# Minimal Controllability of Conjunctive Boolean Networks is NP-Complete

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#### Abstract

Given a conjunctive Boolean network (CBN) with n state-variables, we consider the problem of finding a minimal set of state-variables to directly affect with an input so that the resulting conjunctive Boolean control network (CBCN) is controllable. We give a necessary and sufficient condition for controllability of a CBCN; an  $O(n^2)$ -time algorithm for testing controllability; and prove that nonetheless the minimal controllability problem for CBNs is NP-hard.

#### **Index Terms**

Logical systems, controllability, Boolean control networks, computational complexity, minimum dominating set, systems biology.

## I. INTRODUCTION

Many modern networked systems include a large number of nodes (or state-variables). Examples range from the electric grid to complex biological processes. If the system includes control inputs then a natural question is whether the system is *controllable*, that is, whether the control authority is powerful enough to steer the system from any initial condition to any desired final condition. Controllability is an important property of control systems, and it plays a crucial role in many control problems, such as stabilization of unstable systems by feedback, and optimal control [1].

If the system is not controllable (and in particular if there are no control inputs) then an important problem is what is the *minimal* number of control inputs that should be added to the network so that it becomes controllable. This calls for finding the key locations within the system such that controlling them allows driving the entire system to any desired state. This problem is interesting both theoretically and for applications, as in many real-world systems it is indeed possible to add control actuators, but naturally this may be timely and costly, so minimizing the number of added controls is desirable.

Several recent papers studied minimal controllability in networks with a linear and time-invariant (LTI) dynamics (see, e.g. [2], [3] and the references therein). In particular, Olshevsky [2] considered the following problem. Given the *n*-dimensional LTI system  $\dot{x}_i(t) = \sum_{j=1}^n a_{ij}x_j(t)$ , i = 1, ..., n, determine a *minimal* set of indices  $\mathcal{I} \subseteq \{1, ..., n\}$ , such that the modified system:

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij} x_j(t) + u_i(t), \qquad i \in \mathcal{I},$$
  
$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij} x_j(t), \qquad i \notin \mathcal{I},$$

is controllable. Olshevsky [2] showed, using a reduction to the minimum hitting set problem, that this problem is NP-hard (in the number of state-variables n). For a general survey on the computational complexity of various problems in systems and control theory, see [4].

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Boolean control networks (BCNs) are discrete-time dynamical systems with Boolean state-variables and Boolean control inputs. A BCN without inputs is called a Boolean network (BN). BCNs date back to the early days of digital switching networks and neural network models with on-off type neurons. They have also been used to model many other important phenomena such as social networks (see, e.g. [5], [6]), the spread of epidemics [7], etc.

More recently, BCNs have been extensively used to model biological processes and networks where the possible set of states is assumed to be finite (see, e.g. [8], [9], [10]). For example, in gene regulation networks one may assume that each gene may be either "on" or "off" (i.e. expressed or not expressed). Then the state of each gene can be modeled as a state-variable in a BN and the interactions between the genes (e.g., via the proteins that they encode) determine the Boolean update function for each state-variable.

A BCN with state-vector  $X(k) = [X_1(k) \dots X_n(k)]'$  is said to be controllable if for any pair of states  $a, b \in \{0, 1\}^n$  there exists an integer  $N \ge 0$  and a control sequence  $u(0), \dots, u(N-1)$  steering the state from X(0) = a to X(N) = b.

A natural representation of a BCN is via its graph of states G = (V, E), where the vertices  $V = \{1, \ldots, 2^n\}$  represent all the possible  $2^n$  states, and a directed edge  $e = (v_i \rightarrow v_j) \in E$  means that there is a control such that  $X(k) = v_i$  and  $X(k+1) = v_j$ . Then clearly a BCN is controllable if and only if G is strongly connected [11]. Since testing strong connectivity of a digraph takes linear time in the number of its vertices and directed edges (see, e.g. [12]), one may expect that verifying controllability of a general BCN is intractable.

