On quotients of Boolean control networks

Rui Li^a, Qi Zhang^b, Tianguang Chu^c

^aSchool of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China

^bSchool of Information Technology & Management, University of International Business & Economics, Beijing 100029, China

^cState Key Laboratory for Turbulence and Complex Systems, College of Engineering, Peking University, Beijing 100871, China

Abstract

In this paper, we focus on the study of quotients of Boolean control networks (BCNs) with the motivation that they might serve as smaller models that still carry enough information about the original network. Given a BCN and an equivalence relation on the state set, we consider a labeled transition system that is generated by the BCN. The resulting quotient transition system then naturally captures the quotient dynamics of the BCN concerned. We therefore develop a method for constructing a Boolean system that behaves equivalently to the resulting quotient transition system. The use of the obtained quotient system for control design is discussed and we show that for BCNs, controller synthesis can be done by first designing a controller for a quotient and subsequently lifting it to the original model. We finally demonstrate the applicability of the proposed techniques on a biological example.

Key words: Boolean control networks, quotient transition systems, control design, stabilization, optimal control.

1 Introduction

Boolean networks (BNs) and Boolean control networks (BCNs), wherein each component is characterized with a binary variable, have been widely employed in modeling biological regulatory networks. After assembling the components of a system as well as their regulatory interactions, BN/BCN models can nicely describe the qualitative temporal behavior of the system [1]. They can also illuminate how perturbations may disrupt normal behavior and yield testable predictions which are particularly valuable in less well understood biological systems [2]. As a nice framework for modeling dynamical processes on networks, especially in biological context, BN/BCN models have led to fruitful insights for unicellular organisms [3], plants [4], animals [5], and humans [6], especially human signaling networks implicated in diseases [7]. A BN/BCN is typically placed in the form of a nonlinear (control) system; while interestingly, based on an algebraic state representation approach, the Boolean dynamics can be mapped exactly into a standard discrete-time linear dynamics [8]. This formal simplicity makes it relatively easy to formulate and solve classical control-theoretic problems for BNs/BCNs, and thereby lays a suitable foundation for a series of subsequent studies. Examples include recent investigations of dynamical properties [9], network synchronization [10], controllability [11, 12] and stabilizability [13–16], observability [17–20] and reconstructibility [21], disturbance or input-output decoupling [22–24], optimal control [25, 26], and more [27–31]. The size of the linear system that describes a BN with n state variables is 2^n . Thus, any algorithm based on this algebraic set-up has an exponential time complexity in the worst case. On the other hand, it has shown that for several control problems, the complexity curse can be alleviated or even removed if the structure of BNs is appropriately constrained [32, 33]. These positive developments notwithstanding, it still seems computationally challenging to solve control-related problems in general BNs/BCNs. since many such problems have shown to be NP-hard [34–37]. The hardness results justify the use of exponential time algorithms and exponential size systems suggested by the algebraic state-space representation.

In this paper, we focus on studying quotients of BCNs since they can be seen as lower dimensional models that may still contain enough information about the original model (whose algebraic representation is of exponential size). We consider quotient systems for BCNs in the exact sense that the notion is used in the control community [38–40]. Precisely, given a BCN and an equivalence

Email addresses: rui_li@dlut.edu.cn (Rui Li),

zhangqi@uibe.edu.cn (Qi Zhang), chutg@pku.edu.cn (Tianguang Chu).

relation on its state set, we consider a (labeled) transition system generated by the BCN and partition the state set based on the relation. The resulting quotient system then naturally captures the quotient dynamics of this BCN, so we propose to develop a Boolean system that generates the transitions of the quotient transition system (Theorem 1). Of course, it is not surprising that additional constraints need to be placed on the equivalence relation to ensure that the quotient dynamics can indeed be generated from some Boolean system. A subsequent question is then, how to obtain an equivalence relation which allows the construction of a quotient BCN. We fully answer this question by giving a procedure that converges in a finite number of iterations to a satisfactory equivalence relation (Theorem 2). As applications of the study, we show how the resulting quotient can be used for controller synthesis. The results tell us that synthesizing controllers for a BCN can be easily done by first controlling the quotient system and then lifting the control law back to the original Boolean model (see Propositions 2 and 3).

Notation. The symbol δ_k^i denotes the *i*th canonical basis vector of length k, Δ_k denotes the set consisting of the canonical vectors $\delta_k^1, \ldots, \delta_k^k$, and $\mathcal{L}^{k \times r}$ denotes the set of all $k \times r$ matrices whose columns are canonical vectors of length k. Elements of $\mathcal{L}^{k \times r}$ are called logical matrices (of size $k \times r$). A (0, 1)-matrix is a matrix that consists solely of the 0 and 1 entries. The (i, j)-entry of a matrix A is invariably denoted by $(A)_{ij}$. If A and B are $k \times r$ (0, 1)matrices, the meet of A and B, denoted by $A \wedge B$, is the (0, 1)-matrix with the (i, j)-entry equal to $(A)_{ij} \wedge (B)_{ij}$. For a $k \times l$ (0, 1)-matrix C and an $l \times r$ (0, 1)-matrix D, the Boolean product of C and D, denoted by $C \odot D$, is the $k \times r$ matrix with the (i, j)-entry $\bigvee_{s=1}^{l} [(C)_{is} \wedge (D)_{sj}]$. Given two relations \mathcal{R}_1 and \mathcal{R}_2 , $\mathcal{R}_2 \circ \mathcal{R}_1$ denotes the composition of \mathcal{R}_1 and \mathcal{R}_2 , i.e., the relation defined by $(a, c) \in \mathcal{R}_2 \circ \mathcal{R}_1$ if and only if there exists b with $(a, b) \in$ \mathcal{R}_1 and $(b, c) \in \mathcal{R}_2$.

2 Preliminaries

2.1 Algebraic representation of Boolean control networks

A BCN is a discrete-time dynamical system with binary state variables and binary control variables, i.e.,

$$x_{1}(t+1) = f_{1}(x_{1}(t), \dots, x_{n}(t), u_{1}(t), \dots, u_{m}(t)),$$

$$\vdots$$

$$x_{n}(t+1) = f_{n}(x_{1}(t), \dots, x_{n}(t), u_{1}(t), \dots, u_{m}(t)),$$

(1)

with $x_i, u_j \in \{1, 0\}$ and $f_i: \{1, 0\}^{n+m} \to \{1, 0\}$. The dynamics (1) can be recast into a form similar to that of a discrete-time linear system, using the semitensor product of matrices [8]. To be more precise, we recall

that the (left) semitensor product of two matrices A and B of sizes $n_1 \times m_1$ and $n_2 \times m_2$, respectively, denoted by $A \ltimes B$, is defined by $A \ltimes B = (A \otimes I_{l/m_1})(B \otimes I_{l/n_2})$, where \otimes is the Kronecker product of matrices, and I_{l/m_1} and I_{l/n_2} are the identity matrices of orders l/m_1 and l/n_2 , respectively, with l being the least common multiple of m_1 and n_2 . If we identify the Boolean values 1 and 0 with the canonical vectors δ_1^2 and δ_2^2 , respectively (so x_i and u_j in (1) are vectors in Δ_2), and if we let $x(t) = x_1(t) \ltimes \cdots \ltimes x_n(t)$ and $u(t) = u_1(t) \ltimes \cdots \ltimes u_m(t)$, then the Boolean dynamics (1) can be represented by an equation of the form

$$x(t+1) = F \ltimes u(t) \ltimes x(t), \tag{2}$$

where $F \in \mathcal{L}^{2^n \times 2^{n+m}}$. (The expression on the right-hand side of (2) is unambiguous, since the semitensor product is associative.) For more information on converting a BCN in the form of (1) to its algebraic representation (2), as well as more information regarding the properties of the semitensor product, the reader is referred to, e.g., [8] and [41].

2.2 Transition systems

Our discussion of quotients of BCNs will be based on the notion of quotient transition systems. We first recall the concept of a (labeled) transition system.

Definition 1 (See, e.g., [42]). A (*labeled*) transition system is a tuple $\mathcal{T} = (Q, L, \rightarrow)$ that consists of a set of states Q, a set of labels L, and a transition relation $\rightarrow \subseteq Q \times L \times Q$.

