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Optimal switching for linear quadratic problem of switched systems in discrete time ⋆

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Abstract

The optimal switching problem is attracting plenty of attention. This problem can be considered as a special type of discrete optimization problem and is NP complete. In this paper, a class of optimal switching problem involving a family of linear subsystems and a quadratic cost functional is considered in discrete time, where only one subsystem is active at each time point. By deriving a precise lower bound expression and applying the branch and bound method, a computational method is developed for solving this discrete optimization problem. Numerical examples have been implemented to demonstrate the efficiency and effectiveness of the proposed method.

Key words: Switched system; lower bound dynamic system; positive semi-definite.

1 Introduction

Switched system is an important class of hybrid systems. It is usually composed of multiple governed subsystems and a switching law among them. There are many real world applications of this system, such as automotive systems, electrical circuit systems, aircraft and traffic control, and so on.

In finding the optimal switching law for a switched system, the switching sequence can be sought such that a given cost functional is minimized. The characteristics of the optimal law has been studied. For example, Sussmann [1] presented a maximum principle for hybrid optimal control problems. For a pre-specified sequence of active subsystems, Shaikh [2] proposed a class of general hybrid maximum principle. Some stability results can be found in [3,4].

In the literature, when the sequence of the deployed system is planned in advance, there are many computational results proposed to optimize on the time when the system switches. For example, in [5–8], the control problem was formulated as determining the optimal switching times so as to minimize a quadratic objective functional. Then the optimal switching times are sought via the derivatives of the objective functional with respect to the switching times. Furthermore, to seek both optimal switching times and optimal continuous inputs, Xu and Antsaklis [9] presented an optimal control framework of the switched system based on a two-stage optimization. In [10], the control parameterization enhancing transform (CPET)[11–13] is applied to find the optimal switching times. In [14], a neighboring extremal solution is considered for a class of optimal switched impulsive control problems with perturbations, where the switching sequence is pre-specified. In this way, the problem of determining the optimal switching times for a given sequence of active subsystems can be transformed into a nonlinear programming problem which can be solved by existing gradient based techniques.

However, different from the switching time optimization which is essential continuous, the determination of the optimal switching sequence of the deployed system is combinatorial in nature. Because it is a discrete optimization problem, the search for all possible switching sequences could have an exponential complexity. In or-

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der to obtain the optimal switching sequence efficiently, Bengea and DeCarlo explored the optimal control problem of a two-switched system in [15], where the switched system was embedded into a larger family of systems. In [16], the ineffectiveness of the above method due to a possible infinite-loop procedure at each step was shown and the algorithm was improved by using other optimization techniques. At the same time, Wardi and Egerstedt proposed an adaptive-precision algorithm in [17] which is based on simultaneous swapping of subsystems at uncountable time-sets whose Lebesgue measures are determined by the Armijio step size to solve the above problem. The similar problem is also considered in [18], where an optimal switching sequence is designed for jump linear systems with given Gaussian initial state uncertainty. Gradient-based methods can also be found in [19,20] for solving discrete-valued optimal control problem, where an equivalent penalty problem was constructed to solve the optimal control problem with piecewise constant controls and fixed switching times. Some stochastic methods such as evolutionary algorithms and simulated annealing algorithm were also developed in [21–23].

Although the continuous system is popular in practice, it should be discretized into discrete system to find the numerical solution. For example, Murphey et al. [24] considered the discretized hybrid dynamical systems, and proposed an optimization method to determine the optimal switching time. A linear-quadratic control problem for discrete time switched systems with uncertain subsystems is considered in [25] and the analytical solution is derived. However, the optimal switching problem of discrete switched system is NP-complete, and there is no literature which can consider the global solution of this problem now, even in its simple case. For this, we consider the discrete system case, where the switched subsystems are linear and the cost functional is quadratic. We aim to propose an efficient method to find the global solution of this problem.

The rest of the paper is organized as follows. In Section 2, the optimal switching problem of switched systems in discrete time is formulated, where the subsystems are linear and the cost functional is quadratic. In Section 3, we analyze the positive semi-definite property and construct a lower bound dynamic system, which is used to compute the lower bound. Then, a branch and bound method is proposed to solve this problem in Section 4. For illustration, two numerical examples are implemented in Section 5 to demonstrate the efficiency of the method.

2 Problem Formulation

For a discrete time switched problem with N subsystems, we consider a dynamic system governed by the following family of linear difference equations

$$\mathbf{x}(t+1) = \mathbf{A}_i(t)\mathbf{x}(t), \quad t \in I,\, i = 1, 2, \ldots, N,$$

with initial condition

$$\mathbf{x}(0) = \mathbf{x}_0,$$

where $I = \{0, 1, \cdots, T-1\}$. For each $t \in I$, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $i = 1, 2, \cdots, N$. The state $\mathbf{x}(t)$ and initial state $\mathbf{x}_0$ are n-dimensional column vectors.

A switching sequence is denoted by a function $u : I \to \Lambda = \{1, 2, ..., N\}$. In particular, $u(t) = i$ means that the i-th subsystem is active at time t. Let U be the set of all such switching sequences.

We formulate a discrete-time optimal switching problem as follows:

Problem 1. *Find a switching sequence $u \in U$ such that*

$$J(u) = \sum_{t=1}^{T} \mathbf{x}^{\intercal}(t)\mathbf{Q}(t)\mathbf{x}(t)$$

is minimized, subject to the linear dynamic constraint:

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}_{u(t)}(t)\mathbf{x}(t), \quad t \in I, \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned}$$

where, for each $t \in I$, $\mathbf{Q}(t+1)$ is an $n \times n$ positive semi-definite matrix, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $i \in \Lambda$. The initial state $\mathbf{x}_0$ is a given n-dimensional column vector.

