# Improved Razumikhin and Krasovskii Stability Criteria for Time-Varying Stochastic Time-Delay Systems

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#### Abstract

The problem of *p*th moment stability for time-varying stochastic time-delay systems with Markovian switching is investigated in this paper. Some novel stability criteria are obtained by applying the generalized Razumikhin and Krasovskii stability theorems. Both *p*th moment asymptotic stability and (integral) input-to-state stability are considered based on the notion and properties of uniformly stable functions and the improved comparison principles. The established results show that time-derivatives of the constructed Razumikhin functions and Krasovskii functionals are allowed to be indefinite, which improve the existing results on this topic. By applying the obtained results for stochastic systems, we also analyze briefly the stability of time-varying deterministic time-delay systems. Finally, examples are provided to illustrate the effectiveness of the proposed results.

**Keywords:** *p*th moment stability, Input-to-state stability, Razumikhin-type theorems, Krasovskii-type theorems, Stochastic time-delay systems

#### 1 Introduction

Practical control systems need to meet several important features, for example, time delays, stochastic perturbations, and time-varying parameters. Time-delay systems refer to a class of dynamics systems whose rate of the current states is affected by their past states. In most cases, time delays are negative for the analysis and design of control systems since they may be the source of performance degradation and instability. On the other hand, stochastic perturbations are very common in practical problems since many engineering systems are subject to external random fluctuations including environmental noise and Markovian switching in the system parameters. Finally, time-varying control systems are of great importance since in most cases the system parameters are changing with time and do not fulfill the usual stationary assumption. There are plenty of papers that deal with control systems having at least one of these three features. For example, stability analysis of control systems with time-varying coefficients and/or time-delays were considered in [1, 4, 17] and [36]; stability analysis of delay-free time-varying stochastic systems was investigated in [12] and [30]; stability analysis of stochastic time-delay systems was studied in [2, 13] and [29]; and stability of stochastic neutral time-delay systems was discussed in [11] and [25].

There are only a few papers that deal with control problems, especially, stability and stabilization, for systems having all of these three features mentioned in the above. The Krasovskii functional approach is effective for handling this class of systems. For example, by this approach, stability analysis,  $H_{\infty}$  control, and  $H_{\infty}$  filtering for stochastic systems with time-delays were respectively investigated in [35], [33] and [34], and the results were expressed by linear matrix inequalities. Compared with the Krasovskii functional approach, the study of the Razumikhin approach was relatively retarded. Razumikhin-type stability theorems for *p*th moment exponential stability of time-varying stochastic interval systems were originally established in [15] and were then utilized to study hybrid stochastic interval systems in [16]. Later on, Razumikhin-type theorems for general *p*th moment input-to-state stability and asymptotic stability of time-varying stochastic time-delay systems were respectively established in [5] and [6]. Some other recently published work related to this issue can be found in [8, 32] and the references therein.

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However, in all the mentioned references in the above, as commonly required in the classical Lyapunov stability theory, time-derivatives of both the Krasovskii functional and the Razumikhin function need to be negative definite (under the Razumikhin condition for the later case), which may be conservative. Such a requirement was relaxed in [30] where a time-varying function was introduced so that the time-derivative of the Lyapunov function along the trajectories of a delay-free stochastic system can be indefinite. This idea was also utilized in [22] and [23] to build Razumikhin-type stability theorems for time-varying stochastic time-delay systems. This method, together with some other new ideas, has also been utilized in [18] and [19, 20] to study stability of deterministic time-varying nonlinear systems and time-delay systems, respectively. A different method that also allows indefinite time-derivatives of the Lyapunov function was established recently in [31].

Very recently, to weaken the condition required by the classical Lyapunov stability theory that the timederivative of the Lyapunov function along the trajectories of a delay-free linear time-varying system is negative definite, a new Lyapunov approach that allows the time-derivative of the Lyapunov function to take both negative and positive values was proposed in [37] by using the notion of scalar stable scalar functions. Motivated by the results of [37], the classical Razumikhin and Krasovskii stability theorems were generalized in [39] for time-varying time-delay systems by allowing time-derivatives of the Razumikhin functions and the Krasovskii functionals to be indefinite. The approach was also utilized in [38] to study input-to-state stability of general nonlinear time-varying systems. As mentioned in [38] and [39], the main advantage of the approach in [38] and [39] is that the scalar function there does not need to satisfy some restrictive conditions that were required in [18, 19, 20, 22, 30] and [31] (see detailed explanations at the end of Section 3).

Motivated by these existing work, especially our recent work [37, 38] and [39], in the present paper we will provide some new Razumikhin-type and Krasovskii-type stability theorems for time-varying stochastic time-delay systems. Both *p*th moment (integral) input-to-state stability and asymptotic stability will be considered. The established results possess the significant feature that the time-derivatives of the Razumikhin functions and Krasovskii functionals are not required to be negative definite, which relaxes the existing results on the same problems. To derive our results, the comparison lemmas build in [38] and [39] were generalized to the case that the right hand side of the comparison differential equation contains a drift term. With this new comparison lemma, stability theorems are established based on the generalized Itô formula and stochastic analysis theory. The improvement of the proposed results over the existing ones, especially those provided in [22] and [31] will be made clear. Applications of the established theorems to some special systems such as time-varying stochastic time-delay systems without Markovian jumping and time-varying deterministic time-delay systems are also discussed in detail. We also provide some numerical examples to illustrate the effectiveness of the proposed theorems.

The remaining of the paper is organized as follows. Problem formulation and some preliminary results will be presented in Section 2. Razumikhin-type and Krasovskii-type stability theorems for time-varying stochastic time-delay systems are respectively presented in Sections 3 and 4. Applications of the theory established in Sections 3 and 4 to time-varying deterministic time-delay systems will be shown in Section 5. Numerical examples are given in Section 6 and finally Section 7 concludes the paper.