For any $q, q' \in Q$ and any $l \in L$, a transition $(q, l, q') \in \to$ means that it is possible to move from state q to state q'under the action labeled by l. Following standard practice, we denote $q \stackrel{l}{\to} q'$ if $(q, l, q') \in \to$.

Recall that an equivalence relation \mathcal{R} on Q is a reflexive, symmetric, and transitive binary relation on Q. Given a transition system \mathcal{T} , if \mathcal{R} is an equivalence relation on the state set of \mathcal{T} , then it naturally induces a quotient transition system, as follows.

Definition 2 (See, e.g., [40]). Let $\mathcal{T} = (Q, L, \rightarrow)$ be a transition system and let \mathcal{R} be an equivalence relation on Q. The quotient transition system \mathcal{T}/\mathcal{R} is defined by $\mathcal{T}/\mathcal{R} = (Q/\mathcal{R}, L, \rightarrow_{\mathcal{R}})$, where Q/\mathcal{R} is the quotient set (i.e., the set of all equivalence classes $[q] = \{p \in Q: (q, p) \in \mathcal{R}\}$ for $q \in Q$), and for all $[q], [q'] \in Q/\mathcal{R}, [q] \stackrel{l}{\rightarrow}_{\mathcal{R}} [q']$ if and only if there exist $p \in [q]$ and $p' \in [q']$ such that $p \stackrel{l}{\rightarrow} p'$.

That is, a state [q] in \mathcal{T}/\mathcal{R} can make a transition to another state [q'] under an action l, if some $p \in [q]$ can make a transition to some $p' \in [q']$ when taking the action l. In what follows, we will use a similar framework to study quotients of a BCN.

3 Quotients of Boolean control networks

3.1 Constructing quotient Boolean systems

Let us consider a BCN described by the algebraic representation

$$\Sigma: \quad x(t+1) = F \ltimes u(t) \ltimes x(t), \quad x \in \Delta_N, \quad u \in \Delta_M, \\ F \in \mathcal{L}^{N \times NM}.$$
(3)

(Note that, in the above, N and M are in fact certain powers of 2, but we do not need this fact for our argument.) In order to investigate quotients of (3), we first turn our attention to the equivalence relations on its state set Δ_N . An immediate observation is that every such equivalence relation \mathcal{R} can be viewed as induced by a logical matrix C with N columns, by saying

$$(x, x') \in \mathcal{R} \iff Cx = Cx'. \tag{4}$$

Furthermore, the logical matrix C can be chosen of full row rank (hence in particular having no zero rows). We remark that such a full row rank matrix can be directly derived from the *matrix representation* of \mathcal{R} . In fact, let $A_{\mathcal{R}}$ be the $N \times N$ matrix whose entries are given by

$$(A_{\mathcal{R}})_{ij} = \begin{cases} 1 & \text{if } (\delta_N^i, \delta_N^j) \in \mathcal{R}, \\ 0 & \text{otherwise.} \end{cases}$$

If C is a matrix which has the same set of rows as $A_{\mathcal{R}}$ but with no rows repeated, then it must be a logical matrix with full row rank and fulfill condition (4) [43, Lemma 4.6].

Example 1. To illustrate this fact, as well as the main idea behind obtaining an algebraic representation, we consider a BCN as in (1), with n = 3 and m = 1. The corresponding Boolean functions are given by the truth table shown in Table 1. Since n = 3 and m = 1, the size of the matrix F in the algebraic representation is 8×16 . To find this matrix, we see from Table 1 that if $u(t) = x_1(t) = x_2(t) = x_3(t) = 1$, we have $x_1(t+1) = x_2(t+1) = 1$, and $x_3(t+1) = 0$. In the algebraic framework, this corresponds to $u(t) = x_1(t) = x_2(t) = x_3(t) = \delta_2^1$, and $x_3(t+1) = \delta_2^2$, so

$$\begin{split} x(t+1) &= \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^2 = \delta_8^2, \\ u(t) \ltimes x(t) &= \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^1 = \delta_{16}^1. \end{split}$$

Substituting these to the left- and right-hand sides of (2) yields

$$\delta_8^2 = F \ltimes \delta_{16}^1 = F \delta_{16}^1. \tag{5}$$

Table 1 Truth table for Example 1.

$u x_1 x_2 x_3$	f_1	f_2	f_3	$u x_1 x_2 x_3$	f_1	f_2	f_3
1 1 1 1	1	1	0	0 1 1 1	1	1	1
$1 \ 1 \ 1 \ 0$	1	1	1	$0\ 1\ 1\ 0$	1	1	1
$1 \ 1 \ 0 \ 1$	1	1	1	$0\ 1\ 0\ 1$	1	1	1
$1 \ 1 \ 0 \ 0$	0	1	1	$0\ 1\ 0\ 0$	0	0	0
$1 \ 0 \ 1 \ 1$	0	1	0	$0 \ 0 \ 1 \ 1$	0	1	0
$1 \ 0 \ 1 \ 0$	0	0	1	$0 \ 0 \ 1 \ 0$	0	0	1
$1 \ 0 \ 0 \ 1$	0	0	0	$0 \ 0 \ 0 \ 1$	0	0	0
$1 \ 0 \ 0 \ 0$	0	1	1	0000	0	0	1

The second equality follows since the semitensor product is nothing but the standard product if the multiplied matrices (or vectors) have compatible sizes [8]. From (5), and considering that right-multiplying a matrix by a canonical vector yields the corresponding column of the matrix, we know that the first column of F is δ_8^2 . Repeating a similar argument for each combination in the truth table, we can determine all the columns of F, i.e., we determine the second column of F by considering the case when $u(t) = x_1(t) = x_2(t) = 1$ and $x_3(t) =$ 0, the third column by considering $u(t) = x_1(t) = 1$, $x_2(t) = 0$, and $x_3(t) = 1$, and so on. The matrix we get is

$$F = \begin{bmatrix} \delta_8^2 & \delta_8^1 & \delta_8^1 & \delta_8^5 & \delta_8^6 & \delta_8^7 & \delta_8^8 & \delta_8^5 & \delta_8^1 & \delta_8^1 & \delta_8^1 & \delta_8^8 & \delta_8^6 & \delta_8^7 & \delta_8^8 & \delta_8^7 \end{bmatrix}.$$
(6)

Consequently, the algebraic representation of this BCN is given by

$$x(t+1) = F \ltimes u(t) \ltimes x(t), \ x(t) \in \Delta_8, \ u(t) \in \Delta_2, \ (7)$$

with F found above. Note that system (7) evolves on the set $\Delta_8 = \{\delta_8^1, \ldots, \delta_8^8\}$, and each canonical vector δ_8^i corresponds to a possible configuration of the BCN (e.g., δ_8^1 corresponds to [1, 1, 1] since $\delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^1 = \delta_8^1$, δ_8^2 corresponds to [1, 1, 0] since $\delta_2^1 \ltimes \delta_2^1 \ltimes \delta_2^2 = \delta_8^2$, etc.). The trajectories of (7) are shown in Fig. 1. Now let \mathcal{R} be the equivalence relation produced by the partition $\{\{\delta_8^1\}, \{\delta_8^2, \delta_8^3\}, \{\delta_8^4\}, \{\delta_8^5, \delta_8^6, \delta_8^7, \delta_8^8\}\}$; that is, the pair $(a, b) \in \mathcal{R}$ if and only if a and b are in the same subset of



Fig. 1. Trajectories of system (7), which represents the BCN in Example 1. A solid line denotes the transition corresponding to $u(t) = \delta_2^1$ and a dashed line denotes the transition corresponding to $u(t) = \delta_2^2$.

the partition. By definition, the matrix that represents \mathcal{R} has a 1 as its (i, j)-entry when δ_8^i is related to δ_8^j , and a 0 in this position if δ_8^i is not related to δ_8^j . Accordingly, we get the following matrix for \mathcal{R} :

$$A_{\mathcal{R}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix},$$

where J_k denotes the all-one matrix of size $k \times k$. Collapsing the identical rows of $A_{\mathcal{R}}$ yields

$$C = \begin{bmatrix} \delta_4^1 & \delta_4^2 & \delta_4^2 & \delta_4^3 & \delta_4^4 & \delta_4^4 & \delta_4^4 & \delta_4^4 \end{bmatrix}.$$
(8)

It is clear that C is a full row rank logical matrix and that (4) holds.

