

Power-Controlled Hamiltonian Systems: Application to Electrical Systems with Constant Power Loads [★]

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Abstract

We study a type of port-Hamiltonian system, in which the controller or disturbance is not applied to the flow variables, but to the systems power—a scenario that appears in many practical applications. A suitable framework is provided to model these systems and to investigate their shifted passivity properties, based on which, a stability analysis is carried out. The applicability of the results is illustrated with the important problem of stability analysis of electrical circuits with constant power loads.

Key words: Port-Hamiltonian systems, Passivity theory, Stability of nonlinear systems, Constant power loads

1 Introduction

In recent years, port-Hamiltonian (pH) modeling of physical systems has gained extensive attention. pH systems theory provides a systematic framework for modeling and analysis of physical systems and processes [18, 22, 23, 26, 27]. Typically, the external inputs (controls or disturbances) in pH systems act on the flow variables—that is on the derivative of the energy storing coordinates. However, in some cases of practical interest, these external inputs act on the systems *power*, either as the control variable, or as a power that is extracted from (or injected to) the system. We refer to this kind of systems as *Power-controlled Hamiltonian* (P_wH) systems. P_wH systems cannot be modeled with constant control input matrices, which is the scenario considered in [13, 19], and therefore analyzing their passivity properties is nontrivial.

An example of P_wH systems is electrical systems with instantaneous constant-power loads (CPLs), which model the behavior of some point-of-load converters that are widely used in modern electrical systems (see [17, 24] and

references therein). It is well-known that CPLs introduce a destabilizing effect that gives rise to significant oscillations or to network collapse [11], and hence they are the most challenging component of the standard load model—referred to as ZIP model [10, 25]. Therefore, the study of *stability* of the equilibria of the systems with CPLs is a topic of utmost importance; see [3, 5, 16, 17] for an analysis of *existence* of equilibria.

In [25], sufficient conditions are derived for all operating points of purely resistive networks with CPLs to lie in a desirable set. Stability analysis has been carried out in [2, 3] using linearization methods, see also [17]. In [4], and recently in [8], Brayton-Moser potential theory [6] is employed, however, constraints on individual grid components are imposed. Moreover, as shown in [17], the estimate of the region of attraction (ROA) of the equilibria based on the Brayton-Moser potential is rather conservative.

In this paper, we propose a framework to model P_wH systems, and provide sufficient conditions for shifted passivity and stability. Following [13], we use a shifted storage function to address this issue. This shifted function is closely related to the notion of availability function used in thermodynamics [1, 14], and is associated with the Bregman distance of the Hamiltonian with respect to an equilibrium of the system [7]. Therefore, we use the shifted Hamiltonian as a candidate Lyapunov function, which is based on the physical energy of the system, and unlike the Brayton-Moser potential, is trivially

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computed. Two immediate corollaries of the shifted passivity property are: (i) that their shifted equilibrium can be stabilized with simple PI controllers [13]; (ii) that in the uncontrolled case, when a constant input power or load is imposed, stability of this equilibrium can be established. In this paper we concentrate on the latter issue, that was first studied in the standard pH systems framework in [18]. Interestingly, our framework allows us to give an analytic characterization of an estimate of the ROA, in the case of a quadratic Hamiltonian.

The remainder of this paper is organized as follows. The proposed model for P_wH systems is introduced in Section 2. The main result, that is, the derivation of conditions for their shifted passivity, is provided in Section 3. The stability analysis is given in Section 4. The main result is then illustrated in Section 5 with its application to electrical systems with CPLs, and in Section 6 with the application to synchronous generators. Finally, some concluding remarks are provided in Section 7.

Notation. For $i \in \{1, 2, \dots, n\}$, by $\text{col}(a_i)$ we denote the vector $[a_1 \ a_2, \dots \ a_n]^\top$. For a given vector $a \in \mathbb{R}^n$, the diagonal matrix $\text{diag}\{a_1, a_2, \dots, a_n\}$ is denoted in short by $\langle a \rangle$. The symbol $\mathbf{1}$ denotes the vector of ones with an appropriate dimension, and I_n is the $n \times n$ identity matrix. For a function $\mathcal{H}(x)$ the vector $\frac{\partial \mathcal{H}}{\partial x}^\top$ is denoted in short by $\nabla \mathcal{H}$. For a mapping $G(x) \in \mathbb{R}^{n \times m}$ and the distinguished element $\bar{x} \in \mathbb{R}^n$ we define the constant matrix $\bar{G} := G(\bar{x})$. The largest and smallest eigenvalues of the square, symmetric matrix A are denoted by $\lambda_M\{A\}$, and $\lambda_m\{A\}$, respectively.

