# On quotients of Boolean control networks 

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#### Abstract

In this paper, we focus on the study of quotients of Boolean control networks (BCNs) with the motivation that they might serve as smaller models that still carry enough information about the original network. Given a BCN and an equivalence relation on the state set, we consider a labeled transition system that is generated by the BCN. The resulting quotient transition system then naturally captures the quotient dynamics of the BCN concerned. We therefore develop a method for constructing a Boolean system that behaves equivalently to the resulting quotient transition system. The use of the obtained quotient system for control design is discussed and we show that for BCNs, controller synthesis can be done by first designing a controller for a quotient and subsequently lifting it to the original model. We finally demonstrate the applicability of the proposed techniques on a biological example.


Key words: Boolean control networks, quotient transition systems, control design, stabilization, optimal control.

## 1 Introduction

Boolean networks (BNs) and Boolean control networks (BCNs), wherein each component is characterized with a binary variable, have been widely employed in modeling biological regulatory networks. After assembling the components of a system as well as their regulatory interactions, BN/BCN models can nicely describe the qualitative temporal behavior of the system [1]. They can also illuminate how perturbations may disrupt normal behavior and yield testable predictions which are particularly valuable in less well understood biological systems [2]. As a nice framework for modeling dynamical processes on networks, especially in biological context, BN/BCN models have led to fruitful insights for unicellular organisms [3], plants [4], animals (5], and humans [6], especially human signaling networks implicated in diseases [7]. A $\mathrm{BN} / \mathrm{BCN}$ is typically placed in the form of a nonlinear (control) system; while interestingly, based on an algebraic state representation approach, the Boolean dynamics can be mapped exactly into a standard discrete-time linear dynamics [8]. This formal simplicity makes it relatively easy to formulate and solve classical control-theoretic problems for

[^0]BNs/BCNs, and thereby lays a suitable foundation for a series of subsequent studies. Examples include recent investigations of dynamical properties [9], network synchronization [10], controllability [11, 12] and stabilizability [13-16], observability [17-20] and reconstructibility [21], disturbance or input-output decoupling [22-24], optimal control [25, 26], and more 27 31]. The size of the linear system that describes a BN with $n$ state variables is $2^{n}$. Thus, any algorithm based on this algebraic set-up has an exponential time complexity in the worst case. On the other hand, it has shown that for several control problems, the complexity curse can be alleviated or even removed if the structure of BNs is appropriately constrained 32, 33]. These positive developments notwithstanding, it still seems computationally challenging to solve control-related problems in general BNs/BCNs, since many such problems have shown to be NP-hard [34 37]. The hardness results justify the use of exponential time algorithms and exponential size systems suggested by the algebraic state-space representation.

In this paper, we focus on studying quotients of BCNs since they can be seen as lower dimensional models that may still contain enough information about the original model (whose algebraic representation is of exponential size). We consider quotient systems for BCNs in the exact sense that the notion is used in the control community [38 40]. Precisely, given a BCN and an equivalence
relation on its state set, we consider a (labeled) transition system generated by the BCN and partition the state set based on the relation. The resulting quotient system then naturally captures the quotient dynamics of this BCN, so we propose to develop a Boolean system that generates the transitions of the quotient transition system (Theorem 1). Of course, it is not surprising that additional constraints need to be placed on the equivalence relation to ensure that the quotient dynamics can indeed be generated from some Boolean system. A subsequent question is then, how to obtain an equivalence relation which allows the construction of a quotient BCN. We fully answer this question by giving a procedure that converges in a finite number of iterations to a satisfactory equivalence relation (Theorem 2). As applications of the study, we show how the resulting quotient can be used for controller synthesis. The results tell us that synthesizing controllers for a BCN can be easily done by first controlling the quotient system and then lifting the control law back to the original Boolean model (see Propositions 2 and 3).

Notation. The symbol $\delta_{k}^{i}$ denotes the $i$ th canonical basis vector of length $k, \Delta_{k}$ denotes the set consisting of the canonical vectors $\delta_{k}^{1}, \ldots, \delta_{k}^{k}$, and $\mathcal{L}^{k \times r}$ denotes the set of all $k \times r$ matrices whose columns are canonical vectors of length $k$. Elements of $\mathcal{L}^{k \times r}$ are called logical matrices (of size $k \times r)$. A ( 0,1 )-matrix is a matrix that consists solely of the 0 and 1 entries. The $(i, j)$-entry of a matrix $A$ is invariably denoted by $(A)_{i j}$. If $A$ and $B$ are $k \times r(0,1)$ matrices, the meet of $A$ and $B$, denoted by $A \wedge B$, is the $(0,1)$-matrix with the $(i, j)$-entry equal to $(A)_{i j} \wedge(B)_{i j}$. For a $k \times l(0,1)$-matrix $C$ and an $l \times r(0,1)$-matrix $D$, the Boolean product of $C$ and $D$, denoted by $C \odot D$, is the $k \times r$ matrix with the $(i, j)$-entry $\bigvee_{s=1}^{l}\left[(C)_{i s} \wedge(D)_{s j}\right]$. Given two relations $\mathcal{R}_{1}$ and $\mathcal{R}_{2}, \mathcal{R}_{2} \circ \mathcal{R}_{1}$ denotes the composition of $\mathcal{R}_{1}$ and $\mathcal{R}_{2}$, i.e., the relation defined by $(a, c) \in \mathcal{R}_{2} \circ \mathcal{R}_{1}$ if and only if there exists $b$ with $(a, b) \in$ $\mathcal{R}_{1}$ and $(b, c) \in \mathcal{R}_{2}$.

## 2 Preliminaries

### 2.1 Algebraic representation of Boolean control networks

A BCN is a discrete-time dynamical system with binary state variables and binary control variables, i.e.,

$$
\begin{align*}
x_{1}(t+1) & =f_{1}\left(x_{1}(t), \ldots, x_{n}(t), u_{1}(t), \ldots, u_{m}(t)\right) \\
& \vdots  \tag{1}\\
x_{n}(t+1) & =f_{n}\left(x_{1}(t), \ldots, x_{n}(t), u_{1}(t), \ldots, u_{m}(t)\right)
\end{align*}
$$

with $x_{i}, u_{j} \in\{1,0\}$ and $f_{i}:\{1,0\}^{n+m} \rightarrow\{1,0\}$. The dynamics (1) can be recast into a form similar to that of a discrete-time linear system, using the semitensor product of matrices [8]. To be more precise, we recall
that the (left) semitensor product of two matrices $A$ and $B$ of sizes $n_{1} \times m_{1}$ and $n_{2} \times m_{2}$, respectively, denoted by $A \ltimes B$, is defined by $A \ltimes B=\left(A \otimes I_{l / m_{1}}\right)\left(B \otimes I_{l / n_{2}}\right)$, where $\otimes$ is the Kronecker product of matrices, and $I_{l / m_{1}}$ and $I_{l / n_{2}}$ are the identity matrices of orders $l / m_{1}$ and $l / n_{2}$, respectively, with $l$ being the least common multiple of $m_{1}$ and $n_{2}$. If we identify the Boolean values 1 and 0 with the canonical vectors $\delta_{2}^{1}$ and $\delta_{2}^{2}$, respectively (so $x_{i}$ and $u_{j}$ in (1) are vectors in $\Delta_{2}$ ), and if we let $x(t)=$ $x_{1}(t) \ltimes \cdots \ltimes x_{n}(t)$ and $u(t)=u_{1}(t) \ltimes \cdots \ltimes u_{m}(t)$, then the Boolean dynamics (1) can be represented by an equation of the form

$$
\begin{equation*}
x(t+1)=F \ltimes u(t) \ltimes x(t), \tag{2}
\end{equation*}
$$

where $F \in \mathcal{L}^{2^{n} \times 2^{n+m}}$. (The expression on the right-hand side of (2) is unambiguous, since the semitensor product is associative.) For more information on converting a BCN in the form of (1) to its algebraic representation (2), as well as more information regarding the properties of the semitensor product, the reader is referred to, e.g., [8] and [41].

### 2.2 Transition systems

Our discussion of quotients of BCNs will be based on the notion of quotient transition systems. We first recall the concept of a (labeled) transition system.

Definition 1 (See, e.g., 42]). A (labeled) transition system is a tuple $\mathcal{T}=(Q, L, \rightarrow)$ that consists of a set of states $Q$, a set of labels $L$, and a transition relation $\rightarrow \subseteq Q \times L \times Q$.

For any $q, q^{\prime} \in Q$ and any $l \in L$, a transition $\left(q, l, q^{\prime}\right) \in \rightarrow$ means that it is possible to move from state $q$ to state $q^{\prime}$ under the action labeled by $l$. Following standard practice, we denote $q \xrightarrow{l} q^{\prime}$ if $\left(q, l, q^{\prime}\right) \in \rightarrow$.

Recall that an equivalence relation $\mathcal{R}$ on $Q$ is a reflexive, symmetric, and transitive binary relation on $Q$. Given a transition system $\mathcal{T}$, if $\mathcal{R}$ is an equivalence relation on the state set of $\mathcal{T}$, then it naturally induces a quotient transition system, as follows.