Remark 1. Note that if \mathcal{R} is an equivalence relation on Δ_N induced by a matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$ of full row rank, then the quotient set Δ_N/\mathcal{R} is of cardinality \widetilde{N} , and the correspondence $[x] \mapsto Cx$ gives a bijection between the sets Δ_N/\mathcal{R} and $\Delta_{\widetilde{N}}$.

We now consider quotients of (3). We note that the BCN (3) naturally generates a transition system $\mathcal{T}(\Sigma) = (\Delta_N, \Delta_M, \rightarrow)$, where

$$x \xrightarrow{u} x' \Longleftrightarrow x' = F \ltimes u \ltimes x. \tag{9}$$

(In other words, a transition $x \stackrel{u}{\to} x'$ occurs in $\mathcal{T}(\Sigma)$ if u steers Σ from x to x'.) Let \mathcal{R} be an equivalence relation induced by a full row rank logical matrix Cof size $\widetilde{N} \times N$. Then the quotient transition system $\mathcal{T}(\Sigma)/\mathcal{R} = (\Delta_N/\mathcal{R}, \Delta_M, \to_{\mathcal{R}})$ can be thought of as having the state set $\Delta_{\widetilde{N}}$; and the transition relation is then given by

$$z \xrightarrow{u}_{\mathcal{R}} z' \iff$$
 there exists a transition $x \xrightarrow{u} x'$ of $\mathcal{T}(\Sigma)$
with $z = Cx$ and $z' = Cx'$ (10)

(cf. Definition 2 and Remark 1). For the analysis to remain in the Boolean context, we expect that the transitions of $\mathcal{T}(\Sigma)/\mathcal{R}$ are also generated by a Boolean system. (Here, and below, we use the term "Boolean system" to refer to a system of the form (3) where N and M are not restricted to be powers of 2.) It is readily seen that this is the case if and only if for any $z \in \Delta_{\widetilde{N}}$ and any $u \in \Delta_M$, there is a unique transition $z \xrightarrow{u}_{\mathcal{R}} z'$ of $\mathcal{T}(\Sigma)/\mathcal{R}$.¹ By (4), (9) and (10), the latter is equivalent to the requirement that

$$(a,b) \in \mathcal{R} \iff (F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R}$$

for all $u \in \Delta_M$. (11)

We therefore restrict our attention to those \mathcal{R} satisfying (11).

Remark 2. The meaning of condition (11) is clear: if we think of \mathcal{R} as a partition of Δ_N , then the successor set of each block in this partition is included in a single block of the partition.

The following theorem gives a method for explicitly constructing a Boolean system that generates the transitions of $\mathcal{T}(\Sigma)/\mathcal{R}$.

Theorem 1. Consider a BCN Σ as in (3). Suppose that \mathcal{R} is an equivalence relation on Δ_N induced by a matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$ of full row rank, and that property (11) holds. For each $1 \leq k \leq M$, let F_k be the matrix in $\mathcal{L}^{N \times N}$ defined by $F_k = F \ltimes \delta_M^k$, and let $\widetilde{F}_k = C \odot F_k \odot C^{\top}$. Then:

(a)
$$\widetilde{F}_k \in \mathcal{L}^{\widetilde{N} \times \widetilde{N}}$$
 for $1 \le k \le M$.
(b) Let

$$\Sigma_{\mathcal{R}}: \ x_{\mathcal{R}}(t+1) = \widetilde{F} \ltimes u(t) \ltimes x_{\mathcal{R}}(t), \ x_{\mathcal{R}} \in \Delta_{\widetilde{N}}, \ u \in \Delta_M$$

be the system where $\widetilde{F} = [\widetilde{F}_1 \ \widetilde{F}_2 \ \cdots \ \widetilde{F}_M]$. If an input $u \in \Delta_M$ steers Σ from a state $a \in \Delta_N$ to a state $a' \in \Delta_N$, then it also steers $\Sigma_{\mathcal{R}}$ from Ca to Ca'. Conversely, if u steers $\Sigma_{\mathcal{R}}$ from a state $q \in \Delta_{\widetilde{N}}$ to a state $q' \in \Delta_{\widetilde{N}}$, then there is a one-step transition of Σ from some $a \in \Delta_N$ to some $a' \in \Delta_N$ with Ca = qand Ca' = q', under this input u.

Proof. (a) It is clear that each \widetilde{F}_k is a (0, 1)-matrix of size $\widetilde{N} \times \widetilde{N}$. So we need only show that, for $1 \leq k \leq M$, every column of \widetilde{F}_k contains exactly one 1. Let $1 \leq k \leq M$ and $1 \leq j \leq \widetilde{N}$ be fixed. Since C (being logical) has no zero rows, there exists $1 \leq s \leq N$ such that $(C)_{js} = 1$. Choose $1 \leq r \leq N$ so that $\delta_N^r = F \ltimes \delta_M^k \ltimes \delta_N^s$. Then $(F_k)_{rs} = 1$. For this r, let $1 \leq i \leq \widetilde{N}$ be such that $(C)_{ir} = 1$. Then, by the definition of Boolean matrix multiplication, the (i, j)-entry of \widetilde{F}_k is equal to $\bigvee_{p=1}^N \bigvee_{l=1}^N [(C)_{ip} \wedge (F_k)_{pl} \wedge (C)_{jl}]$, and hence equal to 1 (since $(C)_{ir} = (F_k)_{rs} = (C)_{js} = 1$). This means that each column of \widetilde{F}_k has at least one 1. Now suppose that there is another i' with $1 \leq i' \leq \widetilde{N}$ such that $(\widetilde{F}_k)_{i'j} =$

¹ Note that this is equivalent to only requiring $\mathcal{T}(\Sigma)/\mathcal{R}$ to be *deterministic* (i.e., there do not exist transitions of the

form $z \xrightarrow{u}_{\mathcal{R}} z'$ and $z \xrightarrow{u}_{\mathcal{R}} z''$ with $z' \neq z''$), since for any $z \in \Delta_{\widetilde{N}}$ and $u \in \Delta_M$ there always exists at least one $z' \in \Delta_{\widetilde{N}}$ such that $z \xrightarrow{u}_{\mathcal{R}} z'$.



Fig. 2. Trajectories of the Boolean system $\Sigma_{\mathcal{R}}$ defined in Example 2. A solid (resp. dashed) line represents the transition resulting from $u(t) = \delta_2^1$ (resp. $u(t) = \delta_2^2$).

1. Then we must have $(C)_{i'r'} = 1$, $(F_k)_{r's'} = 1$, and $(C)_{js'} = 1$ for some $1 \leq r', s' \leq N$. These imply that $C\delta_N^{r'} = \delta_{\widetilde{N}}^{i'}, C\delta_N^{s'} = \delta_{\widetilde{N}}^{j}$, and $\delta_N^{r'} = F \ltimes \delta_M^k \ltimes \delta_N^{s'}$. Since $(C)_{js} = 1$, we have $C\delta_N^s = \delta_{\widetilde{N}}^{j}$ and, thus, $(\delta_N^s, \delta_N^{s'}) \in \mathcal{R}$. By (11), it follows that $(F \ltimes \delta_M^k \ltimes \delta_N^s, F \ltimes \delta_M^k \ltimes \delta_N^{s'}) \in \mathcal{R}$, that is, $(\delta_N^r, \delta_N^{r'}) \in \mathcal{R}$. Hence, $\delta_{\widetilde{N}}^i = C\delta_N^r = C\delta_N^{r'} = \delta_{\widetilde{N}}^{i'}$, which shows that i = i'. Thus, there is a unique 1 in each column of F_k .