Remark 1. *If the linear dynamic constraint in Problem 1 is given by*

$$\mathbf{x}(t+1) = \mathbf{A}_i(t)\mathbf{x}(t) + \mathbf{b}_i(t), \quad i \in \Lambda, \tag{1}$$

where for each $t \in I$ and $i \in \Lambda$, $\mathbf{A}_i(t) \in \mathbb{R}^{n\times n}$, $\mathbf{b}_i(t) \in \mathbb{R}^n$, we can solve the problem similar to Problem 1 through a suitable transformation. For this, we introduce some new symbols as follows:

$$\tilde{\mathbf{A}}_i(t) = \begin{bmatrix} \mathbf{A}_i(t) & \mathbf{b}_i(t) \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$\tilde{\mathbf{Q}}(t) = \begin{bmatrix} \mathbf{Q}(t) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 0 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ 1 \end{bmatrix} \in \mathbb{R}^{(n+1)}.$$

Then, the optimal switching problem with linear dynamic constraint (1) is equivalent to a new problem which is of *the form Problem 1, where the cost functional is given by*

$$J(u) = \sum_{t=1}^{T} \mathbf{y}^{\mathsf{T}}(t)\tilde{\mathbf{Q}}(t)\mathbf{y}(t),$$

and the dynamic constraint becomes

$$\begin{aligned}\mathbf{y}(t+1) &= \tilde{\mathbf{A}}_{u(t)}(t)\mathbf{y}(t), \quad t \in I,\\ \mathbf{y}^{\mathsf{T}}(0) &= \begin{bmatrix} \mathbf{x}_0 & 1 \end{bmatrix}^{\mathsf{T}}.\end{aligned}$$

Remark 2. *If the cost functional in Problem 1 becomes*

$$J(u) = \sum_{t=1}^{T} (\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{r}^{\mathsf{T}}(t)\mathbf{x}(t) + s(t)), \tag{2}$$

where for each $t \in I$, $\mathbf{Q}(t+1) \in \mathbb{R}^{n\times n}$ is positive semi-definite, $\mathbf{r}(t+1) \in \mathbb{R}^{n\times 1}$ is a column vector function, $s(t+1) \in \mathbb{R}$ is a function. This problem can also be transformed into a new problem which is of the form Problem 1 by introducing some new symbols as follows:

$$\hat{\mathbf{Q}}(t) = \begin{bmatrix} \mathbf{Q}(t) & \frac{1}{2}\mathbf{r}(t) \\ \frac{1}{2}\mathbf{r}^{\mathsf{T}}(t) & w(t) \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$\hat{\mathbf{A}}_i(t) = \begin{bmatrix} \mathbf{A}_i(t) & \mathbf{0}_{n\times 1} \\ \mathbf{0}_{1\times n} & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)},$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ 1 \end{bmatrix} \in \mathbb{R}^{(n+1)},$$

where $w(t)$ is any positive function such that $\hat{\mathbf{Q}}(t)$ is positive semi-definite. Then, the cost functional becomes

$$\begin{aligned}\tilde{J}(u) &= \sum_{t=1}^{T} \left(\mathbf{x}^{\mathsf{T}}(t)\mathbf{Q}(t)\mathbf{x}(t) + \mathbf{r}^{\mathsf{T}}(t)\mathbf{x}(t) + s(t)\right)\\ &= \sum_{t=1}^{T} \mathbf{y}^{\mathsf{T}}(t)\hat{\mathbf{Q}}(t)\mathbf{y}(t) + \sum_{t=1}^{T} \left(s(t) - w(t)\right).\end{aligned}$$

Note that $\sum_{t=1}^{T}(s(t) - w(t))$ is a constant. The cost functional is equivalent to

$$\hat{J}(u) = \sum_{t=1}^{T} \mathbf{y}^{\mathsf{T}}(t)\hat{\mathbf{Q}}(t)\mathbf{y}(t). \tag{3}$$

The dynamic constraint becomes

$$\begin{aligned}\mathbf{y}(t+1) &= \hat{\mathbf{A}}_{u(t)}(t)\mathbf{y}(t), \quad t \in I,\\ \mathbf{y}^{\mathsf{T}}(0) &= \begin{bmatrix} \mathbf{x}_0 & 1 \end{bmatrix}^{\mathsf{T}}.\end{aligned}$$

Since U is a finite set, the existence of optimal solutions for Problem 1 is obvious. Completing enumeration of all switching sequences is feasible when the numbers of time points and subsystems are small. However, the number of switching sequences will grow exponentially with respect to the number of switching time points. Thus, efficient method should be developed to solve this problem.

3 Lower bound analysis

Note that this problem is discrete and NP-complete. There are very few methods which can solve the discrete optimization problem efficiently. Branch and bound method has been a popular technique for discrete optimization and is one of the efficient methods. To implement this method, we have to compute a precise lower bound at first for any current switching sequence $(u(1), u(2), \ldots, u(t))$.

3.1 Upper bound of positive semi-definite matrix

Before constructing an efficient lower bound expression, we need to analyze some properties of the positive semi-definite matrix. First, we have the definition below.

Definition 1. *Given two real symmetric matrices $\mathbf{P}_1$ and $\mathbf{P}_2$ with the same dimension, if $\mathbf{P}_1 - \mathbf{P}_2$ is a positive semi-definite matrix, that is $\mathbf{P}_1 - \mathbf{P}_2 \geq \mathbf{0}$, then we call that $\mathbf{P}_1$ greater than or equal to $\mathbf{P}_2$, which is denoted by $\mathbf{P}_1 \geq \mathbf{P}_2$.*

For any switching sequence u, denote an $n \times n$ matrix function by

$$\mathbf{P}_u(t) = \mathbf{A}^{\mathsf{T}}_{u(0)}(t)\cdots\mathbf{A}^{\mathsf{T}}_{u(t)}(t)\mathbf{Q}(t+1)\mathbf{A}_{u(t)}(t)\cdots\mathbf{A}_{u(0)}(t).$$