# 2 Problem Formulation and Preliminaries

#### 2.1 Problem Formulation

Throughout this paper, unless otherwise specified,  $\mathbf{R}^n$  and  $\mathbf{R}^{n \times m}$  denote, respectively, the *n*-dimensional Euclidean space and the set of  $n \times m$  real matrices. Let  $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq t_0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}\}_{t \geq t_0}$  satisfying the usual conditions (*i.e.*, it is right continuous and  $\mathcal{F}_{t_0}$  contains all  $\mathbb{P}$ -null sets) and  $\mathbf{E}\{\cdot\}$  be the expectation operator with respect to the probability measure. Let  $w(t) = [w_1(t), \cdots, w_m(t)]^T$  be an *m*-dimensional Brownian motion defined on the probability space. Let  $J = [0, \infty), |\cdot|$  denote the Euclidean norm in  $\mathbf{R}^n$ , and  $||f||_{[t_0,t]} = \sup\{|f(s)|, s \in [t_0,t] \subset J\}$ . Let  $\mathbf{C}([-\tau,0],\mathbf{R}^n)$ , where  $\tau \geq 0$ , denote the family of all continuous  $\mathbf{R}^n$ -valued function  $\varepsilon$  defined on  $[-\tau,0]$  with the norm  $||\epsilon|| = \sup_{-\tau \leq \theta \leq 0} |\epsilon(\theta)|$ . Let  $\mathbf{C}_{\mathcal{F}_{t_0}}^{\mathrm{b}}([-\tau,0],\mathbf{R}^n)$  be the family of all  $\mathcal{F}_{t_0}$ -measurable bounded  $\mathbf{C}([-\tau,0],\mathbf{R}^n)$ -valued random variables  $\xi = \{\xi(\theta): -\tau \leq \theta \leq 0\}$ . For p > 0 and  $t \geq t_0$ , denote by  $L_{\mathcal{F}_t}^p([-\tau,0],\mathbf{R}^n)$  the family of all  $\mathcal{F}_t$ -measurable  $\mathbf{C}([-\tau,0],\mathbf{R}^n)$ -valued random processes  $\eta = \{\eta(\theta): -\tau \leq \theta \leq 0\}$  such that

 $\sup_{-\tau \leq \theta \leq 0} \mathbf{E} \{ |\eta(\theta)|^p \} < \infty. \text{ We denote by } \lambda \in \mathcal{VK}_{\infty} \text{ and } \lambda \in \mathcal{CK}_{\infty} \text{ if } \lambda \in \mathcal{K}_{\infty} \text{ and } \lambda \text{ is convex and concave, respectively. We use } \mathbf{C}(J, \mathbf{R}^n) \text{ and } \mathbf{PC}(J, \mathbf{R}^n) \text{ to denote respectively the space of } \mathbf{R}^n\text{-valued continuous functions and piecewise continuous functions defined on } J, \text{ and } J_{\tau} = [-\tau, \infty). \text{ We also denote } \mathbf{L}^l_{\infty}(J) = \{f(t) : J \to \mathbf{R}^l, |\sup_{t \in J} \{ |f(t)| \} < \infty \}.$ 

Let  $r(t), t \in J$  be a right-continuous Markov chain on the probability space taking values in a finite state  $\mathbf{S} = \{1, 2, \dots, N\}$  with generator  $\Gamma = (\gamma_{ij})_{N \times N}$  given by

$$\mathbf{P}\left\{r\left(t+\Delta\right)=j|r\left(t\right)=i\right\}=\left\{\begin{array}{cc}\gamma_{ij}\Delta+o\left(\Delta\right), & \text{if } i\neq j,\\ 1+\gamma_{ij}\Delta+o\left(\Delta\right), & \text{if } i=j,\end{array}\right.$$

where  $\Delta > 0$  and  $\gamma_{ij} \ge 0$  is the transition rate from *i* to *j* if  $i \ne j$  while  $\gamma_{ii} = -\sum_{i \ne j} \gamma_{ij}$ . Assume that the Markov chain  $r(\cdot)$  is independent of the Brownian motion  $w(\cdot)$ . It is known that almost all sample paths of r(t) are right-continuous step functions with a finite number of simple jumps in any finite subinterval of *J*.

We consider the following time-varying stochastic functional differential equation (SFDE) with Markovian switching [5]:

$$dx(t) = f(t, x_t, r(t), u(t)) dt + g(t, x_t, r(t), u(t)) dw(t), \ t \in J,$$
(1)

where the initial state is  $x_{t_0} = \xi \in \mathbf{C}^b_{\mathcal{F}_{t_0}}([-\tau, 0], \mathbf{R}^n)$ , the input  $u : J \to \mathbf{R}^l$  is assumed to be locally essentially bounded, and  $x_t = x(t + \theta), \theta \in [-\tau, 0]$  is regarded as a **C**-valued stochastic process. We assume that

$$f: J \times \mathbf{C} \left( \left[ -\tau, 0 \right], \mathbf{R}^n \right) \times \mathbf{S} \times \mathbf{R}^l \to \mathbf{R}^n,$$
  
$$g: J \times \mathbf{C} \left( \left[ -\tau, 0 \right], \mathbf{R}^n \right) \times \mathbf{S} \times \mathbf{R}^l \to \mathbf{R}^{n \times m},$$

are measurable functions with  $f(t, 0, i, 0) \equiv 0$  and  $g(t, 0, i, 0) \equiv 0$  for all  $t \in J$  and  $i \in \mathbf{S}$ , and are sufficiently smooth so that (1) only has continuous solution on J. Thus, (1) admits a trivial solution  $x(t, t_0, 0) \equiv 0, t \geq t_0 \in J$ . In the absence of Markovian jumping, the SFDE (1) degrades into the following SFDE [7]

$$dx(t) = f_0(t, x_t, u(t)) dt + g_0(t, x_t, u(t)) dw(t), \ t \in J,$$
(2)

where  $f_0: J \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^n$  and  $g_0: J \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \to \mathbf{R}^{n \times m}$  are measurable functions with  $f_0(t, 0, 0) \equiv 0$  and  $g_0(t, 0, 0) \equiv 0$  for all  $t \in J$ , and are sufficiently smooth so that (2) only has continuous solution on J.

In this paper we are interested in the stability analysis of the SFDE (1) and (2). To this end, we give the following definitions.

**Definition 1** Let p > 0 be a constant. The trivial solution of the SFDE (1) or (2) is said to be

1. pth moment input-to-state stable (ISS) if there exist  $\sigma \in \mathcal{KL}$  and  $\gamma_1 \in \mathcal{K}$  such that, for any  $u \in \mathbf{L}^1_{\infty}$ ,

$$\mathbf{E}\{|x(t)|^{p}\} \le \sigma(\mathbf{E}\{\|\xi\|^{p}\}, t-t_{0}) + \gamma_{1}\left(\|u\|_{[t_{0},t]}\right), \ t,t_{0} \in J, \ t \ge t_{0}.$$
(3)

2. pth moment integral input-to-state stable (iISS) if there exist  $\sigma \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that, for any u,

$$\mathbf{E}\{|x(t)|^{p}\} \le \sigma(\mathbf{E}\{||\xi||^{p}\}, t-t_{0}) + \gamma_{1}\left(\int_{t_{0}}^{t} \gamma_{2}(|u(s)|) \,\mathrm{d}s\right), \ t, t_{0} \in J, \ t \ge t_{0}.$$

**Definition 2** Let p > 0 be a constant. The trivial solution of the SFDE (1) or (2) with  $u(t) \equiv 0$  is said to be