Akutsu et al. [13] showed, using a reduction to the 3SAT problem, that determining if there exists a control sequence steering a BCN between two given states is NP-hard, and that this holds even for BCNs with restricted network structures. This implies in particular that verifying controllability is NP-hard. Of course, it is still possible that the problems of verifying controllability and finding the minimal number of controls needed to make a BN controllable are tractable in some special classes of Boolean networks.

An important special class of BNs are those comprised of nested canalyzing functions (NCFs) [14]. A Boolean function is called canalyzing if there exists a certain value, called the canalyzing value, such that any input with this value uniquely determines the output of the function regardless of the other variables. For example, 0 is a canalyzing value for the function AND, as  $AND(0, x_1, ..., x_k) = 0$  for any  $x_i \in \{0, 1\}$ . BNs with nested canalyzing functions are often used to model genetic networks [15], [16], [17].

Here, we consider the subclass consisting of those NCFs that are constructed only with the AND operator, the conjunctive functions. A BN is called a *conjunctive Boolean network* (CBN) if every update function includes only AND operations, i.e., the state-variables satisfy an equation of the form:

$$X_i(k+1) = \prod_{j=1}^n (X_j(k))^{\epsilon_{ji}}, \quad i = 1, \dots, n,$$
(1)

where  $\epsilon_{ji} \in \{0, 1\}$  for all i, j.

Recall that a BN is called a disjunctive Boolean network (DBN) if every update function includes only OR operations. By applying De Morgan Law's, one can reduce DBNs to CBNs, so all the results in this note hold also for DBNs.

A CBN with n state-variables can be represented by its *dependency graph* (or wiring diagram) that has n vertices corresponding to the Boolean state-variables. There is a directed edge  $(i \rightarrow j)$  if  $X_i(k)$  appears in the update function of  $X_j(k+1)$ . That is, the dependency graph encodes the variable dependencies in the update functions. We will assume from here on that none of the update functions is constant, so every vertex in the dependency graph has a positive in-degree. In this case, there is a one-to-one correspondence between the CBN and its dependency graph, and this allows a graph-theoretic analysis of the CBN. For example, the problem of characterizing all the periodic orbits of a CBN with a strongly connected dependency graph has been solved in [18], and the robustness of these orbits has been studied in [19].

In the context of modeling gene regulation, CBNs encode synergistic regulation of a gene by several transcription factors [18], and there is increasing evidence that this type of mechanism is common in regulatory networks [20], [21], [22].

Here, we consider the following problem.

**Problem 1.** Given a CBN with n state-variables, suppose that for any  $i \in \{1, ..., n\}$  we can replace the update function of  $X_i(k+1)$  by an independent Boolean control  $U_i(k)$ . Determine a minimal<sup>1</sup> set of indices  $\mathcal{I} \subseteq \{1, ..., n\}$ , such that the modified system:

$$X_{i}(k+1) = U_{i}(k), \qquad i \in \mathcal{I},$$
  

$$X_{i}(k+1) = \prod_{j=1}^{n} (X_{j}(k))^{\epsilon_{ji}}, \qquad i \notin \mathcal{I},$$
(2)

is controllable.

We refer to a BCN in the form (2) as a *conjunctive Boolean control network* (CBCN).

Problem 1 is important because it calls for finding a minimal set of key variables in the CBN such that controlling them makes the system controllable. Of course, an efficient algorithm for solving this problem must encapsulate an efficient algorithm for testing controllability of a CBCN.

**Example 1.** Consider Problem 1 for the CBN

$$X_1(k+1) = X_2(k), X_2(k+1) = X_1(k)X_2(k)$$

Suppose that we replace the update function for  $X_2(k)$  by a control  $U_2(k)$  so that the network becomes:

$$X_1(k+1) = X_2(k),$$
  
 $X_2(k+1) = U_2(k).$ 

This CBCN is clearly controllable. Indeed, given a desired final state  $s = \begin{bmatrix} s_1 & s_2 \end{bmatrix}' \in \{0, 1\}^2$ , the control sequence  $U_2(0) = s_1$ ,  $U_2(1) = s_2$ , steers the CBCN from an arbitrary initial condition  $X_1(0), X_2(0)$  to  $\begin{bmatrix} X_1(2) & X_2(2) \end{bmatrix}' = s$ . Thus, in this case a solution to Problem 1 is to replace the update function of  $X_2$  by a control.