Then, it follows by Definition 1 that we have

Lemma 1. *Consider Problem 1. Suppose that there are two switching sequences u_1 and u_2 such that*

$$\mathbf{P}_{u_1}(t) \leq \mathbf{P}_{u_2}(t), \quad t \in I.$$

Then, we have

$$J(u_1) \leq J(u_2).$$

Proof. Since $\mathbf{P}_{u_1}(t) \leq \mathbf{P}_{u_2}(t)$, $\forall t \in I$, we have $\mathbf{P}_{u_1}(t) - \mathbf{P}_{u_2}(t) \leq \mathbf{0}$, that is, $\mathbf{P}_{u_2}(t) - \mathbf{P}_{u_1}(t)$ is a positive semi-definite matrix function. Then, for any nonzero n-dimensional vector $\mathbf{x}$, we have $\mathbf{x}^{\mathsf{T}}[\mathbf{P}_{u_2}(t) - \mathbf{P}_{u_1}(t)]\mathbf{x} \geq 0$, that is $\mathbf{x}^{\mathsf{T}}\mathbf{P}_{u_2}(t)\mathbf{x} \geq \mathbf{x}^{\mathsf{T}}\mathbf{P}_{u_1}(t)\mathbf{x}$, $t \in I$. It is clear to see that $J(u) = \sum_{t=0}^{T-1} \mathbf{x}^{\mathsf{T}}(0)\mathbf{P}_u(t)\mathbf{x}(0)$. Therefore, $J(u_1) \leq J(u_2)$. $\square$

To compute a lower bound for a switching sequence, it follows from Lemma 1 that we need to construct a lower bound dynamic system $\Psi(t)$ such that for each $t \in I_1$,

$$\Psi(t) \le \mathbf{P}_u(t), \quad \forall u \in U. \tag{4}$$

Next, the property of the positive definite matrix can be used to obtain a lower bound $\Psi(t)$. Note that the choice of $\Psi(t)$ is not unique.

Lemma 2. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are* n*-dimensional positive definite matrices, then* $\mathbf{A} > \mathbf{B}$ *is equivalent to* $\rho(\mathbf{B}\mathbf{A}^{-1}) < 1$*, where* $\rho(\mathbf{A})$ *is the spectral radius of* $\mathbf{A}$.

Proof. Since $\mathbf{A}$ and $\mathbf{B}$ are positive definite matrices with the same dimension, the matrices $\mathbf{A}$ and $\mathbf{B}$ can be diagonalized at the same time. Hence, we can find a nonsingular matrix $\mathbf{C}$, such that

$$\mathbf{A} = \mathbf{C}\mathbf{I}\mathbf{C}^{\intercal}, \quad \mathbf{B} = \mathbf{C}\mathbf{D}\mathbf{C}^{\intercal},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{D} = diag(d_1, d_2, \ldots, d_n)$. Hence, $\mathbf{A} > \mathbf{B}$ if and only if $\mathbf{C}(\mathbf{I} - \mathbf{D})\mathbf{C}^{\intercal} > \mathbf{0}$, that is, $0 < d_i < 1$, $i = 1, 2, \ldots, n$. However, note that

$$\mathbf{B}\mathbf{A}^{-1} = \mathbf{C}\mathbf{D}\mathbf{C}^{\intercal}(\mathbf{C}^{\intercal})^{-1}\mathbf{I}\mathbf{C}^{-1} = \mathbf{C}\mathbf{D}\mathbf{C}^{-1},$$

then the eigenvalue of $\mathbf{B}\mathbf{A}^{-1}$ happens to be d_i, $i = 1, 2, \ldots, n$. That is, $\rho(\mathbf{B}\mathbf{A}^{-1}) < 1$. Thus, $\mathbf{A} > \mathbf{B}$ is equivalent to $\rho(\mathbf{B}\mathbf{A}^{-1}) < 1$.

This completes the proof. □

Lemma 3. *Suppose that* $\mathbf{A}$ *and* $\mathbf{B}$ *are positive definite matrices with the same dimension, then* $\mathbf{A} > \mathbf{B}$ *if and only if* $\mathbf{B}^{-1} > \mathbf{A}^{-1}$.

Proof. It follows by Lemma 2 that $\mathbf{A} > \mathbf{B}$ if and only if $\rho(\mathbf{B}\mathbf{A}^{-1}) < 1$. Since $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{B}\mathbf{A}^{-1}$ have the same eigenvalues, we have $\rho(\mathbf{B}\mathbf{A}^{-1}) = \rho(\mathbf{A}^{-1}\mathbf{B})$. Then, we have $\rho(\mathbf{A}^{-1}\mathbf{B}) < 1$, and $\mathbf{A} > \mathbf{B}$ if and only if $\mathbf{B}^{-1} > \mathbf{A}^{-1}$. □

We can compute a lower bound for any current switching sequence in Problem 1, based on the theorem below.

Theorem 1. *Suppose that there is an* $n \times n$ *symmetric matrix* $\mathbf{M}$*. If a matrix* Ψ *is constructed by*

$$\Psi = \begin{bmatrix} \sum_{j=1}^{n} |M_{1j}| & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sum_{j=1}^{n} |M_{nj}| \end{bmatrix}, \tag{5}$$

then, we have $\mathbf{M} \le \Psi$.

Proof. We prove that $\Psi - \mathbf{M}$ is a positive semi-definite matrix.

Denote a matrix $\mathbf{P}$ by

$$\begin{aligned} \mathbf{P} =& \Psi - \mathbf{M} \\ =& \begin{bmatrix} \sum_{j=1}^{n} |M_{1j}| - M_{11} & \cdots & -M_{1n} \\ \vdots & \ddots & \vdots \\ -M_{n1} & \cdots & \sum_{j=1}^{n} |M_{nj}| - M_{nn} \end{bmatrix}. \end{aligned}$$

Obviously, $\mathbf{P}$ is also an $n \times n$ symmetry matrix, and

$$\sum_{j=1}^{n} |P_{ij}| = \sum_{j=1, j\neq i}^{n} |M_{ij}| + \left| \sum_{j=1}^{n} |M_{ij}| - M_{ii} \right|, \quad \forall i = 1, 2, \ldots, n.$$

For this matrix $\mathbf{P}$, we have

$$\sum_{j=1}^{n} |P_{ij}| \le 2P_{ii}, \quad \forall i = 1, 2, \ldots, n. \tag{6}$$

This is because for any i, there are two cases $M_{ii} \ge 0$ and $M_{ii} < 0$. If $M_{ii} \ge 0$, we have

$$\sum_{j=1}^{n} |P_{ij}| = 2 \sum_{j=1, j\neq i}^{n} |M_{ij}| = 2P_{ii}, \quad \forall i = 1, 2, \ldots, n.$$

If $M_{ii} < 0$, we have

$$\sum_{j=1}^{n} |P_{ij}| = 2 \sum_{j=1}^{n} |M_{ij}| < 2P_{ii}, \quad \forall i = 1, 2, \ldots, n.$$

Next, we prove that the symmetric matrix $\mathbf{P}$ which satisfies condition (6) is a positive semi-definite matrix by mathematical induction.