Definition 2 (See, e.g., [40]). Let $\mathcal{T}=(Q, L, \rightarrow)$ be a transition system and let $\mathcal{R}$ be an equivalence relation on $Q$. The quotient transition system $\mathcal{T} / \mathcal{R}$ is defined by $\mathcal{T} / \mathcal{R}=\left(Q / \mathcal{R}, L, \rightarrow_{\mathcal{R}}\right)$, where $Q / \mathcal{R}$ is the quotient set (i.e., the set of all equivalence classes $[q]=\{p \in$ $Q:(q, p) \in \mathcal{R}\}$ for $q \in Q)$, and for all $[q],\left[q^{\prime}\right] \in Q / \mathcal{R}$, $[q] \xrightarrow{l}_{\mathcal{R}}\left[q^{\prime}\right]$ if and only if there exist $p \in[q]$ and $p^{\prime} \in\left[q^{\prime}\right]$ such that $p \xrightarrow{l} p^{\prime}$.

That is, a state $[q]$ in $\mathcal{T} / \mathcal{R}$ can make a transition to another state $\left[q^{\prime}\right]$ under an action $l$, if some $p \in[q]$ can
make a transition to some $p^{\prime} \in\left[q^{\prime}\right]$ when taking the action $l$. In what follows, we will use a similar framework to study quotients of a BCN.

## 3 Quotients of Boolean control networks

### 3.1 Constructing quotient Boolean systems

Let us consider a BCN described by the algebraic representation

$$
\begin{align*}
& \Sigma: x(t+1)=F \ltimes u(t) \ltimes x(t), \quad \\
& x \in \Delta_{N}, \quad u \in \Delta_{M},  \tag{3}\\
& F \in \mathcal{L}^{N \times N M} .
\end{align*}
$$

(Note that, in the above, $N$ and $M$ are in fact certain powers of 2 , but we do not need this fact for our argument.) In order to investigate quotients of (3), we first turn our attention to the equivalence relations on its state set $\Delta_{N}$. An immediate observation is that every such equivalence relation $\mathcal{R}$ can be viewed as induced by a logical matrix $C$ with $N$ columns, by saying

$$
\begin{equation*}
\left(x, x^{\prime}\right) \in \mathcal{R} \Longleftrightarrow C x=C x^{\prime} \tag{4}
\end{equation*}
$$

Furthermore, the logical matrix $C$ can be chosen of full row rank (hence in particular having no zero rows). We remark that such a full row rank matrix can be directly derived from the matrix representation of $\mathcal{R}$. In fact, let $A_{\mathcal{R}}$ be the $N \times N$ matrix whose entries are given by

$$
\left(A_{\mathcal{R}}\right)_{i j}= \begin{cases}1 & \text { if }\left(\delta_{N}^{i}, \delta_{N}^{j}\right) \in \mathcal{R} \\ 0 & \text { otherwise }\end{cases}
$$

If $C$ is a matrix which has the same set of rows as $A_{\mathcal{R}}$ but with no rows repeated, then it must be a logical matrix with full row rank and fulfill condition (4) [43, Lemma 4.6].

Example 1. To illustrate this fact, as well as the main idea behind obtaining an algebraic representation, we consider a BCN as in (1), with $n=3$ and $m=1$. The corresponding Boolean functions are given by the truth table shown in Table 1 . Since $n=3$ and $m=1$, the size of the matrix $F$ in the algebraic representation is $8 \times 16$. To find this matrix, we see from Table 1 that if $u(t)=$ $x_{1}(t)=x_{2}(t)=x_{3}(t)=1$, we have $x_{1}(t+1)=x_{2}(t+$ $1)=1$, and $x_{3}(t+1)=0$. In the algebraic framework, this corresponds to $u(t)=x_{1}(t)=x_{2}(t)=x_{3}(t)=\delta_{2}^{1}$, $x_{1}(t+1)=x_{2}(t+1)=\delta_{2}^{1}$, and $x_{3}(t+1)=\delta_{2}^{2}$, so

$$
\begin{aligned}
& x(t+1)=\delta_{2}^{1} \ltimes \delta_{2}^{1} \ltimes \delta_{2}^{2}=\delta_{8}^{2} \\
& u(t) \ltimes x(t)=\delta_{2}^{1} \ltimes \delta_{2}^{1} \ltimes \delta_{2}^{1} \ltimes \delta_{2}^{1}=\delta_{16}^{1} .
\end{aligned}
$$

Substituting these to the left- and right-hand sides of (2) yields

$$
\begin{equation*}
\delta_{8}^{2}=F \ltimes \delta_{16}^{1}=F \delta_{16}^{1} \tag{5}
\end{equation*}
$$

Table 1
Truth table for Example 1.

| $u$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $f_{1}$ | $f_{2}$ | $f_{3}$ | $u$ | $x_{1}$ | $x_{2}$ | $x_{3}$ | $f_{1}$ | $f_{2}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 1 | 1 | 1 | 1 | 0 | $f_{3}$ |  |  |  |  |  |
| 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 |  | 1 |
| 1 | 1 | 1 | 1 | 0 |  | 1 | 1 | 1 |  |  |  |  |
| 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 |  | 1 |
| 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |  |  |  |
| 1 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 |  | 0 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |  | 0 | 1 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 |  | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |  | 0 |

The second equality follows since the semitensor product is nothing but the standard product if the multiplied matrices (or vectors) have compatible sizes [8]. From (5), and considering that right-multiplying a matrix by a canonical vector yields the corresponding column of the matrix, we know that the first column of $F$ is $\delta_{8}^{2}$. Repeating a similar argument for each combination in the truth table, we can determine all the columns of $F$, i.e., we determine the second column of $F$ by considering the case when $u(t)=x_{1}(t)=x_{2}(t)=1$ and $x_{3}(t)=$ 0 , the third column by considering $u(t)=x_{1}(t)=1$, $x_{2}(t)=0$, and $x_{3}(t)=1$, and so on. The matrix we get is

$$
F=\left[\begin{array}{llllllllll}
\delta_{8}^{2} & \delta_{8}^{1} & \delta_{8}^{1} & \delta_{8}^{5} & \delta_{8}^{6} & \delta_{8}^{7} & \delta_{8}^{8} & \delta_{8}^{5} & \delta_{8}^{1} & \delta_{8}^{1} \tag{6}
\end{array} \delta_{8}^{1} \delta_{8}^{8} \delta_{8}^{6} \delta_{8}^{7} \delta_{8}^{8} \delta_{8}^{7}\right]
$$

Consequently, the algebraic representation of this BCN is given by

$$
\begin{equation*}
x(t+1)=F \ltimes u(t) \ltimes x(t), \quad x(t) \in \Delta_{8}, \quad u(t) \in \Delta_{2}, \tag{7}
\end{equation*}
$$

with $F$ found above. Note that system (7) evolves on the set $\Delta_{8}=\left\{\delta_{8}^{1}, \ldots, \delta_{8}^{8}\right\}$, and each canonical vector $\delta_{8}^{i}$ corresponds to a possible configuration of the BCN (e.g., $\delta_{8}^{1}$ corresponds to $[1,1,1]$ since $\delta_{2}^{1} \ltimes \delta_{2}^{1} \ltimes \delta_{2}^{1}=\delta_{8}^{1}$, $\delta_{8}^{2}$ corresponds to $[1,1,0]$ since $\delta_{2}^{1} \ltimes \delta_{2}^{1} \ltimes \delta_{2}^{2}=\delta_{8}^{2}$, etc.). The trajectories of (7) are shown in Fig. 1. Now let $\mathcal{R}$ be the equivalence relation produced by the partition $\left\{\left\{\delta_{8}^{1}\right\},\left\{\delta_{8}^{2}, \delta_{8}^{3}\right\},\left\{\delta_{8}^{4}\right\},\left\{\delta_{8}^{5}, \delta_{8}^{6}, \delta_{8}^{7}, \delta_{8}^{8}\right\}\right\}$; that is, the pair $(a, b) \in \mathcal{R}$ if and only if $a$ and $b$ are in the same subset of


Fig. 1. Trajectories of system (7), which represents the BCN in Example 1. A solid line denotes the transition corresponding to $u(t)=\delta_{2}^{1}$ and a dashed line denotes the transition corresponding to $u(t)=\delta_{2}^{2}$.
the partition. By definition, the matrix that represents $\mathcal{R}$ has a 1 as its $(i, j)$-entry when $\delta_{8}^{i}$ is related to $\delta_{8}^{j}$, and a 0 in this position if $\delta_{8}^{i}$ is not related to $\delta_{8}^{j}$. Accordingly, we get the following matrix for $\mathcal{R}$ :

$$
A_{\mathcal{R}}=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & J_{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & J_{4}
\end{array}\right]
$$

where $J_{k}$ denotes the all-one matrix of size $k \times k$. Collapsing the identical rows of $A_{\mathcal{R}}$ yields

$$
C=\left[\begin{array}{lllllll}
\delta_{4}^{1} & \delta_{4}^{2} & \delta_{4}^{2} & \delta_{4}^{3} & \delta_{4}^{4} & \delta_{4}^{4} & \delta_{4}^{4} \tag{8}
\end{array} \delta_{4}^{4}\right] .
$$

It is clear that $C$ is a full row rank logical matrix and that (4) holds.

Remark 1. Note that if $\mathcal{R}$ is an equivalence relation on $\Delta_{N}$ induced by a matrix $C \in \mathcal{L}^{\tilde{N} \times N}$ of full row rank, then the quotient set $\Delta_{N} / \mathcal{R}$ is of cardinality $\widetilde{N}$, and the correspondence $[x] \mapsto C x$ gives a bijection between the sets $\Delta_{N} / \mathcal{R}$ and $\Delta_{\widetilde{N}}$.