The main contributions of this note are:

- 1) a necessary and sufficient condition for the controllability of a CBCN;
- 2) a polynomial-time algorithm for determining whether a CBCN is controllable (more specifically the time complexity of this algorithm is  $O(n^2)$ , where n is the number of state-variables in the BCN);
- 3) a proof that Problem 1 is NP-hard.

Together, these results imply that checking the controllability of a given CBCN is "easy", yet there does not exist a polynomial-time algorithm for solving Problem 1 (unless P=NP).

The next section reviews definitions and notations from graph theory that will be used later on. Section III describes our main results. Section IV concludes and presents several directions for further research.

## II. PRELIMINARIES

Let G = (V, E) be an undirected graph, where V is the set of vertices, and E is the set of edges. If two vertices  $v_i, v_j$  are connected by an edge then we denote this edge by  $e_{ij}$  or by  $(v_i, v_j)$ , and say that  $v_i$ and  $v_j$  are *neighbors*. The set of neighbors of  $v_i$  is denoted by  $\mathcal{N}(v_i)$ , and the *degree* of  $v_i$  is  $|\mathcal{N}(v_i)|$ .

<sup>&</sup>lt;sup>1</sup>In computer science, this is usually called a *minimum* cardinality set rather than a minimal set. We follow the terminology used in control theory.

**Problem 2 (Dominating set problem).** Given a graph G = (V, E) and a positive integer  $k \le |V|$ , does there exist a dominating set D of V with  $|D| \le k$ ?

This is known to be an NP-complete decision problem (see, e.g. [23]).

Let G = (V, E) be a directed graph (digraph), with V the set of vertices, and E the set of directed edges (arcs). Let  $e_{i \to j}$  (or  $(v_i \to v_j)$ ) denote the arc from  $v_i$  to  $v_j$ . When such an arc exists, we say that  $v_i$ is an *in-neighbor* of  $v_j$ , and  $v_j$  as an *out-neighbor* of  $v_i$ . The set of in-neighbors and out-neighbors of a vertex  $v_i$  is denoted by  $\mathcal{N}_{in}(v_i)$  and  $\mathcal{N}_{out}(v_i)$ , respectively. The *in-degree* and *out-degree* of  $v_i$  are  $|\mathcal{N}_{in}(v_i)|$ and  $|\mathcal{N}_{out}(v_i)|$ , respectively.

Let  $v_i$  and  $v_j$  be two vertices in V. A walk from  $v_i$  to  $v_j$ , denoted by  $w_{ij}$ , is a sequence:  $v_{i_0}v_{i_1} \dots v_{i_q}$ , with  $v_{i_0} = v_i$ ,  $v_{i_q} = v_j$ , and  $e_{i_k \to i_{k+1}} \in E$  for all  $k \in \{0, 1, \dots, q-1\}$ . A simple path is a walk with pairwise distinct vertices. We say that  $v_i$  is reachable from  $v_j$  if there exists a simple path from  $v_j$  to  $v_i$ . A closed walk is a walk that starts and finishes at the same vertex. A closed walk is called a cycle if all the vertices in the walk are distinct, except for the start-vertex and the end-vertex. A strongly connected digraph is a digraph for which every vertex in the graph is reachable from any other vertex in the graph.

Recall that given the CBN (1), the associated *dependency graph* is a digraph G = (V, E) with n vertices, such that  $e_{i\to j} \in E$  if and only if  $\epsilon_{ij} = 1$ . A CBN is uniquely determined by its dependency graph, and for this reason we interchangeably refer to the *i*th state-variable in the CBN and the *i*th vertex in its dependency graph. We extend the definition of a dependency graph to a CBCN in a natural way: upon replacing the update equation for  $X_i(k+1)$  to  $X_i(k+1) = U_i(k)$  we remove all the arcs pointing to  $v_i$ , introduce a new vertex  $v_{U_i}$  for the new control input, and add an arc  $e_{U_i \to i}$ .