(b) We first note that the system $\Sigma_{\mathcal{R}}$ is well defined since, by (a), \widetilde{F} is a logical matrix of size $\widetilde{N} \times \widetilde{N}M$. Let $1 \leq r, s \leq N$, let $1 \leq k \leq M$, and assume that the input $u = \delta_M^k$ steers Σ from δ_N^s to δ_N^r . We have $(F_k)_{rs} = 1$. Suppose that $C\delta_N^s = \delta_{\widetilde{N}}^j$ and $C\delta_N^r = \delta_{\widetilde{N}}^i$. Then $(C)_{js} = (C)_{ir} = 1$ and, hence, $(\widetilde{F}_k)_{ij} = 1$ by the definition of the Boolean product. This combined with (a) implies that $\delta_{\widetilde{N}}^i = \widetilde{F} \ltimes \delta_M^k \ltimes \delta_{\widetilde{N}}^j$; in other words, the input $u = \delta_M^k$ steers $\Sigma_{\mathcal{R}}$ from $\delta_{\widetilde{N}}^j$ to $\delta_{\widetilde{N}}^i$.

Conversely, let $1 \leq i, j \leq \tilde{N}$ and suppose that the input $u = \delta_M^k$ takes $\Sigma_{\mathcal{R}}$ from $\delta_{\tilde{N}}^j$ to $\delta_{\tilde{N}}^i$. Then $(\tilde{F}_k)_{ij} = 1$, and hence there must be some $1 \leq r, s \leq N$ such that $(C)_{ir} = 1, (F_k)_{rs} = 1$, and $(C)_{js} = 1$. Thus, $C\delta_N^s = \delta_{\tilde{N}}^j$, $C\delta_N^r = \delta_{\tilde{N}}^i$, and Σ can be driven from δ_N^s to δ_N^r with the input $u = \delta_M^k$.

Since, by the above theorem, $\Sigma_{\mathcal{R}}$ generates the transitions of $\mathcal{T}(\Sigma)/\mathcal{R}$ (cf. (10)), it can be interpreted as a quotient of the BCN Σ .

Example 2. Consider the BCN in Example 1. The matrix F in the algebraic representation is given by (6). Let C be as in (8) and let \mathcal{R} be the equivalence relation defined in Example 1, induced by C. It is easy to check that \mathcal{R} satisfies (11). Set $F_1 = F \ltimes \delta_2^1$ and $F_2 = F \ltimes \delta_2^2$. A calculation yields

$$\widetilde{F}_1 = C \odot F_1 \odot C^{\top} = \begin{bmatrix} \delta_4^2 & \delta_4^1 & \delta_4^4 & \delta_4^4 \end{bmatrix},$$

$$\widetilde{F}_2 = C \odot F_2 \odot C^{\top} = \begin{bmatrix} \delta_4^1 & \delta_4^1 & \delta_4^4 & \delta_4^4 \end{bmatrix}.$$

Fig. 2 shows the trajectories of $\Sigma_{\mathcal{R}}$ with $\widetilde{N} = 4$, M = 2, and $\widetilde{F} \in \mathcal{L}^{4 \times 8}$ given by $\widetilde{F} = [\widetilde{F}_1 \ \widetilde{F}_2]$. We see from the figure that $\Sigma_{\mathcal{R}}$ is indeed a quotient of the original BCN, which does not distinguish between states related by \mathcal{R} . Using Theorem 1 one can obtain a quotient Boolean system, once an equivalence relation satisfying property (11) is found. In the next subsection, we will address the issue of computing equivalence relations which allow the construction of quotient Boolean systems.

3.2 Computing equivalence relations

Precisely, in this subsection we are concerned with the following problem: given an equivalence relation S on Δ_N , determine the maximal (with respect to set inclusion) equivalence relation \mathcal{R} on Δ_N such that $\mathcal{R} \subseteq S$ and (11) holds. Here the relation S may be interpreted as a preliminary classification of the states of a BCN; see Section 4 below for specific instances. We are interested in finding the maximal equivalence relation since in many cases we want the size of the quotient system to be as small as possible.

First, we remark that such a maximal equivalence relation always exists and it is unique, as shown in the following proposition.

Proposition 1. Let S be an equivalence relation on Δ_N . Then the set of all relations $\mathcal{R} \subseteq \Delta_N \times \Delta_N$ that are contained in S and satisfy property (11) has a unique maximal element (with respect to set inclusion), and the maximal element is an equivalence relation on Δ_N .

Proof. Note that the identity relation $\mathcal{R}_{id} = \{(a, a) : a \in \Delta_N\}$ satisfies (11) and $\mathcal{R}_{id} \subseteq \mathcal{S}$ (since \mathcal{S} is reflexive). Also note that if two relations $\mathcal{R}_1 \subseteq \mathcal{S}$ and $\mathcal{R}_2 \subseteq \mathcal{S}$ both satisfy property (11), then the same is true for their union $\mathcal{R}_1 \cup \mathcal{R}_2$. The first statement follows immediately.

The maximal element $\widetilde{\mathcal{R}}$ is reflexive since it contains the identity relation \mathcal{R}_{id} . To show the symmetry and transitivity of $\widetilde{\mathcal{R}}$, consider the inverse relation $\widetilde{\mathcal{R}}^{-1} = \{(b, a) : (a, b) \in \widetilde{\mathcal{R}}\}$ and the composition $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}} =$ $\{(a, c) :$ there exists $b \in \Delta_N$ such that $(a, b) \in \widetilde{\mathcal{R}}$ and $(b, c) \in \widetilde{\mathcal{R}}\}$. It is easy to see that both $\widetilde{\mathcal{R}}^{-1}$ and $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}$ satisfy (11), and are contained in \mathcal{S} since $\widetilde{\mathcal{R}} \subseteq \mathcal{S}$ and \mathcal{S} is symmetric and transitive. Hence, $\widetilde{\mathcal{R}}$ contains $\widetilde{\mathcal{R}}^{-1}$ and $\widetilde{\mathcal{R}} \circ \widetilde{\mathcal{R}}$, implying that $\widetilde{\mathcal{R}}$ is symmetric and transitive. The second statement is proved.

The following theorem suggests a way of computing such an equivalence relation.

Theorem 2. Let $F \in \mathcal{L}^{N \times NM}$, and let S be an equivalence relation on Δ_N . For each $u \in \Delta_M$ define a relation S_u on Δ_N by: $(a, a') \in S_u$ if and only if $a' = F \ltimes u \ltimes a$. Define a sequence of relations \mathcal{R}_k by

$$\mathcal{R}_1 = \mathcal{S} \ and \ \mathcal{R}_{k+1} = \left(\bigcap_{u \in \Delta_M} (\mathcal{S}_u^{-1} \circ \mathcal{R}_k \circ \mathcal{S}_u)\right) \cap \mathcal{R}_k.$$

Then:

- (a) The sequence of relations $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_k, \dots$ satisfies $\mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \dots \supseteq \mathcal{R}_k \supseteq \dots$.
- (b) There is an integer $\overline{k^*}$ such that $\mathcal{R}_{k^*+1} = \mathcal{R}_{k^*}$.
- (c) \mathcal{R}_{k^*} is the maximal equivalence relation on Δ_N such that $\mathcal{R}_{k^*} \subseteq S$ and property (11) holds.

Proof. Part (a) is quite trivial. Part (b) follows from (a) and the finiteness of each \mathcal{R}_k .

We turn to the proof of (c). By Proposition 1, it suffices to show that $\mathcal{R}_{k^*} \subseteq \Delta_N \times \Delta_N$ is the maximal relation satisfying $\mathcal{R}_{k^*} \subseteq S$ and condition (11). The relation \mathcal{R}_{k^*} is clearly a subset of S. To show that (11) holds true, suppose that $(a, b) \in \mathcal{R}_{k^*}$ and $u \in \Delta_M$. Since $\mathcal{R}_{k^*} = \mathcal{R}_{k^*+1} \subseteq \mathcal{S}_u^{-1} \circ \mathcal{R}_{k^*} \circ \mathcal{S}_u$, there exist $a', b' \in \Delta_N$ such that $(a, a') \in \mathcal{S}_u$, $(b', b) \in \mathcal{S}_u^{-1}$, and $(a', b') \in \mathcal{R}_{k^*}$. It follows from the definition of \mathcal{S}_u that $a' = F \ltimes u \ltimes a$ and $b' = F \ltimes u \ltimes b$. Hence, $(F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R}_{k^*}$.

