\top} & Q^{(k)} v \geq v^{\top} \mathbb{E}\left[\check{A}_{l}^{(k)} \check{A}_{l}^{(k) \top}\right] v=\mathbb{E}\left[\operatorname{tr}\left(v v^{\top} \check{A}_{l}^{(k)} \check{A}_{l}^{(k) \top}\right)\right] \\
& =\mathbb{E}\left[\check{\theta}_{l}^{(k) \top}\left(v v^{\top} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right) \check{\theta}_{l}^{(k)}\right] \\
& =\operatorname{tr}\left(\left(v v^{\top} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right) \mathbb{E}\left[\check{\theta}_{l}^{(k)} \check{\theta}_{l}^{(k) \top}\right]\right) \\
& =\operatorname{tr}\left(\left(v v^{\top} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right)\left(\lambda^{(k)} I_{p^{2}} \otimes P_{\mathcal{I}_{l}}^{-1}+\Lambda^{(k)^{-1}} \otimes I_{p\left|\mathcal{I}_{l}\right|}\right)^{-1}\right) \\
& \geq v^{\top}\left(\lambda^{(k)} \mu I_{p}+\Lambda^{(k)^{-1}}\right)^{-1} v p\left|\mathcal{I}_{l}\right| \\
& =v^{\top}\left(\Lambda^{(k)}-\Lambda^{(k)}\left(\Lambda^{(k)}+\frac{1}{\lambda^{(k)} \mu} I_{p}\right)^{-1} \Lambda^{(k)}\right) v p\left|\mathcal{I}_{l}\right|>0 \tag{74}
\end{align*}
$$

where we exploited (68) and $\mu>0$ is the maximum eigenvalue of $P_{\mathcal{I}_{l}}$. Accordingly, if $v \in \mathbf{Q}$ then $\Lambda^{(k)} v \rightarrow 0$ as $k \rightarrow \infty$.

## Proof of Proposition 10

We use the same decomposition for $\breve{P}$ exploited in the proof of Proposition 4. Thus, we have

$$
\begin{align*}
Q & \geq \mathbb{E}\left[A_{l}\left(P^{-1} \otimes I_{p}\right) A_{l}^{\top}\right] \geq \mathbb{E}\left[A_{l}\left(\breve{P}^{-1} \otimes I_{p}\right) A_{l}^{\top}\right] \\
& \geq \mathbb{E}\left[A_{l}\left(D^{-1} \otimes I_{p}\right)\left(\bar{P}^{-1} \otimes I_{p}\right)\left(D^{-1} \otimes I_{p}\right) A_{l}^{\top}\right] \\
& \geq \lambda^{-1} \mathbb{E}\left[A_{l}\left(D^{-2} \otimes I_{p}\right) A_{l}^{\top}\right] \\
& \geq \lambda^{-1} \sum_{t=1}^{\infty} t^{-(\alpha+\varepsilon)} e^{\beta t} \mathbb{E}\left[L_{t} L_{t}^{\top}\right] \tag{75}
\end{align*}
$$

where $\lambda$ is the maximum eigenvalue of $\bar{P}$ which is bounded. Hence, condition (43) implies that

$$
\begin{equation*}
\sum_{t=1}^{\infty} t^{-(\alpha+\varepsilon)} e^{\beta t} \mathbb{E}\left[\left|\left[L_{t}\right]_{i j}\right|^{2}\right] \tag{76}
\end{equation*}
$$

is bounded for $i, j=1 \ldots p$. Accordingly, $\mathbb{E}\left[\left|\left[L_{t}\right]_{i j}\right|^{2}\right] \leq$ $\bar{\kappa}_{\varepsilon} t^{\alpha+\varepsilon} e^{-\beta t}, t \in \mathbb{N}$, for some $\bar{\kappa}_{\varepsilon}>0$. Last, let $v$ be in the null space of $Q$, i.e. $v^{\top} Q v=0$. It follows from (43) that

$$
\mathbb{E}\left[v^{\top} A_{l}\left(P^{-1} \otimes I_{p}\right) A_{l}^{\top} v\right]=v^{\top} Q v=0
$$

Since $\left(P^{-1} \otimes I_{p}\right)$ is positive definite this implies that $\mathbb{E}\left[v^{\top} A_{l} A_{l}^{\top} v\right]=0$, which completes the proof.