For the case $n = 1$, it follows from (6) that $P = P_{11} \ge 0$ is a positive number, then P is a positive semi-definite matrix obviously.

Suppose that any symmetric matrix which satisfies the condition (6) is a positive semi-definite matrix if its dimension is $n = k$. We point out that the symmetric matrix $\mathbf{P}$ which satisfies the condition (6) is also a positive semi-definite matrix if its dimension is $n = k + 1$.

According to the condition (6), we have $P_{ii} \ge 0$, $\forall i = 1, \ldots, k+1$. If $P_{ii} = 0$, $\forall i = 1, \ldots, k+1$, then $\mathbf{P}$ is a zero matrix, which is a positive semi-definite. Suppose that there exists an $i \in \{1, 2, \ldots, k+1\}$, such that $P_{ii} > 0$. Without loss of generality, let $P_{11} > 0$, we have

$$\begin{bmatrix} 1 & \mathbf{0} \\ -\frac{\mathbf{S}}{P_{11}} & I_k \end{bmatrix} \begin{bmatrix} P_{11} & \mathbf{S}^\intercal \\ \mathbf{S} & \mathbf{Q} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\mathbf{S}^\intercal}{P_{11}} \\ \mathbf{0} & I_k \end{bmatrix} = \begin{bmatrix} P_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}} \end{bmatrix},$$

where

$$\mathbf{S} = (P_{21}, P_{31}, \ldots, P_{k+1,1})^\intercal,$$

$$\mathbf{Q} = \begin{bmatrix} P_{22} & \ldots & P_{2,k+1} \\ \vdots & \ddots & \vdots \\ P_{k+1,2} & \ldots & P_{k+1,k+1} \end{bmatrix}.$$

Then, it follows from the condition (6) that for each $i = 1, 2, \ldots, k$, we have

$$\begin{aligned} \left(\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}\right)_{ii} &= \frac{1}{P_{11}}(P_{11}P_{i+1,i+1} - P_{i+1,1}^2) \\ &\geq \frac{1}{P_{11}}(|P_{i+1,1}|^2 - P_{i+1,1}^2) \geq 0. \end{aligned}$$

Hence, we have

$$\begin{aligned} &2\left(\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}\right)_{ii} - \sum_{j=1}^{k} \left|\left(\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}\right)_{ij}\right| \\ =&2\left(P_{i+1,i+1} - \frac{P_{i+1,1}^2}{P_{11}}\right) - \sum_{j=1}^{k} \left|P_{i+1,j+1} - \frac{P_{i+1,1}P_{j+1,1}}{P_{11}}\right| \\ =&P_{i+1,i+1} - \frac{P_{i+1,1}^2}{P_{11}} - \sum_{j=1, j\neq i}^{k} \left|P_{i+1,j+1} - \frac{P_{i+1,1}P_{j+1,1}}{P_{11}}\right| \\ \geq&P_{i+1,i+1} - \sum_{j=1, j\neq i}^{k} |P_{i+1,j+1}| - |P_{i+1,1}| \sum_{j=1}^{k} \frac{|P_{j+1,1}|}{P_{11}} \\ \geq&P_{i+1,i+1} - |P_{i+1,1}| - \sum_{j=2, j\neq i+1}^{k+1} |P_{i+1,j}| \\ =&P_{i+1,i+1} - \sum_{j=1, j\neq i+1}^{k} |P_{i+1,j}| \geq 0. \end{aligned}$$

That is, for each $i = 1, 2, \ldots, k$,

$$2\left(\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}\right)_{ii} \geq \sum_{j=1}^{k} \left|\left(\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}\right)_{ij}\right|.$$

So, $\mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}}$ is a positive semi-definite matrix. Therefore, $\begin{bmatrix} P_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} - \frac{\mathbf{S}\mathbf{S}^\intercal}{P_{11}} \end{bmatrix}$ is a positive semi-definite matrix. Then, $\begin{bmatrix} P_{11} & \mathbf{S}^\intercal \\ \mathbf{S} & \mathbf{Q} \end{bmatrix}$ is a positive semi-definite matrix with the dimension $k+1$. It follows from mathematical induction that $\mathbf{P}$ is a positive semi-definite matrix, that is, $\Psi \geq \mathbf{M}$.

This completes the proof. $\square$

3.2 Lower bound expression

In this section, we derive the lower bound expression. For a given switching sequence u, if $u(0), u(1), \ldots, u(j-1)$ are chosen and $u(j), u(1), \ldots, u(T-1)$ are not determined, we should compute the lower bound to decide whether $u(j), u(1), \ldots, u(T-1)$ should be chosen or eliminated. For this, we will construct a positive definite matrix $\Psi(t)$, such that $\forall i_1, i_2, \ldots, i_{t-j} \in \{1, 2, \ldots, N\}$,

$$\begin{aligned} \Psi(t) \leq& \mathbf{A}_{i_1}^\intercal(j+1)\mathbf{A}_{i_2}^\intercal(j+2)\cdots\mathbf{A}_{i_{t-j}}^\intercal(t)\mathbf{Q}(t+1)\cdot \\ &\cdot \mathbf{A}_{i_{t-j}}(t)\cdots\mathbf{A}_{i_2}(j+2)\mathbf{A}_{i_1}(j+1), \end{aligned} \tag{7}$$

where $t = j+1, j+2, \ldots, T-1$.