To prove the maximality of \mathcal{R}_{k^*} , let $\mathcal{R} \subseteq \Delta_N \times \Delta_N$ be another relation which is contained in \mathcal{S} and satisfies (11). We claim that $\mathcal{R} \subseteq \mathcal{R}_k$ for all k. The case $k = k^*$ completes the proof. We shall use induction on k. The case k = 1 is trivial, so we take k > 1 and assume that $\mathcal{R} \subseteq \mathcal{R}_{k-1}$. Let $(a, b) \in \mathcal{R}$. Then for any $u \in \Delta_M$, we have $(F \ltimes u \ltimes a, F \ltimes u \ltimes b) \in \mathcal{R} \subseteq \mathcal{R}_{k-1}$. By the definition of \mathcal{S}_u , it follows that $(a, F \ltimes u \ltimes a) \in \mathcal{S}_u$ and $(F \ltimes u \ltimes b, b) \in \mathcal{S}_u^{-1}$. Hence, $(a, b) \in \mathcal{S}_u^{-1} \circ \mathcal{R}_{k-1} \circ \mathcal{S}_u$, and consequently $(a, b) \in \mathcal{R}_k$ since u was arbitrary. This shows that $\mathcal{R} \subseteq \mathcal{R}_k$, and our claim follows. \Box

For applications, it is convenient to reformulate Theorem 2 in terms of (0, 1)-matrices. Recall that a relation \mathcal{R} on Δ_N can be represented by an $N \times N$ matrix, whose (i, j)-entry is 1 if $(\delta_N^i, \delta_N^j) \in \mathcal{R}$ and 0 otherwise. So if $A_{\mathcal{R}}$ is the matrix representing \mathcal{R} , then the inverse relation \mathcal{R}^{-1} has $A_{\mathcal{R}}^{\top}$ as the matrix representation. Moreover, if \mathcal{R}' is another relation on Δ_N represented by $A_{\mathcal{R}'}$, then the matrices representing $\mathcal{R} \cap \mathcal{R}'$ and $\mathcal{R}' \circ \mathcal{R}$ are $A_{\mathcal{R}} \wedge A_{\mathcal{R}'}$ and $A_{\mathcal{R}} \odot A_{\mathcal{R}'}$ (see, e.g., [44, Section 9.3]). Note that if \mathcal{S}_u is the relation defined in Theorem 2 and if $u = \delta_M^k$, then

$$(\delta_N^i, \delta_N^j) \in \mathcal{S}_u \iff F_k \delta_N^i = \delta_N^j \iff (F_k)_{ji} = 1,$$

where $F_k = F \ltimes \delta_M^k$, and thus F_k^{\top} is the matrix representing S_u . From these facts and Theorem 2, the following corollary follows immediately.

Corollary 1. Suppose that S is an equivalence relation on Δ_N represented by a matrix A_S , and suppose that $F \in \mathcal{L}^{N \times NM}$. For each $1 \leq i \leq M$, let F_i be the matrix $F_i = F \ltimes \delta^i_M$. Define a sequence of (0, 1)-matrices by

$$A_1 = A_S$$
 and $A_{k+1} = A_k \wedge (F_1^\top \odot A_k \odot F_1) \wedge \cdots \wedge (F_M^\top \odot A_k \odot F_M).$

Then there is an integer k^* such that $A_{k^*+1} = A_{k^*}$, and A_{k^*} is the matrix representing the maximal equivalence relation on Δ_N that is contained in S and satisfies property (11).

Example 3. Consider again the BCN in Example 1. If we let S be the equivalence relation induced by the partition $\{\{\delta_8^1\}, \{\delta_8^2, \delta_8^3, \delta_8^4\}, \{\delta_8^5, \delta_8^7, \delta_8^8, \delta_8^7, \delta_8^8\}\}$, then

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_4 \end{bmatrix},$$

and a short computation yields

$$A_2 = A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J_4 \end{bmatrix},$$

which is exactly the matrix representing the relation given in Example 1. So the relation \mathcal{R} presented in Example 1 is the maximal equivalence relation contained in \mathcal{S} and satisfying property (11).

4 Control design via quotients

This section discusses the application of quotient systems for control design. We consider two typical control problems in BCNs and show how these problems can be solved through the use of a quotient Boolean system.

4.1 Stabilization

Consider a BCN Σ as given in (3). Let $\mathcal{M} \subseteq \Delta_N$ be a target set of states. We say that Σ is *stabilizable* to \mathcal{M} if for every $x(0) \in \Delta_N$ there exists a control sequence $\{u(0), u(1), u(2), \ldots\}$, with $u(i) \in \Delta_M$, and a positive integer τ such that $x(t) \in \mathcal{M}$ for all $t \geq \tau$ (see, e.g., [8]). The following result shows that, by defining the equivalence relation appropriately, we can easily obtain a stabilizing controller for Σ on the basis of a stabilizer for its quotient system.

Proposition 2. Consider a BCN Σ as given in (3). Let $\mathcal{M} \subseteq \Delta_N$ and let S be the equivalence relation on Δ_N determined by the partition $\{\mathcal{M}, \Delta_N - \mathcal{M}\}$. Suppose that \mathcal{R} is an equivalence relation on Δ_N induced by a matrix

 $C \in \mathcal{L}^{\widetilde{N} \times N}$ of full row rank, $\mathcal{R} \subseteq S$, and condition (11) holds. Suppose $\Sigma_{\mathcal{R}}$ is defined as in Theorem 1. If $\Sigma_{\mathcal{R}}$ can be stabilized to the set $\mathcal{M}_{\mathcal{R}} = \{Cx: x \in \mathcal{M}\}$ via a feedback law $(x_{\mathcal{R}}, t) \mapsto u(x_{\mathcal{R}}, t)$, then Σ can be stabilized to \mathcal{M} using the feedback law $(x, t) \mapsto u(Cx, t)$.

Proof. We first show that if the initial states of Σ and $\Sigma_{\mathcal{R}}$ satisfy $Cx(0) = x_{\mathcal{R}}(0)$, then the feedback laws $(x, t) \mapsto u(Cx, t)$ and $(x_{\mathcal{R}}, t) \mapsto u(x_{\mathcal{R}}, t)$ generate the same input sequence, and the trajectories satisfy $Cx(t) = x_{\mathcal{R}}(t)$ for all $t = 0, 1, 2, \ldots$ It is clearly true that the two feedback laws generate the same input, say u_0 , at t = 0. By the second part of Theorem 1(b), there is a one-step transition of Σ from some $a \in \Delta_N$ to some $a' \in \Delta_N$ with $Ca = x_{\mathcal{R}}(0)$ and $Ca' = x_{\mathcal{R}}(1)$, under this input u_0 . Since Cx(0) = Ca, it follows from (4) that $(x(0), a) \in \mathcal{R}$, and then by (11) we have $(x(1), a') \in \mathcal{R}$. Thus $Cx(1) = Ca' = x_{\mathcal{R}}(1)$ again by (4). The fact we want now follows by induction.

Now we can prove the proposition. Assume that the feedback laws $(x,t) \mapsto u(Cx,t)$ and $(x_{\mathcal{R}},t) \mapsto u(x_{\mathcal{R}},t)$ are applied to Σ and $\Sigma_{\mathcal{R}}$, respectively. Let $p \in \Delta_N$ and let q = Cp. Then there is a τ such that the trajectory of $\Sigma_{\mathcal{R}}$ with $x_{\mathcal{R}}(0) = q$ satisfies $x_{\mathcal{R}}(t) \in \mathcal{M}_{\mathcal{R}}$ for all $t \geq \tau$. Since the trajectory of Σ with x(0) = p always satisfies $Cx(t) = x_{\mathcal{R}}(t)$, to each $t \geq \tau$ there corresponds some $b \in \mathcal{M}$ such that Cx(t) = Cb, and hence $(x(t), b) \in \mathcal{R} \subseteq S$. This forces $x(t) \in \mathcal{M}$ whenever $t \geq \tau$, since S is the equivalence relation yielded by the partition $\{\mathcal{M}, \Delta_N - \mathcal{M}\}$. Since p was arbitrary, we conclude that stabilization of Σ to \mathcal{M} is achieved, via the feedback law $(x, t) \mapsto u(Cx, t)$.

Remark 3. Note that in Proposition 2 we do not assume \mathcal{R} to be maximal, although that will be the case in most applications of the proposition. A similar remark applies to Proposition 3 below.

4.2 Optimal control

As another example of application we consider the following finite-horizon optimal control problem, introduced in [25].