We now consider quotients of (3). We note that the BCN (3) naturally generates a transition system $\mathcal{T}(\Sigma)=$ $\left(\Delta_{N}, \Delta_{M}, \rightarrow\right)$, where

$$
\begin{equation*}
x \xrightarrow{u} x^{\prime} \Longleftrightarrow x^{\prime}=F \ltimes u \ltimes x . \tag{9}
\end{equation*}
$$

(In other words, a transition $x \xrightarrow{u} x^{\prime}$ occurs in $\mathcal{T}(\Sigma)$ if $u$ steers $\Sigma$ from $x$ to $x^{\prime}$.) Let $\mathcal{R}$ be an equivalence relation induced by a full row rank logical matrix $C$ of size $\widetilde{N} \times N$. Then the quotient transition system $\mathcal{T}(\Sigma) / \mathcal{R}=\left(\Delta_{N} / \mathcal{R}, \Delta_{M}, \rightarrow_{\mathcal{R}}\right)$ can be thought of as having the state set $\Delta_{\widetilde{N}}$; and the transition relation is then given by

$$
\begin{align*}
z \xrightarrow{u} \mathcal{R} & z^{\prime} \Longleftrightarrow \\
& \text { there exists a transition } x \xrightarrow{u} x^{\prime} \text { of } \mathcal{T}(\Sigma)  \tag{10}\\
& \text { with } z=C x \text { and } z^{\prime}=C x^{\prime}
\end{align*}
$$

(cf. Definition 2 and Remark 1). For the analysis to remain in the Boolean context, we expect that the transitions of $\mathcal{T}(\Sigma) / \mathcal{R}$ are also generated by a Boolean system. (Here, and below, we use the term "Boolean system" to refer to a system of the form (3) where $N$ and $M$ are not restricted to be powers of 2.) It is readily seen that this is the case if and only if for any $z \in \Delta_{\widetilde{N}}$ and any $u \in \Delta_{M}$, there is a unique transition $z{ }_{\mathcal{R}}^{u} z^{\prime}$ of $\mathcal{T}(\Sigma) / \mathcal{R} \square$ By (4), (9) and (10), the latter is equivalent

[^1]to the requirement that
\[

$$
\begin{align*}
(a, b) \in \mathcal{R} \Longleftrightarrow & (F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R} \\
& \text { for all } u \in \Delta_{M} . \tag{11}
\end{align*}
$$
\]

We therefore restrict our attention to those $\mathcal{R}$ satisfying (11).

Remark 2. The meaning of condition (11) is clear: if we think of $\mathcal{R}$ as a partition of $\Delta_{N}$, then the successor set of each block in this partition is included in a single block of the partition.

The following theorem gives a method for explicitly constructing a Boolean system that generates the transitions of $\mathcal{T}(\Sigma) / \mathcal{R}$.

Theorem 1. Consider a $B C N \Sigma$ as in (3). Suppose that $\mathcal{R}$ is an equivalence relation on $\Delta_{N}$ induced by a matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$ of full row rank, and that property (11) holds. For each $1 \leq k \leq M$, let $F_{k}$ be the matrix in $\mathcal{L}^{N \times N}$ defined by $F_{k}=F \ltimes \delta_{M}^{k}$, and let $\widetilde{F}_{k}=C \odot F_{k} \odot C^{\top}$. Then:
(a) $\widetilde{F}_{k} \in \mathcal{L}^{\widetilde{N} \times \widetilde{N}}$ for $1 \leq k \leq M$.
(b) Let
$\Sigma_{\mathcal{R}}: \quad x_{\mathcal{R}}(t+1)=\widetilde{F} \ltimes u(t) \ltimes x_{\mathcal{R}}(t), \quad x_{\mathcal{R}} \in \Delta_{\widetilde{N}}, u \in \Delta_{M}$
be the system where $\widetilde{F}=\left[\begin{array}{llll}\widetilde{F}_{1} & \widetilde{F}_{2} & \ldots & \widetilde{F}_{M}\end{array}\right]$. If an input $u \in \Delta_{M}$ steers $\Sigma$ from a state $a \in \Delta_{N}$ to a state $a^{\prime} \in \Delta_{N}$, then it also steers $\Sigma_{\mathcal{R}}$ from $C a$ to $C a^{\prime}$. Conversely, if u steers $\Sigma_{\mathcal{R}}$ from a state $q \in \Delta_{\widetilde{N}}$ to a state $q^{\prime} \in \Delta_{\widetilde{N}}$, then there is a one-step transition of $\Sigma$ from some $a \in \Delta_{N}$ to some $a^{\prime} \in \Delta_{N}$ with $C a=q$ and $C a^{\prime}=q^{\prime}$, under this input $u$.

Proof. (a) It is clear that each $\widetilde{F}_{k}$ is a $(0,1)$-matrix of size $\widetilde{N} \times \widetilde{N}$. So we need only show that, for $1 \leq k \leq M$, every column of $\widetilde{F}_{k}$ contains exactly one 1 . Let $1 \leq k \leq$ $M$ and $1 \leq j \leq \widetilde{N}$ be fixed. Since $C$ (being logical) has no zero rows, there exists $1 \leq s \leq N$ such that $(C)_{j s}=1$. Choose $1 \leq r \leq N$ so that $\overline{\delta_{N}^{r}}=F \ltimes \delta_{M}^{k} \ltimes$ $\delta_{N}^{s}$. Then $\left(F_{k}\right)_{r s}=1$. For this $r$, let $1 \leq i \leq \widetilde{N}$ be such that $(C)_{i r}=1$. Then, by the definition of Boolean matrix multiplication, the $(i, j)$-entry of $\widetilde{F}_{k}$ is equal to $\bigvee_{p=1}^{N} \bigvee_{l=1}^{N}\left[(C)_{i p} \wedge\left(F_{k}\right)_{p l} \wedge(C)_{j l}\right]$, and hence equal to 1 (since $(C)_{i r}=\left(F_{k}\right)_{r s}=(C)_{j s}=1$ ). This means that each column of $\widetilde{F}_{k}$ has at least one 1 . Now suppose that there is another $i^{\prime}$ with $1 \leq i^{\prime} \leq \widetilde{N}$ such that $\left(\widetilde{F}_{k}\right)_{i^{\prime} j}=$

[^2]

Fig. 2. Trajectories of the Boolean system $\Sigma_{\mathcal{R}}$ defined in Example 2. A solid (resp. dashed) line represents the transition resulting from $u(t)=\delta_{2}^{1}\left(\right.$ resp. $\left.u(t)=\delta_{2}^{2}\right)$.

1. Then we must have $(C)_{i^{\prime} r^{\prime}}=1,\left(F_{k}\right)_{r^{\prime} s^{\prime}}=1$, and $(C)_{j s^{\prime}}=1$ for some $1 \leq r^{\prime}, s^{\prime} \leq N$. These imply that $C \delta_{N}^{r^{\prime}}=\delta_{\widetilde{N}}^{i^{\prime}}, C \delta_{N}^{s^{\prime}}=\delta_{\widetilde{N}}^{j}$, and $\delta_{N}^{r^{\prime}}=F \ltimes \delta_{M}^{k} \ltimes \delta_{N}^{s^{\prime}}$. Since $(C)_{j s}=1$, we have $C \delta_{N}^{s}=\delta_{\widetilde{N}}^{j}$ and, thus, $\left(\delta_{N}^{s}, \delta_{N}^{s^{\prime}}\right) \in \mathcal{R}$. By (11), it follows that $\left(F \ltimes \delta_{M}^{k} \ltimes \delta_{N}^{s}, F \ltimes \delta_{M}^{k} \ltimes \delta_{N}^{s^{\prime}}\right) \in \mathcal{R}$, that is, $\left(\delta_{N}^{r}, \delta_{N}^{r^{\prime}}\right) \in \mathcal{R}$. Hence, $\delta_{\widetilde{N}}^{i}=C \delta_{N}^{r}=C \delta_{N}^{r^{\prime}}=\delta_{\widetilde{N}}^{i^{\prime}}$, which shows that $i=i^{\prime}$. Thus, there is a unique 1 in each column of $F_{k}$.
(b) We first note that the system $\Sigma_{\mathcal{R}}$ is well defined since, by (a), $\widetilde{F}$ is a logical matrix of size $\widetilde{N} \times \widetilde{N} M$. Let $1 \leq r, s \leq N$, let $1 \leq k \leq M$, and assume that the input $u=\delta_{M}^{k}$ steers $\Sigma$ from $\delta_{N}^{s}$ to $\delta_{N}^{r}$. We have $\left(F_{k}\right)_{r s}=1$. Suppose that $C \delta_{N}^{s}=\delta_{\widetilde{N}}^{j}$ and $C \delta_{N}^{r}=\delta_{\widetilde{N}}^{i}$. Then $(C)_{j s}=(C)_{i r}=1$ and, hence, $\left(\widetilde{F}_{k}\right)_{i j}=1$ by the definition of the Boolean product. This combined with (a) implies that $\delta_{\widetilde{N}}^{i}=\widetilde{F} \ltimes \delta_{M}^{k} \ltimes \delta_{\widetilde{N}}^{j}$; in other words, the input $u=\delta_{M}^{k}$ steers $\Sigma_{\mathcal{R}}$ from $\delta_{\widetilde{N}}^{j}$ to $\delta_{\widetilde{N}}^{i}$.