An *m*-layer graph is a digraph G = (V, E) for which each layer  $L_k$ , k = 1, ..., m, is a subset of V, every vertex in V belongs to a single layer, and any arc  $e_{i \to j} \in E$  is such that  $v_i \in L_k$  and  $v_j \in L_{k+1}$ , for some  $k \in \{1, 2, ..., m - 1\}$ .

## **III. MAIN RESULTS**

## A. Complexity Analysis

We begin by analyzing the computational complexity of Problem 1. We will prove a hardness result for a CBCN whose dependency graph is a 3-layer graph. Our first result uses the special structure of this CBCN to provide a simple necessary and sufficient condition for controllability. We will later use this condition to relate controllability analysis for this CBCN to the dominating set problem.

**Lemma 1.** Consider a CBCN with a dependency graph that is a 3-layer graph satisfying: every vertex in layer-1 is a control input to a vertex in layer-2, and every vertex in layer-2 has **its own** control input (in layer-1). This CBCN is controllable if and only if every vertex in layer-3 has an in-neighbor (in layer-2) with out-degree equal to one.

Proof of Lemma 1. Assume that the CBCN is controllable. Seeking a contradiction, suppose that there exists a vertex  $v_i$  in layer 3 that has no in-neighbor with out-degree one. Let  $X_i$  denote the corresponding state-variable. If  $v_i$  has zero in-neighbors then  $X_i(k)$  is constant, contradicting the assumption of controllability. Hence  $v_i$  must have at least one in-neighbor and each of them with out-degree greater than one. From the controllability of the CBCN, it follows that for any initial state X(0), there exists a time  $T \ge 0$  and a control sequence  $\{u(0), \ldots, u(T-1)\}$  steering the CBCN to the final state:

$$X_i(T) = 0,$$
  

$$X_j(T) = 1 \text{ for all } j \neq i.$$
(3)

In other words, the state-variables corresponding to all the nodes in layers 2 and 3, except for  $X_i$ , are one at time T. But for (3) to hold at least one of the in-neighbors of  $v_i$  must be zero at time T-1. Since

To prove the converse implication, suppose that every vertex in layer 3 of the CBCN has an in-neighbor (in layer 2) with out-degree equal to one. We need to prove that the CBCN is controllable. Denote the nodes in layer 3 by  $w_1, \ldots, w_q$ . For any  $i \in \{1, \ldots, q\}$  the node  $w_i$  in layer 3 has an in-neighbor  $v_i$  in layer 2 such that  $v_i$  has out degree one. Let  $p \ge q$  denote the number of nodes in layer 2, so that the nodes in layer 2 are  $v_1, \ldots, v_q, v_{q+1}, \ldots, v_p$ , and their corresponding controls in layer 1 are  $u_1, \ldots, u_p$ .

Fix arbitrary  $a \in \{0, 1\}^p$  and  $b \in \{0, 1\}^q$ . We will show that for any initial condition there exists a control sequence that steers the network to the state v(4) = a and w(4) = b, where  $v = [v_1 \dots v_p]'$ ,  $w = [w_1 \dots w_q]'$ , thus proving controllability. In time steps 0 and 1, set all the inputs to one. Then  $v_i(2) = w_j(2) = 1$  for all i, j. Now let  $u_i(2) = b_i$  for all  $i \in \{1, \dots, q\}$ , and  $u_i(2) = 1$  for all i > q. Then  $v_i(3) = b_i$  for all  $i \in \{1, \dots, q\}$ , and  $u_i(3) = a_i$  for all  $i \in \{1, \dots, p\}$ . Then  $w_i(4) = v_i(3) = b_i$  for all  $i \in \{1, \dots, q\}$  (because AND(1, z) = z for all  $z \in \{0, 1\}$ ) and  $v_i(4) = a_i$  for all  $i \in \{1, \dots, p\}$ . This completes the proof of Lemma 1.