We rewrite the cost functional as

$$\sum_{t=1}^{j} \mathbf{x}^\intercal(t)\mathbf{Q}(t)\mathbf{x}(t) + \sum_{t=j+1}^{T} \mathbf{x}^\intercal(t)\mathbf{Q}(t)\mathbf{x}(t).$$

The first term is fixed and the second term is not fixed. Hence, we need to find the lower bound of the second term. For this, we first find the lower bound of $\mathbf{x}^\intercal(j+1)\mathbf{Q}(j+1)\mathbf{x}(j+1)$. Note that the vector $\mathbf{x}(j)$ is fixed, we have

$$\begin{aligned} &\mathbf{x}^\intercal(j+1)\mathbf{Q}(j+1)\mathbf{x}(j+1) \\ =&\mathbf{x}^\intercal(j)\mathbf{A}_{u(j)}^\intercal(j)\mathbf{Q}(j+1)\mathbf{A}_{u(j)}(j)\mathbf{x}(j). \end{aligned}$$

Since $u(j)$ can be any value in $\{1, 2, \ldots, N\}$, suppose that we find a matrix $\Psi_1(j+1)$ such that

$$\Psi_1(j+1) \leq \mathbf{A}_k^\intercal(j)\mathbf{Q}(j+1)\mathbf{A}_k(j), \forall k = 1, \ldots, N. \tag{8}$$

Then, a lower bound of the term $\mathbf{x}^\intercal(j+1)\mathbf{Q}(j+1)\mathbf{x}(j+1)$ is given by $\mathbf{x}^\intercal(j)\Psi_1(j+1)\mathbf{x}(j)$.

Next, we consider the lower bound of the term $\mathbf{x}^\intercal(j+2)\mathbf{Q}(j+2)\mathbf{x}(j+2)$. Note that

$$\begin{aligned} &\mathbf{x}^\intercal(j+2)\mathbf{Q}(j+2)\mathbf{x}(j+2) \\ =&\mathbf{x}^\intercal(j)\mathbf{A}_{u(j)}^\intercal(j)\mathbf{A}_{u(j+1)}^\intercal(j+1)\mathbf{Q}(j+2) \\ &\cdot \mathbf{A}_{u(j+1)}(j+1)\mathbf{A}_{u(j)}(j)\mathbf{x}(j). \end{aligned}$$

Suppose that we can find a matrix $\Psi_2(j+2)$ such that

$$\begin{aligned} \Psi_2(j+2) \leq& \mathbf{A}_k^\intercal(j)\mathbf{A}_l^\intercal(j+1)\mathbf{Q}(j+2)\mathbf{A}_k(j+1)\mathbf{A}_k(j), \\ &\forall k, l = 1, 2, \ldots, N. \end{aligned} \tag{9}$$

Then, a lower bound of the term $\mathbf{x}^\intercal(j+2)\mathbf{Q}(j+2)\mathbf{x}(j+2)$ is given by $\mathbf{x}^\intercal(j)\Psi_2(j+2)\mathbf{x}(j)$.

Similarly, for the term $\mathbf{x}^\intercal(j+i)\mathbf{Q}(j+i)\mathbf{x}(j+i)$, its lower bound is given by $\mathbf{x}^\intercal(j)\Psi_i(j+i)\mathbf{x}(j)$, where the matrix $\Psi_i(j+i)$ satisfies

$$\begin{aligned}\Psi_i(j+i) \leq & \mathbf{A}_{k_1}^\intercal(j)\mathbf{A}_{k_2}^\intercal(j+1)\cdots\mathbf{A}_{k_i}^\intercal(j+i-1)\mathbf{Q}(j+i) \\ & \cdot\mathbf{A}_{k_i}(j+i-1)\cdots\mathbf{A}_{k_2}(j+1)\mathbf{A}_{k_1}(j), \\ & \forall k_1,\ldots,k_i = 1,2,\ldots,N. \end{aligned} \tag{10}$$

Combining (8), (9) and (10), the lower bound of the term $\sum_{t=j+1}^{T}\mathbf{x}^\intercal(t)\mathbf{Q}(t)\mathbf{x}(t)$ is given by

$$\mathbf{x}^\intercal(j)\left(\sum_{t=j+1}^{T}\Psi_{t-j}(t)\right)\mathbf{x}(j).$$

Then, we need to find the expression of $\Psi_i(j+i)$, where the conditions (8), (9) and (10) should be satisfied. For this, we first find the expression of $\Psi_1(t)$. Note that (8) is equivalent to

$$\Psi_1^{-1}(t) \geq \left(\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1)\right)^{-1}, \forall k = 1,\ldots,N.$$

However, the matrix $\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1)$ can be positive semi-definite and is not invertible. Then, the equation above does not hold. For this, we add a sufficiently small matrix such that this matrix is positive definite and rewrite (8) as

$$\Psi_1(t) + \varepsilon I \leq \mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1) + \varepsilon I,$$

where $\varepsilon > 0$ is a small number. Then, we have

$$\left(\Psi_1(t) + \varepsilon I\right)^{-1} \geq \left(\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1) + \varepsilon I\right)^{-1},$$

Let $\varepsilon > 0$ approaches to zero, it follows from Lemma 3 that for any $k = 1,2,\ldots,N$, we have

$$\begin{aligned}\Psi_1(t) &= \lim_{\varepsilon\to 0+}\Psi_1(t) + \varepsilon I = \lim_{\varepsilon\to 0+}\left((\Psi_1(t)+\varepsilon I)^{-1}\right)^{-1} \\ &\leq \lim_{\varepsilon\to 0+}\left(\left(\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1)+\varepsilon I\right)^{-1}\right)^{-1}.\end{aligned}$$

Hence, it follows from Theorem 1 that the matrix $\Psi_1(t)$ can be chosen as a diagonal matrix by

$$\Psi_1(t) = \begin{bmatrix} \max\limits_k \phi_{k1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \max\limits_k \phi_{kn} \end{bmatrix}^{-1},$$

where $\phi_{kl} = \sum_{j=1}^{n}|\Phi_{lj}|$, and for each k, Φ_k is given by

$$\Phi_k = \left(\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1) + \varepsilon I\right)^{-1},$$

where $\varepsilon > 0$ is a sufficiently small number.

