

Stochastic Predictive Control under Intermittent Observations and Unreliable Actions

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Abstract

We propose a provably stabilizing and tractable approach for control of constrained linear systems under intermittent observations and unreliable transmissions of control commands. A smart sensor equipped with a Kalman filter is employed for the estimation of the states from incomplete and corrupt measurements, and an estimator at the controller side optimally feeds the intermittently received sensor data to the controller. The remote controller iteratively solves constrained stochastic optimal control problems and transmits the control commands according to a carefully designed transmission protocol through an unreliable channel. We present a (globally) recursively feasible quadratic program, which is solved online to yield a stabilizing controller for Lyapunov stable linear time invariant systems under any positive bound on control values and any non-zero transmission probabilities of Bernoulli channels.

Key words: stochastic predictive control, packet dropouts, output feedback, Kalman filtering, networked systems.

1 Introduction

Predictive techniques for networked control systems (NCSs) have significantly advanced over the past decade. Emphasis has been placed on information loss [1, 2], recursive feasibility issue [3], stability analysis [4, 5], and focus on applications such as drinking water networks [6, 7]. In addition, NCSs have been constructed in diverse, interesting and important applications including haptic collaboration over the internet [8–10], building automation [11], vehicle control [12], mobile sensor networks [13] to name only a few. Predictive techniques make the controller capable of handling constraints while optimizing a desired performance objective.

Most of the predictive control techniques neglect uncertainties when designing the objective function, e.g., [4], or consider only the worst case scenario over uncertainties, e.g., [6]. In both cases the resulting controllers are conservative and do not take advantage of the available statistics. However, while controller design without incorporating such statistical information is easier and simpler than otherwise, this results in a decline in the desired performance [14]. Stochastic predictive controllers offer a way out of such conservatism. Two major challenges need to be overcome in the stochastic controller design – tractability of the underlying constrained stochastic optimal control problem (CSOCP) and

guaranteeing stability in some suitable sense. We refer the readers to [15, 16] for discussions on the underlying challenges in stochastic predictive control under the settings of perfect channels. Recently, [14, 17, 18] designed tractable and stabilizing controllers for networked systems that take the probability distributions of the uncertainties and network induced effects into account at the synthesis stage. However, [14, 17, 18] rely on the assumption that the sensor channel is perfect and there are dropouts only in the control channel. This assumption is reasonable for applications as mentioned in [19, 20], but in a wide range of applications available sensor-communication channels are unreliable. Therefore, an extension of [14, 17, 18] to the setting of an unreliable sensor channel is important for a large class of applications, and lies at the heart of the current work.

Apart from unreliable communication channels, typically measurements are corrupted by sensor noise and/or full state information are not available. Moreover, since actuators are physical devices, constraints on the control actions must be satisfied for all realizations of the uncertainties. Although the synthesis of stabilizing constrained control under incomplete and corrupt observations is an interesting problem, the literature on this topic is sparse, apart from [21, 22]. In both of these articles a stochastic predictive controller was proposed over ideal communication channels with an affine saturated innovation feedback policy employing a Kalman

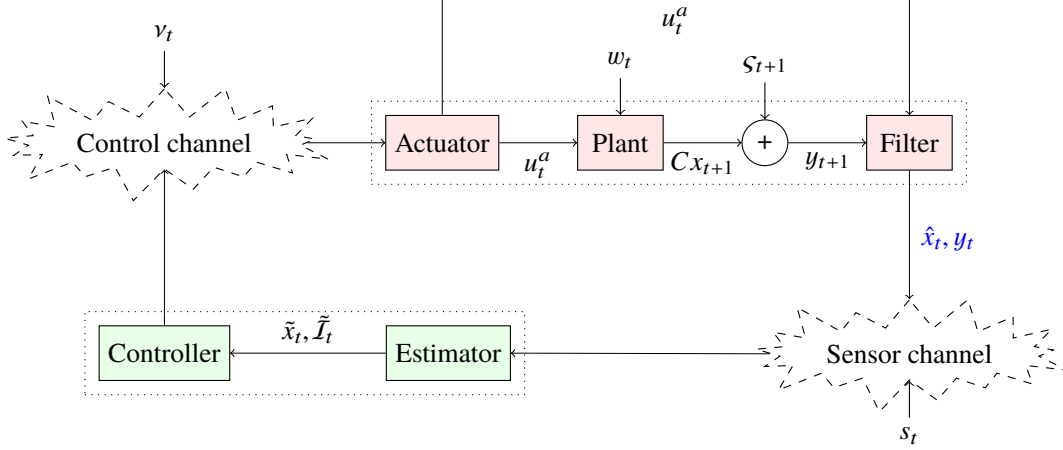


Fig. 1. The red blocks are situated at the plant and the green blocks at the remote controller. The red and the green blocks communicate via erasure channels. The Kalman filter employs the measurement y_t and the applied control u_t^a to estimate the state \hat{x}_{t+1} . The acknowledgements of successful transmissions of control commands are causally available to the controller and the estimator. The estimator is also aware of the sensor channel dropout s_t at time t .

filter and constant negative drift conditions. However, their implementation over unreliable networks is non-trivial due to boundedness issues in Kalman filtering with intermittent observations [23, 24].

Several interesting methods have been proposed to estimate the system states over unreliable sensor channels [23–34]. However, to the best of our knowledge the case of sensor channel erasure under the settings of stochastic predictive constrained control has not been examined so far due to, as already mentioned, boundedness issues involved in Kalman filtering. To be precise, the conditional error covariance matrix exhibits unbounded oscillations almost surely, and is therefore computationally difficult to deal with. In this article we adapt the idea of smart sensors [33, 34] and present a controller that has a computationally tractable underlying optimal control problem. The proposed controller ensures stability of the closed-loop states in a suitable sense under any positive bound on control and any successful transmission probabilities of the sensor and the control channels. Our results hold for the largest class (to date) of linear time invariant (LTI) discrete dynamical systems known to be stabilizable under bounded control actions even without dropouts. For that purpose, we generalize the approach of [14, 17, 18, 21, 22] by considering the unreliable sensor channel in the network system architecture.

The main features and contributions of the proposed approach are as follows:

- We consider an LTI system, with incomplete and corrupt measurements, remotely controlled over unreliable channels.
- We present a tractable and recursively feasible quadratic program to be solved periodically online, which provides optimal control that minimizes the expected quadratic cost function and ensures a good closed-loop behaviour of states.

- In order to compensate the effect of packet dropouts in the sensor channel, we employ a remote estimator. We show that the remote estimator provides the conditional expectation of states given causally available information at the estimator.
- We employ a novel class of feedback policies and show convexity of the underlying optimization program.
- For systems having all eigenvalues inside the unit circle and semi-simple eigenvalues on the boundary (if any), we show mean-square boundedness of the controlled system for any positive bound on the control actions. Mean-square boundedness of the controlled states is guaranteed for any non-zero probabilities of successful transmissions in both the sensor channel and the control channel.

This article exposes as follows: In Section 2 we discuss the basic assumptions and formally define the problem statement. The system setup is presented in Section 3. We develop our main results on tractability and stability in Section 4 and Section 5, respectively. We validate our theoretical results by numerical experiments in Section 6 and conclude in Section 7. Some proofs are presented in the Appendix.

Notation

Let $\mathbb{R}, \mathbb{N}_0, \mathbb{Z}_+$ denote the set of real numbers, the non-negative integers and the positive integers, respectively. We use the symbol $\mathbf{0}$ to denote a matrix of appropriate dimensions with all elements 0. For any vector sequence $(v_n)_{n \in \mathbb{N}_0}$ taking values in some Euclidean space, let $v_{n:k}$ denote the vector $\begin{bmatrix} v_n^\top & v_{n+1}^\top & \cdots & v_{n+k-1}^\top \end{bmatrix}^\top$, $k \in \mathbb{Z}_+$. The notations $\mathbb{E}_z[\cdot]$ and $\mathbb{E}[\cdot | z]$ are interchangeably used for the conditional expectation with given z . For a vector v , its i^{th} element is denoted by $v^{(i)}$. Similarly, $M^{(j, \cdot)}$ denotes the j^{th} row of a given matrix M . Let $\sigma_1(M)$ denote the largest singular value of M , and M^\dagger its Moore-Penrose pseudo inverse. A

block diagonal matrix M with diagonal entries M_1, \dots, M_n is represented as $M = \text{bdiag}\{M_1, \dots, M_n\}$ and I_d is the $d \times d$ identity matrix. For a real quantity ξ , its positive component ξ_+ and negative component ξ_- are defined to be $\max\{0, \xi\}$ and $\max\{0, -\xi\}$, respectively. For a given positive semi-definite (or definite) matrix P and a vector v , the notation $\|v\|_P^2$ is used to denote the scalar $v^\top P v$.

2 Problem Statement

Let us consider a discrete time dynamical system

$$x_{t+1} = Ax_t + Bu_t^a + w_t \quad (1a)$$

$$y_t = Cx_t + \zeta_t, \quad (1b)$$

where $t \in \mathbb{N}_0$, and $x_t \in \mathbb{R}^d$, $u_t^a \in \mathbb{R}^m$, $y_t \in \mathbb{R}^q$ are the states, the applied control to the plant and the measurements, respectively, at time t . The additive process noise $w_t \in \mathbb{R}^d$ and the measurement noise $\zeta_t \in \mathbb{R}^q$ are zero-mean Gaussian. System matrices A, B and C are known matrices of appropriate dimensions and the matrix pair (A, B) is stabilizable. At each time t the control u_t^a is constrained to take values in the admissible set

$$\mathbb{U} := \{v \in \mathbb{R}^m \mid \|v\|_\infty \leq u_{\max}\}, \quad (2)$$

where the (uniform) bound $u_{\max} > 0$ is preassigned.

Remark 1 Admissible control set of the form

$$\mathbb{U}' := \left\{v \in \mathbb{R}^m \mid |v^{(i)}| \leq U_i \text{ for } i = 1, \dots, m\right\},$$

for not necessarily equal values U_i , can be transformed easily into \mathbb{U} as in (2). Please see [35, Remark 1] for more details.

The sensor channel and the control channel both are unreliable with successful transmission probabilities, p_s and p_c , respectively. We represent the dropout at time t in the sensor and the control channel by i.i.d. Bernoulli random variables s_t and v_t , respectively, see Fig. 1. We have the following assumption:

(A1) The dropout processes $(v_t)_{t \in \mathbb{N}_0}$ and $(s_t)_{t \in \mathbb{N}_0}$ are mutually independent and individually i. i. d. They are also independent of the process noise and the measurement noise processes $(w_t)_{t \in \mathbb{N}_0}$ and $(\zeta_t)_{t \in \mathbb{N}_0}$, respectively.

The inclusion of the constraint (2) at the synthesis stage is achieved by the following constrained stochastic optimal control problem (CSOCP) that is solved iteratively over time, and that constitutes the backbone of predictive control techniques:

$$\begin{aligned} & \underset{\text{control policies}}{\text{minimize}} && \text{a quadratic objective function} \\ & \text{subject to} && \begin{cases} \text{system dynamics (1),} \\ \text{hard constraint on control (2).} \end{cases} \end{aligned} \quad (3)$$

It is well known that the stochastic optimal control problem in the form of (3) is computationally intractable even in the presence of the full state information and perfect channels. An affine feedback policy based approach is often used to present a tractable surrogate of CSOCP (3) when an expected quadratic cost is used as objective function [36]. In order to mitigate the effect of incomplete and corrupt measurements, a Kalman filter (see Section 3.2) at the sensor with the following assumptions is employed.

(A2) The matrix pair (A, C) is observable.

(A3) The initial condition x_0 , the process and the measurement noise vectors are normally distributed and mutually independent, i.e. $x_0 \sim N(0, \Sigma_{x_0})$, $w_t \sim N(0, \Sigma_w)$, $\zeta_t \sim N(0, \Sigma_\zeta)$, with $\Sigma_{x_0} \geq 0$, $\Sigma_w \geq 0$ and $\Sigma_\zeta > 0$.

(A4) The matrix pair $(A, \Sigma_w^{1/2})$ is controllable.

At the controller end, an optimal estimator is employed to mitigate the effects of sensor channel dropouts; see Fig. 1 for a schematic. In one hop communication links, it is a standard practice to transmit error free receipt acknowledgements (or negative acknowledgements) [37, page 207]. Therefore, the estimator is aware of the previously applied controls to the plant. Accordingly, we have the following assumption:

(A5) The acknowledgements of the successfully transmitted control commands are causally available to the controller and the estimator.

The objective of this paper is to present a tractable surrogate of CSOCP (3) and ensure stability (in some suitable sense) of the closed-loop system (1a) when control is computed by iteratively solving (3). Since in the examined situation asymptotic stability of the origin cannot be achieved due to the unbounded support of the additive process noise and the measurement error, we are interested in the notion of mean square boundedness, as defined below:

Definition 1 (Mean square boundedness) An \mathbb{R}^d -valued random process $(x_t)_{t \in \mathbb{N}_0}$ is said to be mean-square bounded (MSB) if there exists some $\gamma > 0$ such that

$$\sup_{t \in \mathbb{N}_0} \mathbb{E}_{\mathfrak{Y}_0^s} [\|x_t\|^2] \leq \gamma,$$

where \mathfrak{Y}_0^s is the information available at $t = 0$.