We are now ready to present our main complexity result.

## **Theorem 2.** The decision version of Problem 1 is NP-hard.

Proof of Thm. 2. Given an undirected graph G = (V, E), and a positive integer k, consider Problem 2. We will show that we can solve this instance of the dominating set problem by solving a minimal controllability problem for a 3-layer CBCN with dependency graph  $\tilde{G} = (\tilde{V}, \tilde{E})$  constructed as follows. Define  $L_2 := E$ ,  $L_3 := V$ , and let  $\tilde{V} := L_2 \bigcup L_3$ . In other words,  $\tilde{G}$  has |E| + |V| nodes. The nodes in  $L_3$  are denoted  $v_1, \ldots, v_s$ , where s := |V| and those in  $L_2$  are denoted by  $e_{uv}$ . The set  $\tilde{E}$  includes two sets of directed edges. First, every  $e_{uv} \in L_2$  induces an arc  $(e_{uv} \to e_{uv})$  (i.e., a self loop). Second, every edge  $e \in E$  induces two directed edges in  $\tilde{E}$ :  $(e_{uv} \to v)$  and  $(e_{uv} \to u)$ . Thus,  $|\tilde{E}| = 3|E|$  (Example 2 below demonstrates this construction). Note that this construction is polynomial in |V|, |E|.

Consider the CBN induced by  $\tilde{G}$  as a dependency graph, and the minimal controllability problem for this CBN. A solution for this problem includes adding controls to some of the state-variables. Since every node in  $L_2$  has a self-loop in  $\tilde{G}$ , a control input must be added to each node  $e \in L_2$ . Denoting by  $L_1$ the set of nodes corresponding to the control inputs added to the CBN, it is evident that we obtained a 3-layer graph. The solution to the controllability problem may also add control inputs to nodes in  $L_3$ . Let  $Y \subseteq L_3$  denote the set of these nodes. Note that Y is also a set of nodes in the original graph G. We require the following result.

# **Lemma 3.** The set Y is a minimum dominating set of G.

Proof of Lemma 3. Let  $\overline{G}$  denote the dependency graph of the controllable CBCN obtained by solving the minimal controllability problem described above. Define a new dependency graph  $\overline{G}'$  by removing the nodes in Y, their adjacent directed edges, and the nodes of the control inputs added to them in  $\overline{G}$ . Then  $\overline{G}'$  is a 3-layer controllable CBCN: the first layer consists of the controls that were added to the nodes in  $L_2$  (namely the nodes of  $L_1$ ), the second layer is composed of the nodes in  $L_2$ , and the third is composed of the nodes in  $V \setminus Y$ . It follows from Lemma 1 that for every vertex  $v \in (V \setminus Y)$  there exists a vertex  $e \in L_2$  such that e is an in-neighbor of v and e is *not* an in-neighbor of any other vertex in  $V \setminus Y$ . Thus, e must be an in-neighbor of a node in Y. We conclude that Y is a dominating set of G. Since the construction solves the minimal controllability problem, it is clear that Y is a minimum dominating set of G.

We now consider two cases. If  $|Y| \le k$  then we found a dominating set with cardinality  $\le k$  and thus the answer to Problem 2 for the given instance is yes. If |Y| > k then the argument above implies that there does not exist a dominating set with cardinality  $\le k$  and thus the answer to Problem 2 for the given instance is no. Since Problem 2 is NP-hard, this completes the proof of Theorem 2.



Fig. 1. An example demonstrating the construction in the proof of Thm. 2. Left: a graph G. The unique solution to the minimal dominating set problem for G is  $D = \{v_3\}$ . Middle: the digraph  $\tilde{G}$  described in the proof of Thm. 2. Right: the digraph  $\bar{G}$  describing the unique solution to the minimal controllability problem for  $\tilde{G}$  obtained by adding a control to each node  $e_{ij}$  and to the node  $v_3$  in  $\tilde{G}$ .