Next, we consider the choice of $\Psi_2(t)$. It will be complicated if we derive it directly from (9). To simplify the computation of $\Psi_2(t)$, the equation (9) can be derived by the equation below.

$$\Psi_2(t) \leq \mathbf{A}_k^\intercal(t-2)\Psi_1(t)\mathbf{A}_k(t-2), \quad \forall k = 1,2,\ldots,N.$$

Similarly, the general case (10) can be derived from the equation below.

$$\Psi_i(t) \leq \mathbf{A}_k^\intercal(t-i)\Psi_{i-1}(t)\mathbf{A}_k(t-i), \quad \forall k = 1,2,\ldots,N.$$

Hence, we can choose the matrices $\Psi_i(t)$ by Algorithm 1.

Algorithm 1 (Calculating the matrix $\Psi_i(t)$)

---

(1) Choose a sufficiently small number $\varepsilon > 0$.
(2) For each k, calculate Φ_k by

$$\begin{cases} \left(\mathbf{A}_k^\intercal(t-1)\mathbf{Q}(t)\mathbf{A}_k(t-1) + \varepsilon I\right)^{-1}, & \text{if } i = 1, \\ \left(\mathbf{A}_k^\intercal(t-i)\Psi_{i-1}(t)\mathbf{A}_k(t-i) + \varepsilon I\right)^{-1}, & \text{if } i > 1. \end{cases}$$

(3) Let $\phi_{kl} = \sum_{j=1}^{n}|\Phi_{lj}|$.
(4) Choose the matrix as

$$\Psi_i(t) = \begin{bmatrix} \max\limits_k \phi_{k1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \max\limits_k \phi_{kn} \end{bmatrix}^{-1}.$$

For a given current switching sequence $u(0),\ldots,u(j-1)$, we need to compute all the values $\Psi_i(t), t = j+1,\ldots,T, i = 1,\ldots,t-j$. For this, the procedure above start from $i = 1$ to T. Then, a lower bound can be computed as

$$\begin{aligned}&L(u(0),u(1),\ldots,u(j-1)) \\ &= \sum_{t=1}^{j}\mathbf{x}^\intercal(t)\mathbf{Q}(t)\mathbf{x}(t) + \sum_{t=j+1}^{T}\mathbf{x}^\intercal(j)\Psi_{t-j}(t)\mathbf{x}(j).\end{aligned} \tag{11}$$

Let

$$\Theta_j(t) = \begin{cases} \Psi_{t-j}(t), & \text{if } t = j+1, \\ \Psi_{t-j}(t)(\Psi_{t-1-j}(t-1))^{-1}, & \text{if } t > j+1. \end{cases}$$

Obviously, $\Theta_j(t), \forall t > j$ are diagonal matrices. Then, for a given switching sequence $u(0), u(1), \ldots, u(j-1)$, a lower bound dynamic system is given by

$$\mathbf{y}(t+1) = \begin{cases} \mathbf{A}_{u(t)}(t)\mathbf{y}(t), & \text{if } t \le j-1, \\ (\Theta_j(t))^{1/2}\mathbf{y}(t), & \text{if } t \ge j, \end{cases}$$

with initial condition

$$\mathbf{y}(0) = \mathbf{x}_0.$$

For this system, it can be seen that if $t \le j$, we have

$$\mathbf{y}(t) = \mathbf{x}(t), \quad t \le j.$$

If $t = j+1$, we have

$$\begin{aligned} &\mathbf{y}^\intercal(t)\mathbf{y}(t) \\ =&\mathbf{y}^\intercal(t-1)(\Theta_j(t-1))^{1/2}(\Theta_j(t-1))^{1/2}\mathbf{y}(t-1) \\ =&\mathbf{y}^\intercal(t-1)(\Psi_{t-2-j}(t-2))^{-1}\Psi_{t-1-j}(t-1)\mathbf{y}(t-1) \\ =&\mathbf{y}^\intercal(j)\Psi_1(j+1)\cdot(\Psi_1(j+1))^{-1}\Psi_2(j+2)\cdots \\ &\qquad\cdots(\Psi_{t-1-j}(t-1))^{-1}\Psi_{t-j}(t)\mathbf{y}(j). \end{aligned}$$

If $t > j+1$, we have

$$\begin{aligned} &\mathbf{y}^\intercal(t)\mathbf{y}(t) \\ =&\mathbf{y}^\intercal(t-1)(\Theta_j(t-1))^{1/2}(\Theta_j(t-1))^{1/2}\mathbf{y}(t-1) \\ =&\mathbf{y}^\intercal(t-1)(\Psi_{t-2-j}(t-2))^{-1}\Psi_{t-1-j}(t-1)\mathbf{y}(t-1) \\ =&\mathbf{y}^\intercal(j)\Psi_1(j+1)\cdot(\Psi_1(j+1))^{-1}\Psi_2(j+2)\cdots \\ &\qquad\cdots(\Psi_{t-1-j}(t-1))^{-1}\Psi_{t-j}(t)\mathbf{y}(j). \end{aligned}$$

The lower bound (11) is rewritten as

$$\begin{aligned} &L(u(0), u(1), \ldots, u(j-1)) \\ =&\sum_{t=1}^{j}\mathbf{y}^\intercal(t)\mathbf{Q}(t)\mathbf{y}(t)+ \\ &\sum_{t=j+1}^{T}\mathbf{y}^\intercal(j)\Psi_1(j+1)\cdot(\Psi_1(j+1))^{-1}\Psi_2(j+2)\cdots \\ &\qquad\cdots(\Psi_{t-1-j}(t-1))^{-1}\Psi_{t-j}(t)\mathbf{y}(j) \\ =&\sum_{t=1}^{j}\mathbf{y}^\intercal(t)\mathbf{Q}(t)\mathbf{y}(t) + \sum_{t=j+1}^{T}\mathbf{y}^\intercal(t)\mathbf{y}(t). \end{aligned} \tag{12}$$

4 Branch and bound method

A switching sequence of a switched system is denoted by

$$u = \{u(0), u(1), \ldots, u(T-1)\},$$

where $u(t) \in \{1, 2, \ldots, N\}$, $\forall t \in I$. In general, enumeration of all switching sequences is very expensive. Suppose that the number of the switching time points is T, we have to search for all possible switching sequences whose number is N^T. To obtain the global optimal solution, it is necessary to reduce the search region to improve the computational complexity.