It is well known that, even with perfect communication channels, an LTI system of the form (1) with system matrix A having eigenvalues outside the unit circle cannot be (globally) stabilized by any control technique with help of bounded control actions [38, Abstract], [39, Theorem 1.7]. Consequently, for the present networked case, we impose the following assumption:

(A6) The system matrix A has all eigenvalues on the unit disk and those on the unit circle are semi-simple.¹

Dynamical systems satisfying Assumption (A6) are well-studied as Lyapunov stable systems [40]. In the presence of unbounded disturbances, an open-loop Lyapunov stable (but not asymptotically stable) system leads to unstable trajectories with probability one, unless the control is designed with sufficient care. Such designs are non-trivial if hard constraints on the control have to be satisfied at all times. Traditional MPC for LTI systems under hard bounds on control inputs has been widely studied [41]. Unfortunately, the approaches in [41] do not guarantee mean square boundedness in the presence of unbounded disturbances. The control strategy developed in the present article is for stochastic systems (additive disturbance has unbounded support as well as observations are intermittent and control commands are unreliable) where most of the well developed tools of deterministic MPC do not carry over. In particular, tools developed assuming deterministic settings mostly rely on terminal set and cost methods [4, 42]. Further, the construction of suitable positively invariant sets in the presence of disturbances with unbounded support is impossible [43, §3]; please also see [44, Lemma 1]. In order to transcend beyond the regime of terminal set and cost method drift conditions are used for stochastic systems. We do not invoke martingale arguments directly for our drift conditions. Instead, our result is based on the approach in [45, Theorem 1]. The latter work contains a delicate proof of their main result relying on the Burkholder's inequality, Doob decomposition in the theory of martingales and some other arguments. We recall the following result:

Theorem 2 ([45, Theorem 1, Corollary 2]) *Let $(X_t)_{t \in \mathbb{N}_0}$ be a family of real valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a filtration $(\mathcal{F}_t)_{t \in \mathbb{N}_0}$. Suppose that there exist scalars $a, b, c > 0$ such that*

$$\begin{aligned} \mathbb{E}_{\mathcal{F}_t}[X_{t+1} - X_t] &\leq -a \quad \text{on the event } X_t > b, \\ \mathbb{E}[|X_{t+1} - X_t|^4 \mid X_0, \dots, X_t] &\leq c \quad \text{for all } t \in \mathbb{N}_0. \end{aligned}$$

Then there exists a constant $\gamma > 0$ such that

$$\sup_{t \in \mathbb{N}_0} \mathbb{E}[(X_t)_+^2 \mid \mathcal{F}_0] \leq \gamma.$$

The first condition of the above theorem is called the *constant negative drift condition* and it is active when X_t is larger than some $b > 0$. The second condition is called *skip-free condition* and is needed to avoid long jumps when $X_t \leq b$. The above theorem gives sufficient conditions for the mean

¹ The stabilizability of LTI systems with non-semi-simple eigenvalues on the unit circle under bounded control actions remains an open problem [39]. Therefore, the considered class of systems is the largest class of LTI systems, known till date, stabilizable under bounded control actions.

square boundedness of the positive component $(X_t)_+$ of a scalar process $(X_t)_{t \in \mathbb{N}_0}$ and extended for the vector processes in [39] utilizing the Assumption (A6) and decomposition of the system dynamics into orthogonal and Schur stable subsystems, which is given explicitly in Section 5. We utilized the above theorem and the idea of decomposition in our previous works [14, 35] in a limited context. A part of our stability result is along the lines of [39, Theorem 1.2] but satisfaction of the second condition of Theorem 2 is non-trivial in the setting of the present article. In particular, [39, Theorem 1.2] shows the existence of a stabilizing history dependent feedback policy by considering a κ -subsampled process when the system matrix A is orthogonal with reachability index κ and channels are perfect. In this article we extend the stability analysis of [39, 46] under the settings of unreliable channels.

Remark 2 The recursive feasibility of stochastic predictive control techniques under state constraints is challenging whenever involved noise processes have unbounded support [47]. The inclusion of state constraints within our framework can be investigated along the lines of [48, 49]. For simplicity of the presentation we have not considered state constraints in this article.

The problem statement of the present article is formally given below:

PROBLEM STATEMENT 1 Present a tractable, stabilizing and recursively feasible surrogate of CSOCP (3) under the assumptions (A1) – (A6).

3 Setup

As illustrated in Fig. 1, we employ a Kalman filter at the sensor and an estimator at the controller. The information of the past outputs and previously applied control is available at the filter. At each time t the filtered state \hat{x}_t and output y_t are transmitted through the sensor channel. Since, the sensor channel is affected by Bernoulli dropouts, either the transmitted information reaches the estimator or it is lost. For the sake of computational tractability we consider quadratic cost functions, which are minimized over a class of policies. Control commands obtained by solving the optimization programs are transmitted through an erasure channel. To mitigate the effects of dropouts in the control channel we employ ad-hoc transmission strategies. A detailed discussion on the above features is presented below:

3.1 Expected quadratic cost

Let us fix an optimization horizon $N \in \mathbb{Z}_+$ and recalculation interval (control horizon) $N_r \leq N$. Let \mathcal{Y}_t^s be the information available at the controller/estimator at the time of optimization t (see Section 3.3 for a precise definition). The cost V_t is defined to be the conditional expectation of the quadratic

cost in one optimization horizon with given information \mathfrak{Y}_t^s . We define V_t as

$$V_t := \mathbb{E}_{\mathfrak{Y}_t^s} \left[\sum_{k=0}^{N-1} (\|x_{t+k}\|_Q^2 + \|u_{t+k}^a\|_R^2) + \|x_{t+N}\|_{Q_N}^2 \right], \quad (4)$$

where, Q, Q_N are given symmetric positive semi-definite matrices of appropriate dimensions and R is given symmetric positive definite matrix. The compact form representation of the system (1) over one optimization horizon is as follows:

$$x_{t:N+1} = \mathcal{A}x_t + \mathcal{B}u_{t:N}^a + \mathcal{D}w_{t:N} \quad (5a)$$

$$y_{t:N+1} = \mathcal{C}x_{t:N+1} + \zeta_{t:N+1}, \quad (5b)$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} are standard matrices of appropriate dimensions. The cost function (4) can also be written in a compact form for later use as follows:

$$V_t = \mathbb{E}_{\mathfrak{Y}_t^s} \left[\|x_{t:N+1}\|_Q^2 + \|u_{t:N}^a\|_R^2 \right], \quad (6)$$

where Q and R are standard block diagonal matrices of appropriate dimensions.

3.2 Kalman filter

In this section we recall the framework of stochastic predictive control using Kalman filtering. The detailed discussion is available in [21]. For each t let $\mathfrak{Y}_t := \{y_0, \dots, y_t, u_0^a, \dots, u_{t-1}^a\}$ denote the set of observations up to time t . For $t, s \in \mathbb{N}_0, t \geq s$, let us define $\hat{x}_{t|s} := \mathbb{E}_{\mathfrak{Y}_s} [x_t]$ and $P_{t|s} := \mathbb{E}_{\mathfrak{Y}_s} [(x_t - \hat{x}_{t|s})(x_t - \hat{x}_{t|s})^\top]$, and for brevity of notation, we denote $\hat{x}_{t|t}$ by \hat{x}_t and $P_{t|t}$ by P_t . We need the following result related to Kalman filtering for which recursions are [50, p.102]:

$$\begin{aligned} \hat{x}_{t+1} &= \hat{x}_{t+1|t} + K_t(y_{t+1} - C\hat{x}_{t+1|t}) \\ P_{t+1} &= P_{t+1|t} - K_t C P_{t+1|t} \end{aligned} \quad (7)$$

where $\hat{x}_{t+1|t} = A\hat{x}_t + Bu_t^a$, $P_{t+1|t} = AP_tA^\top + \Sigma_w$ and $K_t = P_{t+1|t}C^\top (CP_{t+1|t}C^\top + \Sigma_\zeta)^{-1}$. We initialize the Kalman filter by setting $\hat{x}_{0|-1} = \mathbf{0}, P_{0|-1} = \Sigma_{x_0}$, we get $\hat{x}_0 = K_0(Cx_0 + \zeta_0)$. Let us recall the results of [21, Lemma 8] and [51, Lemma 4.2.2] that there exists a constant $\rho > 0$ such that

$$\mathbb{E}_{\mathfrak{Y}_t} [\|x_t - \hat{x}_t\|^2] \leq \rho \quad \text{for all } t. \quad (8)$$

Let us define filtered output $\hat{y}_t := C\hat{x}_t$, the innovation term $\bar{I}_t := y_t - \hat{y}_t$ and Kalman filter error $e_t := x_t - \hat{x}_t$. A straightforward calculation gives the innovation term for one optimization horizon as follows:

$$\bar{I}_{t:N+1} = C\mathcal{F}_t e_t + \mathcal{O}_t w_{t:N} + (I - C\mathcal{H}_t)\zeta_{t:N+1} \quad (9)$$

where

$$\begin{aligned} \mathcal{F}_t &:= \begin{bmatrix} I_d \\ \phi_t \\ \phi_{t+1}\phi_t \\ \vdots \\ \phi_{t+N-1} \cdots \phi_t \end{bmatrix} \\ \mathcal{O}_t &:= \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \Gamma_t & \cdots & 0 & 0 \\ \phi_{t+1}\Gamma_t & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ \phi_{t+N-2} \cdots \phi_{t+1}\Gamma_t & \cdots & \Gamma_{t+N-1} & 0 \\ \phi_{t+N-1} \cdots \phi_{t+1}\Gamma_t & \cdots & \phi_{t+N-1}\Gamma_{t+N-2} & \Gamma_{t+N-1} \end{bmatrix} \\ \mathcal{H}_t &:= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & K_t & \cdots & 0 & 0 \\ 0 & \phi_{t+1}K_t & \cdots & 0 & 0 \\ 0 & \vdots & \cdots & \vdots & \vdots \\ 0 & \phi_{t+N-2} \cdots \phi_{t+1}K_t & \cdots & K_{t+N-1} & 0 \\ 0 & \phi_{t+N-1} \cdots \phi_{t+1}K_t & \cdots & \phi_{t+N-1}K_{t+N-2} & K_{t+N-1} \end{bmatrix} \end{aligned}$$

The dynamics of \hat{x}_t can be given by

$$\hat{x}_{t+1} = A\hat{x}_t + Bu_t^a + \hat{w}_t, \quad (10)$$

where $\hat{w}_t = K_t(CAe_t + Cw_t + \zeta_{t+1})$. The above equations (9) and (10) are standard in literature and can be found by using (1) and (7). Since \hat{w}_t is a linear sum of three mutually independent Gaussian random variables, it is also Gaussian with a time varying and bounded variance due to K_t .

3.3 Optimal estimator

For each t let $\mathcal{M}_t := \{s_t y_t, s_t \hat{x}_t, s_t, u_{t-1}^a\}$ and $\mathfrak{Y}_t^s := \{\mathcal{M}_0, \dots, \mathcal{M}_t\}$ denote the set of data available at the estimator up to time t , with the convention that $u_{-1}^a = \mathbf{0}$. We have the following results:

Lemma 3 Let $\tilde{x}_t := \mathbb{E}[x_t | \mathfrak{Y}_t^s]$. Then

$$\tilde{x}_t = s_t \hat{x}_t + (1 - s_t)(A\tilde{x}_{t-1} + Bu_{t-1}^a). \quad (11)$$

A proof of Lemma 3 is given in the appendix. The estimator (11) has been widely used in the literature. The above Lemma 3 proves that it is optimal under the settings of the present article. We define the estimation error $\tilde{e}_t := \hat{x}_t - \tilde{x}_t$ and initialize $\tilde{x}_{-1} = \mathbf{0}, \tilde{e}_{-1} = \mathbf{0}$.

3.4 Affine saturated received innovation feedback policy

Affine feedback parametrizations in terms of the innovation sequence are standard in the literature [52]. Since the optimization is carried over a particular class of policies, the obtained solution is sub-optimal. In order to satisfy hard bounds on the control actions while retaining computational tractability, affine saturated innovation feedback policies are used in [21, 22]. Since innovation terms are affected by sensor channel dropouts under the settings of this article, a received innovation term is employed in the feedback parametrization used in the present article. We consider the following causal feedback policy class for $\ell = 0, \dots, N-1$,

$$\begin{aligned} u_{t+\ell} &= \eta_{t+\ell} + \sum_{i=0}^{\ell} \theta_{\ell,t+i} \psi_i(s_{t+i} \tilde{\mathcal{I}}_{t+i}) \\ &= \eta_{t+\ell} + \sum_{i=0}^{\ell} s_{t+i} \theta_{\ell,t+i} \psi_i(\tilde{\mathcal{I}}_{t+i}), \end{aligned} \quad (12)$$

where $\psi_i : \mathbb{R}^q \rightarrow \mathbb{R}^q$ is a measurable map for each i such that $\|\psi_i(y_{t+i} - \hat{y}_{t+i})\|_{\infty} \leq \psi_{\max}$. Let us denote $\tilde{\mathcal{I}}_t := s_t \mathcal{I}_t$ for brevity. The above control policy class (12) can be represented in a compact form as follows:

$$u_{t:N} = \eta_t + \Theta_t \psi(\tilde{\mathcal{I}}_{t:N}) \quad (13)$$

where $\eta_t \in \mathbb{R}^{mN}$, $\psi := [\psi_1^\top \dots \psi_N^\top]^\top$ and Θ_t is the following block triangular matrix

$$\Theta_t = \begin{bmatrix} \theta_{0,t} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \theta_{1,t} & \theta_{1,t+1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \theta_{N-1,t} & \theta_{N-1,t+1} & \cdots & \theta_{N-1,t+N-2} & \theta_{N-1,t+N-1} \end{bmatrix}, \quad (14)$$

with each $\theta_{k,\ell} \in \mathbb{R}^{m \times q}$ and

$$\|\psi(\tilde{\mathcal{I}}_{t:N})\|_{\infty} \leq \psi_{\max}.$$

We choose ψ_i 's to be component-wise odd functions, e.g., standard saturation function, sigmoidal function, etc.

Lemma 4 *The hard constraint on control (2) under the class of control policies (12) is equivalent to the following constraint:*

$$\|\eta_t^{(i)} + \|\Theta_t^{(i,:)}\|_1 \varphi_{\max} \leq u_{\max} \quad (15)$$

PROOF. In view of the dropout process $v_t \in \{0, 1\}$,

$$\begin{aligned} \|u_{t:N}^a\|_{\infty} \leq u_{\max} &\iff \|u_{t:N}\|_{\infty} \leq u_{\max} \\ &\iff \|\eta_t + \Theta_t \psi(\tilde{\mathcal{I}}_{t:N})\|_{\infty} \leq u_{\max}. \end{aligned}$$

Now the assertion follows from [53, Proposition 3].