1. pth moment globally uniformly asymptotically stable (GUAS), if there exists a  $\mathcal{KL}$ -function  $\sigma$  such that

$$\mathbf{E}\{|x(t)|^{p}\} \le \sigma (\mathbf{E}\{\|\xi\|^{p}\}, t-t_{0}), \ t, t_{0} \in J, \ t \ge t_{0}.$$

2. pth moment globally uniformly exponentially stable (GUES), if there exist two constants  $\alpha > 0$  and  $\beta > 0$  such that

$$\mathbf{E}\{|x(t)|^{p}\} \leq \beta e^{-\alpha(t-t_{0})} \mathbf{E}\{\|\xi\|^{p}\}, t, t_{0} \in J, t \geq t_{0}.$$

The concepts of ISS and iISS, originally introduced in [26], have received much attention due to their wide usages in characterizing the effects of external inputs on a control system. The (i)ISS implies that, no matter what the size of the initial state is, the state will eventually approach to a neighborhood of the origin whose size is proportional to the magnitude of the input. The ISS property is frequently characterized by the ISS-Lyapunov function [27]. As usual, the time-derivative of the ISS-Lyapunov function is required to be negative definite under some additional condition on the input signal (*e.g.*, [5, 32]). In this paper, we will weaken such a condition.

At the end of this section, we introduce the definition of infinitesimal operator for a Lyapunov functional. For a given functional  $V: J \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \times \mathbf{S} \to \mathbf{R}$ , its infinitesimal operator  $\mathcal{L}$  is defined by [13]

$$\mathcal{L}_{V}(t, x_{t}, i) = \lim_{\Delta \to 0^{+}} \frac{1}{\Delta} \left[ \mathbf{E} \left\{ V(t, x_{t+\Delta}, r(t+\Delta)) | x_{t}, r(t) = i \right\} - V(t, x_{t}, i) \right]$$

From the above definition, if

$$V(t, \phi, i) = V(t, \phi(0), i), \ \phi \in \mathbf{C}([-\tau, 0], \mathbf{R}^n),$$

where  $V(t, x, i) \in \mathbb{C}^{2,1}(J \times \mathbb{R}^n \times S, J)$ , in which  $\mathbb{C}^{2,1}(J \times \mathbb{R}^n \times S, J)$  denotes the family of all nonnegative functions V(t, x, i) on  $J \times \mathbb{R}^n \times S$  that are twice continuously differentiable in x and once in t, then the infinitesimal operator of V(t, x, i) along the SFDE (1) is given by [15]

$$\mathcal{L}_{V}(t, x_{t}, i) = V_{t}(t, x, i) + V_{x}(t, x, i) f(t, x_{t}, i, u) + \frac{1}{2} \operatorname{trace} \left( g^{\mathrm{T}}(t, x_{t}, i, u) V_{xx}(t, x, i) g(t, x_{t}, i, u) \right) + \sum_{j=1}^{N} \gamma_{ij} V(t, x, j), \qquad (4)$$

where

$$V_t(t, x, i) = \frac{\partial V(t, x, i)}{\partial t},$$
  

$$V_x(t, x, i) = \left[\frac{\partial V(t, x, i)}{\partial x_1}, \cdots, \frac{\partial V(t, x, i)}{\partial x_n}\right],$$
  

$$V_{xx}(t, x, i) = \left(\frac{\partial^2 V(t, x, i)}{\partial x_i \partial x_j}\right)_{n \times n}.$$

#### 2.2 USF and Comparison Principles

To build our results, we need the following concept of uniformly stable functions which are recalled from [37] and [39].

**Definition 3** [37] A function  $\mu \in \mathbf{C}(J, \mathbf{R})$  is said to be a uniformly stable function (USF) if the following linear time-varying (LTV) system is GUAS:

$$\dot{y}(t) = \mu(t) y(t), \ t \in J.$$

Hence,  $\mu(t)$  is a USF if and only if there exist two constants  $\varepsilon > 0$  and  $\delta \ge 0$  such that

$$\int_{t_0}^t \mu(s) \,\mathrm{d}s \le -\varepsilon \,(t-t_0) + \delta, \ t, t_0 \in J, t \ge t_0.$$
(5)

The following two concepts are also important in developing our Razumikhin-type stability theorems.

**Definition 4** [39] Let  $\mu(t)$  be a USF. Then the set

$$\Omega_{\mu} = \left\{ T > 0 : \sup_{t \in J} \left\{ \int_{t}^{t+T} \mu(s) \, \mathrm{d}s \right\} < 0 \right\},\,$$

is said to be the uniform convergence set (UCS) of  $\mu(t)$ .

**Definition 5** [39] Let  $\mu(t)$  be a USF. Then for any given  $T \ge 0$  the overshoot  $\varphi_{\mu}(T)$  of  $\mu(t)$  is defined as

$$\varphi_{\mu}(T) = \sup_{t \in J} \left\{ \max_{\theta \in [0,T]} \left\{ \int_{t}^{t+\theta} \mu(s) \, \mathrm{d}s \right\} \right\}.$$

From Proposition 2 in [39], we can see that  $\varphi_{\mu}(T)$  is a non-decreasing function of T and  $0 \leq \varphi_{\mu}(T) \leq \delta < \infty$ , where  $\delta$  is from (5).

Next, we introduce some improved comparison principles for time-varying systems.

**Lemma 1** [3] Assume that  $\pi \in \mathbf{PC}(J, \mathbf{R}), \mu \in \mathbf{C}(J, \mathbf{R})$  and  $y: J \to J$  is such that

$$\mathbf{D}^{+}y(t) \leq \mu(t)y(t) + \pi(t), \ t \in J.$$

Then, for any  $t \ge s \in J$ , the following inequality holds true

$$y(t) \le y(s) \exp\left(\int_{s}^{t} \mu(r) dr\right) + \int_{s}^{t} \exp\left(\int_{\lambda}^{t} \mu(r) dr\right) \pi(\lambda) d\lambda.$$

**Lemma 2** Let  $y(t) : J \to J$  be a continuous function such that

$$\mathbf{D}^{+}y(t) \le \mu(t) y(t) + \pi(t) \quad \text{whenever } y(t) \ge \psi(t), \tag{6}$$

where  $\mu \in \mathbf{C}(J, \mathbf{R})$  is a USF,  $\pi \in \mathbf{PC}(J, J)$  and  $\psi(\cdot)$  is a  $\mathcal{K}$ -function. Then, for any given constant T > 0 and any  $t \geq T$ , there holds

$$y(t) \le \max\left\{y(t-T)\exp\left(\int_{t-T}^{t}\mu(s)\,\mathrm{d}s\right), \sup_{t-T\le s\le t}\left\{\psi(s)\right\}\mathrm{e}^{\varphi_{\mu}(T)}\right\} + \int_{t-T}^{t}\exp\left(\int_{s}^{t}\mu(r)\,\mathrm{d}r\right)\pi(s)\,\mathrm{d}s.$$

Letting  $\pi(t) = 0$  and  $\mathbf{D}^+ y(t) = \dot{y}(t)$  in Lemma 2 gives immediately Lemma 4 in [39]. These two lemmas in the above will play critical roles in establishing respectively Krasovskii-type and Razumikhin-type stability theorems in the next two sections.