Conversely, let $1 \leq i, j \leq \widetilde{N}$ and suppose that the input $u=\delta_{M}^{k}$ takes $\Sigma_{\mathcal{R}}$ from $\delta_{\widetilde{N}}^{j}$ to $\delta_{\widetilde{N}}^{i}$. Then $\left(\widetilde{F}_{k}\right)_{i j}=1$, and hence there must be some $1 \leq r, s \leq N$ such that $(C)_{i r}=1,\left(F_{k}\right)_{r s}=1$, and $(C)_{j s}=1$. Thus, $C \delta_{N}^{s}=\delta_{\widetilde{N}}^{j}$, $C \delta_{N}^{r}=\delta_{\widetilde{N}}^{i}$, and $\Sigma$ can be driven from $\delta_{N}^{s}$ to $\delta_{N}^{r}$ with the input $u=\delta_{M}^{k}$.

Since, by the above theorem, $\Sigma_{\mathcal{R}}$ generates the transitions of $\mathcal{T}(\Sigma) / \mathcal{R}$ (cf. (10)), it can be interpreted as a quotient of the BCN $\Sigma$.

Example 2. Consider the BCN in Example 1. The matrix $F$ in the algebraic representation is given by (6). Let $C$ be as in (8) and let $\mathcal{R}$ be the equivalence relation defined in Example 1, induced by $C$. It is easy to check that $\mathcal{R}$ satisfies (11). Set $F_{1}=F \ltimes \delta_{2}^{1}$ and $F_{2}=F \ltimes \delta_{2}^{2}$. A calculation yields

$$
\begin{aligned}
& \widetilde{F}_{1}=C \odot F_{1} \odot C^{\top}=\left[\begin{array}{llll}
\delta_{4}^{2} & \delta_{4}^{1} & \delta_{4}^{4} & \delta_{4}^{4}
\end{array}\right], \\
& \widetilde{F}_{2}=C \odot F_{2} \odot C^{\top}=\left[\begin{array}{llll}
\delta_{4}^{1} & \delta_{4}^{1} & \delta_{4}^{4} & \delta_{4}^{4}
\end{array}\right] .
\end{aligned}
$$

Fig. 2 shows the trajectories of $\Sigma_{\mathcal{R}}$ with $\widetilde{N}=4, M=2$, and $\widetilde{F} \in \mathcal{L}^{4 \times 8}$ given by $\widetilde{F}=\left[\begin{array}{ll}\widetilde{F}_{1} & \widetilde{F}_{2}\end{array}\right]$. We see from the figure that $\Sigma_{\mathcal{R}}$ is indeed a quotient of the original BCN, which does not distinguish between states related by $\mathcal{R}$.

Using Theorem 1 one can obtain a quotient Boolean system, once an equivalence relation satisfying property (11) is found. In the next subsection, we will address the issue of computing equivalence relations which allow the construction of quotient Boolean systems.

### 3.2 Computing equivalence relations

Precisely, in this subsection we are concerned with the following problem: given an equivalence relation $\mathcal{S}$ on $\Delta_{N}$, determine the maximal (with respect to set inclusion) equivalence relation $\mathcal{R}$ on $\Delta_{N}$ such that $\mathcal{R} \subseteq \mathcal{S}$ and (11) holds. Here the relation $\mathcal{S}$ may be interpreted as a preliminary classification of the states of a BCN; see Section 4 below for specific instances. We are interested in finding the maximal equivalence relation since in many cases we want the size of the quotient system to be as small as possible.

First, we remark that such a maximal equivalence relation always exists and it is unique, as shown in the following proposition.

Proposition 1. Let $\mathcal{S}$ be an equivalence relation on $\Delta_{N}$. Then the set of all relations $\mathcal{R} \subseteq \Delta_{N} \times \Delta_{N}$ that are contained in $\mathcal{S}$ and satisfy property (11) has a unique maximal element (with respect to set inclusion), and the maximal element is an equivalence relation on $\Delta_{N}$.

Proof. Note that the identity relation $\mathcal{R}_{\mathrm{id}}=\{(a, a): a \in$ $\left.\Delta_{N}\right\}$ satisfies (11) and $\mathcal{R}_{\text {id }} \subseteq \mathcal{S}$ (since $\mathcal{S}$ is reflexive). Also note that if two relations $\mathcal{R}_{1} \subseteq \mathcal{S}$ and $\mathcal{R}_{2} \subseteq \mathcal{S}$ both satisfy property (11), then the same is true for their union $\mathcal{R}_{1} \cup \mathcal{R}_{2}$. The first statement follows immediately.

The maximal element $\widetilde{\mathcal{R}}$ is reflexive since it contains the identity relation $\mathcal{R}_{\text {id }}$. To show the symmetry and transitivity of $\widetilde{\mathcal{R}}$, consider the inverse relation $\widetilde{\mathcal{R}}^{-1}=\{(b, a):(a, b) \in \widetilde{\mathcal{R}}\}$ and the composition $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}=$ $\left\{(a, c)\right.$ : there exists $b \in \Delta_{N}$ such that $(a, b) \in \widetilde{\mathcal{R}}$ and $(b$, $c) \in \widetilde{\mathcal{R}}\}$. It is easy to see that both $\widetilde{\mathcal{R}}^{-1}$ and $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}$ satisfy (11), and are contained in $\mathcal{S}$ since $\widetilde{\mathcal{R}} \subseteq \mathcal{S}$ and $\mathcal{S}$ is symmetric and transitive. Hence, $\widetilde{\mathcal{R}}$ contains $\widetilde{\mathcal{R}}^{-1}$ and $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}$, implying that $\widetilde{\mathcal{R}}$ is symmetric and transitive. The second statement is proved.

The following theorem suggests a way of computing such an equivalence relation.

Theorem 2. Let $F \in \mathcal{L}^{N \times N M}$, and let $\mathcal{S}$ be an equivalence relation on $\Delta_{N}$. For each $u \in \Delta_{M}$ define a relation $\mathcal{S}_{u}$ on $\Delta_{N}$ by: $\left(a, a^{\prime}\right) \in \mathcal{S}_{u}$ if and only if $a^{\prime}=F \ltimes u \ltimes a$. Define a sequence of relations $\mathcal{R}_{k}$ by
$\mathcal{R}_{1}=\mathcal{S}$ and $\mathcal{R}_{k+1}=\left(\bigcap_{u \in \Delta_{M}}\left(\mathcal{S}_{u}^{-1} \circ \mathcal{R}_{k} \circ \mathcal{S}_{u}\right)\right) \cap \mathcal{R}_{k}$.

Then:
(a) The sequence of relations $\mathcal{R}_{1}, \mathcal{R}_{2}, \ldots, \mathcal{R}_{k}, \ldots$ satisfies $\mathcal{R}_{1} \supseteq \mathcal{R}_{2} \supseteq \cdots \supseteq \mathcal{R}_{k} \supseteq \cdots$.
(b) There is an integer $\bar{k}^{*}$ such that $\mathcal{R}_{k^{*}+1}=\mathcal{R}_{k^{*}}$.
(c) $\mathcal{R}_{k^{*}}$ is the maximal equivalence relation on $\Delta_{N}$ such that $\mathcal{R}_{k^{*}} \subseteq \mathcal{S}$ and property (11) holds.

Proof. Part (a) is quite trivial. Part (b) follows from (a) and the finiteness of each $\mathcal{R}_{k}$.

We turn to the proof of (c). By Proposition 1, it suffices to show that $\mathcal{R}_{k^{*}} \subseteq \Delta_{N} \times \Delta_{N}$ is the maximal relation satisfying $\mathcal{R}_{k^{*}} \subseteq \mathcal{S}$ and condition (11). The relation $\mathcal{R}_{k^{*}}$ is clearly a subset of $\mathcal{S}$. To show that (11) holds true, suppose that $(a, b) \in \mathcal{R}_{k^{*}}$ and $u \in \Delta_{M}$. Since $\mathcal{R}_{k^{*}}=\mathcal{R}_{k^{*}+1} \subseteq \mathcal{S}_{u}^{-1} \circ \mathcal{R}_{k^{*}} \circ \mathcal{S}_{u}$, there exist $a^{\prime}, b^{\prime} \in \Delta_{N}$ such that $\left(a, a^{\prime}\right) \in \mathcal{S}_{u},\left(b^{\prime}, b\right) \in \mathcal{S}_{u}^{-1}$, and $\left(a^{\prime}, b^{\prime}\right) \in \mathcal{R}_{k^{*}}$. It follows from the definition of $\mathcal{S}_{u}$ that $a^{\prime}=F \ltimes u \ltimes a$ and $b^{\prime}=F \ltimes u \ltimes b$. Hence, $(F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R}_{k^{*}}$.

To prove the maximality of $\mathcal{R}_{k^{*}}$, let $\mathcal{R} \subseteq \Delta_{N} \times \Delta_{N}$ be another relation which is contained in $\mathcal{S}$ and satisfies (11). We claim that $\mathcal{R} \subseteq \mathcal{R}_{k}$ for all $k$. The case $k=k^{*}$ completes the proof. We shall use induction on $k$. The case $k=1$ is trivial, so we take $k>1$ and assume that $\mathcal{R} \subseteq \mathcal{R}_{k-1}$. Let $(a, b) \in \mathcal{R}$. Then for any $u \in \Delta_{M}$, we have $(F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R} \subseteq \mathcal{R}_{k-1}$. By the definition of $\mathcal{S}_{u}$, it follows that $(a, F \ltimes u \ltimes a) \in \mathcal{S}_{u}$ and $(F \ltimes u \ltimes b, b) \in \mathcal{S}_{u}^{-1}$. Hence, $(a, b) \in \mathcal{S}_{u}^{-1} \circ \mathcal{R}_{k-1} \circ \mathcal{S}_{u}$, and consequently $(a, b) \in \mathcal{R}_{k}$ since $u$ was arbitrary. This shows that $\mathcal{R} \subseteq \mathcal{R}_{k}$, and our claim follows.