2 Model

The dynamics of the pH system investigated in this paper is given by

$$\dot{x} = (J - R)\nabla \mathcal{H}(x) + G(x)u, \quad x \in \Omega^+ \quad (1)$$

where x is the system state, $u \in \mathbb{R}^m$ is an external signal that represents, either a control input or a constant disturbance, \mathcal{H} is the system Hamiltonian (energy) function, $G \in \mathbb{R}^{n \times m}$ is the input matrix, the $n \times n$ constant, matrices $J = -J^\top$ and $R \geq 0$, are the structure and the dissipation matrices, respectively, and the set Ω^+ will be defined later.

Now let $\mathcal{I} := \{i \in \{1, \dots, n\} : u_i = 0\}$, where u_i is the i th element of the vector u . It is assumed that the input matrix $G(x) \in \mathbb{R}^{n \times m}$ may be written in the form

$$G(x) := \text{diag}\{g_1, \dots, g_i, \dots, g_n\}, \quad (2)$$

where

$$g_i = \begin{cases} 0 & i \in \mathcal{I} \\ \frac{1}{\nabla \mathcal{H}(x)_i} & i \notin \mathcal{I} \end{cases},$$

and the set where the system lives is defined as

$$\Omega^+ := \{x \in \mathbb{R}^n : \nabla \mathcal{H}(x)_i > 0, \forall i \notin \mathcal{I}\},$$

where $\nabla \mathcal{H}(x)_i$ is the i th element of the vector $\nabla \mathcal{H}(x)$. Although this—admittedly cryptic—assumption seems rather restrictive, it turns out to hold for many widely accepted models of physical systems. In fact, the motivation for such an assumption comes from the fact that, with the input matrix (2), the external input of the system (1), acts *directly* on the power (rate of change of the Hamiltonian), *i.e.*,

$$\dot{\mathcal{H}} = -\nabla \mathcal{H}^\top(x) R \nabla \mathcal{H}(x) + \mathbf{1}^\top u.$$

This is in contrast with standard pH systems where the product of input and the natural output, *i.e.*, $G^\top(x) \nabla \mathcal{H}(x)$, appears in the Hamiltonian rate of change.

Defining the steady-state relation

$$\mathcal{E} := \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid (J - R)G(x)u = 0\},$$

we can write the *shifted* model for the system as:

Lemma 1 [Shifted model]

Fix $(\bar{x}, \bar{u}) \in \mathcal{E}$, then the system (1), (2) can be rewritten as

$$\dot{x} = (J - (R + Z(x)))\nabla \mathcal{S}(x) + G(x)(u - \bar{u}), \quad (3)$$

where

$$Z(x) := \bar{G}\langle \bar{u} \rangle G(x), \quad (4)$$

and \mathcal{S} is the shifted Hamiltonian [13]

$$\mathcal{S}(x) := \mathcal{H}(x) - (x - \bar{x})^\top \nabla \mathcal{H}(\bar{x}) - \mathcal{H}(\bar{x}). \quad (5)$$

PROOF. Subtracting the steady-state equation from (1) gives

$$\dot{x} = (J - R)(\nabla \mathcal{H}(x) - \nabla \mathcal{H}(\bar{x})) + G(x)u - \bar{G}\bar{u}.$$

Bearing in mind that $\nabla \mathcal{S}(x) = \nabla \mathcal{H}(x) - \nabla \mathcal{H}(\bar{x})$, we have

$$\begin{aligned} \dot{x} &= (J - R)\nabla \mathcal{S}(x) + G(x)u - \bar{G}\bar{u} \\ &\quad - G(x)\bar{u} + G(x)\bar{u} \\ &= (J - R)\nabla \mathcal{S}(x) + G(x)(u - \bar{u}) \\ &\quad + (G(x) - \bar{G})\bar{u} \\ &= (J - R)\nabla \mathcal{S}(x) + G(x)(u - \bar{u}) \\ &\quad - (\langle \nabla \mathcal{H}(x) \rangle - \langle \nabla \mathcal{H}(\bar{x}) \rangle)G(x)\bar{G}\bar{u} \\ &= (J - R)\nabla \mathcal{S}(x) + G(x)(u - \bar{u}) \\ &\quad - G(x)\bar{G}\langle \bar{u} \rangle (\nabla \mathcal{H}(x) - \nabla \mathcal{H}(\bar{x})) \\ &= (J - (R + Z(x)))\nabla \mathcal{S}(x) + G(x)(u - \bar{u}), \end{aligned}$$

where we used the fact that for all $a, b \in \mathbb{R}^n$, we have $\langle a \rangle b = \langle b \rangle a$. This completes the proof.