## 3 Razumikhin-Type Stability Theorems

This section is concerned with Razumikhin-type stability theorems for the time-varying SFDE (1). We first give the following result regarding ISS of (1).

**Theorem 1** (Razumikhin Theorem for ISS) The SFDE (1) is pth moment ISS if there exist a  $V(t, x, i) \in \mathbb{C}^{2,1}(J_{\tau} \times \mathbb{R}^n \times \mathbb{S}, J)$ , a USF  $\mu(t)$ ,  $\alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}, q \in \mathcal{K}_{\infty}, \varpi \in \mathcal{K}_{\infty}$ , a number  $T \in \Omega_{\mu}$ , and a constant  $\rho \in (0, 1)$  satisfying

$$q(s) \ge \frac{\mathrm{e}^{\varphi_{\mu}(T)}}{\rho} s, \quad \forall s \ge 0,$$
(7)

such that the following conditions are met for all  $t \in J$ :

$$\begin{aligned} (\mathbf{A}_{1}). \ \alpha_{1}\left(|x|^{p}\right) &\leq V\left(t, x, i\right) \leq \alpha_{2}\left(|x|^{p}\right), \quad \forall i \in \mathbf{S}. \\ (\mathbf{B}_{1}). \ \mathbf{E}\left\{\mathcal{L}_{V}\left(t, x_{t}, i\right)\right\} &\leq \mu(t) \mathbf{E}\left\{V\left(t, x(t), i\right)\right\}, \ \forall i \in \mathbf{S}, \\ & if \max_{i \in \mathbf{S}}\left\{\mathbf{E}\left\{V\left(t, x(t), i\right)\right\}\right\} \geq \max\left\{\varpi\left(|u\left(t\right)|\right), q^{-1}\left(\min_{i \in \mathbf{S}}\left\{\mathbf{E}\left\{V\left(t+\theta, x(t+\theta), i\right)\right\}\right\}\right)\right\}, \ \forall \theta \in [-\tau, 0] \end{aligned}$$

$$(8)$$

We next present the following result regarding the characterization of iISS.

**Theorem 2** (Razumikhin Theorem for iISS) The SFDE (1) is pth moment iISS if there exist a  $V(t, x, i) \in \mathbb{C}^{2,1}(J_{\tau} \times \mathbb{R}^n \times \mathbb{S}, J)$ , a USF  $\mu(t)$ ,  $\alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}, \varpi \in \mathcal{K}_{\infty}$ , a number  $T \in \Omega_{\mu}, q \in \mathcal{K}_{\infty}$  and a constant  $\rho \in (0, 1)$  satisfying (7) such that  $(A_1)$  and the following conditions are met for all  $t \in J$ :

(B<sub>2</sub>). 
$$\mathbf{E} \{ \mathcal{L}_V(t, x_t, i) \} \leq \mu(t) \mathbf{E} \{ V(t, x(t), i) \} + \varpi(|u(t)|), \forall i \in \mathbf{S},$$
  
 $if \min_{j \in \mathbf{S}} \{ \mathbf{E} \{ V(t + \theta, x(t + \theta), j) \} \} \leq q \left( \max_{i \in \mathbf{S}} \{ \mathbf{E} \{ V(t, x(t), i) \} \} \right), \forall \theta \in [-\tau, 0].$ 

Setting  $u(t) \equiv 0$  in Theorems 1 and 2 gives immediately the following corollary regarding Razumikhin-type theorem for the *p*th moment GUAS of the SFDE (1).

**Corollary 1** (Razumikhin Theorem for GUAS) The SFDE (1) is pth moment GUAS if there exist a  $V(t, x, i) \in \mathbb{C}^{2,1}(J_{\tau} \times \mathbb{R}^n \times \mathbb{S}, J)$ , a USF  $\mu(t)$ ,  $\alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}, q \in \mathcal{K}_{\infty}$ , a number  $T \in \Omega_{\mu}$ , and a constant  $\rho \in (0, 1)$  satisfying (7) such that  $(A_1)$  and the following conditions are met for all  $t \in J$ :

$$(\mathbf{B}_{3}). \mathbf{E} \left\{ \mathcal{L}_{V} \left( t, x_{t}, i \right) \right\} \leq \mu(t) \mathbf{E} \left\{ V \left( t, x(t), i \right) \right\}, \ \forall i \in \mathbf{S}, \\ if \min_{j \in \mathbf{S}} \left\{ \mathbf{E} \left\{ V \left( t + \theta, x(t + \theta), j \right) \right\} \right\} \leq q \left( \max_{i \in \mathbf{S}} \left\{ \mathbf{E} \left\{ V \left( t, x(t), i \right) \right\} \right), \ \forall \theta \in [-\tau, 0].$$

$$(9)$$

The SFDE is furthermore pth moment GUES if there exist three positive numbers  $\beta_i$ , i = 0, 1, 2 such that the following condition is satisfied:

(D). 
$$\alpha_1(s) = \beta_1 s^{\beta_0}, \ \alpha_2(s) = \beta_2 s^{\beta_0}.$$

Applying Theorems 1 and 2 and Corollary 1 on system (2) gives the following corollary.