For applications, it is convenient to reformulate Theorem 2 in terms of $(0,1)$-matrices. Recall that a relation $\mathcal{R}$ on $\Delta_{N}$ can be represented by an $N \times N$ matrix, whose $(i, j)$-entry is 1 if $\left(\delta_{N}^{i}, \delta_{N}^{j}\right) \in \mathcal{R}$ and 0 otherwise. So if $A_{\mathcal{R}}$ is the matrix representing $\mathcal{R}$, then the inverse relation $\mathcal{R}^{-1}$ has $A_{\mathcal{R}}^{\top}$ as the matrix representation. Moreover, if $\mathcal{R}^{\prime}$ is another relation on $\Delta_{N}$ represented by $A_{\mathcal{R}^{\prime}}$, then the matrices representing $\mathcal{R} \cap \mathcal{R}^{\prime}$ and $\mathcal{R}^{\prime} \circ \mathcal{R}$ are $A_{\mathcal{R}} \wedge A_{\mathcal{R}^{\prime}}$ and $A_{\mathcal{R}} \odot A_{\mathcal{R}^{\prime}}$ (see, e.g., [44, Section 9.3]). Note that if $\mathcal{S}_{u}$ is the relation defined in Theorem 2 and if $u=\delta_{M}^{k}$, then

$$
\left(\delta_{N}^{i}, \delta_{N}^{j}\right) \in \mathcal{S}_{u} \Longleftrightarrow F_{k} \delta_{N}^{i}=\delta_{N}^{j} \Longleftrightarrow\left(F_{k}\right)_{j i}=1
$$

where $F_{k}=F \ltimes \delta_{M}^{k}$, and thus $F_{k}^{\top}$ is the matrix representing $\mathcal{S}_{u}$. From these facts and Theorem 2, the following corollary follows immediately.

Corollary 1. Suppose that $\mathcal{S}$ is an equivalence relation on $\Delta_{N}$ represented by a matrix $A_{\mathcal{S}}$, and suppose that $F \in \mathcal{L}^{N \times N M}$. For each $1 \leq i \leq M$, let $F_{i}$ be the matrix
$F_{i}=F \ltimes \delta_{M}^{i}$. Define a sequence of $(0,1)$-matrices by

$$
\begin{aligned}
A_{1}=A_{\mathcal{S}} \text { and } A_{k+1}=A_{k} \wedge( & \left.F_{1}^{\top} \odot A_{k} \odot F_{1}\right) \wedge \cdots \\
& \wedge\left(F_{M}^{\top} \odot A_{k} \odot F_{M}\right)
\end{aligned}
$$

Then there is an integer $k^{*}$ such that $A_{k^{*}+1}=A_{k^{*}}$, and $A_{k^{*}}$ is the matrix representing the maximal equivalence relation on $\Delta_{N}$ that is contained in $\mathcal{S}$ and satisfies property (11).

Example 3. Consider again the BCN in Example 1. If we let $\mathcal{S}$ be the equivalence relation induced by the partition $\left\{\left\{\delta_{8}^{1}\right\},\left\{\delta_{8}^{2}, \delta_{8}^{3}, \delta_{8}^{4}\right\},\left\{\delta_{8}^{5}, \delta_{8}^{6}, \delta_{8}^{7}, \delta_{8}^{8}\right\}\right\}$, then

$$
A_{1}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & J_{3} & 0 \\
0 & 0 & J_{4}
\end{array}\right]
$$

and a short computation yields

$$
A_{2}=A_{3}=\left[\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & J_{2} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & J_{4}
\end{array}\right]
$$

which is exactly the matrix representing the relation given in Example 1. So the relation $\mathcal{R}$ presented in Example 1 is the maximal equivalence relation contained in $\mathcal{S}$ and satisfying property (11).

## 4 Control design via quotients

This section discusses the application of quotient systems for control design. We consider two typical control problems in BCNs and show how these problems can be solved through the use of a quotient Boolean system.

### 4.1 Stabilization

Consider a BCN $\Sigma$ as given in (3). Let $\mathcal{M} \subseteq \Delta_{N}$ be a target set of states. We say that $\Sigma$ is stabilizable to $\mathcal{M}$ if for every $x(0) \in \Delta_{N}$ there exists a control sequence $\{u(0), u(1), u(2), \ldots\}$, with $u(i) \in \Delta_{M}$, and a positive integer $\tau$ such that $x(t) \in \mathcal{M}$ for all $t \geq \tau$ (see, e.g., [8]). The following result shows that, by defining the equivalence relation appropriately, we can easily obtain a stabilizing controller for $\Sigma$ on the basis of a stabilizer for its quotient system.

Proposition 2. Consider a BCN $\Sigma$ as given in (3). Let $\mathcal{M} \subseteq \Delta_{N}$ and let $\mathcal{S}$ be the equivalence relation on $\Delta_{N}$ determined by the partition $\left\{\mathcal{M}, \Delta_{N}-\mathcal{M}\right\}$. Suppose that $\mathcal{R}$ is an equivalence relation on $\Delta_{N}$ induced by a matrix
$C \in \mathcal{L}^{\widetilde{N} \times N}$ of full row rank, $\mathcal{R} \subseteq \mathcal{S}$, and condition (11) holds. Suppose $\Sigma_{\mathcal{R}}$ is defined as in Theorem 1. If $\Sigma_{\mathcal{R}}$ can be stabilized to the set $\mathcal{M}_{\mathcal{R}}=\{C x: x \in \mathcal{M}\}$ via a feedback law $\left(x_{\mathcal{R}}, t\right) \mapsto u\left(x_{\mathcal{R}}, t\right)$, then $\Sigma$ can be stabilized to $\mathcal{M}$ using the feedback law $(x, t) \mapsto u(C x, t)$.

Proof. We first show that if the initial states of $\Sigma$ and $\Sigma_{\mathcal{R}}$ satisfy $C x(0)=x_{\mathcal{R}}(0)$, then the feedback laws $(x, t) \mapsto$ $u(C x, t)$ and $\left(x_{\mathcal{R}}, t\right) \mapsto u\left(x_{\mathcal{R}}, t\right)$ generate the same input sequence, and the trajectories satisfy $C x(t)=x_{\mathcal{R}}(t)$ for all $t=0,1,2, \ldots$ It is clearly true that the two feedback laws generate the same input, say $u_{0}$, at $t=0$. By the second part of Theorem 1(b), there is a one-step transition of $\Sigma$ from some $a \in \Delta_{N}$ to some $a^{\prime} \in \Delta_{N}$ with $C a=x_{\mathcal{R}}(0)$ and $C a^{\prime}=x_{\mathcal{R}}(1)$, under this input $u_{0}$. Since $C x(0)=C a$, it follows from (4) that $(x(0), a) \in \mathcal{R}$, and then by (11) we have $\left(x(1), a^{\prime}\right) \in \mathcal{R}$. Thus $C x(1)=C a^{\prime}=x_{\mathcal{R}}(1)$ again by (4). The fact we want now follows by induction.

Now we can prove the proposition. Assume that the feedback laws $(x, t) \mapsto u(C x, t)$ and $\left(x_{\mathcal{R}}, t\right) \mapsto u\left(x_{\mathcal{R}}, t\right)$ are applied to $\Sigma$ and $\Sigma_{\mathcal{R}}$, respectively. Let $p \in \Delta_{N}$ and let $q=C p$. Then there is a $\tau$ such that the trajectory of $\Sigma_{\mathcal{R}}$ with $x_{\mathcal{R}}(0)=q$ satisfies $x_{\mathcal{R}}(t) \in \mathcal{M}_{\mathcal{R}}$ for all $t \geq \tau$. Since the trajectory of $\Sigma$ with $x(0)=p$ always satisfies $C x(t)=x_{\mathcal{R}}(t)$, to each $t \geq \tau$ there corresponds some $b \in \mathcal{M}$ such that $C x(t)=C b$, and hence $(x(t), b) \in \mathcal{R} \subseteq \mathcal{S}$. This forces $x(t) \in \mathcal{M}$ whenever $t \geq \tau$, since $\mathcal{S}$ is the equivalence relation yielded by the partition $\left\{\mathcal{M}, \Delta_{N}-\mathcal{M}\right\}$. Since $p$ was arbitrary, we conclude that stabilization of $\Sigma$ to $\mathcal{M}$ is achieved, via the feedback law $(x, t) \mapsto u(C x, t)$.

Remark 3. Note that in Proposition 2 we do not assume $\mathcal{R}$ to be maximal, although that will be the case in most applications of the proposition. A similar remark applies to Proposition 3 below.

### 4.2 Optimal control

As another example of application we consider the following finite-horizon optimal control problem, introduced in [25].

Problem 1. Consider a BCN $\Sigma$ as in (3). Given an initial state $x_{0}$ and a finite time horizon $T \in \mathbb{Z}^{+}$, find a control sequence that minimizes the cost function

$$
\begin{equation*}
J=\sum_{t=0}^{T-1} l(u(t), x(t))+g(x(T)) \tag{12}
\end{equation*}
$$

where $l(u, x)$ and $g(x)$ are functions defined on $\Delta_{M} \times \Delta_{N}$ and $\Delta_{N}$, respectively.