With the lower bound computed by the lower bound dynamic system, we propose a branch and bound algorithm to solve Problem 1.

First, we introduce the current switching sequence as

$$u = (u(0), u(1), \ldots, u(t)), \quad t < T-1.$$

That is, a current switching sequence is that part of the switching sequence $\{u(0), u(1), \ldots, u(t)\}$ have been chosen, and others $\{u(t+1), \ldots, u(T-1)\}$ are not determined.

To search all the switching sequences, we choose the depth first tree search. The subsystem $u(t)$ is chosen from $t = 0$ to $T-1$. If $u(t)$ is chosen at the current time $t = j$, $j \le T-1$, then the current switching sequence is $\{u(0), u(1), \ldots, u(j-1)\}$. Next, we compute the lower bounds $L(u(0), \ldots, u(j-1), v)$, $v = 1, 2, \ldots, N$, and arrange these N lower bounds by ascending rule, that is,

$$\begin{aligned} &L(u(0), \ldots, u(j-1), v_j(k)) \le \\ &\quad L(u(0), \ldots, u(j-1), v_j(k+1)), k = 1, \ldots, N-1. \end{aligned}$$

There are two cases for the lower bounds. The first case is that $L(u(0), \ldots, u(j-1), v_j(k)) \le J^*$, where J^* is the optimal value. It means that some switching sequence $(u(0), \ldots, u(j-1), v_j(k), *, \ldots, *)$ may be better than or equals to the optimal solution, where $*$ denotes some subsystem. Hence, the tree should be branched to find a possible better solution, that is, we choose $u(j) = v_j(k)$ and set the current time as the next time point $t = j+1$.

The second case is that $L(u(0), \ldots, u(j-1), v_j(k)) > J^*$. Then, all the switching sequences $(u(0), \ldots, u(j-1), v_j(k), *, \ldots, *)$ can be eliminated, where $*$ denotes any one subsystem. Hence, the tree is not necessary to branch further and can be pruned. Furthermore, it follows from the ascending rule that we have

$$\begin{aligned} &L(u(0), \ldots, u(j-1), v_j(l)) \\ \ge&L(u(0), \ldots, u(j-1), v_j(k)) > J^*, \quad \forall l > k. \end{aligned} \tag{13}$$

Hence, we can eliminate all the switching sequences $(u(0), \ldots, u(j-1), v_j(l), *, \ldots, *)$, where $*$ denotes any one subsystem. Then, we have finished the search of $u(j)$ at time j, and the current time goes back and becomes $t = j-1$. We choose $u(j-2)$ as another subsystem to continue the search.

However, the optimal value is not known in advance during the tree search. For this, we replace the optimal value with a current optimal value J_{min}, which is an upper bound of J^*. Then, the two cases above are replaced by $L(u(0), \ldots, u(j-1), v_j(k)) \leq J_{min}$ and $L(u(0), \ldots, u(j-1), v_j(k)) > J_{min}$, which can be depicted in Figure 1 and 2.

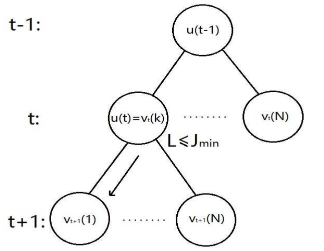

Fig. 1. Branch the tree when the lower bound is less than or equals to the current optimal value.

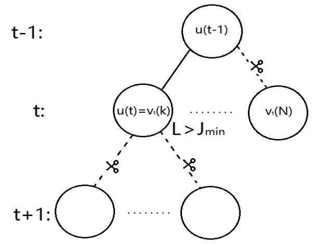

Fig. 2. Prune the tree when the lower bound is greater than the current optimal value.

A current optimal value J_{min} can be obtained by computing the cost functional value of a feasible switching sequence, since $J_{min} \geq J^*$. If $t = T-1$, $(u(0), \ldots, u(T-1))$ is no more a current switching sequence. It becomes a complete switching sequence. Then, the cost functional value $J(u(0), \ldots, u(T-1))$ can be computed and compared with the current optimal value J_{min}. If $J(u(0), \ldots, u(T-1)) < J_{min}$, then the current optimal value is updated as $J_{min} = J(u(0), \ldots, u(T-1))$, and the corresponding current solution is stored. Detail of updating J_{min} can also be depicted in Figure 3.

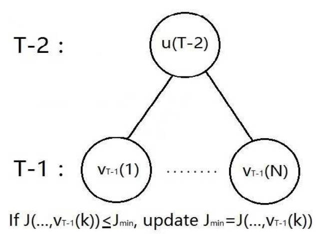

Fig. 3. Update the current optimal value when $t = T-1$.

If all the subsystems at time 0 have been chosen as $u(0)$, then all the switching sequences have been searched or branched. The current time goes back and becomes -1, and the search is finished.

Thus, we summarize the method as Algorithm 2.

Note that the initial value of J_{min} is $+\infty$, the lower bounds at the beginning are less than J_{min}, until the first switching sequence is obtained. Hence, every updated values of J_{min} correspond to at least one switching sequence.

It follows by Algorithm 2 that all the feasible solutions which are not computed are eliminated in Step 2(d). For any switching sequence u which is eliminated in Step 2(d) at current time t, we have

$$J(u) \geq L(u(0), \ldots, u(t)) > J_{min} \geq J^*.$$

Then, the switching sequence u which is eliminated in Step 2(d) can not be the global optimal solution. Hence, the global optimal solution u^* can be achieved.