**Corollary 2** Consider the SFDE (2). Let  $V(t,x) \in \mathbb{C}^{2,1}(J_{\tau} \times \mathbb{R}^n, J)$ ,  $\mu(t)$  be a USF,  $\alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}, \varpi \in \mathcal{K}_{\infty}, q \in \mathcal{K}_{\infty}, T \in \Omega_{\mu}$ , and  $\rho \in (0,1)$  be a constant satisfying (7). Consider the following conditions, where  $t \in J$ :

$$\begin{aligned} &(\mathbf{A}_{2}). \ \alpha_{1}\left(|x|^{p}\right) \leq V\left(t,x\right) \leq \alpha_{2}\left(|x|^{p}\right). \\ &(\mathbf{B}_{4}). \ \mathbf{E}\left\{\mathcal{L}_{V}\left(t,x_{t}\right)\right\} \leq \mu(t)\mathbf{E}\left\{V\left(t,x(t)\right)\right\}, \\ & if \mathbf{E}\left\{V\left(t,x(t)\right)\right\} \geq \max\left\{\varpi\left(|u\left(t\right)|\right),q^{-1}\left(\mathbf{E}\left\{V\left(t+\theta,x(t+\theta)\right)\right\}\right)\right\}, \ \forall \theta \in [-\tau,0] \\ &(\mathbf{B}_{5}). \ \mathbf{E}\left\{\mathcal{L}_{V}\left(t,x_{t}\right)\right\} \leq \mu(t)\mathbf{E}\left\{V\left(t,x(t)\right)\right\} + \varpi\left(|u\left(t\right)|\right), \\ & if \mathbf{E}\left\{V\left(t+\theta,x(t+\theta)\right)\right\} \leq q\left(\mathbf{E}\left\{V\left(t,x(t)\right)\right\}\right), \ \forall \theta \in [-\tau,0]. \\ &(\mathbf{B}_{6}). \ \mathbf{E}\left\{\mathcal{L}_{V}\left(t,x_{t}\right)\right\} \leq \mu(t)\mathbf{E}\left\{V\left(t,x(t)\right)\right\}, \\ & if \mathbf{E}\left\{V\left(t+\theta,x(t+\theta)\right)\right\} \leq q\left(\mathbf{E}\left\{V\left(t,x(t)\right)\right\}\right), \ \forall \theta \in [-\tau,0]. \end{aligned}$$

Then the SFDE (2)

- 1. is pth moment ISS if  $(A_2)$  and  $(B_4)$  are satisfied.
- 2. is pth moment iISS if  $(A_2)$  and  $(B_5)$  are satisfied.
- 3. is pth moment GUAS if  $(A_2)$  and  $(B_6)$  are satisfied.
- 4. is pth moment GUES if  $(A_2)$ ,  $(B_6)$  and (D) are satisfied.

The classical Razumikhin-type stability theorems for deterministic systems were generalized to SFDEs in [5, 6, 7, 15]. In these Razumikhin-stability theorems, time-derivatives of the Razumikhin functions must be negative definite. Such a requirement was relaxed in [30], where the *p*th moment stability for delay-free stochastic differential equations was investigated by allowing time-derivatives of Lyapunov functions to be indefinite. The same idea was utilized in [22] to study asymptotic stability of SFDEs and some

improved Razumikhin-type theorems, which take similar forms as Corollary 1, were established. However, the corresponding function  $\mu(t)$  there needs to satisfy

$$\int_{0}^{\infty} \max\left\{\mu\left(t\right), 0\right\} \mathrm{d}t < \infty.$$

$$\tag{10}$$

This condition is very restrictive since it cannot be satisfied by any continuous periodic functions having positive values (see more explanations in Remark 3 in [39] and the examples given in Section 6). Though the restrictive condition (10) was not required in [31], another condition

$$\mu(t) \ge -\frac{\ln q}{\tau}, \ t \ge 0,\tag{11}$$

was imposed there, where q > 1 is some constant and  $\tau$  is the delay. We mention that the restrictive condition (10) was also required in some other related references such as [18, 19] and [20]. In the proposed generalized Razumikhin-type stability theorems (Theorem 1, 2, and Corollary 1), both of these two restrictions (10)-(11) are not required. In this sense we emphasize that Corollary 1 generalizes Theorem 3.2 in [22] and Item 3 of Corollary 2 generalizes the results in [14]. Finally, we mention that, to the best of our knowledge, the Razumikhin-type stability criteria for (i)ISS of SFDFs without Markovian switching shown in Corollary 2 was not available in the literature even if  $\mu(t)$  is a negative constant.

## 4 Krasovskii-Type Stability Theorems

In this section, the Krasovskii functionals with indefinite time-derivatives are employed to derive pth moment ISS, iISS, GUAS and GUES of systems (1) and (2).

**Theorem 3** (Krasovskii Theorem for ISS) The SFDE (1) is pth moment ISS if there exist a continuous functional  $V(t, \phi, i) : J_{\tau} \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \times \mathbf{S} \to J$ , a USF  $\mu(t), \alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}$ , and  $\varpi \in \mathcal{K}_{\infty}$ , such that the following conditions are met for all  $t \in J$ :

(a<sub>1</sub>). 
$$\alpha_1 (|\phi(0)|^p) \le V(t, \phi, i) \le \alpha_2 (||\phi||^p), \forall i \in \mathbf{S}.$$
  
(b<sub>1</sub>).  $\mathbf{E} \{ \mathcal{L}_V(t, x_t, i) \} \le \mu(t) \mathbf{E} \{ V(t, x_t, i) \}, \forall i \in \mathbf{S}, \text{ if } \max_{i \in \mathbf{S}} \{ \mathbf{E} \{ V(t, x_t, i) \} \} \ge \varpi (|u(t)|).$ 

**Theorem 4** (Krasovskii Theorem for iISS) The SFDE (1) is pth moment iISS if there exist a continuous functional  $V(t, \phi, i) : J_{\tau} \times \mathbb{C}([-\tau, 0], \mathbb{R}^n) \times \mathbb{S} \to J$ , a USF  $\mu(t), \alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}$ , and  $\varpi_i \in \mathcal{K}, i = 1, 2$  such that  $(a_1)$  and the following conditions are met for all  $t \in J$ :

(b<sub>2</sub>). 
$$\mathbf{E} \{ \mathcal{L}_V(t, x_t, i) \} \le (\varpi_1 (|u(t)|) + \mu(t)) \mathbf{E} \{ V(t, x_t, i) \} + \varpi_2 (|u(t)|), \forall i \in \mathbf{S}.$$

Setting  $u(t) \equiv 0$  in Theorems 3 and 4 gives immediately the following corollary regarding Krasovskii-type theorem for the *p*th moment GUAS of the SFDE (1).

**Corollary 3** (Krasovskii Theorem for GUAS) The SFDE (1) is pth moment GUAS if there exist a continuous functional  $V(t, \phi, i) : J_{\tau} \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \times \mathbf{S} \to J$ , a USF  $\mu(t), \alpha_1 \in \mathcal{VK}_{\infty}$  and  $\alpha_2 \in \mathbf{CK}_{\infty}$ , such that (a<sub>1</sub>) and the following conditions are met for all  $t \in J$ :

(b<sub>3</sub>). 
$$\mathbf{E} \{ \mathcal{L}_V(t, x_t, i) \} \leq \mu(t) \mathbf{E} \{ V(t, x_t, i) \}, \forall i \in \mathbf{S}.$$

If, in addition, Condition (D) is satisfied, then the system is pth moment GUES.