To complete the description of the P_wH system, we define the output of (1) as

$$y = G^\top(x) \nabla \mathcal{S}(x). \quad (6)$$

In the next section we will investigate the shifted passivity properties of the P_wH system (1), (6).

3 Main Result: Shifted Passivity

To establish the shifted passivity property we further restrict the trajectories to be inside the set

$$\bar{\Omega}_p := \{x \in \Omega^+ : R + Z(x) \geq 0\},$$

that is the closure of the open set

$$\Omega_p := \{x \in \Omega^+ : R + Z(x) > 0\}, \quad (7)$$

where we assume that Ω_p is non-empty.

Theorem 1 [Shifted passivity]

Consider the system (1), (6). For all trajectories $x \in \bar{\Omega}_p$ we have that

$$\dot{\mathcal{S}} \leq (y - \bar{y})^\top (u - \bar{u}). \quad (8)$$

Moreover, if \mathcal{H} is convex, the system is shifted passive [26], i.e. the mapping $(u - \bar{u}) \mapsto (y - \bar{y})$ is passive.¹

PROOF. Using Lemma 1 we can rewrite the system as in (3). Therefore, we have

$$\dot{\mathcal{S}} = -\nabla \mathcal{S}^\top (R + Z(x)) \nabla \mathcal{S} + y^\top (u - \bar{u}).$$

Now, note that y given in (6) can be written as

$$y = G(x) (\nabla \mathcal{H}(x) - \nabla \mathcal{H}(\bar{x})),$$

and hence $\bar{y} = 0$. The proof of (8) is completed restricting the trajectories to satisfy $x \in \bar{\Omega}_p$. To establish the passivity claim, note that since \mathcal{H} is convex, $\mathcal{S}(x)$ has an isolated minimum at \bar{x} , and hence is (locally, around \bar{x}) non-negative; see [13].

Remark 1 [Constant Power Sources]

In case we have constant power sources, i.e., $\langle \bar{u} \rangle \geq 0$, we see from (4) and the fact that $x \in \Omega^+$, that $Z(x) \geq 0$, and hence $\bar{\Omega}_p = \Omega^+$.

¹ This property is called passivity of the incremental model in [13].

Remark 2 [Additional Constant Input]

Theorem 1 holds also for the systems with an additional constant input, i.e.,

$$\dot{x} = (J - R) \nabla \mathcal{H}(x) + G(x)u + \bar{u}_c,$$

since the constant input $\bar{u}_c \in \mathbb{R}^n$ disappears in the shifted model (3).

4 Stability Analysis for Constant Inputs

Consider the system (1) with a constant input $u = \bar{u}$. Then the dynamics reads as

$$\dot{x} = (J - R) \nabla \mathcal{H}(x) + G(x)\bar{u}. \quad (9)$$

In this section, we first investigate the local stability of the equilibria of the system (9), that is, points \bar{x} such that $(\bar{x}, \bar{u}) \in \mathcal{E}$. Then, we give an estimate of their region of attraction (ROA). To establish these results we impose the stronger assumption that $x \in \Omega_p$ and, naturally, restrict ourselves to equilibrium points $\bar{x} \in \Omega_p$.

4.1 Local stability

Using the result of Theorem 1, we have the following corollary:

Corollary 1 [Local Stability]

Consider the system (9) having a point $\bar{x} \in \Omega_p$ such that $(\bar{x}, \bar{u}) \in \mathcal{E}$ and $\nabla^2 \mathcal{H}(\bar{x}) > 0$. Then, the equilibrium $x = \bar{x}$ of the system (9) is asymptotically stable.

PROOF. Since $R + Z(\bar{x}) > 0$ and $\nabla^2 \mathcal{H}(\bar{x}) > 0$, there exists a ball $\mathcal{B}(\bar{x})$, centered in \bar{x} , such that $\mathcal{S}(x) > 0$ and $R + Z(x) > 0$ for all $x \in \mathcal{B}(\bar{x})$. Moreover, \mathcal{S} satisfies

$$\dot{\mathcal{S}} = -\nabla \mathcal{S}^\top (R + Z(x)) \nabla \mathcal{S} < 0, \quad \forall x \in \mathcal{B}(\bar{x}), x \neq \bar{x},$$

making it a strict Lyapunov function. This completes the proof.

Note that the result of Corollary 1 applies also to an equilibrium point $x \in \bar{\Omega}_p$, if a *detectability* condition is satisfied, guaranteeing asymptotic stability by the use of LaSalle's Invariance principle; see [27], Ch. 8.