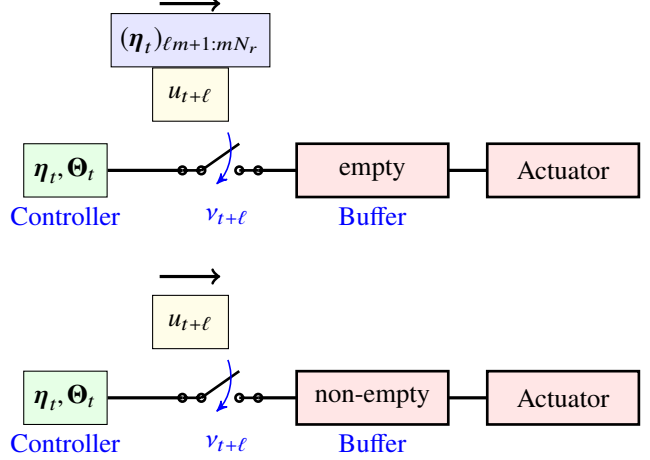


Fig. 2. Control channel and buffer at time $t + \ell$ for (TP): The blue blocks are transmitted only if the buffer is empty and the yellow blocks are transmitted at each time.

3.5 Transmission protocol

In order to mitigate the effects of packet dropouts in the control channel, several transmission protocols are discussed in [14]. We consider one of them, which is formally defined below. Our approach remains valid for the other protocols as well, provided minor adjustments are made.

(TP) At the beginning of each optimization instant ($t = 0, N_r, 2N_r, \dots$), the buffer is emptied. For $\ell \leq N_r - 1$, $u_{t+\ell}$ is transmitted and directly applied to the plant if successfully received at the actuator. In addition, $\eta_{t+\ell+1}, \dots, \eta_{t+N_r-1}$ are also transmitted until the first successful reception to store in a buffer near the actuator. In case of the loss of $u_{t+\ell}$ at $t + \ell$, $\eta_{t+\ell}$ is applied to the plant if it is already present in buffer, otherwise null control is applied.

Since the optimization problem is solved after each recalculation interval, only $N_r \leq N$ blocks of controls in (13) are applied to the plant, the rest of them are discarded. Therefore, the above protocol transmits $(\eta_t)_{\ell m+1:mN_r} = [\eta_{t+\ell+1}^\top \dots \eta_{t+N_r-1}^\top]^\top$ repetitively until the first successful transmission and store them in a buffer at the actuator node. In order to avoid any addition operation at the actuator, $u_{t+\ell}$ is transmitted at each $t + \ell$ and applied to the plant if successfully received at the actuator otherwise the corresponding $\eta_{t+\ell}$ from the buffer is applied. In a worst case, if the buffer is empty and packets are also lost, then null control is applied to the plant. The plant input sequence using (TP) can therefore be represented in compact form as:

$$u_{t:N}^a := \mathcal{G}_t \eta_t + \mathcal{S}_t \Theta_t \psi(\tilde{\mathcal{I}}_{t:N}), \quad (16)$$

where

$$\mathcal{S}_t := \begin{bmatrix} I_m \otimes v_t & & & \\ & \ddots & & \\ & & I_m \otimes v_{t+\kappa-1} & \\ & & & I_{m(N-\kappa)} \end{bmatrix},$$

and the matrix \mathcal{G}_t has $(N \times N)$ blocks in total, each of dimension $m \times m$. For $i = 1, \dots, N$ and $j = 1, \dots, N$, the matrix \mathcal{G}_t can be given in terms of the blocks $\mathcal{G}_t^{(i,j)}$ each of dimension $m \times m$ as follows:

$$\mathcal{G}_t^{(i,j)} := \begin{cases} g_{t+i-1} I_m & \text{if } i = j \leq N_r, \\ I_m & \text{if } i = j > N_r, \\ 0_m & \text{otherwise,} \end{cases} \quad (17)$$

where $g_t = v_t$, $g_{t+\ell} = g_{t+\ell-1} + (1 - g_{t+\ell-1})v_{t+\ell}$, and Θ_t and $\psi(\tilde{I}_{t:N})$ are as defined in (13). The term $g_{t+\ell}$ captures the effect of (TP). Note that (TP) requires storage at the actuator, but no advanced computation capacity.

4 Tractability

In this section we present a tractable surrogate of the optimal control problem (3) under the system setup discussed in Section 3. Let us define $\mu_{\mathcal{G}_t} := \mathbb{E}[\mathcal{G}_t]$, $\Sigma_{\mathcal{G}_t} := \mathbb{E}[\mathcal{G}_t^\top \alpha \mathcal{G}_t]$, $\alpha := \mathcal{B}^\top Q \mathcal{B} + \mathcal{R}$, $\mu_{S_t} := \mathbb{E}[S_t]$, $\Sigma_{S_t} := \mathbb{E}[S_t^\top \alpha S_t]$, $\Sigma_{\mathcal{G}_t S_t} := \mathbb{E}[\mathcal{G}_t^\top \alpha S_t]$, $\Pi_{y_t} := \psi_0(\tilde{I}_t) \psi_0(\tilde{I}_t)^\top$, $\Sigma_\psi := \mathbb{E}[\psi'(\tilde{I}_{t+1:N-1}) \psi'(\tilde{I}_{t+1:N-1})^\top]$, $\Sigma_{\psi'w} := \mathbb{E}[\psi'(\tilde{I}_{t+1:N-1}) w_{t:N}^\top]$, $\Sigma_{e\psi'} := \mathbb{E}[\psi'(\tilde{I}_{t+1:N-1}) e_t^\top]$, $\Theta_t^{(:,t)} := \begin{bmatrix} \theta_{0,t}^\top & \theta_{1,t}^\top & \dots & \theta_{N-1,t}^\top \end{bmatrix}^\top$, $Q_{\mathcal{A}} := \mathcal{A}^\top Q \mathcal{B}$ and $Q_{\mathcal{D}} := \mathcal{D}^\top Q \mathcal{B}$. We have the following Lemma:

Lemma 5 *The objective function (6) can be represented in terms of the decision variables as follows:*

$$\begin{aligned} V_t' &= \eta_t^\top \Sigma_{\mathcal{G}_t} \eta_t + \text{tr}(\Sigma_{S_t} \Theta_t^{(:,t)} \Pi_{y_t} (\Theta_t^{(:,t)})^\top) + \text{tr}(\Sigma_{S_t} \Theta_t' \Sigma_\psi (\Theta_t')^\top) \\ &+ 2(\eta_t^\top \Sigma_{\mathcal{G}_t S_t} + \tilde{x}_t^\top Q_{\mathcal{A}} \mu_{S_t}) \Theta_t^{(:,t)} \psi_0(\tilde{I}_t) + 2\tilde{x}_t^\top Q_{\mathcal{A}} \mu_{\mathcal{G}_t} \eta_t \\ &+ 2 \text{tr}(Q_{\mathcal{D}} \mu_{S_t} \Theta_t' \Sigma_{\psi'w}) + 2 \text{tr}(Q_{\mathcal{A}} \mu_{S_t} \Theta_t' \Sigma_{e\psi'}). \end{aligned} \quad (18)$$

A proof of the Lemma 5 is given in the appendix. At each optimization time t , the above objective function is updated by substituting new information \tilde{I}_t and \tilde{x}_t . The matrices $\mu_{\mathcal{G}_t}$, $\Sigma_{\mathcal{G}_t}$, μ_{S_t} , Σ_{S_t} , $\Sigma_{\mathcal{G}_t S_t}$, Σ_ψ , $\Sigma_{\psi'w}$, $\Sigma_{e\psi'}$ incorporate all elements of the problem setup (Section 3). These matrices are computed offline by using Monte-Carlo simulations to reduce the burden of online computation. We have the following result:

Proposition 6 *For every time $t = 0, N_r, 2N_r, \dots$, the optimal control problem (3) can be written as the following convex quadratic, (globally) feasible program:*

$$\text{minimize} \quad (18)$$

$$\eta_t, \Theta_t$$

$$\text{subject to} \quad (15)$$

$$(19)$$

PROOF. The objective function (18) is convex quadratic in decision variables η_t and Θ_t , and the constraint (15) is a convex affine function of the decision variables. Since (15) does not depend on x_t , the optimization program (19) is feasible for all $x_t \in \mathbb{R}^d$ for all $t \in \mathbb{N}_0$.

5 Stability

In this section, we show that the system setup discussed in Section 3 leads to the mean square boundedness of the controlled states if carefully designed stability constraints are also included in the optimization program (19). Let us first represent the estimator process recursion (11) in terms of the matrix pair (A, B) as follows:

$$\tilde{x}_{t+1} = A \tilde{x}_t + B u_t^a + \tilde{w}_t, \quad (20)$$

where $\tilde{w}_t := s_{t+1}(A \tilde{e}_t + \hat{w}_t)$. Note that a Lyapunov stable system matrix A can be decomposed into a Schur stable component A_s and an orthogonal component A_o as:

$$\begin{bmatrix} \tilde{x}_{t+1}^o \\ \tilde{x}_{t+1}^s \end{bmatrix} = \begin{bmatrix} A_o \tilde{x}_t^o \\ A_s \tilde{x}_t^s \end{bmatrix} + \begin{bmatrix} B_o \\ B_s \end{bmatrix} u_t^a + \begin{bmatrix} \tilde{w}_t^o \\ \tilde{w}_t^s \end{bmatrix}, \quad (21)$$

where $\tilde{x}_t^s \in \mathbb{R}^{d_s}$, $\tilde{x}_t^o \in \mathbb{R}^{d_o}$, and $d = d_o + d_s$. Let us define the reachability matrix

$$R_\kappa(A, B) := \begin{bmatrix} A^{\kappa-1} B & \dots & AB & B \end{bmatrix}. \quad (22)$$

By the stabilizability of (A, B) , there exists an integer κ such that the reachability matrix $R_\kappa(A_o, B_o)$ has full row rank. The integer κ is called reachability index of the matrix pair (A_o, B_o) . The reachability index is important in our approach. We can choose $N_r \geq \kappa$ but for simplicity, in our subsequent analysis we use $N_r = \kappa$. We consider the orthogonal component of the κ -subsampled process of (20), which is given by

$$\tilde{x}_{\kappa(t+1)}^o = A_o^\kappa \tilde{x}_{\kappa t}^o + R_\kappa(A_o, B_o) u_{\kappa t:\kappa}^a + R_\kappa(A_o, I) \tilde{w}_{\kappa t:\kappa}^o. \quad (23)$$

Let us define $z_t := ((A_o^\kappa)^\top \tilde{x}_{\kappa t}^o)$ then the process $(z_t)_{t \in \mathbb{N}_0}$ can be considered as a d -dimensional random walk with recursion

$$z_{t+1} = z_t + (A_o^{\kappa(t+1)})^\top (R_\kappa(A_o, B_o) u_{\kappa t:\kappa}^a + R_\kappa(A_o, I) \tilde{w}_{\kappa t:\kappa}^o). \quad (24)$$

We present the following lemma:

Lemma 7 Consider the recursions (20) and (24). Suppose that u_t^a is constrained in the set \mathbb{U} for each t and that there exist $a, r > 0$ such that for $j = 1, 2, \dots, d_o$, the following conditions hold

$$\mathbb{E}_{\mathbb{Y}_{\kappa t}^s} \left[(z_{t+1})^{(j)} - (z_t)^{(j)} \right] \leq -a \text{ whenever } (z_t)^{(j)} > r, \quad (25a)$$

$$\mathbb{E}_{\mathbb{Y}_{\kappa t}^s} \left[(z_{t+1})^{(j)} - (z_t)^{(j)} \right] \geq a \text{ whenever } (z_t)^{(j)} < -r, \quad (25b)$$

$$\mathbb{E} \left[\left| (z_{t+1})^{(j)} - (z_t)^{(j)} \right|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \leq M \text{ for all } t. \quad (25c)$$

Then there exists $\bar{\gamma} > 0$ such that $\mathbb{E}_{\mathbb{Y}_0^s} [\|\tilde{x}_t\|^2] \leq \bar{\gamma}$ for all $t \geq 0$. Moreover, there exists a κ -history dependent class of policies $u_{\kappa t:\kappa}$ of the form (13) such that (25) and (2) are satisfied for $\zeta \in \left[0, \frac{u_{\max}}{\sqrt{d_o} \sigma_1(R_\kappa(A_o, B_o)^\dagger)} \right]$ and $a = \zeta p_c$ under the transmission protocol (TP). Furthermore, for $t = 0, \kappa, 2\kappa, \dots$, the conditions (25) are equivalent to the following conditions:

$$\begin{aligned} & \left((A_o^{t+\kappa})^\top R_\kappa(A_o, B_o) \mathbb{E}_{\mathbb{Y}_t^s} [u_{t:\kappa}] \right)^{(j)} \leq -\zeta \\ & \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} > r, \end{aligned} \quad (26a)$$

$$\begin{aligned} & \left((A_o^{t+\kappa})^\top R_\kappa(A_o, B_o) \mathbb{E}_{\mathbb{Y}_t^s} [u_{t:\kappa}] \right)^{(j)} \geq \zeta \\ & \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} < -r. \end{aligned} \quad (26b)$$

A proof of Lemma 7 is given in the appendix. To embed the drift conditions (26) in a tractable optimization program, we consider the first κ blocks of (13)

$$u_{t:\kappa} = (\eta_t)_{1:\kappa m} + (\Theta_t)_{1:\kappa m} \psi(\tilde{I}_{t:N}) \quad (27)$$

for $t = 0, \kappa, \dots$, and substitute them in (26). We get the following stability constraints:

$$\begin{aligned} & \left((A_o^{t+\kappa})^\top R_\kappa(A_o, B_o) \left((\eta_t)_{1:\kappa m} + (\Theta_t^{(:,t)})_{1:\kappa m} \psi_0(\tilde{I}_t) \right) \right)^{(j)} \leq -\zeta \\ & \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} > r, \end{aligned} \quad (28a)$$

$$\begin{aligned} & \left((A_o^{t+\kappa})^\top R_\kappa(A_o, B_o) \left((\eta_t)_{1:\kappa m} + (\Theta_t^{(:,t)})_{1:\kappa m} \psi_0(\tilde{I}_t) \right) \right)^{(j)} \geq \zeta \\ & \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} < -r, \end{aligned} \quad (28b)$$

where r, ζ and j are as in (26). We have the following result:

Theorem 8 Let the stability constraints (28) be included in the optimization program of Theorem 6 and controls be generated by successively solving the underlying optimization program. Then the application of these controls ensures mean-square boundedness of the states of (1) for any bound $u_{\max} > 0$ and transmission probabilities $p_s, p_c > 0$.