**Example 2.** Consider the graph G = (V, E) shown on the left of Fig. 1, with  $V = \{v_1, v_2, v_3, v_4\}$ and  $E = \{e_{13}, e_{23}, e_{34}\}$ . The unique solution to the minimal dominating set for G is  $D = \{v_3\}$ . The construction in the proof above yields the directed graph  $\tilde{G} = (\tilde{V}, \tilde{E})$  depicted in the middle of Fig. 1. Note that this has two layers of nodes:  $L_2$  and  $L_3$  that include the edges and vertices in G, respectively. The solution of the minimal controllability problem for  $\tilde{G}$  is depicted on the right of Fig. 1. Here  $Y = \{v_3\}$ .

Our next goal is to present a polynomial-time algorithm for determining if (2) is controllable. To do this, we first derive graph-theoretic necessary and sufficient conditions for the controllability of a CBCN.

## B. Necessary and Sufficient Conditions for Controllability of a CBCN

We begin by introducing several definitions that will be used later on. In a digraph, a *source* is a node that has in-degree zero, and a *sink* is a node that has out-degree zero. A digraph that does not contain cycles is called a *directed acyclic graph* (DAG).

We now introduce several definitions referring to the dependency graph of a CBCN. A node that represents a state-variable (SV) is called a *simple node*. A node that represents a control input is called a *generator*. Note that a generator is always a source, and that its set of out-neighbors always contains a single simple node (see (2)). A node that represents an SV that has an added control input is called a *directly controlled node*. Note that a directly controlled node is also a simple node. A simple node with out-degree one, and without a self-loop, is called a *channel*. Note that the only out-neighbor of a channel is another simple node (since a generator is always a source).

We are now ready to derive necessary conditions for the controllability of a CBCN. We will assume throughout that no SV in the network has a constant updating function. Indeed, in this case it is clear that the updating function of such an SV must be replaced by a control input. Under this assumption, a simple node cannot be a source.

## **Proposition 4.** The dependency graph of a controllable CBCN is acyclic.

Proof of Prop. 4. Consider a CBCN with a cycle in its dependency graph. Every vertex in the cycle corresponds to an SV (i.e., it is a simple node), as a generator is a source, so it cannot be part of a cycle. Moreover, any simple node in the cycle is not a directly controlled node, since the only in-neighbor of a directly controlled node is a generator. This implies that if at time 0 the SVs in the cycle are all zero then they can never be steered to the a state where they are all one. Hence, the CBCN is not controllable.

It is natural to expect that in a controllable CBCN every SV has the following property. There exists a path from a control input to the SV that allows to set the SV to zero (the canalyzing value), without affecting the other SVs. To make this precise, we say that a CBCN has *Property P* if every simple node in its dependency graph contains in its set of in-neighbors either a generator or a channel. The next result provides another necessary condition for controllability of a CBCN.

**Proposition 5.** A controllable CBCN has Property P.

Proof of Prop. 5. Seeking a contradiction, assume that the CBCN is controllable and that there exists a simple node v in its dependency graph that does not contain a generator nor a channel in its set of in-neighbors. This means that there does not exist a node w in the graph such that v is the only simple node in the out-neighbors of w. Hence, the SV that corresponds to v cannot change its value to zero (the canalyzing value) without at least one other SV changing its value to zero as well. Consider two states: a corresponding to all SVs being zero, and b corresponding to all SVs being one except for v that is zero. Since the CBCN is controllable it is possible to steer it from a to b. This implies that v has a self-loop, as it holds the value zero while the other SVs change their values. Prop. 4 implies that the CBCN is not controllable.

Props. 4 and 5 provide two necessary conditions for the controllability of a CBCN. The next result shows that together they are also sufficient.

# Theorem 6. A CBCN is controllable if and only if its dependency graph is a DAG and satisfies Property P.

To prove this, we introduce another definition and some auxiliary results. A *controlled path* is an ordered non-empty set of nodes in the dependency graph such that: the first element in the ordered set is a generator, and if the set contains more than one element, then for any i > 1 the *i*th node is a simple node, and is the *only* element in the set of out-neighbors of node i - 1. Controlled paths with non-overlapping nodes are called *disjoint controlled paths*.