4.2 Characterizing an estimate of the ROA

As it is well-known, all bounded level sets of Lyapunov functions are invariant sets. However, our proof of asymptotic stability is restricted to the domain Ω_p . Consequently, to provide an estimate of the ROA of \bar{x} it is necessary to find a constant k such that the corresponding sublevel set of \mathcal{S}

$$\mathcal{L}_k := \{x \in \mathbb{R}^n \mid \mathcal{S}(x) < k, k \in \mathbb{R}_+\}, \quad (10)$$

is bounded and is contained in Ω_p . To solve this, otherwise daunting task, we make some assumptions on the system. First, we assume a positive definite dissipation matrix, that is, $R > 0$. Given this assumption, it is possible to construct a set—defined in terms of lower bounds on $\nabla\mathcal{H}$ —that is strictly contained in Ω_p .

Lemma 2 [Lower Bounds on $\nabla\mathcal{H}$]

If the dissipation matrix R is positive definite, then the set Ω_Γ defined as

$$\Omega_\Gamma := \left\{ x \in \Omega^+ : \langle \nabla\mathcal{H}(x) \rangle > -\frac{\bar{G}\langle \bar{u} \rangle}{\lambda_m\{R\}} \right\}, \quad (11)$$

is contained in Ω_p .

PROOF. For all $x \in \Omega_\Gamma$ we have

$$\lambda_m\{R\}I_n + \langle \nabla\mathcal{H}(x) \rangle^{-1}\bar{G}\langle \bar{u} \rangle > 0.$$

The proof is completed noting that the second left-hand term above is $Z(x)$ and recalling that $R \geq \lambda_m\{R\}I_n$.

Our second assumption is that the Hamiltonian is quadratic of the form

$$\mathcal{H}(x) = \frac{1}{2}x^\top \mathcal{M}x, \quad \mathcal{M} > 0, \quad (12)$$

In this case, the shifted Hamiltonian (5) reduces to

$$\mathcal{S}(x) = \frac{1}{2}(x - \bar{x})^\top \mathcal{M}(x - \bar{x}). \quad (13)$$

Notice that, now, all sublevel sets \mathcal{L}_k , given in (10), are *bounded*. Therefore, in view of Lemma 2, we only need to find a constant $k_c > 0$ such that $\mathcal{L}_{k_c} \subset \Omega_\Gamma$, and this sublevel set provides an estimate of the ROA of \bar{x} .

Theorem 2 [Estimate of the ROA]

Consider the system (9) with the quadratic Hamiltonian (12) and the dissipation matrix $R > 0$. Assume that $\bar{x} \in \Omega_\Gamma$ where Ω_Γ is given by (11). Define

$$k_c := \frac{1}{2\lambda_M\{\mathcal{M}\}} \min_{i=1,\dots,n} \left\{ (\gamma_i - (\mathcal{M}\bar{x})_i)^2 \right\},$$

with

$$\gamma_i := -\frac{\bar{G}_{ii}\bar{u}_i}{\lambda_m\{R\}},$$

and $(\mathcal{M}\bar{x})_i$ being the i th element of the vector $\mathcal{M}\bar{x}$. Then, an estimate of the ROA of the equilibrium \bar{x} is the sublevel set \mathcal{L}_{k_c} of the shifted Hamiltonian function $\mathcal{S}(x)$ defined in (13).

PROOF. From (13) we have

$$\mathcal{S}(x) = \frac{1}{2}(\mathcal{M}x - \mathcal{M}\bar{x})^\top \mathcal{M}^{-1}(\mathcal{M}x - \mathcal{M}\bar{x}).$$

Hence,

$$\mathcal{S}(x) \geq \frac{|\mathcal{M}x - \mathcal{M}\bar{x}|^2}{2\lambda_M\{\mathcal{M}\}}$$

with $|\cdot|$ the Euclidean norm. This bound, together with $\mathcal{S}(x) < k_c$, ensures

$$((\mathcal{M}x)_i - (\mathcal{M}\bar{x})_i)^2 < (\gamma_i - (\mathcal{M}\bar{x})_i)^2.$$

Note that since $\bar{x} \in \Omega_\Gamma$ we have $(\mathcal{M}\bar{x})_i > \gamma_i$. Hence $\gamma_i - (\mathcal{M}\bar{x})_i < 0$. Consequently,

$$\gamma_i - (\mathcal{M}\bar{x})_i < (\mathcal{M}x)_i - (\mathcal{M}\bar{x})_i < -\gamma_i + (\mathcal{M}\bar{x})_i.$$

The left hand side of the inequality above guarantees $(\mathcal{M}x)_i > \gamma_i$. Therefore, using Lemma 2, we have $R + Z(x) > 0$. The proof is completed noting that the latter ensures $\mathcal{S}(x)$ is a strict Lyapunov function of the system.