A proof of Theorem 8 is given in the appendix.

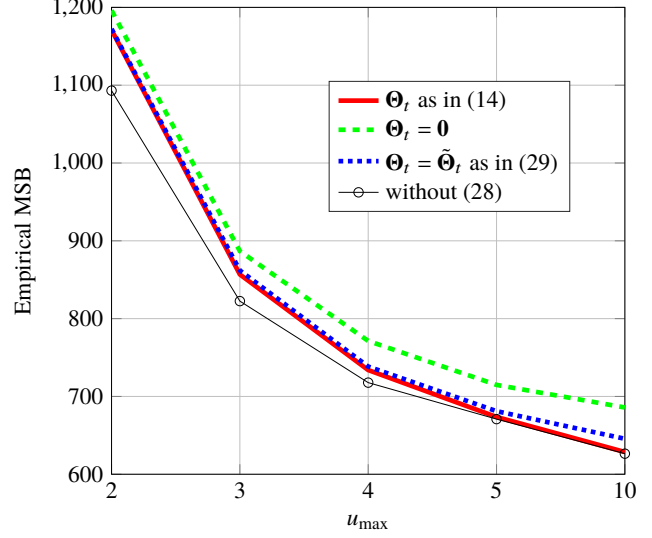


Fig. 3. Empirical MSB when u_{\max} varies in the set $\{2, 3, 4, 5, 10\}$ while $p_c = p_s = 0.8$ remain fixed.

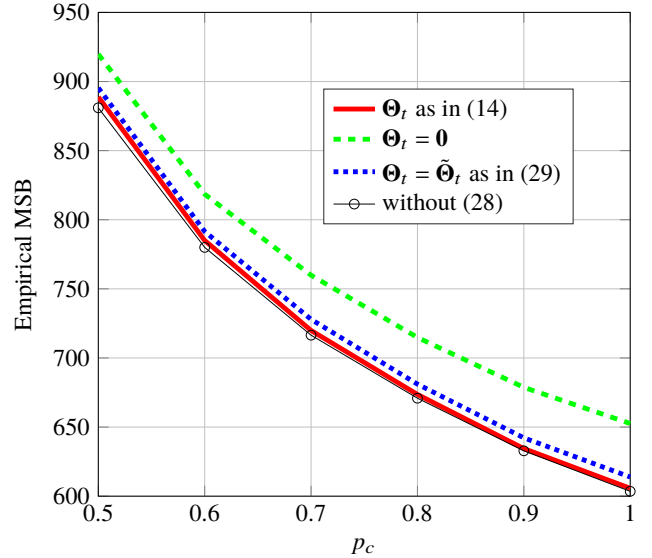


Fig. 4. Empirical MSB when p_c varies from 0.5 to 1 while $u_{\max} = 5$ and $p_s = 0.8$ remain fixed.

6 Numerical Experiments

In this section, we present numerical experiments and record the empirical mean of the quantities of interest to illustrate our results by taking averages over 1000 sample paths for 120 time steps. We consider the four dimensional stochastic LTI system (1) with matrices taken from [21], which can be written in the form of (21) with $A_s = 0.9$, $B_s = 0$,

$$A_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } B_o = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

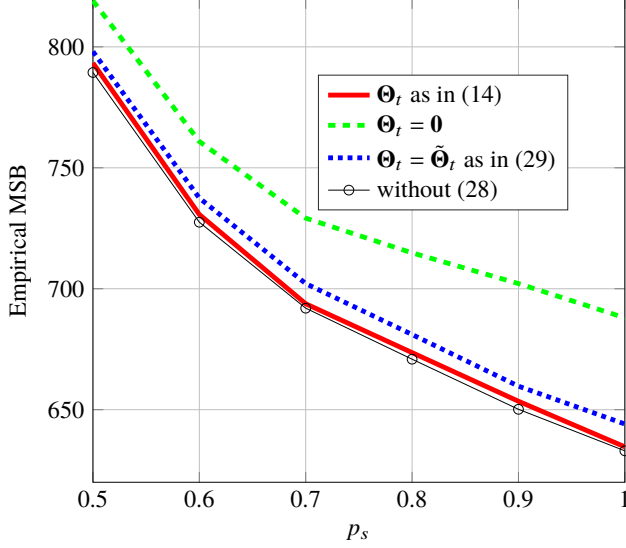


Fig. 5. Empirical MSB when p_s varies from 0.5 to 1 while $u_{\max} = 5$ and $p_c = 0.8$ remain fixed.

The reachability matrix (defined in (22))

$$R_3(A_o, B_o) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

has full row rank for $\kappa = 3$. We repeatedly solved a finite-horizon CSOCP reported in Theorem 8 corresponding to states and control weights $Q = I_4, Q_f = I_4, R = 1$, the optimization horizon, $N = 5$, recalculation interval $N_r = \kappa = 3$ and simulation data $x_0 \sim N(0, I_4)$, $w_t \sim N(0, 10I_4)$, $\varsigma_t \sim N(0, 10I_4)$. We selected the nonlinear bounded term $\psi(\cdot)$ in our policy, similar to [22], to be a vector of scalar sigmoidal functions $\varphi(\xi) = \frac{1-e^{-\xi}}{1+e^{-\xi}}$ applied to each coordinate of the received innovation sequence. We use a MATLAB-based software package YALMIP [54] and a solver SDPT3-4.0 [55] to solve the underlying optimization programs.

We compare the present approach with a simplified version of our main result by setting $\Theta_t = \mathbf{0}$ in (13) and using a modified version of Θ_t as given below:

$$\tilde{\Theta}_t = \begin{bmatrix} \theta_{0,t} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \theta_{1,t+1} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \theta_{N-1,t+N-1} \end{bmatrix}. \quad (29)$$

In particular, $\Theta_t = \mathbf{0}$ represents the optimization over an open-loop control sequence and $\Theta_t = \tilde{\Theta}_t$ represents the optimization over only one causal feedback term. In the above three cases, we have used stability constraints (28) as mentioned in Theorem 8. In the fourth case, we have removed

stability constraints (28) and simulated with the same simulation data. Our observations from numerical experiments are listed below.

- (O1) In Fig. 3, we plot the empirical mean square bound (MSB) with respect to u_{\max} picked from the set $\{2, 3, 4, 5, 10\}$ while $p_s = p_c = 0.8$ remain fixed. The empirical MSB decreases with the increase in u_{\max} . The empirical MSB is less when Θ_t is chosen according to (14) and it further decreases when we remove the constraints (28). The difference due to the removal of (28) is more for the lower values of u_{\max} and it vanishes when we further increase u_{\max} .
- (O2) In Fig. 4, we plot the empirical MSB with respect to p_c picked from the set $\{i/10 \mid i = 5, \dots, 10\}$ while $p_s = 0.8$, $u_{\max} = 5$ remain fixed. The empirical MSB decreases with increase in p_c with higher slope at smaller p_c . The empirical MSB for $\Theta_t = \mathbf{0}$ is the highest of all considered cases and the difference increases with the increase in p_c . The lowest empirical MSB's are achieved in the absence of (28).
- (O3) In Fig. 5, we plot the empirical MSB with respect to p_s picked from the set $\{i/10 \mid i = 5, \dots, 10\}$ while $p_c = 0.8$, $u_{\max} = 5$ remain fixed. With increase in p_s , we observe decrease in empirical MSB. In this case also, the lowest empirical MSB's are achieved in the absence of (28).
- (O4) In Fig. 6, we plot the empirical mean of actuator energy (MAE) per stage with respect to u_{\max} by fixing the parameters as in the observation (O1). The empirical MAE per stage increases with the increase in u_{\max} . The empirical MAE per stage in the absence of (28) is the largest (slightly) for $u_{\max} = 2$ and the lowest for $u_{\max} = 10$ of all considered cases.
- (O5) In Fig. 7, we plot the empirical MAE per stage with respect to p_c by fixing the parameters as in the observation (O2). The empirical MAE per stage increases with the increase in p_c . In this observation, the empirical MAE per stage is the lowest in the absence of (28).
- (O6) In Fig. 8, we plot the empirical MAE per stage with respect to p_s by fixing the parameters as in the observation (O3). The empirical MAE per stage does not vary much with the increase in p_s . The lowest empirical MAE per stage is obtained in the absence of (28).

The proposed class of policies performs better than open loop control sequence (which is obtained by substituting $\Theta_t = \mathbf{0}$) in terms of empirical MSB and empirical MAE. Moreover, numerical experiments demonstrate mean square boundedness of controlled states for all considered cases. The optimization over $\tilde{\Theta}_t$ in place of Θ_t results in significant reduction in the number of decision variables (please also see [40, Remark 1]) with minimal increase in MSB and MAE.

We repeated the above experiments for one sample path on intel i7-8750, 6 cores, 12 threads processor with 16 GB DDR4 RAM without invoking parallel pool in MATLAB.

In one sample path, there are 120 time steps and 40 optimization instants. For each optimization instant, we compute the percentage difference with respect to the case “ Θ_t as in (14)” and then take average over 40 optimization instants. The solver-time for the case “ Θ_t as in (14)” is considered the base-value for the comparison. Our observations are given below:

- (O7) In Fig. 9, we plot the percentage difference in the solver-time with respect to u_{\max} picked from the set $\{2, 3, 4, 5, 10\}$ while $p_s = p_c = 0.8$ remain fixed as in the observation (O1). When $\Theta_t = \mathbf{0}$, the solver-time is around 64% less than the base-value with slight variations when we vary u_{\max} . When $\Theta_t = \tilde{\Theta}_t$, the solver-time is around 44% less than the base-value with slight variations when we vary u_{\max} . The absence of (28) does not affect much.
- (O8) In Fig. 10, we plot the percentage difference in the solver-time with respect to p_c picked from the set $\{i/10 \mid i = 5, \dots, 10\}$ while $p_s = 0.8$, $u_{\max} = 5$ remain fixed as in the observation (O2). In this experiment, the observations are same as in (O7) with a very slight difference.
- (O9) In Fig. 11, we plot the percentage difference in the solver-time with respect to p_s picked from the set $\{i/10 \mid i = 5, \dots, 10\}$ while $p_c = 0.8$, $u_{\max} = 5$ remain fixed as in the observation (O3). In this experiment also, the observations are same as in (O7) with a very slight difference.

In this article, we proposed an analytical framework, which is applicable to control with mutually independent and individually i.i.d. channels. For numerical experiments we can also consider that channels are mutually independent but packet dropouts in each channel are correlated. Such channels are often modelled by the Gilbert-Elliott channel model [56] as given in Fig. 12. Here one assumes that each channel has two states – good and bad, respectively. For simplicity, we model both channels similarly. For both channels, the bad state has successful transmission probability $p_b = 0$, the transition probabilities from good to bad state and bad to good state are $p_{gb} = 0.2$ and $p_{bg} = 0.9$, respectively. The successful transmission probability of the good state of the control channel is p_{gc} and that of the sensor channel is p_{gs} . Since both channels are modelled similarly except their good state, p_{gx} in Fig. 12 takes value of p_{gc} and p_{gs} , respectively, depending upon the channel. By considering the same experimental data as in the first part of this section, we did two experiments. In the first experiment, we fixed the good state of the sensor channel $p_{gx} = p_{gs} = 0.8$ and vary the good state of the control channel $p_{gx} = p_{gc}$ in the set $\{i/10 \mid i = 5, \dots, 10\}$. In the second experiment, we fixed the good state of the control channel $p_{gx} = p_{gc} = 0.8$ and vary the good state of the sensor channel $p_{gx} = p_{gs}$ in the set $\{i/10 \mid i = 5, \dots, 10\}$. We record the following observations:

- (O10) Empirical MSB decreases with increase in p_{gc} and fixed p_{gs} . It also decreases with increase in p_{gs} and fixed p_{gc} . But in the second case the rate is lower than

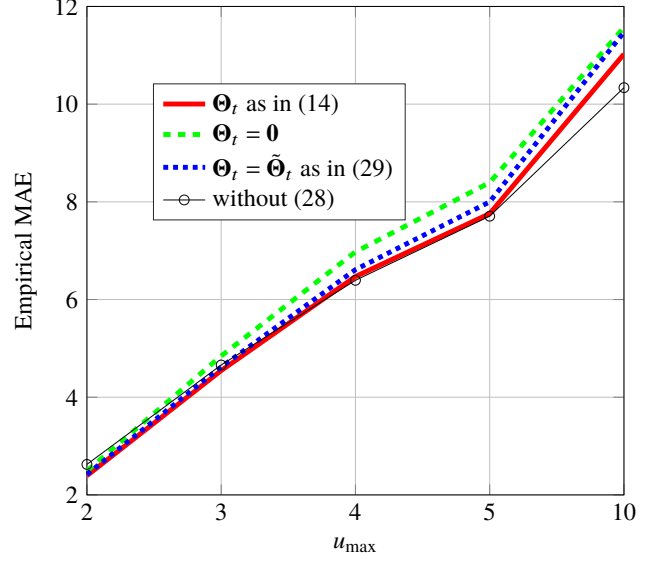


Fig. 6. Empirical MAE per stage when u_{\max} varies in the set $\{2, 3, 4, 5, 10\}$ while $p_c = p_s = 0.8$ remain fixed.