Applying Theorems 3 and 4 and Corollary 3 on system (2) gives the following corollary.

**Corollary 4** Consider the SFDE (2). Let  $V(t,\phi) : J_{\tau} \times \mathbf{C}([-\tau,0], \mathbf{R}^n) \to J$ ,  $\mu(t)$  be a USF,  $\alpha_1 \in \mathcal{VK}_{\infty}, \alpha_2 \in \mathcal{CK}_{\infty}, \varpi \in \mathcal{K}, \varpi_i \in \mathcal{K}_{\infty}, i = 1, 2, q \in \mathcal{K}_{\infty}$ , and  $T \in \Omega_{\mu}$ . Consider the following conditions, where  $t \in J$ :

(a<sub>2</sub>). 
$$\alpha_1(|\phi(0)|^p) \le V(t,\phi) \le \alpha_2(||\phi||^p).$$

$$\begin{aligned} &(\mathbf{b}_{4}). \quad \mathbf{E} \left\{ \mathcal{L}_{V} \left( t, x_{t} \right) \right\} \leq \mu(t) \mathbf{E} \left\{ V \left( t, x_{t} \right) \right\}, \quad if \, \mathbf{E} \left\{ V \left( t, x_{t} \right) \right\} \geq \varpi \left( |u \left( t \right)| \right) \\ &(\mathbf{b}_{5}). \quad \mathbf{E} \left\{ \mathcal{L}_{V} \left( t, x_{t} \right) \right\} \leq \left( \varpi_{1} \left( |u \left( t \right)| \right) + \mu(t) \right) \mathbf{E} \left\{ V \left( t, x_{t} \right) \right\} + \varpi_{2} \left( |u \left( t \right)| \right). \\ &(\mathbf{b}_{6}). \quad \mathbf{E} \left\{ \mathcal{L}_{V} \left( t, x_{t} \right) \right\} \leq \mu(t) \mathbf{E} \left\{ V \left( t, x_{t} \right) \right\}. \end{aligned}$$

Then the SFDE (2)

- 1. is pth moment ISS if  $(a_2)$  and  $(b_4)$  are satisfied.
- 2. is pth moment iISS if  $(a_2)$  and  $(b_5)$  are satisfied.
- 3. is pth moment GUAS if  $(a_2)$  and  $(b_6)$  are satisfied.
- 4. is pth moment GUES if  $(a_2)$ ,  $(b_6)$  and (D) are satisfied.

By applying the Krasovskii functional based approach, the pth moment asymptotic stability of SFDEs without Markovian switching and exponential stability of stochastic delay interval systems with Markovian switching were respectively investigated in [10] and [13], where time-derivatives of the Krasovskii functionals were required to be negative definite. Compared with these results, in this section time-derivatives of Krasovskii functionals can take both negative and positive values along the solutions of SFDEs.

## 5 Stability of Deterministic Time-Delay Systems

In this section, we discuss briefly the stability of the following time-varying functional differential equation [19]

$$\dot{x}(t) = f(t, x_t, u(t)), \ t \in J,$$
(12)

where  $f: J \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \times \mathbf{R}^l \to \mathbf{R}^n$  is assumed to be locally Lipschitz in (t, x) and uniformly continuous in u, and to satisfy  $f(t, 0, 0) = 0, \forall t \in J$ . Clearly, this class of systems is a special case of the SFDE (1) in the absence of stochastic disturbance and Markovian switching. Similarly to the stochastic setting, we give the following definitions [21, 28].

**Definition 6** The trivial solution of system (12) is said to be

1. ISS, if there exist  $\sigma \in \mathcal{KL}$  and  $\gamma_1 \in \mathcal{K}$  such that, for any  $u \in \mathbf{L}^l_{\infty}$ ,

$$|x(t)| \le \sigma(||x_{t_0}||, t - t_0) + \gamma_1(||u||_{[t_0, t]}), \ t, t_0 \in J, \ t \ge t_0.$$

2. *iISS*, if there exist  $\sigma \in \mathcal{KL}$  and  $\gamma_1, \gamma_2 \in \mathcal{K}$  such that, for any u,

$$|x(t)| \le \sigma(||x_{t_0}||, t - t_0) + \gamma_1\left(\int_{t_0}^t \gamma_2(|u(s)|) \,\mathrm{d}s\right), \ t, t_0 \in J, \ t \ge t_0.$$

3. GUAS if there exists a  $\mathcal{KL}$ -function  $\sigma$  such that, when  $u(t) \equiv 0$ ,

$$|x(t)| \leq \sigma(||x_{t_0}||, t-t_0), t, t_0 \in J, t \geq t_0.$$

4. GUES if there exist two constants  $\alpha > 0$  and  $\beta > 0$  such that, when  $u(t) \equiv 0$ ,

$$|x(t)| \leq \beta e^{-\alpha(t-t_0)} ||x_{t_0}||, t, t_0 \in J, t \geq t_0.$$

Applying Theorems 1 and 2 and Corollary 1 on system (12) gives the following Razumikhin-type stability theorem.

**Theorem 5** (Razumikhin Theorem) Let  $V(t, x) : J_{\tau} \times \mathbf{R}^n \to J$  be a continuous function,  $\mu(t)$  be a USF,  $\alpha_i \in \mathcal{K}_{\infty}, i = 1, 2, q \in \mathcal{K}_{\infty}, \varpi \in \mathcal{K}_{\infty}, T \in \Omega_{\mu}$ , and  $\rho \in (0, 1)$  be a constant satisfying (7). Consider the following conditions, where  $t \in J$ :

$$\begin{split} &(\mathbf{H}_{1}). \ \alpha_{1} \left(|x|\right) \leq V\left(t,x\right) \leq \alpha_{2} \left(|x|\right). \\ &(\mathbf{H}_{2}). \ \dot{V}\left(t,x(t)\right) \leq \mu(t)V\left(t,x(t)\right), \\ & \quad if V\left(t,x(t)\right) \geq \max\left\{\varpi\left(|u\left(t\right)|\right), q^{-1}\left(V\left(t+\theta,x(t+\theta)\right)\right)\right\}, \ \forall \theta \in [-\tau,0]. \\ &(\mathbf{H}_{3}). \ \dot{V}\left(t,x(t)\right) \leq \mu(t)V\left(t,x(t)\right) + \varpi\left(|u\left(t\right)|\right), \quad if V\left(t+\theta,x(t+\theta)\right) \leq q\left(V\left(t,x(t)\right)\right), \ \forall \theta \in [-\tau,0]. \\ &(\mathbf{H}_{4}). \ \dot{V}\left(t,x(t)\right) \leq \mu(t)V\left(t,x(t)\right), \ if V\left(t+\theta,x(t+\theta)\right) \leq q\left(V\left(t,x(t)\right)\right), \ \forall \theta \in [-\tau,0]. \end{split}$$

Then the time-varying time-delay system (12) is

- 1. ISS, if  $(H_1)$  and  $(H_2)$  are satisfied.
- 2. *iISS*, if  $(H_1)$  and  $(H_3)$  are satisfied.
- 3. GUAS if  $(H_1)$  and  $(H_4)$  are satisfied.
- 4. GUES if  $(H_1)$ ,  $(H_4)$  and (D) are satisfied.