In the following corollary we show that, in cases where \mathcal{M} and R are diagonal, the largest k in (10)—and hence the largest \mathcal{L}_k contained in Ω_p —can be constructed explicitly.

To streamline the presentation of the result we define the constants

$$\eta_i := -\frac{\bar{G}_{ii}\bar{u}_i}{R_{ii}}, \quad (14)$$

and the constant vectors

$$\ell^i := \text{col}(\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{i-1}, \frac{\eta_i}{\mathcal{M}_{ii}}, \bar{x}_{i+1}, \dots, \bar{x}_n). \quad (15)$$

Corollary 2 [Estimate of the ROA with diagonal \mathcal{M} and R]

Consider the system (9) with the quadratic Hamiltonian (12) with *diagonal* $R > 0$ and $\mathcal{M} > 0$. Assume that $\bar{x} \in \Omega_\Gamma$. Recall $\mathcal{I} := \{i \in \{1, \dots, n\} : u_i = 0\}$, and define

$$k_d := \min_{i \notin \mathcal{I}} \{\mathcal{S}(\ell^i)\},$$

with (14) and (15). Then, an estimate of the ROA of the equilibrium \bar{x} is the sublevel set \mathcal{L}_{k_d} of the shifted Hamiltonian function $\mathcal{S}(x)$ defined in (13).

PROOF. Since both Z and R are diagonal, we have $R + Z(x) > 0$ if and only if $R_{ii} + Z(x)_{ii} > 0$ for all $i = \{1, \dots, n\}$. Therefore, Ω_p is computed as

$$\Omega_p = \{x \in \Omega^+ : R_{ii} + \bar{G}_{ii}\bar{u}_i G_{ii} > 0, \quad \forall i \notin \mathcal{I}\},$$

where we used the fact that for all $i \in \mathcal{I}$, $R_{ii} + Z_{ii} = R_{ii} > 0$. Hence, the set Ω_p can be defined in terms of lower bounds on $\nabla\mathcal{H}$, *i.e.*,

$$\Omega_p = \{x \in \Omega^+ : \nabla\mathcal{H}(x)_i > -\frac{\bar{G}_{ii}\bar{u}_i}{R_{ii}}, \quad \forall i \notin \mathcal{I}\}. \quad (16)$$

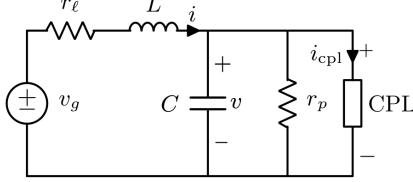


Figure 1. Single-Port DC circuit connected to a CPL.

The rest of the proof follows analogously to the proof of Theorem 2, and is hence, omitted.

Note that the set \mathcal{L}_{k_d} in this case is the ellipsoid

$$\left(\frac{x_1 - \bar{x}_1}{\sqrt{\frac{2k_d}{\mathcal{M}_{11}}}}\right)^2 + \left(\frac{x_2 - \bar{x}_2}{\sqrt{\frac{2k_d}{\mathcal{M}_{22}}}}\right)^2 + \cdots + \left(\frac{x_n - \bar{x}_n}{\sqrt{\frac{2k_d}{\mathcal{M}_{nn}}}}\right)^2 < 1. \quad (17)$$

5 Application to DC Networks with Constant Power Loads (CPL)

In this section, we apply the proposed method to study the stability of equilibria of, single-port and multi-port, DC networks with CPLs.

5.1 Single-port system

A schematic representation of a DC network with a single CPL is shown in Figure 1. Observe that the combination of the resistive load r_p and the CPL, acts as a ZIP load connected to the capacitor C . In view of Remark 2, the current sink is omitted for brevity.

Define the state vector $x = \text{col}(\varphi, q)$, where φ is the inductor flux and q is the capacitor charge. Then the network can be modeled by

$$\dot{x} = (J - R)\nabla\mathcal{H}(x) + G(x)u + u_c, \quad x \in \Omega^+, \quad (18)$$

with $\mathcal{H} = \frac{1}{2}x^\top \mathcal{M}x$ and

$$\mathcal{M} = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & \frac{1}{C} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} r_\ell & 0 \\ 0 & \frac{1}{r_p} \end{bmatrix}, \quad (19)$$

and $u_c = \text{col}(v_g, 0)$, $u = \text{col}(0, -P)$, where $P > 0$ is the power extracted by the CPL. Bearing in mind that the first element of the control input u is zero, the input matrix is

$$G(\varphi, q) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{C}{q} \end{bmatrix},$$

which is well defined in the set

$$\Omega^+ = \{(\varphi, q) \in \mathbb{R}^2 \mid q > 0\}.$$

The equilibria of the system (18), (19) are computed in the following lemma.