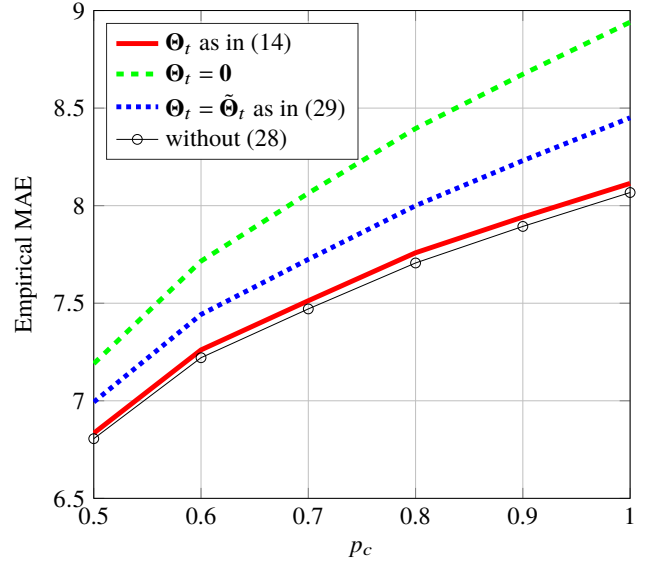


Fig. 7. Empirical MAE per stage when p_c varies from 0.5 to 1 while $u_{\max} = 5$ and $p_s = 0.8$ remain fixed.

that in the first case; See Fig. 13.

- (O11) Empirical MAE per stage increases with increase in p_{gc} and fixed p_{gs} . However, it does not vary much when p_{gs} increases and p_{gc} remain fixed; See Fig. 14.

The above numerical experiments show that our approach is computationally tractable in the presence of correlated channel noise as well.

We further consider a three dimensional stochastic LTI sys-

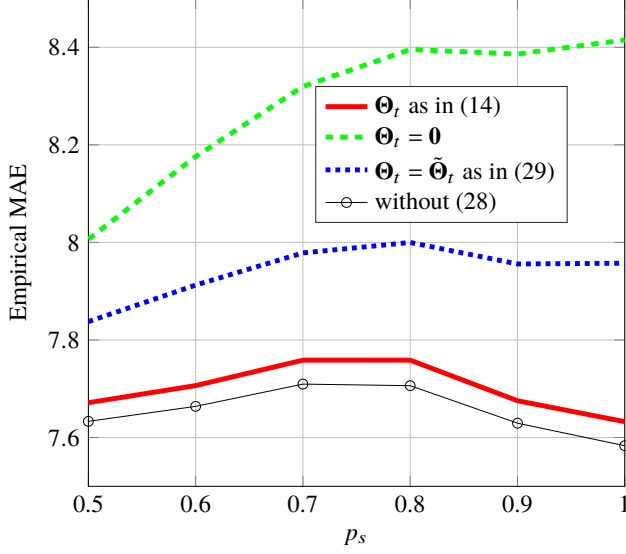


Fig. 8. Empirical MAE per stage when p_s varies from 0.5 to 1 while $u_{\max} = 5$ and $p_c = 0.8$ remain fixed.

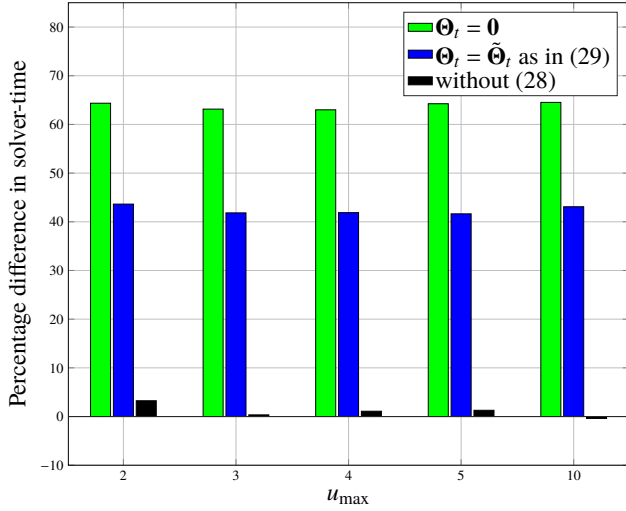


Fig. 9. Percentage difference in solver-time with respect to the Theorem 8 when u_{\max} varies and $p_c = p_s = 0.8$ remain fixed.

tem (1) with matrices taken from [14]:

$$A = \begin{bmatrix} 0 & -0.80 & -0.60 \\ 0.80 & -0.36 & 0.48 \\ 0.60 & 0.48 & -0.64 \end{bmatrix}, B = \begin{bmatrix} 0.16 \\ 0.12 \\ 0.14 \end{bmatrix}, C = I_4,$$

and simulation data $x_0 \sim N(0, I_4)$, $w_t \sim N(0, 2I_3)$, $s_t \sim N(0, 10I_4)$, $p_s = p_c = 0.8$, $u_{\max} = 15$. We repeatedly solved a finite-horizon CSOCP reported in Proposition 6 and Theo-

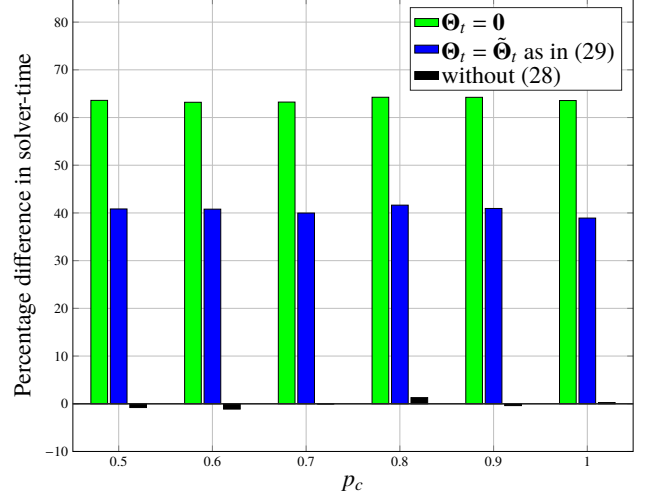


Fig. 10. Percentage difference in solver-time with respect to the Theorem 8 when p_c varies and $p_s = 0.8$, $u_{\max} = 5$ remain fixed.

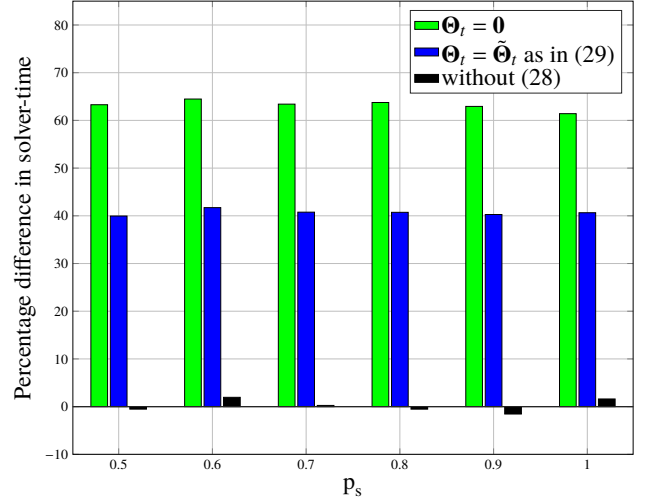


Fig. 11. Percentage difference in solver-time with respect to the Theorem 8 when p_s varies and $p_c = 0.8$, $u_{\max} = 5$ remain fixed.

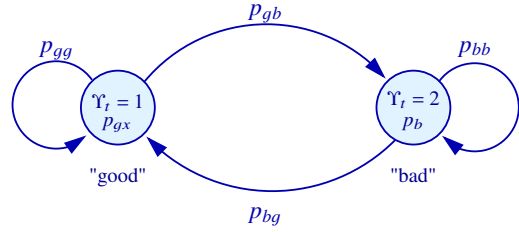


Fig. 12. Transmission dropout model with a binary network state ($\Upsilon_t \in \{1, 2\}$): when $\Upsilon_t = 1$ the channel is reliable with high successful transmission probabilities; $\Upsilon_t = 2$ refers to a situation where the channel is unreliable and transmissions are more likely to be dropped.

rem 8 corresponding to states and control weights

$$Q = I_3, Q_f = \begin{bmatrix} 12 & -0.1 & -0.4 \\ -0.1 & 19 & -0.2 \\ -0.4 & -0.2 & 2 \end{bmatrix}, R = 2,$$

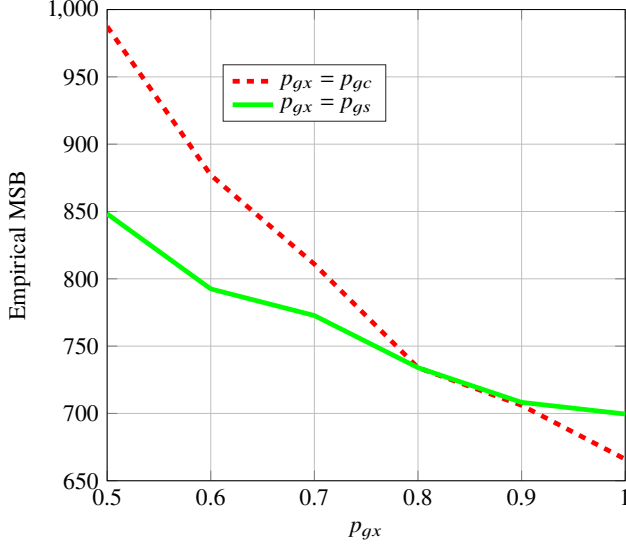


Fig. 13. Empirical MSB with Gilbert-Elliott channel model

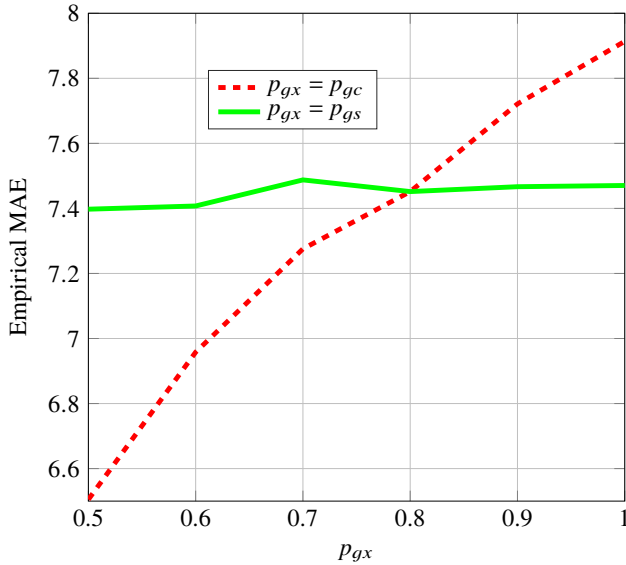


Fig. 14. Empirical MAE with Gilbert-Elliott channel model

the optimization horizon $N = 4$, and compare their corresponding empirical $\|x_t\|$ in Fig. 15. This is important to note that the matrix A in this example is orthogonal and the matrix pair (A, B) has reachability index $\kappa = 3$. Since the stability constraints (28) of Theorem 8 suggest us to choose $N_r \geq \kappa$, we choose $N_r = \kappa = 3$ in our experiment. However, Proposition 6 does not provide any such guideline to choose N_r , we take the standard choice $N_r = 1$ in our experiment. We simulated for 120 time steps and took average of 500 sample paths. We have the following observation:

- (O12) Empirical $\|x_t\|$ in case of Proposition 6 increases almost linearly. However, in case of Theorem 8 it is bounded below 23 in 120 time steps; See Fig. 15.

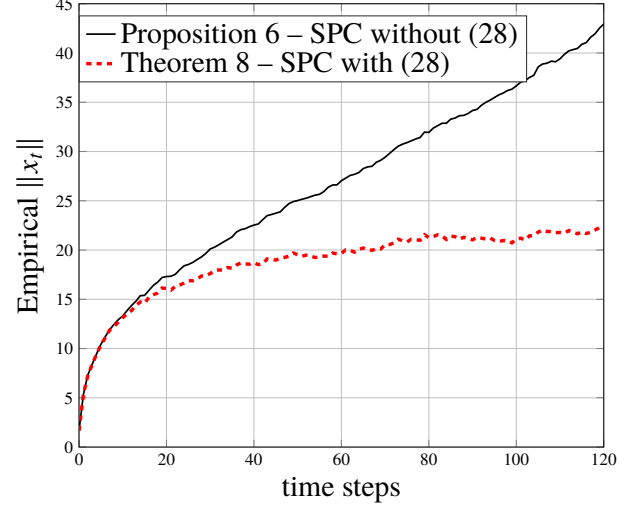


Fig. 15. Comparison between Proposition 6 and Theorem 8

The above experiment demonstrates that there may exist a system, which can become unstable when stability constraints are not embedded in the corresponding optimization program under the given simulation data.

7 Epilogue

We have employed a class of affine saturated received innovation feedback policies for a tractable formulation of the underlying CSOCP under the settings of unreliable channels. We proved that the proposed approach is tractable and ensures mean square boundedness of the controlled states. This approach can be extended for the class of policies parametrized in terms of dropouts along the lines of [35]. Moreover, inclusion of the joint chance constraints along the lines of [49, 57, 58] and the sparsity in the control vector with help of a regularization term as in [35] will also be interesting extensions of the present work.