Problem 1. Consider a BCN Σ as in (3). Given an initial state x_0 and a finite time horizon $T \in \mathbb{Z}^+$, find a control sequence that minimizes the cost function

$$J = \sum_{t=0}^{T-1} l(u(t), x(t)) + g(x(T)), \qquad (12)$$

where l(u, x) and g(x) are functions defined on $\Delta_M \times \Delta_N$ and Δ_N , respectively.

We show that the solution to Problem 1 for Σ can be easily derived on the basis of a solution to Problem 1 for a suitably chosen quotient system. Let S be the equivalence relation on Δ_N given by

$$(x, x') \in \mathcal{S} \iff g(x) = g(x') \text{ and}$$

 $l(u, x) = l(u, x') \text{ for all } u \in \Delta_M.$ (13)

We observe that, for a matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$ with full row rank, if the equivalence relation \mathcal{R} induced by C satisfies $\mathcal{R} \subseteq \mathcal{S}$, then the following two maps are well defined:

$$l_{\mathcal{R}} \colon \Delta_M \times \Delta_{\widetilde{N}} \to \mathbb{R}, \quad (u, a) \mapsto l(u, x)$$

whenever $a = Cx$, (14)
 $g_{\mathcal{R}} \colon \Delta_{\widetilde{N}} \to \mathbb{R}, \quad a \mapsto g(x)$ whenever $a = Cx$. (15)

Based on this observation, we can state the following proposition.

Proposition 3. Let Σ be a BCN described by (3). Suppose that S is the equivalence relation on Δ_N given by (13), \mathcal{R} is an equivalence relation on Δ_N induced by a full row rank logical matrix $C \in \mathcal{L}^{\widetilde{N} \times N}$, $\mathcal{R} \subseteq S$, and (11) holds. Consider Problem 1 with given x_0 , T, and J. Let $\Sigma_{\mathcal{R}}$ be the Boolean system constructed in Theorem 1, and define $J_{\mathcal{R}} = \sum_{t=0}^{T-1} l_{\mathcal{R}}(u(t), x_{\mathcal{R}}(t)) + g_{\mathcal{R}}(x_{\mathcal{R}}(T))$, where $l_{\mathcal{R}}$ and $g_{\mathcal{R}}$ are given by (14) and (15).

- (a) If $U^* = \{u^*(0), \ldots, u^*(T-1)\}$ is an optimal control sequence solving Problem 1 with Σ , x_0 , and J replaced by $\Sigma_{\mathcal{R}}$, $x_{\mathcal{R}}^0 = Cx_0$, and $J_{\mathcal{R}}$, respectively, then U^* is also an optimal control for Σ . Moreover, let J^* be the optimal cost $\min_{u(\cdot)} J$ under the initial condition $x(0) = x_0$ and let $J_{\mathcal{R}}^*$ be the optimal cost $\min_{u(\cdot)} J_{\mathcal{R}}$ under the condition $x_{\mathcal{R}}(0) = Cx_0$. Then $J^* = J_{\mathcal{R}}^*$.
- (b) If $(x_{\mathcal{R}}, t) \mapsto u^*(x_{\mathcal{R}}, t)$ is an optimal control policy² solving Problem 1 with Σ and J replaced by $\Sigma_{\mathcal{R}}$ and $J_{\mathcal{R}}$, respectively, then the control policy given by $(x, t) \mapsto u^*(Cx, t)$ is an optimal control policy for Σ .

Proof. (a) An argument similar to the first paragraph of the proof of Proposition 2 shows that, if the initial states of Σ and $\Sigma_{\mathcal{R}}$ satisfy $Cx(0) = x_{\mathcal{R}}(0)$, then for any control sequence $u(0), \ldots, u(T-1)$, the corresponding trajectories satisfy $Cx(t) = x_{\mathcal{R}}(t)$ for $t = 0, \ldots, T$, and hence $g(x(T)) = g_{\mathcal{R}}(x_{\mathcal{R}}(T))$ and l(u(t), x(t)) = $l_{\mathcal{R}}(u(t), x_{\mathcal{R}}(t))$ for each $0 \leq t \leq T-1$, so that the cost functions J and $J_{\mathcal{R}}$ return the same value. This implies that if U^* minimizes $J_{\mathcal{R}}$ with the initial condition $x_{\mathcal{R}}(0) = Cx_0$, then it also minimizes J subject to $x(0) = x_0$, and moreover, the associated optimal costs J^* and $J_{\mathcal{R}}^*$ are equal.

 $^{^2\,}$ It was shown in [25] that the optimal control input can always be implemented as a time-varying feedback from the states.

Part (b) follows directly from (a) and the fact (explained in the first paragraph of the proof of Proposition 2) that the feedback laws $(x,t) \mapsto u^*(Cx,t)$ and $(x_{\mathcal{R}},t) \mapsto$ $u^*(x_{\mathcal{R}},t)$ generate the same control sequence whenever $Cx(0) = x_{\mathcal{R}}(0)$.

Example 4. To give an intuitive example of the equivalence relation defined by (13), suppose that M = 2, N = 4, and the functions $l: \Delta_2 \times \Delta_4 \to \mathbb{R}$ and $g: \Delta_4 \to \mathbb{R}$ are given by

$$\begin{split} &l(\delta_2^1, \delta_4^1) = 1, \quad l(\delta_2^1, \delta_4^2) = l(\delta_2^1, \delta_4^3) = l(\delta_2^1, \delta_4^4) = 2, \\ &l(\delta_2^2, x) = 3 \quad (x \in \Delta_4), \\ &g(\delta_4^1) = g(\delta_4^2) = g(\delta_4^3) = 1, \quad g(\delta_4^4) = 2. \end{split}$$

First, by definition the relation S contains all pairs of the form (a, a), namely, (δ_4^1, δ_4^1) , (δ_4^2, δ_4^2) , (δ_4^3, δ_4^3) , and (δ_4^4, δ_4^4) . Second, note that $g(\delta_4^2) = g(\delta_4^3) = 1, l(\delta_2^1, \delta_4^2) = l(\delta_2^1, \delta_4^3) = 2$, and $l(\delta_2^2, \delta_4^2) = l(\delta_2^2, \delta_4^3) = 3$. Thus, both pairs (δ_4^2, δ_4^3) and (δ_4^3, δ_4^2) belong to S. Moreover, it is easily checked that they are the only pairs of distinct states that satisfy $g(x) = g(x'), l(\delta_2^1, x) = l(\delta_2^1, x')$, and $l(\delta_2^2, x) = l(\delta_2^2, x')$ simultaneously. Hence no pair other than those listed belongs to S.

Remark 4. It is noted in [25] that the cost function described in (12) can be equivalently expressed in a linear form as $J = \sum_{t=0}^{T-1} \theta \ltimes u(t) \ltimes x(t) + \mu x(T)$, where μ is a row vector of N components and $\theta = [\theta_1, \theta_2, \ldots, \theta_M]$ with each θ_i being an N-component row vector. We remark that the index $J_{\mathcal{R}}$ appearing in Proposition 3 is easily obtained from this expression. In fact, since the function $g_{\mathcal{R}}$ is defined on $\Delta_{\widetilde{N}}$, it can be expressed in the form $g_{\mathcal{R}}(x) = \mu_{\mathcal{R}} x$ for some \widetilde{N} -component row vector $\mu_{\mathcal{R}}$. Let C be as in Proposition 3. Then by (15) we have $\mu_{\mathcal{R}}C = \mu$ and so $\mu_{\mathcal{R}} = \mu C^+$, where $C^+ = C^\top (CC^\top)^{-1}$ is the pseudoinverse of C. In a similar manner, the function $l_{\mathcal{R}}$ defined by (14) can be equivalently expressed as $l_{\mathcal{R}}(u, x) = \theta_{\mathcal{R}} \ltimes u \ltimes x$, where $\theta_{\mathcal{R}} = [\theta'_1, \theta'_2, \ldots, \theta'_M]$ with $\theta'_i = \theta_i C^+$ for each i. Thus the index cost $J_{\mathcal{R}}$ can be rewritten in a linear form as follows: $J_{\mathcal{R}} = \sum_{t=0}^{T-1} \theta_{\mathcal{R}} \ltimes u(t) \ltimes x_{\mathcal{R}}(t) + \mu_{\mathcal{R}} x_{\mathcal{R}}(T)$.