Lemma 3 [Equilibria of the system (18)-(19)]
The system (18)-(19) admits two equilibria given by

$$\bar{\varphi}_s = L \frac{r_p v_g - \sqrt{\Delta}}{r_\ell(r_\ell + r_p)}, \quad \bar{q}_s = Cr_p \frac{v_g + \sqrt{\Delta}}{(r_\ell + r_p)},$$

and

$$\bar{\varphi}_u = L \frac{r_p v_g + \sqrt{\Delta}}{r_\ell(r_\ell + r_p)}, \quad \bar{q}_u = Cr_p \frac{v_g - \sqrt{\Delta}}{(r_\ell + r_p)},$$

where

$$\Delta := v_g^2 - 4 \frac{r_\ell(r_p + r_\ell)}{r_p} P.$$

The equilibrium points are real if and only if $\Delta \geq 0$ or equivalently

$$P \leq P_{\max}^e, \quad P_{\max}^e := \frac{r_p v_g^2}{4r_\ell(r_\ell + r_p)}. \quad (20)$$

Through straightforward computations, it can be shown that the Jacobian of the vector field in the right hand side of (18), has a positive eigenvalue at the equilibrium point $(\bar{\varphi}_u, \bar{q}_u)$ and hence it is unstable. Furthermore, it can be shown that for small values of the load power, the Jacobian is negative definite at the equilibrium point $(\bar{\varphi}_s, \bar{q}_s)$.

Considering the results of Lemma 3, we continue with the equilibrium $(\bar{\varphi}_s, \bar{q}_s)$ as the candidate for nonlinear stability analysis. To use the results of Corollary 1, we first compute

$$R + Z(\varphi, q) = \begin{bmatrix} r_\ell & 0 \\ 0 & \frac{1}{r_p} - \frac{C^2 P}{\bar{q}_s q} \end{bmatrix}.$$

Next, we observe that $R + Z(\bar{\varphi}_s, \bar{q}_s) > 0$ if and only if

$$P < P_{\max}^s, \quad P_{\max}^s := \frac{r_p v_g^2}{(r_p + 2r_\ell)^2}. \quad (21)$$

Hence, according to Corollary 1, if the condition (21) is satisfied, then the equilibrium $(\bar{\varphi}_s, \bar{q}_s)$ is asymptotically stable. Note that if $P < \min\{P_{\max}^e, P_{\max}^s\}$, then the existence of the asymptotically stable equilibrium point $(\bar{\varphi}_s, \bar{q}_s)$, is guaranteed.

Next, using Corollary 2, we derive an estimate of the ROA of $(\bar{\varphi}_s, \bar{q}_s)$. Bearing in mind that the dissipative

matrix R is diagonal, and using Lemma 2, we compute Ω_p as

$$\Omega_p = \{(\varphi, q) \in \mathbb{R}^2 : q > q_{\min}\},$$

where

$$q_{\min} := P r_p \frac{C^2}{\bar{q}_s} = \frac{P}{P_{\max}} \bar{q}_s > 0. \quad (22)$$

The interpretation of (22) is that the closer the load power to P_{\max} is, the smaller the ROA is.

Now assume that (21) holds. Using Corollary 2, the set \mathcal{L}_{k_d} with

$$k_d = \mathcal{S}(\bar{\varphi}_s, q_{\min}),$$

is an estimate of the ROA. Furthermore, using (17), we can rewrite this set as the oval

$$\left(\frac{\varphi - \bar{\varphi}_s}{\sqrt{2Lk_d}} \right)^2 + \left(\frac{q - \bar{q}_s}{\sqrt{2Ck_d}} \right)^2 < 1. \quad (23)$$

This set guarantees $q > q_{\min}$ for all solutions starting within the oval.

We evaluate our results by a numerical example of the network shown in Fig. 1, with the parameters given by Table 1. The maximum power for existence of the equilibrium and its local stability are computed as $P_{\max}^e = 2.57\text{kW}$ and $P_{\max}^s = 2.33\text{kW}$, respectively. Note that the CPL satisfies the conditions (20) and (21), since

$$P < P_{\max}^s < P_{\max}^e.$$

Figure 2 shows the phase plane of the system (18)-(19). The estimate of the ROA (the oval (23)) is shown in blue, and all other converging solutions are shown in light gray. It is evident that the proposed method provides an appropriate estimate of the ROA, as the solutions just beneath this region (in dark gray), diverge from the equilibrium.