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A Appendix

PROOF. [Proof of Lemma 3] On the event $s_t = 1$, we can compute $\mathbb{E}[x_t | \mathcal{Y}_t^s]$ as follows:

$$\begin{aligned}\mathbb{E}[x_t | \mathcal{Y}_t^s] &= s_t \mathbb{E}[x_t | \mathcal{Y}_t^s] \text{ since } s_t = 1 \\ &= s_t \mathbb{E}[\mathbb{E}[x_t | \mathcal{Y}_t, \mathcal{Y}_t^s] | \mathcal{Y}_t^s] = s_t \mathbb{E}[\hat{x}_t | \mathcal{Y}_t^s] \\ &= \mathbb{E}[s_t \hat{x}_t | \mathcal{Y}_t^s] = s_t \hat{x}_t\end{aligned}\quad (\text{A.1})$$

On the event $s_t = 0$, $\mathbb{E}[w_{t-1} | \mathcal{Y}_t^s] = \mathbf{0}$ and $\mathbb{E}[x_{t-1} | \mathcal{Y}_t^s] = \mathbb{E}[x_{t-1} | \mathcal{Y}_{t-1}^s]$. Therefore, we can compute $\mathbb{E}[x_t | \mathcal{Y}_t^s]$ as

follows:

$$\begin{aligned}\mathbb{E}[x_t | \mathcal{Y}_t^s] &= (1 - s_t) \mathbb{E}[x_t | \mathcal{Y}_t^s] \text{ since } s_t = 0 \\ &= (1 - s_t) \mathbb{E}[Ax_{t-1} + Bu_{t-1}^a + w_{t-1} | \mathcal{Y}_t^s] \\ &= (1 - s_t) \mathbb{E}[Ax_{t-1} + Bu_{t-1}^a + w_{t-1} | \mathcal{Y}_t^s] \\ &= (1 - s_t) (\mathbb{E}[Ax_{t-1} | \mathcal{Y}_t^s] + Bu_{t-1}^a) \\ &= (1 - s_t) (\mathbb{E}[Ax_{t-1} | \mathcal{Y}_{t-1}^s] + Bu_{t-1}^a) \\ &= (1 - s_t) (A\tilde{x}_{t-1} + Bu_{t-1}^a) \text{ by definition of } \tilde{x}_t.\end{aligned}\quad (\text{A.2})$$

Since $\mathbb{E}[x_t | \mathcal{Y}_t^s] = s_t [x_t | \mathcal{Y}_t^s] + (1 - s_t) \mathbb{E}[x_t | \mathcal{Y}_t^s]$, the result follows by combining (A.1) and (A.2).

PROOF. [Proof of Lemma 5] Consider the objective function (6). We substitute the stacked state vector (5a) in the objective function.

$$\begin{aligned}V_t &= \mathbb{E}_{\mathcal{Y}_t^s} \left[\sum_{k=0}^{N-1} (\|x_{t+k}\|_{Q_k}^2 + \|u_{t+k}\|_{R_k}^2) + \|x_{t+N}\|_{Q_N}^2 \right] \\ &= \mathbb{E}_{\mathcal{Y}_t^s} \left[\|\mathcal{A}x_t + \mathcal{B}u_{t:N}^a + \mathcal{D}w_{t:N}\|_Q^2 + \|u_{t:N}^a\|_R^2 \right] \\ &= \mathbb{E}_{\mathcal{Y}_t^s} \left[\|\mathcal{A}x_t\|_Q^2 + \|\mathcal{D}w_{t:N}\|_Q^2 + \|u_{t:N}^a\|_R^2 + 2(x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \right. \\ &\quad \left. + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) u_{t:N} + 2x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{D} w_{t:N} \right].\end{aligned}$$

Let $\beta_t := \mathbb{E}_{\mathcal{Y}_t^s} [\|\mathcal{A}x_t\|_Q^2 + \|\mathcal{D}w_{t:N}\|_Q^2 + 2x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{D} w_{t:N}] = \mathbb{E}_{\mathcal{Y}_t^s} [\|\mathcal{A}x_t\|_Q^2 + \|\mathcal{D}w_{t:N}\|_Q^2]$, then

$$V_t = \mathbb{E}_{\mathcal{Y}_t^s} \left[\|u_{t:N}^a\|_R^2 + 2(x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) u_{t:N}^a \right] + \beta_t. \quad (\text{A.3})$$

We now substitute the stacked control vector (16) in (A.3) and for simplicity represent $\tilde{I}_t = y_t - \hat{y}_t$ and $\tilde{J}_t = s_t \tilde{I}_t$ to get the following set of equations:

$$\begin{aligned}V_t &= \mathbb{E}_{\mathcal{Y}_t^s} \left[\|\mathcal{G}_t \eta_t + \mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N})\|_\alpha^2 + 2(x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \right. \\ &\quad \left. + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B})(\mathcal{G}_t \eta_t + \mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N})) \right] + \beta_t \\ &= \eta_t^\top \mathbb{E}_{\mathcal{Y}_t^s} \left[\mathcal{G}_t^\top \alpha \mathcal{G}_t \right] \eta_t + \mathbb{E}_{\mathcal{Y}_t^s} \left[\|\mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N})\|_\alpha^2 \right. \\ &\quad \left. + 2(\eta_t^\top \mathcal{G}_t^\top \alpha + x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) \mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N}) \right] \\ &\quad + \mathbb{E}_{\mathcal{Y}_t^s} \left[2(x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) \mathcal{G}_t \eta_t \right] + \beta_t \\ &= \eta_t^\top \Sigma_{\mathcal{G}_t} \eta_t + \mathbb{E}_{\mathcal{Y}_t^s} \left[\|\mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N})\|_\alpha^2 + 2\eta_t^\top \mathcal{G}_t^\top \alpha \mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N}) \right. \\ &\quad \left. + 2(x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) \mathcal{S}_t \Theta_t \psi(\tilde{J}_{t:N}) \right] \\ &\quad + 2\mathbb{E}_{\mathcal{Y}_t^s} \left[x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathcal{G}_t \eta_t \right] + \beta_t.\end{aligned}$$

Since $\mathbb{E}_{\mathfrak{Y}_t^s} [x_t] = \tilde{x}_t$, we obtain:

$$V_t = \boldsymbol{\eta}_t^\top \Sigma_{\mathcal{G}_t} \boldsymbol{\eta}_t + \mathbb{E}_{\mathfrak{Y}_t^s} \left[\left\| \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right\|_\alpha^2 + 2(\boldsymbol{\eta}_t^\top \mathcal{G}_t^\top \alpha + x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} + w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B}) \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right] + 2\tilde{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathcal{G}_t \boldsymbol{\eta}_t + \beta_t. \quad (\text{A.4})$$

Let us consider the term $\mathbb{E}_{\mathfrak{Y}_t^s} [\boldsymbol{\eta}_t^\top \mathcal{G}_t^\top \alpha \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})]$ on the right hand side of (A.4). By observing $\mathbb{E}_{\mathfrak{Y}_t^s} [\psi_i(\tilde{\mathcal{I}}_{t+i})] = \mathbf{0}$ for each $i = 1, \dots, N$, we get

$$\mathbb{E}_{\mathfrak{Y}_t^s} [\boldsymbol{\eta}_t^\top \mathcal{G}_t^\top \alpha \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})] = \boldsymbol{\eta}_t^\top \Sigma_{\mathcal{G}_t \mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t), \quad (\text{A.5})$$

where $\boldsymbol{\Theta}_t^{(:,t)} := [\theta_{0,t}^\top \ \theta_{1,t}^\top \ \dots \ \theta_{N-1,t}^\top]^\top$ represents the first q columns of the gain matrix $\boldsymbol{\Theta}_t$. Let us consider the term $\mathbb{E}_{\mathfrak{Y}_t^s} [\left\| \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right\|_\alpha^2]$ on the right hand side of (A.4). In order to simplify offline computations, we perform the following manipulation:

$$\begin{aligned} & \mathbb{E}_{\mathfrak{Y}_t^s} \left[\left\| \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right\|_\alpha^2 \right] \\ &= \mathbb{E}_{\mathfrak{Y}_t^s} \left[\left\| \mathbf{S}_t \begin{bmatrix} \boldsymbol{\Theta}_t^{(:,t)} & \boldsymbol{\Theta}_t' \end{bmatrix} \begin{bmatrix} \psi_0(\tilde{\mathcal{I}}_t) \\ \psi'(\tilde{\mathcal{I}}_{t+1:N-1}) \end{bmatrix} \right\|_\alpha^2 \right] \\ &= \text{tr} \left(\Sigma_{\mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t) \psi_0(\tilde{\mathcal{I}}_t)^\top (\boldsymbol{\Theta}_t^{(:,t)})^\top \right. \\ & \quad \left. + \Sigma_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \mathbb{E}_{\mathfrak{Y}_t^s} \left[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) \psi'(\tilde{\mathcal{I}}_{t+1:N-1})^\top \right] (\boldsymbol{\Theta}_t')^\top \right) \\ &= \text{tr}(\Sigma_{\mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \Pi_{y_t} (\boldsymbol{\Theta}_t^{(:,t)})^\top) + \text{tr}(\Sigma_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \Sigma_{\psi'} (\boldsymbol{\Theta}_t')^\top), \quad (\text{A.6}) \end{aligned}$$

where $\Pi_{y_t} = \psi_0(\tilde{\mathcal{I}}_t) \psi_0(\tilde{\mathcal{I}}_t)^\top$ and $\Sigma_{\psi'} = \mathbb{E}[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) \psi'(\tilde{\mathcal{I}}_{t+1:N-1})^\top]$. We simplify the term $\mathbb{E}_{\mathfrak{Y}_t^s} [w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})]$ in (A.4) as follows:

$$\begin{aligned} & \mathbb{E}_{\mathfrak{Y}_t^s} [w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})] \\ &= \mathbb{E}_{\mathfrak{Y}_t^s} \left[w_{t:N}^\top \mathcal{D}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \begin{bmatrix} \boldsymbol{\Theta}_t^{(:,t)} & \boldsymbol{\Theta}_t' \end{bmatrix} \begin{bmatrix} \psi_0(\tilde{\mathcal{I}}_t) \\ \psi'(\tilde{\mathcal{I}}_{t+1:N-1}) \end{bmatrix} \right] \\ &= \text{tr} \left(\mathcal{D}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \mathbb{E}_{\mathfrak{Y}_t^s} \left[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) w_{t:N}^\top \right] \right) \\ &= \text{tr}(\mathcal{D}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \Sigma_{\psi'} w), \quad (\text{A.7}) \end{aligned}$$

where $\Sigma_{\psi' w} = \mathbb{E}[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) w_{t:N}^\top]$. Finally, we consider the term $\mathbb{E}_{\mathfrak{Y}_t^s} [x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})]$ in (A.4) as follows:

$$\mathbb{E}_{\mathfrak{Y}_t^s} [x_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N})]$$

$$\begin{aligned} &= \mathbb{E}_{\mathfrak{Y}_t^s} \left[(x_t - \hat{x}_t)^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right] \\ & \quad + \mathbb{E}_{\mathfrak{Y}_t^s} \left[\hat{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right] \\ &= \mathbb{E}_{\mathfrak{Y}_t^s} \left[e_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right] \\ & \quad + \mathbb{E}_{\mathfrak{Y}_t^s} \left[\hat{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \left(\boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t) + \boldsymbol{\Theta}_t' \psi'(\tilde{\mathcal{I}}_{t+1:N-1}) \right) \right] \\ &= \mathbb{E}_{\mathfrak{Y}_t^s} \left[e_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mathbf{S}_t \boldsymbol{\Theta}_t \psi(\tilde{\mathcal{I}}_{t:N}) \right] + s_t \tilde{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t) \\ &= \text{tr} \left(\mathcal{A}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \mathbb{E}_{\mathfrak{Y}_t^s} \left[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) e_t^\top \right] \right) \\ & \quad + \tilde{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t) \\ &= \text{tr} \left(\mathcal{A}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t' \Sigma_{e\psi'} \right) + \tilde{x}_t^\top \mathcal{A}^\top \mathcal{Q} \mathcal{B} \mu_{\mathbf{S}_t} \boldsymbol{\Theta}_t^{(:,t)} \psi_0(\tilde{\mathcal{I}}_t), \quad (\text{A.8}) \end{aligned}$$

where $\Sigma_{e\psi'} = \mathbb{E}[\psi'(\tilde{\mathcal{I}}_{t+1:N-1}) e_t^\top]$. Expression (18) follows by substituting (A.5), (A.6), (A.7), (A.8) in (A.4), and ignoring the terms independent of the decision variables. Therefore, the objective function in (6) is equivalent to (18) under the constraints (5a) and (13).

We present lemmas 9 - 15 before the proof of Lemma 7.

Lemma 9 Suppose that the estimator is driven by the recursion (11), and let assumptions (A1) – (A6) hold. Then there exists $\tilde{\rho} > 0$ such that

$$\mathbb{E} [\|\tilde{e}_t\|^2 \mid \mathfrak{Y}_0^s] \leq \tilde{\rho} \quad \text{for all } t \geq 0. \quad (\text{A.9})$$

PROOF. The estimation error is given by

$$\tilde{e}_t = \hat{x}_t - \tilde{x}_t = (1 - s_t)(A\tilde{e}_{t-1} + \hat{w}_{t-1}). \quad (\text{A.10})$$

The matrix decomposition (21) allows us to write $\tilde{e}_t = \begin{bmatrix} \tilde{e}_t^o \\ \tilde{e}_t^s \end{bmatrix}$

and $\hat{w}_t = \begin{bmatrix} \hat{w}_t^o \\ \hat{w}_t^s \end{bmatrix}$, therefore

$$\tilde{e}_{t+1}^o = (1 - s_{t+1})(A_o \tilde{e}_t^o + \hat{w}_t^o) \quad (\text{A.11})$$

$$\tilde{e}_{t+1}^s = (1 - s_{t+1})(A_s \tilde{e}_t^s + \hat{w}_t^s). \quad (\text{A.12})$$

Let us first consider the expectation $\mathbb{E}[(\tilde{e}_{t+1}^s)^\top P \tilde{e}_{t+1}^s \mid \tilde{e}_t, \mathfrak{Y}_0^s]$ for some symmetric positive definite matrix $P > 0$. Since \tilde{e}_t and \hat{w}_t are independent at t , we get that

$$\begin{aligned} \mathbb{E}[(\tilde{e}_{t+1}^s)^\top P \tilde{e}_{t+1}^s \mid \tilde{e}_t, \mathfrak{Y}_0^s] &= (1 - p_s)(\tilde{e}_t^s)^\top A_s^\top P A_s \tilde{e}_t^s \\ & \quad + (1 - p_s) \mathbb{E}[(\hat{w}_t^s)^\top P \hat{w}_t^s] \end{aligned}$$