One can obtain analogs of Proposition 3 for other kinds of optimal control problems (such as the infinite-horizon optimal or average-cost optimal problems [25]). The essence of the arguments is the same as that of Proposition 3, and so we omit them.

4.3 Comparative simulations

The proposed methods have been tested on several randomly generated 16-node networks. Recall that a BCN expressed by (1) consists of two types of nodes, namely, internal nodes (x_1, \ldots, x_n) and external control nodes (u_1, \ldots, u_m) . We considered the cases of m = 1, 2, 3, and

Table 2						
Comparison	between	$\operatorname{controller}$	design	done	with	the
quotient base	d method	and done t	he conv	ontions	l wav	

4	Siz	e	CPU tim	ne (sec)
	Orig. BCN	Quotient	Orig. BCN	Quotient
m = 1 $k = 1$	32768	6815	9256.08	454.47
m = 1 k = 100	32768	5807	9845.81	475.89
m = 2 $k = 1$	16384	4014	2327.27	303.04
m = 2 $k = 100$	16384	4647	2401.16	158.76
m = 3 $k = 1$	8192	2793	590.42	53.79
m = 3 k = 100	8192	3071	601.22	65.40
m = 5 $k = 1$	2048	887	37.78	5.50
m = 5 $k = 100$	2048	1015	32.20	6.28

5. When m = 1, there are 15 internal nodes and 1 control nodes; the original network size is $2^{15} = 32768$. When m = 2, there are 14 internal nodes and 2 control nodes; the original network size is $2^{14} = 16384$. When m = 3, there are 13 internal nodes, so the original network size is $2^{13} = 8192$, and when m = 5 the original network size is $2^{11} = 2048$. First, we evaluate the efficiency of the quotient-based method given in Proposition 2. The target sets \mathcal{M} of the stabilization problem were randomly selected, with cardinality k = 1 and k = 100. Table 2 shows the numerical results obtained for different combinations of m and k. The second and third columns give the number of states of the original networks and the number of states of the quotient systems, reflecting the degree of reduction. The fourth column records the CPU time spent for constructing stabilizing controllers directly based on the original networks. Specifically, we followed the design procedure proposed by Fornasini and Valcher [45] and Li et al. [46] when k = 1, and the procedure of Guo et al. [47] when k = 100. The CPU time required for determining stabilizers via Proposition 2 is shown in the last column. Similarly, Table 3 compares the network size and the CPU time to obtain a solution to Problem 1, with T = 40. For the sake of simplicity, we assumed that the function l(u, x) depends only on u, with the value 1 if $u_1 = 1$ and 0 if $u_1 = 0$; the function g(x) was assumed to take the value 5 if $x_1 = 0$ and the value 0 otherwise. (Here we use binary representations of x and u.) The corresponding optimal control problem was solved both by applying the algorithm of Fornasini

Table 3 Comparison between direct and quotient-based methods for solving Problem 1.

	Siz	e	CPU tim	ne (sec)		
	Orig. BCN	Quotient	Orig. BCN	Quotient		
m = 1	32768	6087	490.67	115.87		
m=2	16384	4506	256.51	74.94		
m = 3	8192	3441	130.33	55.50		
m = 5	2048	829	27.96	10.81		

and Valcher [25] directly to the original network, and by using the indirect method given in Proposition 3. It is seen that the proposed methods offer a reduction in computation time compared to the state of the art, and the extent of reduction increases (as a trend) with increasing size of the original network. All computations were run on an Intel Core i7-3.00 GHz personal computer with 8 GB of RAM.

5 A biological example

We apply our methods to a Boolean model for lactose metabolism in the bacterium *E. coli* [48]. The model consists of 13 variables (1 mRNA, 5 proteins, and 7 sugars) denoted by M, P, B, C, R, R_m , A, A_m , L, L_m , L_e , L_{em} and G_e . Here, R and R_m are combined to indicate concentration levels of a specific substance (the repressor protein); that is, the concentration is low when $(R, R_m) = (0, 0)$, medium when $(R, R_m) = (0, 1)$, and high when $(R, R_m) = (1, 1)$. The fourth possibility, $(R, R_m) = (1, 0)$, is meaningless and not allowed. The same situation is for the pairs (A, A_m) , (L, L_m) , and (L_e, L_{em}) (see [48] for more details on this aspect). The equations describing the model are as follows:

$$\begin{split} M(t+1) &= C(t) \land \neg R(t) \land \neg R_m(t), \\ P(t+1) &= M(t), \qquad B(t+1) = M(t), \\ C(t+1) &= \neg G_e(t), \\ R(t+1) &= \neg A(t) \land \neg A_m(t), \\ R_m(t+1) &= (\neg A(t) \land \neg A_m(t)) \lor R(t), \qquad (16) \\ A(t+1) &= B(t) \land L(t), \\ A_m(t+1) &= L(t) \lor L_m(t), \\ L(t+1) &= P(t) \land L_e(t) \land \neg G_e(t), \\ L_m(t+1) &= ((L_{em}(t) \land P(t)) \lor L_e(t)) \land \neg G_e(t). \end{split}$$

We assume that the concentration of extracellular lactose is low $(L_e = L_{em} = 0)$, and treat the extracellular glucose levels (G_e) as input to the model. Then the model can be rewritten as in (3) with ³ N = 432 and M = 2.

The matrix $F \in \mathcal{L}^{432 \times 864}$ is detailed in the Appendix.

(1) Stabilization. When extracellular lactose levels get low, the model is known to exhibit two steady states [48], expressed in the canonical vector form as δ_{432}^{387} and δ_{432}^{414} . Let $\mathcal{M} = \{\delta_{432}^{387}\}$ and let \mathcal{S} be the equivalence relation produced by the partition $\{\mathcal{M}, \Delta_{432} - \mathcal{M}\}$. Then by following the procedure described in Section 3, we get a quotient system $\Sigma_{\mathcal{R}}: x_{\mathcal{R}}(t+1) = \widetilde{F} \ltimes u(t) \ltimes x_{\mathcal{R}}(t)$, with $x_{\mathcal{R}} \in \Delta_8, u \in \Delta_2$, and $\widetilde{F} \in \mathcal{L}^{8 \times 16}$ given by

$$\widetilde{F} = \begin{bmatrix} \delta_8^2 & \delta_8^7 & \delta_8^2 & \delta_8^4 & \delta_8^7 & \delta_8^2 & \delta_8^4 & \delta_8^1 & \delta_8^1 & \delta_8^6 & \delta_8^6 & \delta_8^3 & \delta_8^7 & \delta_8^2 & \delta_8^4 \end{bmatrix}.$$

The matrix C obtained during the procedure (which is of size 8×432 and not given explicitly) satisfies $C\delta_{432}^{387} = \delta_8^1$. It is not hard to see that for any

$$K = \begin{bmatrix} \delta_2^2 & \delta_2^2 & \ast & \ast & \ast & \ast & \ast \end{bmatrix}$$

(* denoting columns that can be either δ_2^1 or δ_2^2), the feedback law given by $x_{\mathcal{R}} \mapsto u(x_{\mathcal{R}}) = K x_{\mathcal{R}}$ stabilizes the quotient system to δ_8^1 . Proposition 2 then ensures that the original model can be globally stabilized to the state δ_{432}^{387} via the feedback law $x \mapsto u(Cx) = KCx$. A similar argument can be made for finding a feedback controller that stabilizes the model to the state δ_{432}^{414} ; the details are not repeated here.

Remark 5. It required about 6.5 s to find the above controller directly based on the procedure described in [45] and [46]. In contrast, it took only 1.16 s to obtain the same stabilizer by using the quotient-based method. Thus in this case there is an increase in speed by a factor of about 5 to 6 when the proposed method is employed.