We mention that Item 3 of Theorem 5 is just Theorem 1 in [39], and Item 1 of this theorem improves Theorem 1 in [19] in the sense that  $\mu(t)$  in Theorem 5 is less restrictive than that in [19] where the function needs to satisfy condition (10).

Similarly, applying Theorems 3 and 4 and Corollary 3 on system (12) gives the following theorem.

**Theorem 6** (Krasovskii Theorem) Let  $V(t, \phi) : J_{\tau} \times \mathbf{C}([-\tau, 0], \mathbf{R}^n) \to J$  be a continuous functional,  $\mu(t)$  be a USF,  $\alpha_i \in \mathcal{K}_{\infty}, i = 1, 2, \varpi \in \mathcal{K}_{\infty}$ , and  $\varpi_i \in \mathcal{K}, i = 1, 2$ . Consider the following conditions, where  $t \in J$ :

 $\begin{aligned} &(\mathbf{h}_{1}). \quad \alpha_{1}\left(|\phi(0)|\right) \leq V\left(t,\phi\right) \leq \alpha_{2}(\|\phi\|). \\ &(\mathbf{h}_{2}). \quad \dot{V}\left(t,x_{t}\right) \leq \mu(t)V\left(t,x_{t}\right), \quad if V\left(t,x_{t}\right) \geq \varpi\left(|u\left(t\right)|\right). \\ &(\mathbf{h}_{3}). \quad \dot{V}\left(t,x_{t}\right) \leq \left(\varpi_{1}\left(|u\left(t\right)|\right) + \mu(t)\right)V\left(t,x_{t}\right) + \varpi_{2}\left(|u\left(t\right)|\right). \\ &(\mathbf{h}_{4}). \quad \dot{V}\left(t,x_{t}\right) \leq \mu(t)V\left(t,x_{t}\right). \end{aligned}$ 

Then the time-varying time-delay system (12) is

- 1. ISS if  $(h_1)$  and  $(h_2)$  are satisfied.
- 2. *iISS if*  $(h_1)$  and  $(h_3)$  are satisfied.
- 3.  $GUAS(h_1)$  and  $(h_4)$  are satisfied.
- 4. GUAS (h<sub>1</sub>), (h<sub>4</sub>) and (D) are satisfied.

Clearly, Item 3 of Theorem 6 is just Theorem 2 in [39]. We point out that Items 1 and 2 of Theorem 6 generalize respectively Theorems 1 and 3 in [20] in the sense that  $\mu(t)$  here does not need to satisfy condition (10) as required in [20].

## 6 Numerical Examples

In this section we use two numerical examples to illustrate the obtained theory. To same spaces, we only illustrate the Razumikhin-type stability theorems.

**Example 1**: Let w(t) be a scalar Brownian motion and r(t) be a right-continuous Markov chain taking values in  $\mathbf{S} = \{1, 2\}$  with generator

$$\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{bmatrix} -1 & 1\\ 2 & -2 \end{bmatrix}.$$

Assume that w(t) and r(t) are independent. Let  $d: J \times \mathbf{S} \to [0, \tau]$  be Borel measurable. Consider the following system

$$dx(t) = f(t, x(t), r(t)) dt + g(t, x(t - d(t, r(t))), r(t)) dw(t),$$
(13)

where  $t \in J$ , and

$$\begin{split} f\left(t,x,1\right) &= -\frac{1}{2}x - |\sin t| \sqrt[4]{|x|}\sqrt[3]{x}, \ f\left(t,x,2\right) = -\frac{1}{2}x, \\ g\left(t,y,1\right) &= -\sqrt{|b\left(t\right)|}y\cos t, \ g\left(t,y,2\right) = \sqrt{|b\left(t\right)|}y\sin t, \end{split}$$

with x = x(t), y = x(t - d(t)) and b(t) being a scalar function to be defined. Observe that

$$2xf(t,x,i) \le -x^2, \ g^2(t,y,i) \le |b(t)| \ y^2, \ i=1,2.$$

To examine the stability of system (13), we construct a Lyapunov function candidate  $V(t, x(t), i) = x^{2}(t), i = 1, 2$  and calculate

$$\mathcal{L}_{V}(t, x_{t}, i) \leq -x^{2} + |b(t)| y^{2}, \ i = 1, 2.$$
(14)

If the classical Razumikhin theorem (Theorem 2 in [15] and Theorem 3.1 in [6]) is applied, it follows from (14) that the system is mean-square GUES if

$$|b(t)| < 1, \quad t \in J. \tag{15}$$

We next show how to improve this result by using Theorem 1. Let q(s) = qs in condition (9). Then, under the condition  $\min_{j \in \mathbf{S}} \{ \mathbf{E} \{ V(t + \theta, x(t + \theta), j) \} \} \leq q (\max_{i \in \mathbf{S}} \{ \mathbf{E} \{ V(t, x(t), i) \} \}), \forall \theta \in [-\tau, 0], we obtain$ from (14) that

$$\mathbf{E} \{ \mathcal{L}_V(t, x_t, i) \} \le \mu(t) \mathbf{E} \{ V(t, x(t), i) \}, \ i = 1, 2$$

where  $\mu(t) = -1 + q |b(t)|$ . Now consider a periodic function b(t) with period  $\omega = 1$ , and [17, 39]

$$b(t) = \begin{cases} 0, \ t \in [0, c), \\ e, \ t \in [c, 1), \end{cases} \quad |b(t)| \le e,$$

where  $c \in (0, 1)$  and e > 0 are some constant. We claim that system (13) is mean-square GUES for arbitrary large e if 1 - c is sufficiently small.

It is easy to see that the function  $\mu(t)$  is USF if and only if [39]

$$\int_{0}^{1} \mu(t) dt = -1 + q (1 - c) e < 0.$$
(16)

If e < 1, then there exists a q > 1 such that  $\mu(t) < 0$ . Consequently,  $\mu(t)$  is USF for any  $\tau$  and system (13) is mean square GUES. Hence, without loss of generality, we assume that  $e \ge 1$ . By (12) in [39], we can compute

$$\varphi_{\mu}(\omega) = \int_{c}^{1} \mu(t) dt = -1 + q(1-c)e + c.$$

Hence Condition (7) is equivalent to

$$q > \exp(-1 + q(1 - c)e + c).$$
 (17)

These two inequalities (16) and (17) are equivalent to

$$e < \frac{1}{(1-c)\exp\left(c\right)}.\tag{18}$$

To summarize, system (13) is mean-square GUES if (18) is satisfied.