Since \hat{w}_t has bounded variance for all $t \geq 0$. We can assume that there exists $C_1 < \infty$ such that $(1 - p_s) \mathbb{E}[(\hat{w}_t^s)^\top P \hat{w}_t^s] \leq C_1$

for all $t \geq 0$. Further, since A_s is stable there exists some $\lambda_1 \in]0, 1[$ such that

$$\begin{aligned} \mathbb{E} \left[(\tilde{e}_{t+1}^s)^\top P \tilde{e}_{t+1}^s \mid \tilde{e}_t, \mathfrak{Y}_0^s \right] &\leq (1 - p_s) \lambda_1 \tilde{e}_t^s P \tilde{e}_t^s + C_1 \\ &=: \lambda \tilde{e}_t^s P \tilde{e}_t^s + C_1 \end{aligned}$$

where $\lambda = (1 - p_s) \lambda_1 < 1$. Since $\mathbb{E} \left[\|\tilde{e}_{t+1}^s\|^2 \mid \tilde{e}_t, \mathfrak{Y}_0^s \right] = \mathbb{E} \left[\mathbb{E} \left[\|\tilde{e}_{t+1}^s\|^2 \mid \tilde{e}_t, \mathfrak{Y}_0^s \right] \mid \tilde{e}_t, \mathfrak{Y}_0^s \right]$, we can show that

$$\mathbb{E} \left[(\tilde{e}_{t+1}^s)^\top P \tilde{e}_{t+1}^s \mid \tilde{e}_0, \mathfrak{Y}_0^s \right] \leq \lambda^{t+1} (\tilde{e}_0^s)^\top P \tilde{e}_0^s + \frac{C_1}{1 - \lambda}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left[\|\tilde{e}_{t+1}^s\|^2 \mid \mathfrak{Y}_0^s \right] &\leq \lambda^{t+1} \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \mathbb{E} \left[\|\tilde{e}_0^s\|^2 \mid \mathfrak{Y}_0^s \right] \\ &\quad + \frac{C_1}{(1 - \lambda) \lambda_{\min}(P)}, \end{aligned}$$

where $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are the largest and the smallest eigenvalues of P , respectively. We can observe that on the event $s_0 = 1$, $\tilde{e}_0 = 0$, whereas on the event $s_0 = 0$, $\tilde{e}_0 = \hat{x}_0$. Under the given initialization of the Kalman filter, we can easily see that $\hat{x}_0 = K_0(Cx_0 + \varsigma_0)$. Since ς_0 is independent of \mathfrak{Y}_0^s on the event $s_0 = 0$, we can conclude that $\mathbb{E} \left[\|\tilde{e}_0^s\|^2 \mid \mathfrak{Y}_0^s \right] = (1 - s_0) \mathbb{E} \left[\|\hat{x}_0\|^2 \right] \leq (1 - s_0) \text{tr}(K_0^\top \Sigma_{x_0} C^\top)$. Therefore, there exists $C_2 > 0$ such that $\mathbb{E} \left[\|\tilde{e}_t^s\|^2 \mid \mathfrak{Y}_0^s \right] \leq C_2$ for all $t \geq 0$. Similarly, we consider the expectation $\mathbb{E} \left[\|\tilde{e}_{t+1}^o\|^2 \mid \tilde{e}_t, \mathfrak{Y}_0^s \right]$ and show that there exists $C_3 > 0$ such that $\mathbb{E} \left[\|\tilde{e}_t^o\|^2 \mid \mathfrak{Y}_0^s \right] \leq C_3$ which implies $\mathbb{E} \left[\|\tilde{e}_t\|^2 \mid \mathfrak{Y}_0^s \right] \leq C_2 + C_3 =: \tilde{\rho}$ for all $t \geq 0$.

Lemma 10 Consider (20) and decomposition (21). If there exist $m_3 > 0$ such that $\mathbb{E} \left[\|\hat{w}_t^o\|^4 \right] \leq m_3$ for every t , then for some $h \in \mathbb{Z}_+$, we have

$$\mathbb{E} \left[\|\mathbf{R}_h(A_o, I_{d_o}) \hat{w}_{t,h}^o\|^4 \right] \leq m_3 h^4 \text{ for each } t. \quad (\text{A.13})$$

PROOF. Let us begin with the term

$$\begin{aligned} \|\mathbf{R}_h(A_o, I_{d_o}) \hat{w}_{t,h}^o\|^4 &= \left(\|A_o^{h-1} \hat{w}_t^o + \dots + \hat{w}_{t+h-1}^o\| \right)^4 \\ &\leq \left(\|\hat{w}_t^o\| + \dots + \|\hat{w}_{t+h-1}^o\| \right)^4 \leq h^2 \left(\|\hat{w}_t^o\|^2 + \dots + \|\hat{w}_{t+h-1}^o\|^2 \right)^2 \\ &\leq h^3 \left(\|\hat{w}_t^o\|^4 + \dots + \|\hat{w}_{t+h-1}^o\|^4 \right). \end{aligned}$$

We take expectation on both sides to get the bound

$$\begin{aligned} \mathbb{E} \left[\|\mathbf{R}_h(A_o, I_{d_o}) \hat{w}_{t,h}^o\|^4 \right] &\leq h^3 \mathbb{E} \left[\|\hat{w}_t^o\|^4 + \dots + \|\hat{w}_{t+h-1}^o\|^4 \right] \\ &\leq h^4 m_3. \end{aligned}$$

Lemma 11 Let us consider z_t as defined in Lemma 7. There exists $m_2 > 0$ such that

$$\mathbb{E} \left[\|\tilde{w}_{\kappa t + \ell}^o\|^4 \mid z_0, \dots, z_t \right] \leq m_2$$

for each t and $\ell = 0, \dots, \kappa - 1$.

PROOF. Let us consider the initialization $\tilde{e}_{-1} = \mathbf{0}$ and define

$$\tau_t := \begin{cases} \sup\{k \leq t \mid s_k = 1\} & \text{if } \max_{k \leq t} s_k = 1, \\ -1 & \text{otherwise,} \end{cases}$$

then $s_{\tau_t} = 1$ and from (A.10) $\tilde{e}_{\tau_t} = 0$. In this way $\tilde{e}_{\tau_t+1} = \hat{w}_{\tau_t}$ and subsequently, $\tilde{e}_t = \mathbf{R}_{t-\tau_t}(A, I) \hat{w}_{\tau_t:t-\tau_t}$ if $\tau_t \neq t$ and $\mathbf{0}$ otherwise. From (20) we get

$$\tilde{w}_t = \begin{cases} \mathbf{R}_{t-\tau_t+1}(A, I) \hat{w}_{\tau_t:t-\tau_t+1} & \text{if } s_{t+1} = 1, \\ \mathbf{0} & \text{otherwise.} \end{cases} \quad (\text{A.14})$$

It is now clear that $\mathbb{E} \left[\|\tilde{w}_{\kappa t + \ell}^o\|^4 \mid z_0, \dots, z_t, s_0, \dots, s_{\kappa t + \ell} \right] = \mathbb{E} \left[\|\tilde{w}_{\kappa t + \ell}^o\|^4 \mid s_{\tau_{\kappa t + \ell}}, \dots, s_{\kappa t + \ell} \right]$. Let $\kappa t + \ell + 1 - \tau_{\kappa t + \ell} := h$, then $\tilde{w}_{\kappa t + \ell}^o = s_{\kappa t + \ell} \mathbf{R}_h(A_o, I_{d_o}) \hat{w}_{\tau_{\kappa t + \ell}:h}^o$. Since \hat{w}_t is Gaussian for each t , there exists $m_3 > 0$ such that $\mathbb{E} \left[\|\hat{w}_t^o\|^4 \right] \leq m_3$ for each t . We can conclude that

$$\begin{aligned} \mathbb{E} \left[\|\tilde{w}_{\kappa t + \ell}^o\|^4 \mid z_0, \dots, z_t \right] &= p_s \sum_{h=1}^{\infty} \mathbb{E} \left[\|\mathbf{R}_h(A_o, I_{d_o}) \hat{w}_{\tau_{\kappa t + \ell}:h}^o\|^4 \mid s_{\kappa t + \ell - h + 1}, \dots, s_{\kappa t + \ell} \right] \\ &\quad p(s_{\kappa t + \ell - h + 1} = 1, s_{\kappa t + \ell - h + 2} = \dots = s_{\kappa t + \ell} = 0) \\ &\leq p_s \sum_{h=1}^{\infty} h^4 m_3 (1 - p_s)^{h-1} p_s = m_3 p_s^2 \sum_{h=1}^{\infty} h^4 (1 - p_s)^{h-1} := m_2. \end{aligned}$$

The last bound is implied by Lemma 10.

Lemma 12 Consider the system (21). If there exists $\gamma_1 > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{x}_{\kappa t}^o\|^2 \right] \leq \gamma_1$ for all t , then there exists $\gamma_o > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{x}_t^o\|^2 \right] \leq \gamma_o$ for all t .

PROOF. This result is standard in literature, i.e. [46]. We provide proof for completeness. Letting $\|M\|_F$ denote the Frobenius norm of a given matrix M , we compute a uniform bound on $\mathbb{E} \left[\|\tilde{x}_{\kappa t + \ell}^o\|^2 \right]$ for $\ell = 0, \dots, \kappa - 1$ as follows:

$$\begin{aligned} \tilde{x}_{\kappa t + \ell}^o &= A_o^\ell \tilde{x}_{\kappa t}^o + \mathbf{R}_\ell(A_o, B_o) u_{\kappa t:\ell}^a + \mathbf{R}_\ell(A_o, I) \tilde{w}_{\kappa t:\ell}^o \\ \|\tilde{x}_{\kappa t + \ell}^o\|^2 &\leq 3 \left(\|\tilde{x}_{\kappa t}^o\|^2 + \|\mathbf{R}_\ell(A_o, B_o) u_{\kappa t:\ell}^a\|^2 + \|\mathbf{R}_\ell(A_o, I) \tilde{w}_{\kappa t:\ell}^o\|^2 \right) \end{aligned}$$

Since $\|\mathbf{R}_\ell(A_o, B_o)u_{\kappa t:\ell}^a\|^2 \leq \|\mathbf{R}_\ell(A_o, B_o)\|_F^2 \|u_{\kappa t:\ell}^a\|^2$
 $\leq \|\mathbf{R}_\ell(A_o, B_o)\|_F^2 \|u_{\kappa t:\ell}^a\|_\infty^2 \ell m \leq \|\mathbf{R}_\ell(A_o, B_o)\|_F^2 u_{\max}^2 \ell m$
 $= (\sum_{i=0}^{\ell-1} \|A_o^i B_o\|_F^2) u_{\max}^2 \ell m = (\sum_{i=0}^{\ell-1} \text{tr}(B_o^\top B_o)) u_{\max}^2 \ell m$
 $\leq (\sum_{i=0}^{\ell-1} m(\sigma_1(B_o))^2) u_{\max}^2 \ell m = \ell m(\sigma_1(B_o))^2 u_{\max}^2 \ell m$, we get

$$\|\tilde{x}_{\kappa t+\ell}^o\|^2 \leq 3 \left(\|\tilde{x}_{\kappa t}^o\|^2 + (\ell m \sigma_1(B_o) u_{\max})^2 + \ell \sum_{i=0}^{\ell-1} \|\tilde{w}_{\kappa t+i}^o\|^2 \right).$$

By taking the conditional expectation on both sides we get

$$\begin{aligned} \mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{x}_{\kappa t+\ell}^o\|^2 \right] &\leq 3 \left(\gamma_1 + (\ell m \sigma_1(B_o) u_{\max})^2 \right) \\ &\quad + 3\ell^2 \max_{i=0, \dots, \ell-1} \mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{w}_{\kappa t+i}^o\|^2 \right] \\ &\leq 3 \left(\gamma_1 + (\ell m \sigma_1(B_o) u_{\max})^2 + \ell^2 \sqrt{m_2} \right), \end{aligned}$$

where m_2 is computed in Lemma 11. Therefore for $\ell = 0, \dots, \kappa-1$ and for each t ,

$$\mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{x}_{\kappa t+\ell}^o\|^2 \right] \leq 3 \left(\gamma_1 + (\kappa m \sigma_1(B_o) u_{\max})^2 + \kappa^2 \sqrt{m_2} \right) =: \gamma_o.$$

Lemma 13 Consider the recursion (20) and expression (24). The constraint (25c) is satisfied when $\|u_t^a\|_\infty \leq u_{\max}$ for all t .

PROOF. Since u_t^a is uniformly bounded there exists $m_1 > 0$ such that $\|\mathbf{R}_\kappa(A_o, B_o)u_{\kappa t:\kappa}^a\|_\infty \leq m_1$ for all t . Let us consider z_t defined in (24). Further, $|(z_{t+1})^{(j)} - (z_t)^{(j)}| \leq \|z_{t+1} - z_t\|_\infty$, the right side of which is bounded by $\|\mathbf{R}_\kappa(A_o, B_o)u_{\kappa t:\kappa}^a + \mathbf{R}_\kappa(A_o, I_{d_o})\tilde{w}_{\kappa t:\kappa}^o\|_\infty$ by the definition of z_t . Therefore, we have $|(z_{t+1})^{(j)} - (z_t)^{(j)}| \leq m_1 + \|\mathbf{R}_\kappa(A_o, I_{d_o})\tilde{w}_{\kappa t:\kappa}^o\|_\infty$ for each $j = 1, \dots, d_o$. Let us consider the left hand side of (25c):

$$\begin{aligned} &\mathbb{E} \left[\left| (z_{t+1})^{(j)} - (z_t)^{(j)} \right|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \\ &\leq \mathbb{E} \left[\left(m_1 + \|\mathbf{R}_\kappa(A_o, I_{d_o})\tilde{w}_{\kappa t:\kappa}^o\| \right)^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \\ &\leq 8 \left(m_1^4 + \mathbb{E} \left[\|\mathbf{R}_\kappa(A_o, I_{d_o})\tilde{w}_{\kappa t:\kappa}^o\|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \right). \end{aligned}$$

Similar to the Lemma 10, we get

$$\begin{aligned} &\mathbb{E} \left[\|\mathbf{R}_\kappa(A_o, I_{d_o})\tilde{w}_{\kappa t:\kappa}^o\|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \\ &\leq \kappa^4 \max_{i=0, \dots, \kappa-1} \mathbb{E} \left[\|\tilde{w}_{\kappa t+i}^o\|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \\ &\leq \kappa^4 m_2, \end{aligned}$$

where the last bound is obtained by Lemma 11. Therefore,

$$\mathbb{E} \left[\left| (z_{t+1})^{(j)} - (z_t)^{(j)} \right|^4 \mid z_0^{(j)}, \dots, z_t^{(j)} \right] \leq 8 \left(m_1^4 + \kappa^4 m_2 \right) =: M.$$

Lemma 14 For a discrete time dynamical system (20),

$$\mathbb{E}[\tilde{w}_{t:\kappa} \mid \mathfrak{Y}_t^s] = \mathbf{0} \text{ for each } t \text{ and } \kappa.$$

PROOF. We recall the expression (A.14)

$$\tilde{w}_t = s_{t+1} \mathbf{R}_{t-\tau_t+1}(A, I) \hat{w}_{\tau_t:t-\tau_t+1}.$$

By taking the conditional expectation, we get

$$\begin{aligned} \mathbb{E}[\tilde{w}_t \mid \mathfrak{Y}_t^s] &= p_s \mathbb{E}[\mathbf{R}_{t-\tau_t+1}(A, I) \hat{w}_{\tau_t:t-\tau_t+1} \mid \mathfrak{Y}_t^s] \\ &= p_s \mathbb{E}[\mathbf{R}_{t-\tau_t+1}(A, I) \mathbb{E}[\hat{w}_{\tau_t:t-\tau_t+1} \mid \mathfrak{Y}_t] \mid \mathfrak{Y}_t^s] = \mathbf{0}. \end{aligned}$$

Similarly, $\mathbb{E}[\tilde{w}_{t+\ell} \mid \mathfrak{Y}_t^s] = \mathbb{E}[\mathbb{E}[\tilde{w}_{t+\ell} \mid \mathfrak{Y}_{t+\ell}^s] \mid \mathfrak{Y}_t^s] = \mathbf{0}$ for $\ell = 0, \dots, \kappa-1$. This completes the proof.