**Proposition 7.** Consider a CBCN with a dependency graph G that is a DAG and satisfies Property P. Then G can be decomposed into disjoint controlled paths, such that every vertex in the graph belongs to a single controlled path (i.e., the union of the disjoint controlled paths forms a vertex cover of G).

Proof of Prop. 7. The proof is based on Algorithm 1 detailed below that accepts such a graph G and terminates after each vertex in the graph belongs to exactly one controlled path.

Algorithm 1 Decompose the nodes in G into disjoint controlled paths	
1: <b>V</b>	<b>vhile</b> there exists a simple node $v \in G$ that is not included in any ordered set <b>do</b>
2:	<i>cnode</i> $\leftarrow v$ and <i>cset</i> $\leftarrow \{v\}$
3:	if $\mathcal{N}_{in}(cnode)$ contains a channel <b>then</b>
4:	pick a channel $u \in \mathcal{N}_{in}(cnode)$
5:	if $u$ does not belong to any ordered set then
6:	insert $u$ to the ordered set <i>cset</i> just before <i>cnode</i>
7:	$cnode \leftarrow u$
8:	goto 3
9:	else
10:	let $H$ denote the ordered set that contains $u$
11:	merge <i>cset</i> into H keeping the order between any two adjacent elements
12:	goto 1
13:	else
14:	pick a generator $u \in \mathcal{N}_{in}(cnode)$
15:	if u does not belong to any ordered set then
16:	insert it to <i>cset</i> just before <i>cnode</i>
17:	goto 1
18:	else
19:	goto 10

Note that the assumption that G has Property P is used in line 14 of the algorithm. The proof that Algorithm 1 terminates in a finite number of steps and divides all the nodes in the graph to a set of disjoint controlled paths is straightforward and thus omitted. The basic idea is that Property P and the

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Fig. 2. Dependency graph of the CBCN (4).

fact that the graph is a DAG implies that we can "go back" from any simple node through a chain of channels ending with a generator, thus creating a controlled path. This completes the proof of Prop. 7. ■

# **Example 3.** Consider the CBCN:

$$X_1(k+1) = U_1(k),$$
  

$$X_2(k+1) = U_2(k),$$
  

$$X_3(k+1) = X_1(k)X_2(k).$$
(4)

It is straightforward to verify that (4) is controllable. Fig. 2 depicts the dependency graph G of this CBCN. Note that G is a DAG and that it satisfies Property P. A decomposition into two disjoint controlled paths is  $C^1 = \{U_1 \to X_1 \to X_3\}, C^2 = \{U_2 \to X_2\}.$ 

We can now prove Thm. 6.

Proof of Thm. 6. Consider a CBCN satisfying the conditions stated in the Theorem. By Prop. 7, there exists a decomposition of its dependency graph into a set of  $m \ge 1$  disjoint controlled paths  $C^1, \ldots, C^m$ , such that every vertex in the graph belongs to a single controlled path. We assume from here on that m = 2(the proof in the general case is a straightforward generalization).

Pick two states  $a, b \in \{0, 1\}^n$ . We prove that the CBCN is controllable by providing a control sequence that steers the CBCN from X(0) = a to X(T) = b, in time  $T \ge 0$ . Since the paths provide a vertex cover, the desired state b can be decomposed into  $b^1$  and  $b^2$  such that when the state is b the simple nodes in  $C^1$  [ $C^2$ ] have state  $b^1$  [ $b^2$ ].

First, feed a control sequence with all ones to both generators until all the SVs reach the value one at some time  $\tau \ge 0$ . Such a  $\tau$  exists, because by the properties of the paths there are no arcs between simple nodes in  $C_1$  and  $C_2$ , except perhaps an arc from the final node in one path, say  $C_1$ , to the other path  $C_2$  (and since the graph is a DAG there is no arc from a node in  $C_2$  to a node in  $C_1$ ).