Clearly, Condition (18) is better than (15) which is equivalent to e < 1, in the sense that e can be arbitrary large in the former condition, by setting 1 - c to be sufficiently small. Finally, we mention that the results in the above cannot be obtained by the approach in [22] since the corresponding function  $\mu(t)$  is periodic and takes positive values in non-empty intervals.

**Example 2**: We consider the following time-varying SFDE

$$dx(t) = f(t, x(t), r(t), u(t)) dt + g(t, x(t - d(t, r(t))), r(t), u(t)) dw(t), t \in J,$$
(19)

where r(t) and d(t, r(t)) are the same as that in Example 1, in which the generator  $\Gamma$  is replaced by

$$\Gamma = (\gamma_{ij})_{2 \times 2} = \begin{bmatrix} -1 & 1\\ 1 & -1 \end{bmatrix}.$$

The corresponding functions f and g are given by

$$\begin{split} f\left(t, x, 1, u\right) &= -\frac{\lambda}{2}x - \frac{1}{2}x^3 + \frac{t\cos t^2}{1 + x^2}u,\\ f\left(t, x, 2, u\right) &= \frac{1}{4}\left(t\cos t^2 - \lambda\right)x + \frac{t\cos t^2}{2\left(1 + x^2\right)}u,\\ g\left(t, x, y, 1, u\right) &= x^2 - \sqrt{\frac{l}{2}}\sin^k\left(t\right)y,\\ g\left(t, y, 2, u\right) &= \frac{\sqrt{2l}}{2}\sin^k\left(t\right)y, \end{split}$$

with x = x(t), y = x(t - d(t)) and u = u(t). Here  $\lambda > \frac{1}{2\pi}$  and l > 0 are constants. Observe that

$$\begin{cases} 2xf(t,x,1) \leq -\lambda x^2 - x^4 + \frac{2t\cos t^2 x}{1+x^2}u, \\ g^2(t,y,1,u) \leq x^4 + l\sin^{2k}(t)y^2, \end{cases}$$

and

$$\begin{cases} 2xf(t,x,1,u) &\leq \frac{-\lambda}{2}x^2 + \frac{t\cos t^2 x}{1+x^2}u, \\ g^2(t,y,1,u) &\leq \frac{1}{2}l\sin^{2k}(t)y^2. \end{cases}$$

To examine the stability of system (19), we construct a Lyapunov function candidate

$$V(t, x(t), i) = \begin{cases} x^2, \ i = 1, \\ \frac{1}{2}x^2, \ i = 2. \end{cases}$$

Let  $|u(t)| \leq V(t, x(t), i), q(s) = qs$  and calculate

$$\mathcal{L}_{V}(t, x_{t}, i) \leq (t \cos t^{2} - \lambda) x^{2} + q l \sin^{2k}(t) y^{2}, \ i = 1, 2.$$

Then, under Condition  $(B_1)$  in Theorem 1, we obtain from (8) that

$$\mathbf{E} \{ \mathcal{L}_{V}(t, x_{t}, i) \} \le \mu(t) \mathbf{E} \{ V(t, x(t), i) \}, \ i = 1, 2,$$

where  $\mu(t) = t \cos t^2 - \lambda + q l \sin^{2k}(t)$ . According to Example 4.3 in [39], if we set  $q = \exp(2\lambda\pi)$  and k to be sufficiently large such that

$$\frac{2k-1}{2k}\frac{2k-3}{2k-2}\cdots\frac{1}{2} < \frac{2\lambda\pi-1}{2\lambda l\exp\left(2\lambda\pi\right)},$$

then  $\mu(t)$  is USF,  $2\pi \in \Omega_{\mu}$ , and  $q > \exp(\varphi_{\mu}(2\pi))$ . Hence system (19) is mean-square ISS by Theorem 1. Again, this result cannot be obtained by the approach in [22] since  $\mu(t)$  cannot satisfy the condition in (10). In fact, we have

$$\int_{0}^{\infty} \max\{\mu(t), 0\} dt > \int_{0}^{\infty} \max\{t \cos t^{2} - \lambda, 0\} dt$$
$$> \sum_{i=1}^{\infty} \int_{\sqrt{2i\pi - \frac{1}{4}\pi}}^{\sqrt{2i\pi + \frac{1}{4}\pi}} \max\{t \cos t^{2} - \lambda, 0\} dt$$

$$\geq \sum_{i=1}^{\infty} \int_{\sqrt{2i\pi + \frac{1}{4}\pi}}^{\sqrt{2i\pi + \frac{1}{4}\pi}} \max\left\{\frac{\sqrt{2}}{2}t - \lambda, 0\right\} dt$$
$$\geq \sum_{i=i^*}^{\infty} \int_{\sqrt{2i\pi - \frac{1}{4}\pi}}^{\sqrt{2i\pi + \frac{1}{4}\pi}} \left(\frac{\sqrt{2}}{2}t - \lambda\right) dt$$
$$= \frac{1}{2}\pi \sum_{i=i^*}^{\infty} \left(\frac{\sqrt{2}}{4} - \frac{\lambda}{\sqrt{2i\pi + \frac{1}{4}\pi} + \sqrt{2i\pi - \frac{1}{4}\pi}}\right)$$
$$\geq \frac{1}{4}\pi \sum_{i=i^*}^{\infty} \left(\frac{\sqrt{2}}{2} - \frac{\lambda}{\sqrt{2i\pi - \frac{1}{4}\pi}}\right)$$
$$= \infty,$$

where  $i^*$  is the minimal integer such that  $\frac{\sqrt{2}}{2}\sqrt{2i^*\pi - \frac{1}{4}\pi} - \lambda \ge 0$ .

## 7 Conclusion

In this paper, generalized Razumikhin-type and Krasovskii-type stability theorems on *p*th moment ISS, iISS, and GUAS were proposed for time-varying stochastic time-delay systems with Markovian switching. Based on the general Itô formula and the stochastic analysis theory, some new Lyapunov function(al)s based stability theorems were proposed by using properties of uniformly stable functions and the improved comparison principle. The most significant feature of the proposed results is that they allow time-derivatives of the Razumikhin functions and Krasovskii functionals to take both negative and positive values. The proposed results also improve the related existing results on the same topic by removing some restrictive conditions. Some improved stability criteria for deterministic time-delay systems were also obtained.

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