## Proof of Theorem 11

The statement is proved by using the duality theory as in the proof of Theorem 6. In particular the Lagrange function is

$$
\begin{align*}
\mathcal{L}(\mathbf{p}, \tilde{\Gamma}, \tilde{\Lambda})= & \mathbf{H}\left(\mathbf{p}_{\mathcal{I}}\right) \\
& +\frac{1}{2} \sum_{i, j=1}^{p} \tilde{\gamma}_{(i-1) p+j}\left(c_{i j}-\mathbb{E}\left[\left\|\check{s}^{[i j]}\right\|_{P_{\mathcal{I}_{s}}^{-1}}^{2}\right]\right) \\
& +\frac{1}{2} \operatorname{tr}\left(\tilde{\Lambda}\left(Q-\mathbb{E}\left[\check{A}_{l}\left(P_{\mathcal{I}_{l}}^{-1} \otimes I_{p}\right) \check{A}_{l}^{\top}\right]\right)\right) \tag{77}
\end{align*}
$$

where $\tilde{\Gamma}=\operatorname{diag}\left(\tilde{\gamma}_{1} \ldots \tilde{\gamma}_{p^{2}}\right)$ with $\tilde{\gamma}_{i} \geq 0, i=1 \ldots p^{2}$, and $\tilde{\Lambda} \in \overline{\mathcal{M}}_{p}^{+}$. Then, it is not difficult to see that $\mathbf{p}_{\mathcal{I}}$ maximizing $\mathcal{L}$ is such that $\mathbf{p}_{\mathcal{I}}=\mathbf{p}_{\mathcal{I}_{s}} \mathbf{p}_{\mathcal{I}_{l}}$ where

$$
\begin{align*}
\mathbf{p}_{s} & =\frac{1}{c_{s}} \exp \left(-\frac{1}{2} \check{\theta}_{s}^{\top}\left(\tilde{\Gamma} \otimes P_{\mathcal{I}_{s}}^{-1}\right) \check{\theta}_{s}\right) \\
\mathbf{p}_{l} & =\frac{1}{c_{l}} \exp \left(-\frac{1}{2} \check{\theta}_{l}^{\top}\left(\tilde{\Lambda} \otimes I_{p} \otimes P_{\mathcal{I}_{l}}^{-1}\right) \check{\theta}_{l}\right) \tag{78}
\end{align*}
$$

with $c_{s}$ and $c_{l}$ normalization constants. Therefore, the optimal solution (if it does exist) is such that $\check{\theta}_{s}$ and $\check{\theta}_{l}$ are
independent, Gaussian, with zero mean and covariance matrix, respectively,

$$
\begin{align*}
K_{S} & =\tilde{\Gamma}^{-1} \otimes P_{\mathcal{I}_{s}} \\
K_{L} & =\tilde{\Lambda}^{-1} \otimes I_{p} \otimes P_{\mathcal{I}_{l}} \tag{79}
\end{align*}
$$

The existence of such a solution is proved by showing that the dual problem admits solution. The latter consists in minimizing the function

$$
\begin{align*}
J(\tilde{\Gamma}, \tilde{\Lambda})=-\left|\mathcal{I}_{s}\right| & \log \operatorname{det}(\tilde{\Gamma})+\operatorname{tr}(\tilde{\Gamma} C) \\
& -p\left|\mathcal{I}_{l}\right| \log \operatorname{det}(\tilde{\Lambda})+\operatorname{tr}(\tilde{\Lambda} Q) \tag{80}
\end{align*}
$$

where $C$ has been defined in (71). In this case the search of the minimum can be restricted to the compact set

$$
\begin{equation*}
\left\{(\tilde{\Gamma}, \tilde{\Lambda}) \text { s.t. } 0<\tilde{\Gamma} \leq M_{1} I_{p^{2}}, \quad 0<\tilde{\Lambda} \leq M_{2} I_{p}\right\} \tag{81}
\end{equation*}
$$

for some $M_{1}, M_{2}>0$. Moreover, $J$ is continuous over this set. Thus, by the Weiestrass Theorem the dual problem admits solution. Finally, setting $\gamma_{i}=\tilde{\gamma}_{i}^{-1}$ and $\Lambda=\tilde{\Lambda}^{-1}$ we obtain the kernel matrices in the statement.

## Proof of Proposition 12

The proof follows the same lines of that of Proposition 7.

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[^0]:    ${ }^{\star}$ This work has been partially supported by the FIRB project "Learning meets time" (RBFR12M3AC) funded by MIUR.

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[^1]:    ${ }^{2}$ Since the initial conditions are set equal to zero, in practice this corresponds to invert a matrix whose dimension is proportional to size of the data.