Algorithm 2 (Branch and bound method)

Step 1(Initialization) Initialize the parameters T, N, $\mathbf{x}_0$. Set $J_{\min} = +\infty$ and the current time as $t = 0$.
Step 2 (Branch and bound search)
If $t = -1$, goes to Step 3. Else goes to Phase t.
Phase t:
(a) (Compute the lower bound and sort)
If $t < T-1$, compute $L(u(0), \ldots, u(t-1), i), i = 1 \ldots, N$ and arrange $v_t(1), \ldots, v_t(N)$ according to the ascending rule. Set $k_t = 1$ for Phase t.
(b) (Compute the cost functional value and update)
If $t = T-1$, compute $J(u(0), \ldots, u(T-1), i)$, $i = 1 \ldots, N$. If $J(u(0), \ldots, u(T-2), k_{T-1}) \leq J_{\min}$, then set $J_{\min}$ to this value and store all the current optimal solutions. Goes back to Phase $t-1$ with current time $t-1$.
(c) (Choose the value of $u(t)$)
If $k_t \leq N$, then choose $u(t) = v_t(k_t)$. Else goes back to Phase $t-1$ with current time $t-1$.
(d) (Condition for further branching)
If $L(u(0), \ldots, u(t)) > J_{\min}$, then break Loop t and goes back to Phase $t-1$ with current time $t-1$. Else goes to Phase $t+1$ with current time $t+1$.
Step 3 (Output and stop)
Output all the optimal switching sequences u and the optimal cost functional value $J^* = J_{\min}$, then stop.

5 Illustrative Example

In this section, the proposed method was implemented in Matlab 2012a. The computations were run on a notebook with the windows system, having a CPU speed of 2.50GHz and equipped with 4G of RAM.
Example 1.

We consider the optimal switching problem with $N = 10$ subsystems, where the parameters $\mathbf{A}_i, i = 1, \ldots, 10$ are

$$\begin{bmatrix} 1&0&1\\0&0&0\\1&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&1\\0&0&0\\0&0&0 \end{bmatrix},$$
$$\begin{bmatrix} 1&0&0\\1&0&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\1&1&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&1\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\1&1&0\\1&1&1 \end{bmatrix}, \begin{bmatrix} 1&1&1\\0&1&1\\0&0&1 \end{bmatrix}.$$

The other parameters are given by

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{Q}(t) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0.5 \\ 1 & 0.5 & 3 \end{bmatrix}.$$

First, we set the terminal time as $T = 5$ such that the branch and bound method can compare with the complete enumeration method. For the complete enumeration method, it is required to search 10^5 feasible solutions and the running time is about 24.37 seconds. The distribution of cost functional values for all feasible solutions can be found in Table 1. Note that most of the values are greater than 100, while the optimal cost functional value is obtained as 64. There are twenty global optimal solutions, which are listed as follow:

$$(2, 6, 5, 8, k), (2, 8, 5, 8, k), \ \forall k \in \{1, \ldots, 10\}.$$

Next, we apply the branch and bound method to solve this problem. We only search 600 feasible solutions and the running time is about 1.30 seconds. The cost functional values of these 600 feasible solutions are depicted in Figure 4. The branch and bound method only search the feasible solutions with the corresponding values a bit larger than the current optimal value. Hence, most of the insignificant switching sequences have been ignored and the branch and bound method is very efficient. The optimal value and corresponding optimal switching sequences are the same as above.

Table 1
The distribution of cost functional values for all switching sequences in the first example, where the first row: range of cost values, and the second row: number of solutions.

$[64, 10^2]$	$(10^2, 10^3]$	$(10^3, 10^4]$	$(10^4, 10^5]$	$(10^5, +\infty)$
1674	40778	52793	4749	6

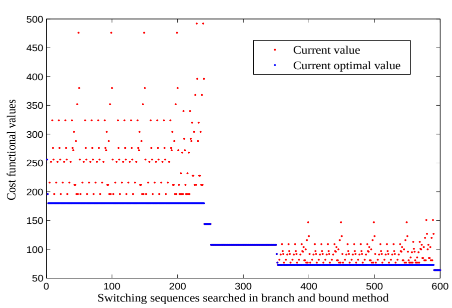

Fig. 4. The cost functional values of 600 feasible solutions and the current optimal values when the branch and bound method is applied in the first example.

Example 2.

We consider the optimal switching problem with $N = 6$ subsystems, where the parameters $\mathbf{A}_i(t), i = 1, \ldots, 6$ are

$$\begin{bmatrix} a(t) & 1 & b(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a(t) & 1 & 0 \\ 0 & 0 & b(t) \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} a(t) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b(t) \end{bmatrix},$$
$$\begin{bmatrix} a(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & b(t) \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a(t) & 1 & b(t) \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a(t) & 1 & 0 \\ 0 & 0 & b(t) \end{bmatrix},$$

where $a(t) = 1 + \sin t$ and $b(t) = 1 + \cos t$. The other parameters are given by

$$\mathbf{x}_0 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{Q}(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix}, \quad T = 8.$$

Note that the number of feasible solutions is $6^8 \approx 1.68 * 10^6$ and is too expensive to apply the enumeration method. For this, we apply the branch and bound method to solve this example. We only search 3672 feasible solutions to obtain the optimal solution as $u^* = (5, 4, 3, 1, 1, 1, 5, 1)$ and the optimal value as $J^* = 136.232245$. Hence, most of the insignificant feasible solutions have been ignored and the branch and bound method is very efficient. The cost functional values of these 3672 feasible solutions are depicted in Figure 5. It can be seen that only the feasible solutions with the corresponding values a bit larger than the current optimal value are searched.

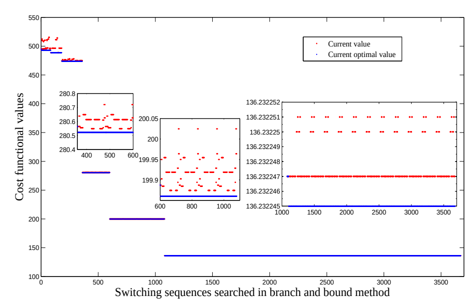

Fig. 5. The cost functional values of 3672 feasible solutions and the current optimal values when the branch and bound method is applied in the second example.

6 Conclusion

In this paper, we have considered the optimal switching problem of switched systems in discrete time, where the subsystems are linear and the cost functional is quadratic. Based on the positive semi-definite property of the matrix, we established the concept of the lower bound dynamic system. Then, for any given current switching sequence, we can construct the lower bound dynamic system which can be used to compute the lower bound. The branch and bound method is proposed to solve the optimal switching sequence problem. We have implemented the proposed method and shown that it is very efficient.

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