We show that the solution to Problem 1 for $\Sigma$ can be easily derived on the basis of a solution to Problem 1 for
a suitably chosen quotient system. Let $\mathcal{S}$ be the equivalence relation on $\Delta_{N}$ given by

$$
\begin{align*}
\left(x, x^{\prime}\right) \in \mathcal{S} \Longleftrightarrow & g(x)=g\left(x^{\prime}\right) \text { and } \\
& l(u, x)=l\left(u, x^{\prime}\right) \text { for all } u \in \Delta_{M} \tag{13}
\end{align*}
$$

We observe that, for a matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$ with full row rank, if the equivalence relation $\mathcal{R}$ induced by $C$ satisfies $\mathcal{R} \subseteq \mathcal{S}$, then the following two maps are well defined:

$$
\begin{array}{r}
l_{\mathcal{R}}: \Delta_{M} \times \Delta_{\widetilde{N}} \rightarrow \mathbb{R}, \quad(u, a) \mapsto l(u, x) \\
\quad \text { whenever } a=C x \\
g_{\mathcal{R}}: \Delta_{\widetilde{N}} \rightarrow \mathbb{R}, \quad a \mapsto g(x) \text { whenever } a=C x \tag{15}
\end{array}
$$

Based on this observation, we can state the following proposition.

Proposition 3. Let $\Sigma$ be a BCN described by (3). Suppose that $\mathcal{S}$ is the equivalence relation on $\Delta_{N}$ given by (13), $\mathcal{R}$ is an equivalence relation on $\Delta_{N}$ induced by a full row rank logical matrix $C \in \mathcal{L}^{\widetilde{N} \times N}, \mathcal{R} \subseteq \mathcal{S}$, and (11) holds. Consider Problem 1 with given $x_{0}, \bar{T}$, and J. Let $\Sigma_{\mathcal{R}}$ be the Boolean system constructed in Theorem 1, and define $J_{\mathcal{R}}=\sum_{t=0}^{T-1} l_{\mathcal{R}}\left(u(t), x_{\mathcal{R}}(t)\right)+g_{\mathcal{R}}\left(x_{\mathcal{R}}(T)\right)$, where $l_{\mathcal{R}}$ and $g_{\mathcal{R}}$ are given by (14) and (15).
(a) If $U^{*}=\left\{u^{*}(0), \ldots, u^{*}(T-1)\right\}$ is an optimal control sequence solving Problem 1 with $\Sigma, x_{0}$, and $J$ replaced by $\Sigma_{\mathcal{R}}, x_{\mathcal{R}}^{0}=C x_{0}$, and $J_{\mathcal{R}}$, respectively, then $U^{*}$ is also an optimal control for $\Sigma$. Moreover, let $J^{*}$ be the optimal cost $\min _{u(\cdot)} J$ under the initial condition $x(0)=x_{0}$ and let $J_{\mathcal{R}}^{*}$ be the optimal cost $\min _{u(\cdot)} J_{\mathcal{R}}$ under the condition $x_{\mathcal{R}}(0)=C x_{0}$. Then $J^{*}=J_{\mathcal{R}}^{*}$.
(b) If $\left(x_{\mathcal{R}}, t\right) \mapsto u^{*}\left(x_{\mathcal{R}}, t\right)$ is an optimal control polic ${ }^{2}$ solving Problem 1 with $\Sigma$ and J replaced by $\Sigma_{\mathcal{R}}$ and $J_{\mathcal{R}}$, respectively, then the control policy given by $(x, t) \mapsto u^{*}(C x, t)$ is an optimal control policy for $\Sigma$.

Proof. (a) An argument similar to the first paragraph of the proof of Proposition 2 shows that, if the initial states of $\Sigma$ and $\Sigma_{\mathcal{R}}$ satisfy $C x(0)=x_{\mathcal{R}}(0)$, then for any control sequence $u(0), \ldots, u(T-1)$, the corresponding trajectories satisfy $C x(t)=x_{\mathcal{R}}(t)$ for $t=0, \ldots, T$, and hence $g(x(T))=g_{\mathcal{R}}\left(x_{\mathcal{R}}(T)\right)$ and $l(u(t), x(t))=$ $l_{\mathcal{R}}\left(u(t), x_{\mathcal{R}}(t)\right)$ for each $0 \leq t \leq T-1$, so that the cost functions $J$ and $J_{\mathcal{R}}$ return the same value. This implies that if $U^{*}$ minimizes $J_{\mathcal{R}}$ with the initial condition $x_{\mathcal{R}}(0)=C x_{0}$, then it also minimizes $J$ subject to $x(0)=x_{0}$, and moreover, the associated optimal costs $J^{*}$ and $J_{\mathcal{R}}^{*}$ are equal.

[^3]Part (b) follows directly from (a) and the fact (explained in the first paragraph of the proof of Proposition 2) that the feedback laws $(x, t) \mapsto u^{*}(C x, t)$ and $\left(x_{\mathcal{R}}, t\right) \mapsto$ $u^{*}\left(x_{\mathcal{R}}, t\right)$ generate the same control sequence whenever $C x(0)=x_{\mathcal{R}}(0)$.

Example 4. To give an intuitive example of the equivalence relation defined by (13), suppose that $M=2$, $N=4$, and the functions $l: \Delta_{2} \times \Delta_{4} \rightarrow \mathbb{R}$ and $g: \Delta_{4} \rightarrow$ $\mathbb{R}$ are given by

$$
\begin{aligned}
& l\left(\delta_{2}^{1}, \delta_{4}^{1}\right)=1, \quad l\left(\delta_{2}^{1}, \delta_{4}^{2}\right)=l\left(\delta_{2}^{1}, \delta_{4}^{3}\right)=l\left(\delta_{2}^{1}, \delta_{4}^{4}\right)=2 \\
& l\left(\delta_{2}^{2}, x\right)=3 \quad\left(x \in \Delta_{4}\right) \\
& g\left(\delta_{4}^{1}\right)=g\left(\delta_{4}^{2}\right)=g\left(\delta_{4}^{3}\right)=1, \quad g\left(\delta_{4}^{4}\right)=2
\end{aligned}
$$

First, by definition the relation $\mathcal{S}$ contains all pairs of the form $(a, a)$, namely, $\left(\delta_{4}^{1}, \delta_{4}^{1}\right),\left(\delta_{4}^{2}, \delta_{4}^{2}\right),\left(\delta_{4}^{3}, \delta_{4}^{3}\right)$, and $\left(\delta_{4}^{4}, \delta_{4}^{4}\right)$. Second, note that $g\left(\delta_{4}^{2}\right)=g\left(\delta_{4}^{3}\right)=1, l\left(\delta_{2}^{1}, \delta_{4}^{2}\right)=$ $l\left(\delta_{2}^{1}, \delta_{4}^{3}\right)=2$, and $l\left(\delta_{2}^{2}, \delta_{4}^{2}\right)=l\left(\delta_{2}^{2}, \delta_{4}^{3}\right)=3$. Thus, both pairs $\left(\delta_{4}^{2}, \delta_{4}^{3}\right)$ and $\left(\delta_{4}^{3}, \delta_{4}^{2}\right)$ belong to $\mathcal{S}$. Moreover, it is easily checked that they are the only pairs of distinct states that satisfy $g(x)=g\left(x^{\prime}\right), l\left(\delta_{2}^{1}, x\right)=l\left(\delta_{2}^{1}, x^{\prime}\right)$, and $l\left(\delta_{2}^{2}, x\right)=l\left(\delta_{2}^{2}, x^{\prime}\right)$ simultaneously. Hence no pair other than those listed belongs to $\mathcal{S}$.

Remark 4. It is noted in [25] that the cost function described in (12) can be equivalently expressed in a linear form as $J=\sum_{t=0}^{T-1} \theta \ltimes u(t) \ltimes x(t)+\mu x(T)$, where $\mu$ is a row vector of $\bar{N}$ components and $\theta=\left[\theta_{1}, \theta_{2}, \ldots, \theta_{M}\right]$ with each $\theta_{i}$ being an $N$-component row vector. We remark that the index $J_{\mathcal{R}}$ appearing in Proposition 3 is easily obtained from this expression. In fact, since the function $g_{\mathcal{R}}$ is defined on $\Delta_{\tilde{N}}$, it can be expressed in the form $g_{\mathcal{R}}(x)=\mu_{\mathcal{R}} x$ for some $\tilde{N}$-component row vector $\mu_{\mathcal{R}}$. Let $C$ be as in Proposition 3. Then by (15) we have $\mu_{\mathcal{R}} C=\mu$ and so $\mu_{\mathcal{R}}=\mu C^{+}$, where $C^{+}=C^{\top}\left(C C^{\top}\right)^{-1}$ is the pseudoinverse of $C$. In a similar manner, the function $l_{\mathcal{R}}$ defined by (14) can be equivalently expressed as $l_{\mathcal{R}}(u, x)=\theta_{\mathcal{R}} \ltimes u \ltimes x$, where $\theta_{\mathcal{R}}=\left[\theta_{1}^{\prime}, \theta_{2}^{\prime}, \ldots, \theta_{M}^{\prime}\right]$ with $\theta_{i}^{\prime}=\theta_{i} C^{+}$for each $i$. Thus the index cost $J_{\mathcal{R}}$ can be rewritten in a linear form as follows: $J_{\mathcal{R}}=\sum_{t=0}^{T-1} \theta_{\mathcal{R}} \ltimes$ $u(t) \ltimes x_{\mathcal{R}}(t)+\mu_{\mathcal{R}} x_{\mathcal{R}}(T)$.

One can obtain analogs of Proposition 3 for other kinds of optimal control problems (such as the infinite-horizon optimal or average-cost optimal problems [25]). The essence of the arguments is the same as that of Proposition 3 , and so we omit them.