5.2 Multi-port networks

In this section, we investigate the stability of a complete multi-port DC network with CPLs. Let $i_M \in \mathbb{R}^l$ represent the currents of the inductors, and $v_C \in \mathbb{R}^c$ denote the voltages of the capacitors, where l and c are the number of inductors and capacitors. Then, the dynamics of the network can be described by [15]

$$\begin{bmatrix} \dot{\mathcal{L}}i_M \\ Cv_C \end{bmatrix} = \begin{bmatrix} -\mathcal{Z} & \Gamma \\ -\Gamma^\top & -\mathcal{Y} \end{bmatrix} \begin{bmatrix} i_M \\ v_C \end{bmatrix} + \begin{bmatrix} 0 \\ -[v_C]^{-1}P \end{bmatrix} + u_c, \quad (24)$$

where $\mathcal{L} > 0 \in \mathbb{R}^{l \times l}$ and $\mathcal{C} > 0 \in \mathbb{R}^{c \times c}$ are matrices associated with the magnitude of inductors (and mutual

Table 1
Simulation Parameters of the Single-Port CPL

$v_g(\text{V})$	$r_e (\Omega)$	$r_p(\Omega)$	$L(\mu\text{H})$	$C(\text{mF})$	$P(\text{kW})$
24	0.04	0.1	78	2	1

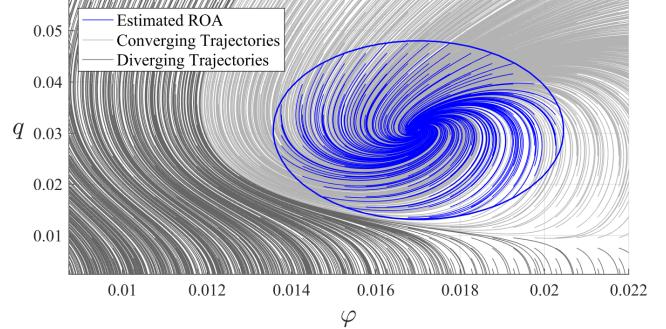


Figure 2. Phase plane of the system (18)-(19) with the parameters given in Table 1.

inductances) and capacitors, $\mathcal{Z} \in \mathbb{R}^{l \times l}$ and $\mathcal{Y} \in \mathbb{R}^{c \times c}$ are positive definite matrices associated with the resistances, and $\Gamma \in \mathbb{R}^{l \times c}$ is the matrix associated with the network topology. Also, the power of the CPLs is denoted by $P = \text{col}(P_1, \dots, P_c)$. The vector $u_c \in \mathbb{R}^{l+c}$ is constant and its components are linear combinations of the voltages and currents of the sources in the network. We assume that the capacitors and the inductors are not ideal, *i.e.* we consider that all the inductors have a resistance in series and the capacitors possess a resistor in parallel. Moreover, we assume that the constant power loads are connected to a capacitor in parallel. This feature amounts for the capacitive effect of the input filters for this type of loads; see [4, 8, 9].

With a little abuse of notation, define the state vector $x = \text{col}(\varphi, q) \in \mathbb{R}^{l+c}$ and the control vector $u = \text{col}(0, -P) \in \mathbb{R}^{l+c}$, where $\varphi \in \mathbb{R}^l$ denotes the magnetic flux of the inductors, and $q \in \mathbb{R}^c$ denotes the electric charge of the capacitors. Then the network dynamics of the multi-port network given in (24) admits a Port-Hamiltonian representation given by

$$\dot{x} = (J - R)\nabla \mathcal{H}(x) + G(x)u + u_c,$$

with $\mathcal{H} = \frac{1}{2}x^\top \mathcal{M}x$ and

$$\mathcal{M} = \begin{bmatrix} \mathcal{L}^{-1} & 0 \\ 0 & \mathcal{C}^{-1} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & \Gamma \\ -\Gamma^\top & 0 \end{bmatrix}, \quad R = \begin{bmatrix} \mathcal{Z} & 0 \\ 0 & \mathcal{Y} \end{bmatrix}.$$

Similar to the case of the single-port *RLC* circuit with a CPL, and using Theorem 2, an ellipsoid can be computed here as an estimate of the ROA.