Lemma 15 There exists $\gamma_s > 0$ such that

$$\sup_{t \in \mathbb{N}_0} \mathbb{E}_{\mathfrak{Y}_0^s} \left[\|\tilde{x}_t^s\|^2 \right] \leq \gamma_s.$$

PROOF. A part of this proof is standard in literature, i.e. [46]. Let $P > 0$ be a symmetric positive definite matrix.

$$\begin{aligned} &\mathbb{E}[(\tilde{x}_{t+1}^s)^\top P \tilde{x}_{t+1}^s \mid \tilde{x}_t, \mathfrak{Y}_0^s] = \|\tilde{x}_t^s\|_{A_s^\top P A_s}^2 + 2(\tilde{x}_t^s)^\top A_s^\top P B_s u_t^a \\ &\quad + \|u_t^a\|_{B_s^\top P B_s}^2 + \mathbb{E}[(\tilde{w}_t^s)^\top P \tilde{w}_t^s]. \end{aligned}$$

Since A_s is Schur stable, there exists some $\lambda \in]0, 1[$ such that $A_s^\top P A_s \leq \lambda P$. We can also assume that there exists $m_4 \geq 0$ such that $\mathbb{E}[(\hat{w}_t^s)^\top P \hat{w}_t^s] \leq m_4$ for each t because \hat{w}_t is Gaussian. We recall the expression (A.14) and compute $\mathbb{E}[(\tilde{w}_t^s)^\top P \tilde{w}_t^s]$ as follows:

$$\begin{aligned} \mathbb{E}[(\tilde{w}_t^s)^\top P \tilde{w}_t^s] &= p_s \mathbb{E} \left[\left\| \mathbf{R}_{t-\tau_t+1}(A_s, I) \hat{w}_{\tau_t:t-\tau_t+1}^s \right\|_P^2 \right] \\ &= p_s \sum_{h=1}^{\infty} \mathbb{E} \left[\left\| \mathbf{R}_h(A_s, I) \hat{w}_{\tau_t:h}^s \right\|_P^2 \mid t - \tau_t + 1 = h \right] \\ &\quad \times p(t - \tau_t + 1 = h) \\ &= p_s \sum_{h=1}^{\infty} \mathbb{E} \left[\left\| \mathbf{R}_h(A_s, I) \hat{w}_{\tau_t:h}^s \right\|_P^2 \mid t - \tau_t + 1 = h \right] p_s (1 - p_s)^{h-1} \\ &\leq p_s^2 \sum_{h=1}^{\infty} h \sum_{i=1}^h \mathbb{E} \left[\left\| A_s^{i-1} \hat{w}_{\tau_t+h-1}^s \right\|_P^2 \mid t - \tau_t + 1 = h \right] (1 - p_s)^{h-1} \\ &\leq p_s^2 \sum_{h=1}^{\infty} h \sum_{i=1}^h \lambda^{i-1} \mathbb{E} \left[\left\| \hat{w}_{\tau_t+h-1}^s \right\|_P^2 \mid t - \tau_t + 1 = h \right] (1 - p_s)^{h-1} \\ &\leq p_s^2 \sum_{h=1}^{\infty} h m_4 (1 - p_s)^{h-1} \sum_{i=1}^h \lambda^{i-1} \\ &\leq \frac{m_4 p_s^2}{1 - \lambda} \sum_{h=1}^{\infty} h (1 - p_s)^{h-1} =: m_5. \end{aligned}$$

Further, for $\varepsilon < \frac{1-\lambda}{\lambda}$ the Peter-Paul inequality provides us the bound $2(\tilde{x}_t^s)^\top A_s^\top P B_s u_t^a \leq \varepsilon \|\tilde{x}_t^s\|_{A_s^\top P A_s}^2 + \frac{1}{\varepsilon} \|u_t^a\|_{B_s^\top P B_s}^2$. Therefore,

$$\begin{aligned} \mathbb{E}[(\tilde{x}_{t+1}^s)^\top P \tilde{x}_{t+1}^s \mid \tilde{x}_t, \mathfrak{Y}_0^s] &\leq (1+\varepsilon) \|\tilde{x}_t^s\|_{A_s^\top P A_s}^2 \\ &\quad + (1 + \frac{1}{\varepsilon}) \|u_t^a\|_{B_s^\top P B_s}^2 + m_5 \\ &\leq (1+\varepsilon) \lambda \|\tilde{x}_t^s\|_P^2 + (1 + \frac{1}{\varepsilon}) m \lambda_{\max}(B_s^\top P B_s) u_{\max}^2 + m_5. \end{aligned}$$

Defining $m_6 := (1 + \frac{1}{\varepsilon}) m \lambda_{\max}(B_s^\top P B_s) u_{\max}^2 + m_5$ and iterating the above expression we get

$$\begin{aligned} \mathbb{E}_{\mathfrak{Y}_0^s} [(\tilde{x}_{t+1}^s)^\top P \tilde{x}_{t+1}^s] &\leq ((1+\varepsilon)\lambda)^{t+1} \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t^s\|_P^2] + \frac{m_6}{1-(1+\varepsilon)\lambda} \\ &\leq ((1+\varepsilon)\lambda)^{t+1} \lambda_{\max}(P) s_0 \mathbb{E}_{\mathfrak{Y}_0^s} [\|\hat{x}_t^s\|^2] + \frac{m_6}{1-(1+\varepsilon)\lambda} \\ &\leq ((1+\varepsilon)\lambda)^{t+1} \lambda_{\max}(P) s_0 \text{tr}(K_0^\top \Sigma_{x_0} C^\top) + \frac{m_6}{1-(1+\varepsilon)\lambda} \end{aligned}$$

Since $(1+\varepsilon)\lambda < 1$, there exists $\gamma_s > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t^s\|^2] \leq \gamma_s$ for all t .

PROOF. [Proof of Lemma 7] The equations in (25) are derived from Theorem 2 by substituting $z_t^{(j)}$ in place of X_t in Theorem 2. Therefore, there exists $\gamma_1 > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} [\|z_t\|^2] = \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_{\kappa t}^o\|^2] \leq \gamma_1$ for all t . Then by Lemma 12 there exists $\gamma_o > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t^o\|^2] \leq \gamma_o$ for all t . Defining $\tilde{\gamma} := \gamma_o + \gamma_s$ and by using Lemma 15 we get $\mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t\|^2] \leq \tilde{\gamma}$ for all t . Now consider (25) and substitute (24) to get that

$$\begin{aligned} \mathbb{E}_{\mathfrak{Y}_{\kappa t}^s} [(z_{t+1} - z_t)^{(j)}] &= \mathbb{E}_{\mathfrak{Y}_{\kappa t}^s} \left[\left((A_o^{\kappa(t+1)})^\top (\mathbf{R}_\kappa(A_o, B_o) u_{\kappa t:\kappa}^a + \mathbf{R}_\kappa(A_o, I_{d_o}) \tilde{w}_{\kappa t:\kappa}^o) \right)^{(j)} \right] \\ &= \mathbb{E}_{\mathfrak{Y}_{\kappa t}^s} \left[\left((A_o^{\kappa(t+1)})^\top \mathbf{R}_\kappa(A_o, B_o) u_{\kappa t:\kappa}^a \right)^{(j)} \right], \end{aligned}$$

where the last equality is due to Lemma 14. Since (25c) is satisfied by Lemma 13, (25) are equivalent to the following conditions for $t = 0, \kappa, 2\kappa, \dots$, and $j = 1, \dots, d_o$:

$$\begin{aligned} \left((A_o^{t+\kappa})^\top \mathbf{R}_\kappa(A_o, B_o) \mathbb{E}_{\mathfrak{Y}_t^s} [u_{t:\kappa}^a] \right)^{(j)} &\leq -a \\ \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} &> r, \end{aligned} \quad (\text{A.15a})$$

$$\begin{aligned} \left((A_o^{t+\kappa})^\top \mathbf{R}_\kappa(A_o, B_o) \mathbb{E}_{\mathfrak{Y}_t^s} [u_{t:\kappa}^a] \right)^{(j)} &\geq a \\ \text{whenever } ((A_o^t)^\top \tilde{x}_t^o)^{(j)} &< -r. \end{aligned} \quad (\text{A.15b})$$

Let us define the component-wise saturation function $\mathbb{R}^{d_o} \ni z \mapsto \text{sat}_{r,\zeta}^\infty(z) \in \mathbb{R}^{d_o}$ to be

$$(\text{sat}_{r,\zeta}^\infty(z))^{(i)} = \begin{cases} z^{(i)} \zeta / r & \text{if } |z^{(i)}| \leq r, \\ \zeta & \text{if } z^{(i)} > r, \text{ and} \\ -\zeta & \text{otherwise,} \end{cases}$$

for each $i = 1, \dots, d_o$. Now consider (27) with $\Theta_t = \mathbf{0}$ and $(\eta_t)_{1:\kappa m} = -\mathbf{R}_\kappa(A_o, B_o)^\dagger A_o^{t+\kappa} \text{sat}_{r,\zeta}^\infty((A_o^\top)^t \tilde{x}_t^o)$. It is clear that $\|u_{t:\kappa}\|_\infty \leq \|u_{t:\kappa}\|_2 \leq \sigma_1(\mathbf{R}_\kappa(A_o, B_o)^\dagger) \sqrt{d_o} \zeta \leq u_{\max}$. Therefore, the given feedback policy satisfies (2). The control sequence $u_{t:\kappa}^a$ under the transmission protocol (TP) is given by

$$u_{t:\kappa}^a = \tilde{\mathcal{G}}_t u_{t:\kappa}, \quad (\text{A.16})$$

where $\tilde{\mathcal{G}}_t \in \mathbb{R}^{\kappa m \times \kappa m}$ is the principal submatrix of the diagonal matrix \mathcal{G}_t and $\mathbb{E}[\tilde{\mathcal{G}}_t^{(i,i)}] \geq p_c$ for each $i = 1, \dots, \kappa m$. Therefore, the given policy also satisfies (A.15) with $a = \zeta p_c$. We substitute $\mathbb{E}[u_{t:\kappa}^a] \geq p_c \mathbb{E}[u_{t:\kappa}]$ in (A.15) to get (26). This completes the proof.

Lemma 16 Consider the system (1). There exists $\rho > 0$ such that $\mathbb{E}_{\mathfrak{Y}_0^s} [\|e_t\|^2] \leq \rho$ for all $t \geq 0$.

PROOF. Let us first observe that

$$\mathbb{E}_{\mathfrak{Y}_0^s} [\|e_t\|^2] = \begin{cases} \mathbb{E}_{\mathfrak{Y}_0} [\|e_t\|^2] & \text{for } s_0 = 1 \\ \mathbb{E}_{\mathfrak{Y}_0^s} [\mathbb{E}_{\mathfrak{Y}_0} [\|e_t\|^2]] & \text{otherwise.} \end{cases}$$

Since $\mathfrak{Y}_0^s \subset \mathfrak{Y}_t^s$ for $t \geq 0$, the claim is implied by [51, Lemma 4.2.2] by using the tower property of the conditional expectation.

PROOF. [Proof of Theorem 8] Since $x_t = x_t - \hat{x}_t + \hat{x}_t - \tilde{x}_t + \tilde{x}_t$, let us consider the inequality

$$\|x_t\|^2 \leq 3 \left(\|x_t - \hat{x}_t\|^2 + \|\hat{x}_t - \tilde{x}_t\|^2 + \|\tilde{x}_t\|^2 \right).$$

Let us apply the conditional expectation on the above inequality to obtain the bound:

$$\begin{aligned} \mathbb{E}_{\mathfrak{Y}_0^s} [\|x_t\|^2] &\leq 3 \left(\mathbb{E}_{\mathfrak{Y}_0^s} [\|x_t - \hat{x}_t\|^2] + \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{e}_t\|^2] + \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t\|^2] \right) \\ &\leq 3 \left(\rho + \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{e}_t\|^2] + \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t\|^2] \right) \text{ from Lemma 16} \\ &\leq 3 \left(\rho + \tilde{\rho} + \mathbb{E}_{\mathfrak{Y}_0^s} [\|\tilde{x}_t\|^2] \right) \text{ from Lemma 9} \\ &\leq 3(\rho + \tilde{\rho} + \tilde{\gamma}) \text{ from Lemma 7} \\ &=: \gamma \text{ for all } t \geq 0. \end{aligned}$$