By adding a chain of dummy control inputs at the beginning of the shorter path, if needed, we may assume that  $|C^1| = |C^2|$ . Now we may view each path as a shift register (with all SVs initiated to one) and it is straightforward to feed each path with a suitable sequence of controls to obtain the desired states  $b^1$ and  $b^2$  at some time  $T \ge 0$ . Thus, the CBCN is controllable.

Note that this proof also provides the sequence of controls needed to steer the CBCN from a to b, i.e. it solves the *control synthesis problem* (given the decomposition into a set of disjoint controlled paths). Combining this with Prop. 7 implies the following.

**Corollary 8.** A CBCN is controllable if and only if its dependency graph can be decomposed into a set of disjoint controlled paths.

Using the necessary and sufficient condition for controllability it is possible to derive an efficient algorithm for determining if a CBCN is controllable.



# C. An efficient algorithm for determining controllability

Algorithm 2 below tests if a CBCN is controllable. It is based on the condition in Thm. 6.

Algorithm 2 Testing controllability of a CBCN in the form (2) with n SVs and  $q \le n$  control inputs 1: generate the dependency graph G = (V, E)2: if G is not a DAG then return ("not controllable") 3: create a list L of n bits 4: set all bits in L to 0 5: for all nodes  $v \in V$  do if  $|\mathcal{N}_{out}(v)| \neq 1$  then return ("not controllable") 6: else 7:  $j \leftarrow$  the element in  $\mathcal{N}_{out}(v)$ 8:  $L(j) \leftarrow 1$ 9: endfor 10: if all bits in L are 1 then return ("controllable") 11: 12: **else** return ("not controllable") 13:

The input to Algorithm 2 is a CBCN with n SVs and  $q \le n$  control inputs. The first step is to build the dependency graph G = (V, E). The complexity of this step is  $O(n^2)$ , as this requires going through n updating functions, and each function has at most n arguments. The resulting graph satisfies  $|V| = n+q \le 2n$ , and  $|E| \le n^2$ .

Checking if G is a DAG in line 2 can be done using a topological sort algorithm. The complexity is linear in |V|, |E| (see, e.g. [24]), i.e. it is  $O(n^2)$ .

Lines 3-13 use the list L to check Property P, that is, to verify that the set of in-neighbors of every SV contains either a generator or a channel. This part has complexity O(n + q) = O(n).

The total time-complexity of the algorithm is thus  $O(n^2)$ . More precisely, the complexity of the algorithm is linear in the length of the description of the CBCN, and the latter is  $O(n^2)$ .

# **IV. CONCLUSIONS**

Minimal controllability problems for dynamical systems are important both theoretically and for realworld applications where actuators can be added to control the SVs. Here, we considered a minimal controllability problem for an important subclass of BNs, namely, CBNs. Using a graph-theoretic approach we derived: a necessary and sufficient condition for the CBCN (2) to be controllable, and a polynomial-time algorithm for testing controllability. We also showed that the minimal controllability problem is NP-hard.

Our approach is based on the new concept of a controlled path and the decomposition of the dependency graph of the CBCN into disjoint controlled paths. Roughly speaking, this corresponds to decomposing the CBCN into a set of shift registers that are almost decoupled.

Recall that given a digraph G = (V, E) a *path cover* is a set of directed paths such that every  $v \in V$  belongs to at least one path. A *vertex-disjoint path cover* is a set of paths such that every  $v \in V$  belongs to exactly one path. The *minimum path cover problem* (MPCP) consists of finding a vertex-disjoint path cover having the minimal number of paths. This problem has applications in software validation [25]. The MPCP may seem closely related to the problem studied here, but this is not necessarily so. First, the MPCP for a DAG can be solved in polynomial time (see, e.g. [26]). Second, the solution of the MPCP does not provide enough information on the controllability of a CBCN. For this, we need the more specific properties of controlled paths.

We believe that the notions introduced here will find more applications in other control-theoretic problems for CBNs. An interesting direction for further research is to derive an efficient algorithm that

is guaranteed to approximately solve the minimal controllability problem for CBCNs, with a guaranteed approximation error in the number of needed control inputs.

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