### 4.3 Comparative simulations

The proposed methods have been tested on several randomly generated 16 -node networks. Recall that a BCN expressed by (1) consists of two types of nodes, namely, internal nodes $\left(x_{1}, \ldots, x_{n}\right)$ and external control nodes $\left(u_{1}, \ldots, u_{m}\right)$. We considered the cases of $m=1,2,3$, and

Table 2
Comparison between controller design done with the quotient-based method and done the conventional way.

|  | Size |  | CPU time (sec) |  |
| :--- | :--- | :--- | :--- | :--- |
|  | Orig. BCN | Quotient | Orig. BCN | Quotient |
| $m=1$ <br> $k=1$ | 32768 | 6815 | 9256.08 | 454.47 |
| $m=1$ <br> $k=100$ | 32768 | 5807 | 9845.81 | 475.89 |
| $m=2$ <br> $k=1$ | 16384 | 4014 | 2327.27 | 303.04 |
| $m=2$ <br> $k=100$ | 16384 | 4647 | 2401.16 | 158.76 |
| $m=3$ <br> $k=1$ | 8192 | 2793 | 590.42 | 53.79 |
| $m=3$ <br> $k=100$ | 8192 | 3071 | 601.22 | 65.40 |
| $m=5$ <br> $k=1$ | 2048 | 887 | 37.78 | 5.50 |
| $m=5$ <br> $k=100$ | 2048 | 1015 | 32.20 | 6.28 |

5 . When $m=1$, there are 15 internal nodes and 1 control nodes; the original network size is $2^{15}=32768$. When $m=2$, there are 14 internal nodes and 2 control nodes; the original network size is $2^{14}=16384$. When $m=3$, there are 13 internal nodes, so the original network size is $2^{13}=8192$, and when $m=5$ the original network size is $2^{11}=2048$. First, we evaluate the efficiency of the quotient-based method given in Proposition 2. The target sets $\mathcal{M}$ of the stabilization problem were randomly selected, with cardinality $k=1$ and $k=100$. Table 2 shows the numerical results obtained for different combinations of $m$ and $k$. The second and third columns give the number of states of the original networks and the number of states of the quotient systems, reflecting the degree of reduction. The fourth column records the CPU time spent for constructing stabilizing controllers directly based on the original networks. Specifically, we followed the design procedure proposed by Fornasini and Valcher [45] and Li et al. [46] when $k=1$, and the procedure of Guo et al. 47] when $k=100$. The CPU time required for determining stabilizers via Proposition 2 is shown in the last column. Similarly, Table 3 compares the network size and the CPU time to obtain a solution to Problem 1, with $T=40$. For the sake of simplicity, we assumed that the function $l(u, x)$ depends only on $u$, with the value 1 if $u_{1}=1$ and 0 if $u_{1}=0$; the function $g(x)$ was assumed to take the value 5 if $x_{1}=0$ and the value 0 otherwise. (Here we use binary representations of $x$ and $u$.) The corresponding optimal control problem was solved both by applying the algorithm of Fornasini

Table 3
Comparison between direct and quotient-based methods for solving Problem 1.

|  | Size |  | CPU time (sec) |  |
| :--- | :--- | :--- | :--- | :--- |
|  | Orig. BCN | Quotient | Orig. BCN | Quotient |
| $m=1$ | 32768 | 6087 | 490.67 | 115.87 |
| $m=2$ | 16384 | 4506 | 256.51 | 74.94 |
| $m=3$ | 8192 | 3441 | 130.33 | 55.50 |
| $m=5$ | 2048 | 829 | 27.96 | 10.81 |

and Valcher [25] directly to the original network, and by using the indirect method given in Proposition 3. It is seen that the proposed methods offer a reduction in computation time compared to the state of the art, and the extent of reduction increases (as a trend) with increasing size of the original network. All computations were run on an Intel Core i7-3.00 GHz personal computer with 8 GB of RAM.

## 5 A biological example

We apply our methods to a Boolean model for lactose metabolism in the bacterium E. coli [48]. The model consists of 13 variables ( $1 \mathrm{mRNA}, 5$ proteins, and 7 sugars) denoted by $M, P, B, C, R, R_{m}, A, A_{m}, L$, $L_{m}, L_{e}, L_{e m}$ and $G_{e}$. Here, $R$ and $R_{m}$ are combined to indicate concentration levels of a specific substance (the repressor protein); that is, the concentration is low when $\left(R, R_{m}\right)=(0,0)$, medium when $\left(R, R_{m}\right)=(0,1)$, and high when $\left(R, R_{m}\right)=(1,1)$. The fourth possibility, $\left(R, R_{m}\right)=(1,0)$, is meaningless and not allowed. The same situation is for the pairs $\left(A, A_{m}\right),\left(L, L_{m}\right)$, and $\left(L_{e}, L_{e m}\right)$ (see [48] for more details on this aspect). The equations describing the model are as follows:

$$
\begin{align*}
& M(t+1)=C(t) \wedge \neg R(t) \wedge \neg R_{m}(t) \\
& P(t+1)=M(t), \quad B(t+1)=M(t) \\
& C(t+1)=\neg G_{e}(t), \\
& R(t+1)=\neg A(t) \wedge \neg A_{m}(t), \\
& R_{m}(t+1)=\left(\neg A(t) \wedge \neg A_{m}(t)\right) \vee R(t),  \tag{16}\\
& A(t+1)=B(t) \wedge L(t) \\
& A_{m}(t+1)=L(t) \vee L_{m}(t) \\
& L(t+1)=P(t) \wedge L_{e}(t) \wedge \neg G_{e}(t), \\
& L_{m}(t+1)=\left(\left(L_{e m}(t) \wedge P(t)\right) \vee L_{e}(t)\right) \wedge \neg G_{e}(t)
\end{align*}
$$

We assume that the concentration of extracellular lactose is low ( $L_{e}=L_{e m}=0$ ), and treat the extracellular glucose levels $\left(G_{e}\right)$ as input to the model. Then the model can be rewritten as in (3) with $3=432$ and $M=2$.

[^4]The matrix $F \in \mathcal{L}^{432 \times 864}$ is detailed in the Appendix.
(1) Stabilization. When extracellular lactose levels get low, the model is known to exhibit two steady states [48], expressed in the canonical vector form as $\delta_{432}^{387}$ and $\delta_{432}^{414}$. Let $\mathcal{M}=\left\{\delta_{432}^{387}\right\}$ and let $\mathcal{S}$ be the equivalence relation produced by the partition $\left\{\mathcal{M}, \Delta_{432}-\mathcal{M}\right\}$. Then by following the procedure described in Section 3, we get a quotient system $\Sigma_{\mathcal{R}}: x_{\mathcal{R}}(t+1)=\widetilde{F} \ltimes u(t) \ltimes x_{\mathcal{R}}(t)$, with $x_{\mathcal{R}} \in \Delta_{8}, u \in \Delta_{2}$, and $\widetilde{F} \in \mathcal{L}^{8 \times 16}$ given by
$\widetilde{F}=\left[\begin{array}{llllllllllllllll}\delta_{8}^{2} & \delta_{8}^{2} & \delta_{8}^{7} & \delta_{8}^{2} & \delta_{8}^{4} & \delta_{8}^{7} & \delta_{8}^{2} & \delta_{8}^{4} & \delta_{8}^{1} & \delta_{8}^{1} & \delta_{8}^{6} & \delta_{8}^{6} & \delta_{8}^{3} & \delta_{8}^{7} & \delta_{8}^{2} & \delta_{8}^{4}\end{array}\right]$.

The matrix $C$ obtained during the procedure (which is of size $8 \times 432$ and not given explicitly) satisfies $C \delta_{432}^{387}=\delta_{8}^{1}$. It is not hard to see that for any

$$
K=\left[\begin{array}{llllllll}
\delta_{2}^{2} & \delta_{2}^{2} & * & * & * & * & *
\end{array}\right]
$$

(* denoting columns that can be either $\delta_{2}^{1}$ or $\delta_{2}^{2}$ ), the feedback law given by $x_{\mathcal{R}} \mapsto u\left(x_{\mathcal{R}}\right)=K x_{\mathcal{R}}$ stabilizes the quotient system to $\delta_{8}^{1}$. Proposition 2 then ensures that the original model can be globally stabilized to the state $\delta_{432}^{387}$ via the feedback law $x \mapsto u(C x)=K C x$. A similar argument can be made for finding a feedback controller that stabilizes the model to the state $\delta_{432}^{414}$; the details are not repeated here.