6 Application to Synchronous Generators connected to a CPL

In this section, we apply the results to the case of a synchronous generator connected to a CPL. This system can be modeled by²

$$\dot{p} = (J - R)\nabla\mathcal{H}(p) + G(p)u + u_c, \quad (25)$$

with

$$\begin{aligned} p &= M\omega, J = 0, R = D_m + D_d, G(p) = \frac{M}{p} = \omega^{-1} \\ \mathcal{H} &= \frac{1}{2M}p^2, u = -P_e, u_c = \tau_m + D_d\omega^*, \end{aligned} \quad (26)$$

where $p \in \mathbb{R}_+$ is the angular momentum, $M > 0$ is the total moment of inertia of the turbine and generator rotor, $\omega \in \mathbb{R}_+$ is the rotor shaft velocity, $\omega^* > 0$ is the nominal frequency of 50 Hz, $D_m > 0$ is the damping coefficient of the mechanical losses, $D_d > 0$ is the damping-torque coefficient of the damper windings, $\tau_m > 0$ is the constant mechanical torque (physical input), and P_e is the constant power load.

Assume that

$$P < \frac{(D_d\omega^* + \tau_m)^2}{4(D_d + D_m)}.$$

Then the dynamics (25), (26) has the following two equilibria

$$\bar{\omega}_s = \frac{D_d\omega^* + \tau_m + \sqrt{\Delta}}{2(D_d + D_m)}, \quad \bar{\omega}_u = \frac{D_d\omega^* + \tau_m - \sqrt{\Delta}}{2(D_d + D_m)},$$

with $\Delta := (D_d\omega^* + \tau_m)^2 - 4(D_d + D_m)P_e$. We have

$$R + Z(\omega) = D_d + D_m - \frac{P_e}{\bar{\omega}_s \omega}.$$

The equilibrium point $\omega = \bar{\omega}_s$ is asymptotically stable since

$$R + Z(\bar{\omega}_s) = \frac{\tau_m + D_d\omega^*}{\bar{\omega}_s} > 0.$$

Through straightforward computations, it can be shown that the set Ω_p in (16) can be written as

$$\Omega_p = \{\omega \in \mathbb{R}_+ : \omega > \bar{\omega}_u\}. \quad (27)$$

In this set, the shifted Hamiltonian $\mathcal{S} = \frac{1}{2}M(\omega - \bar{\omega}_s)^2$ is strictly decreasing. Therefore the solutions get closer to

² This model is called *improved swing equation* in [20,28]. An inverter with a capacitive inertia can be modeled by similar dynamics; see [21].

Table 2
Simulation Parameters of the Synchronous Generator (p.u.)

M	D_m	D_d	P_e	τ_m
0.2	10^{-6}	10^{-4}	3	0.0027

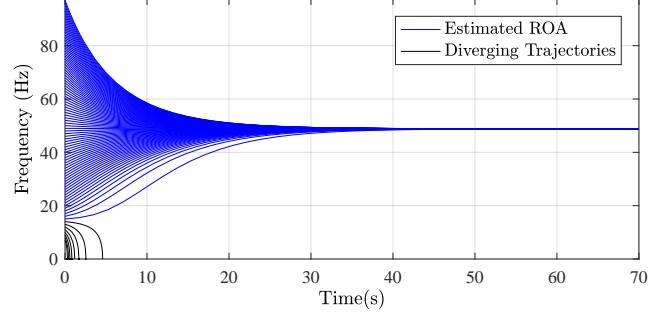


Figure 3. Solutions of the system (25),(26) with different initial conditions.

the equilibrium $\bar{\omega}_s$ and move away from the point $\bar{\omega}_u$ as time goes by. Consequently the set Ω_p in (27) is forward invariant and represents the estimate of the ROA.

Figure 3 shows the trajectories of a number of solutions of the system (25)-(26), with the parameters given by Table 2, and with different initial conditions. It is clear that the proposed method successfully identifies a very precise estimate of the ROA (blue), as all the solutions starting from outside the ROA estimate (black) diverge from the equilibrium.

7 Conclusion and future works

In this paper, a class of pH systems was investigated where the control input/disturbance acts on the power of the system. We refer to these systems as Power-controlled Hamiltonian ($P_w H$) systems. First, a model for such systems was proposed, and second, the condition on which the system is shifted passive was computed. Using these results, the stability of equilibria was investigated. Furthermore, an estimate of the region of attraction was derived for $P_w H$ systems with quadratic Hamiltonian. The proposed modeling and conditions were derived and computed for two cases of interest in practice: A DC circuit and a synchronous generator, both connected to constant power loads. Finally, the validity and utility of the proposed method was confirmed by numerical examples of these case studies. Future work includes design of high-performance controllers with guaranteed stability domains, and investigation over the applicability of the proposed method for AC circuits with constant power loads, higher order models of the synchronous generator [12], and state-dependant structure and dissipation matrices [19].

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