Remark 5. It required about 6.5 s to find the above controller directly based on the procedure described in [45] and [46]. In contrast, it took only 1.16 s to obtain the same stabilizer by using the quotient-based method. Thus in this case there is an increase in speed by a factor of about 5 to 6 when the proposed method is employed.
(2) Optimal control. Assume that $T=3$, the initial condition $x(0)=\delta_{432}^{10}$, and the functions $l(u, x)$ and $g(x)$ are given by

$$
\begin{aligned}
l\left(\delta_{2}^{1}, x\right)=1, \quad l\left(\delta_{2}^{2}, x\right)=2 \quad\left(x \in \Delta_{432}\right) & \\
g\left(\delta_{432}^{1}\right)=\cdots=g\left(\delta_{432}^{54}\right)=0, \quad g\left(\delta_{432}^{55}\right)= & = \\
& =g\left(\delta_{432}^{432}\right)=5
\end{aligned}
$$

Here we remark that the states $\delta_{432}^{1}, \ldots, \delta_{432}^{54}$ correspond to the lac operon, which is responsible for the metabolism of lactose, being ON (induced); cf. 48]. The above choice of $g(x)$ then indicates that the operon is desired to be in an ON state after intervention. By proceeding as in Section 4.2, one can obtain a quotient
the total number of states of (16) is equal to $2^{4} \cdot 3^{3}=432$; thus $N=432$ in the algebraic representation.
system $\Sigma_{\mathcal{R}}$ with $N=12, M=2$, and the matrix

$$
\begin{aligned}
& \widetilde{F}=\left[\begin{array}{llllll}
\delta_{12}^{7} & \delta_{12}^{7} & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^{7}
\end{array} \delta_{12}^{7} \delta_{12}^{12} \delta_{12}^{12} \delta_{12}^{12} \delta_{12}^{7} \delta_{12}^{7}\right. \\
&\left.\delta_{12}^{3} \delta_{12}^{4} \delta_{12}^{12} \delta_{12}^{8} \delta_{12}^{9} \delta_{12}^{1} \delta_{12}^{7} \delta_{12}^{12} \delta_{12}^{8} \delta_{12}^{9} \delta_{12}^{1} \delta_{12}^{7}\right] .
\end{aligned}
$$

The matrix $C$ satisfies $C x(0)=\delta_{12}^{11}$, and the induced functions $l_{\mathcal{R}}$ and $g_{\mathcal{R}}$ are defined by

$$
\begin{aligned}
l_{\mathcal{R}}\left(\delta_{2}^{1}, x_{\mathcal{R}}\right)=1, \quad l_{\mathcal{R}}\left(\delta_{2}^{2}, x_{\mathcal{R}}\right)=2 \quad\left(x_{\mathcal{R}}\right. & \left.\in \Delta_{12}\right) \\
g_{\mathcal{R}}\left(\delta_{12}^{1}\right)=\cdots=g_{\mathcal{R}}\left(\delta_{12}^{7}\right)=5, \quad g_{\mathcal{R}}\left(\delta_{12}^{8}\right) & =\cdots \\
& =g_{\mathcal{R}}\left(\delta_{12}^{12}\right)=0
\end{aligned}
$$

It is straightforward to see that the input sequence

$$
u^{*}(0)=u^{*}(1)=\delta_{2}^{2}, \quad u^{*}(2)=\delta_{2}^{1}
$$

is optimal for $\Sigma_{\mathcal{R}}$, with the optimal cost $J_{\mathcal{R}}^{*}=5$, so it also solves the optimal control problem for the original model, and the optimal cost is $J^{*}=J_{\mathcal{R}}^{*}=5$. Moreover, we see from the value of $J^{*}$ that the optimal input indeed steers the model to an ON state, as desired.

Remark 6. As for the time comparison, we report that it took about 1.5 s to solve this problem directly by the method of Fornasini and Valcher [25], while the above indirect procedure took only 0.79 s . Thus there is about 2 times saving in speed when the quotient-based method is employed.

## 6 Discussions

The paper has considered quotients of BCNs. Two possible applications of the quotient description have been presented in Section 4, where we have seen that the stabilization and optimal control problems of the original BCNs can be boiled down to those of the quotient systems. Let us mention that we have presented only a few examples of such applications, and there are quite a few other problems such as output tracking and observability checking that can also be dealt with in this manner. We do not include the details of these applications for reasons of space.

Since the number of states of the quotient $\Sigma_{\mathcal{R}}$ is precisely the number of the equivalence classes generated by $\mathcal{R}$, the coarser the relation $\mathcal{R}$, the smaller is $\Sigma_{\mathcal{R}}$ and, thus, the greater is the degree of reduction. Recall that the relation $\mathcal{R}$ is required to satisfy (11), which is related to the dynamics of $\Sigma$. Thus, the degree of reduction is affected by the specific dynamics of the original network. Also, since in practice different relations are required for different applications (cf. Sections 4.1 and 4.2), despite the same original network, the reduction degree may still be different, depending on the specific problems to be solved. The size of the quotient systems appearing in the
numerical experiments reported in Section 4.3 is about $50-20 \%$ when compared to the original networks. In the biological example presented in Section 5, the size of the reduced state space is less than $3 \%$ of that of the original one.

In Section 4.3, we have limited the discussion to networks with 16 nodes, since we would like to compute the control policy on each originally generated network and list the exact time that the standard methods require, in order to make the comparisons. Here we report that besides these simulations, we also tested our methods on networks with about 20-23 nodes. We observed that for most instances, the standard methods ran out of memory whereas the proposed methods were able to obtain a solution in a matter of minutes to hours. We do not present the detailed numerical results due to the limitations on the paper length.

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## Appendix

The matrix $F$ for the biological model discussed in Section 5 is

$$
F=\delta_{432}\left[\begin{array}{lllllllllllllllllllll}
255 & 258 & 261 & 255 & 258 & 261 & 246 & 249 & 252 & 264 & 267 & 270 & 264 & 267 & 270 & 246 & 249 & 252 & 48 & 51 & 54
\end{array}\right.
$$ $48 \quad 51 \quad 54 \quad 30 \quad 33$ $\begin{array}{lllllllllllllllllllllllll}246 & 249 & 252 & 264 & 267 & 270 & 264 & 267 & 270 & 246 & 249 & 252 & 258 & 258 & 261 & 258 & 258 & 261 & 249 & 249 & 252\end{array}$ $\begin{array}{lllllllllllllllllll}267 & 267 & 270 & 267 & 267 & 270 & 249 & 249 & 252 & 51 & 51 & 54 & 51 & 51 & 54 & 33 & 33 & 36 & 258 \\ 258 & 261\end{array}$ $\begin{array}{lllllllllllllllllllllllllllll}258 & 258 & 261 & 249 & 249 & 252 & 267 & 267 & 270 & 267 & 267 & 270 & 249 & 249 & 252 & 267 & 267 & 270 & 267\end{array}$ 249249252255258261255258261246249252264267270264267270246249252 $48 \quad 5154 \quad 48 \quad 51 \quad 54$

 $\begin{array}{lllllllllllllllllll}249 & 249 & 252 & 267 & 267 & 270 & 267 & 267 & 270 & 249 & 249 & 252 & 51 & 51 & 54 & 51 & 51 & 54 & 33 \\ 33 & 36\end{array}$
 267267270249249252417420423417420423408411414426429432426429432 408411414210213216210213216192195198417420423417420423408411414
 420420423411411414429429432429429432411411414213121321621321316
 429429432429429432411411414417420423417420423408411414426429432 426429432408411414210213216210213216192195198417420423417420423 408411414426429432426429432408411414426429432426429432408411414 420420423420420423411411414429429432429429432411411414213213216 213213216195195198420420423420420423411411414429429432429429432 411411414429429432429429432411411414228231234228231234219422225 $\begin{array}{lllllllllllllllllll}237 & 240 & 243 & 237 & 240 & 243 & 219 & 222 & 225 & 21 & 24 & 27 & 21 & 24 & 27 & 3 & 6 & 9 & 228 \\ 231 & 234\end{array}$ 228231234219222225237240243237240243219222225237240243237240243 219222225231231234231231234222222225240240243240240243222222225

 $\left.\begin{array}{lllllllllllllllllll}219 & 222 & 225 & 237 & 240 & 243 & 237 & 240 & 243 & 219 & 222 & 225 & 21 & 24 & 27 & 21 & 24 & 27 & 3\end{array}\right) 6$

 $\begin{array}{llllllllllllllllllll}222 & 222 & 225 & 24 & 24 & 27 & 24 & 24 & 27 & 6 & 6 & 9 & 231 & 231 & 234 & 231 & 231 & 234 & 222 & 222 \\ 225\end{array}$ 240240243240240243222222225240240243240240243222222225390393396 390393396381384387399402405399402405381384387183186189183186189 165168171390393396390393396381384387399402405399402405381384387 399402405399402405381384387393393396393393396384384387402402405 402402405384384387186186189186186189168168171393393396393393396 384384387402402405402402405384384387402402405402402405384384387


 4024024054024024053843843871861861891861861891681681713933931396 393393396384384387402402405402402405384384387402402405402402405 384384 387].

The above notation means that the first column of $F$ is $\delta_{432}^{255}$, the second column is $\delta_{432}^{258}$, and so on.


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[^1]:    ${ }^{1}$ Note that this is equivalent to only requiring $\mathcal{T}(\Sigma) / \mathcal{R}$ to be deterministic (i.e., there do not exist transitions of the

[^2]:    $\overline{\text { form } z \xrightarrow{u}} \mathcal{R} z^{\prime}$ and $z \xrightarrow{u}_{\mathcal{R}} z^{\prime \prime}$ with $z^{\prime} \neq z^{\prime \prime}$ ), since for any $z \in \Delta_{\tilde{N}}$ and $u \in \Delta_{M}$ there always exists at least one $z^{\prime} \in \Delta_{\tilde{N}}$ such that $z{ }_{\rightarrow}^{u} z^{\prime}$.

[^3]:    ${ }^{2}$ It was shown in [25] that the optimal control input can always be implemented as a time-varying feedback from the states.

[^4]:    ${ }^{3}$ Here, $N$ is not a power of 2 , since for some Boolean pairs in the model only three of the four values are admissible. More precisely, since each of the variables $M, P, B$, and $C$ has two possible values, whereas each of the pairs $\left(R, R_{m}\right)$, $\left(A, A_{m}\right)$, and $\left(L, L_{m}\right)$ takes on only three possible values,