(2) Optimal control. Assume that T = 3, the initial condition $x(0) = \delta_{432}^{10}$, and the functions l(u, x) and g(x) are given by

$$\begin{split} l(\delta_{1}^{2}, x) &= 1, \quad l(\delta_{2}^{2}, x) = 2 \quad (x \in \Delta_{432}), \\ g(\delta_{432}^{1}) &= \cdots = g(\delta_{432}^{54}) = 0, \quad g(\delta_{432}^{55}) = \cdots \\ &= g(\delta_{432}^{432}) = 5. \end{split}$$

Here we remark that the states $\delta_{432}^1, \ldots, \delta_{432}^{54}$ correspond to the *lac* operon, which is responsible for the metabolism of lactose, being ON (induced); cf. [48]. The above choice of g(x) then indicates that the operon is desired to be in an ON state after intervention. By proceeding as in Section 4.2, one can obtain a quotient

³ Here, N is not a power of 2, since for some Boolean pairs in the model only three of the four values are admissible. More precisely, since each of the variables M, P, B, and Chas two possible values, whereas each of the pairs (R, R_m) , (A, A_m) , and (L, L_m) takes on only three possible values,

the total number of states of (16) is equal to $2^4 \cdot 3^3 = 432$; thus N = 432 in the algebraic representation.

system $\Sigma_{\mathcal{R}}$ with N = 12, M = 2, and the matrix

$$\widetilde{F} = \begin{bmatrix} \delta_{12}^7 & \delta_{12}^7 & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^7 & \delta_{12}^7 & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^{12} & \delta_{12}^7 & \delta_{12}^7 \\ \delta_{12}^3 & \delta_{12}^4 & \delta_{12}^{12} & \delta_{12}^8 & \delta_{12}^9 & \delta_{12}^1 & \delta_{12}^7 & \delta_{12}^{12} & \delta_{12}^8 & \delta_{12}^9 & \delta_{12}^1 & \delta_{12}^7 \end{bmatrix}$$

The matrix C satisfies $Cx(0) = \delta_{12}^{11}$, and the induced functions $l_{\mathcal{R}}$ and $g_{\mathcal{R}}$ are defined by

$$l_{\mathcal{R}}(\delta_{1}^{1}, x_{\mathcal{R}}) = 1, \quad l_{\mathcal{R}}(\delta_{2}^{2}, x_{\mathcal{R}}) = 2 \quad (x_{\mathcal{R}} \in \Delta_{12}), \\ g_{\mathcal{R}}(\delta_{12}^{1}) = \dots = g_{\mathcal{R}}(\delta_{12}^{7}) = 5, \quad g_{\mathcal{R}}(\delta_{12}^{8}) = \dots \\ = g_{\mathcal{R}}(\delta_{12}^{12}) = 0.$$

It is straightforward to see that the input sequence

$$u^*(0) = u^*(1) = \delta_2^2, \quad u^*(2) = \delta_2^1$$

is optimal for $\Sigma_{\mathcal{R}}$, with the optimal cost $J_{\mathcal{R}}^* = 5$, so it also solves the optimal control problem for the original model, and the optimal cost is $J^* = J_{\mathcal{R}}^* = 5$. Moreover, we see from the value of J^* that the optimal input indeed steers the model to an ON state, as desired.

Remark 6. As for the time comparison, we report that it took about 1.5 s to solve this problem directly by the method of Fornasini and Valcher [25], while the above indirect procedure took only 0.79 s. Thus there is about 2 times saving in speed when the quotient-based method is employed.

6 Discussions

The paper has considered quotients of BCNs. Two possible applications of the quotient description have been presented in Section 4, where we have seen that the stabilization and optimal control problems of the original BCNs can be boiled down to those of the quotient systems. Let us mention that we have presented only a few examples of such applications, and there are quite a few other problems such as output tracking and observability checking that can also be dealt with in this manner. We do not include the details of these applications for reasons of space.

Since the number of states of the quotient $\Sigma_{\mathcal{R}}$ is precisely the number of the equivalence classes generated by \mathcal{R} , the coarser the relation \mathcal{R} , the smaller is $\Sigma_{\mathcal{R}}$ and, thus, the greater is the degree of reduction. Recall that the relation \mathcal{R} is required to satisfy (11), which is related to the dynamics of Σ . Thus, the degree of reduction is affected by the specific dynamics of the original network. Also, since in practice different relations are required for different applications (cf. Sections 4.1 and 4.2), despite the same original network, the reduction degree may still be different, depending on the specific problems to be solved. The size of the quotient systems appearing in the numerical experiments reported in Section 4.3 is about 50-20% when compared to the original networks. In the biological example presented in Section 5, the size of the reduced state space is less than 3% of that of the original one.

In Section 4.3, we have limited the discussion to networks with 16 nodes, since we would like to compute the control policy on each originally generated network and list the exact time that the standard methods require, in order to make the comparisons. Here we report that besides these simulations, we also tested our methods on networks with about 20–23 nodes. We observed that for most instances, the standard methods ran out of memory whereas the proposed methods were able to obtain a solution in a matter of minutes to hours. We do not present the detailed numerical results due to the limitations on the paper length.

References

- R. Albert, J. Thakar, Boolean modeling: a logic-based dynamic approach for understanding signaling and regulatory networks and for making useful predictions, WIREs Systems Biology and Medicine 6(5) (2014) 353– 369.
- [2] S. M. Assmann, R. Albert, Discrete dynamic modeling with asynchronous update, or how to model complex systems in the absence of quantitative information, Methods in Molecular Biology 553 (2009) 207–225.
- [3] T. S. Christensen, A. P. Oliveira, J. Nielsen, Reconstruction and logical modeling of glucose repression signaling pathways in *Saccharomyces cerevisiae*, BMC Systems Biology 3 (2009) 7.
- [4] O. E. Akman, S. Watterson, A. Parton, N. Binns, A. J. Millar, P. Ghazal, Digital clocks: simple Boolean models can quantitatively describe circadian systems, Journal of the Royal Society Interface 9(74) (2012) 2365–2382.
- [5] M. Chaves, R. Albert, Studying the effect of cell division on expression patterns of the segment polarity genes, Journal of the Royal Society Interface 5 (2008) S71–S84.
- [6] R. Schlatter, N. Philippi, G. Wangorsch, R. Pick, O. Sawodny, C. Borner, J. Timmer, M. Ederer, T. Dandekar, Integration of Boolean models exemplified on hepatocyte signal transduction, Briefings in Bioinformatics 13(3) (2012) 365–376.
- [7] R. Zhang, M. V. Shah, J. Yang, S. B. Nyland, X. Liu, J. K. Yun, R. Albert, T. P. Loughran, Jr., Network model of survival signaling in large granular lymphocyte leukemia, Proceedings of the National Academy of Sciences 105(42) (2008) 16308–16313.
- [8] D. Cheng, H. Qi, Z. Li, Analysis and Control of Boolean Networks: A Semi-Tensor Product Approach, Springer-Verlag, 2011.
- [9] G. Hochma, M. Margaliot, E. Fornasini, M. E. Valcher, Symbolic dynamics of Boolean control networks, Automatica 49(8) (2013) 2525–2530.
- [10] J. Zhong, J. Lu, Y. Liu, J. Cao, Synchronization in an array of output-coupled Boolean networks with time delay, IEEE Transactions on Neural Networks and Learning Systems 25(12) (2014) 2288–2294.

- [11] J. Liang, H. Chen, J. Lam, An improved criterion for controllability of Boolean control networks, IEEE Transactions on Automatic Control 62(11) (2017) 6012– 6018.
- [12] J. Lu, J. Zhong, C. Huang, J. Cao, On pinning controllability of Boolean control networks, IEEE Transactions on Automatic Control 61(6) (2016) 1658–1663.
- [13] N. Bof, E. Fornasini, M. E. Valcher, Output feedback stabilization of Boolean control networks, Automatica 57 (2015) 21–28.
- [14] F. Li, Y. Tang, Set stabilization for switched Boolean control networks, Automatica 78 (2017) 223–230.
- [15] H. Li, Y. Wang, Further results on feedback stabilization control design of Boolean control networks, Automatica 83 (2017) 303–308.
- [16] M. Meng, J. Lam, J. Feng, K. C. Cheung, Stability and stabilization of Boolean networks with stochastic delays, IEEE Transactions on Automatic Control 64(2) (2019) 790–796.
- [17] D. Cheng, H. Qi, T. Liu, Y. Wang, A note on observability of Boolean control networks, Systems & Control Letters 87 (2016) 76–82.
- [18] Y. Yu, M. Meng, J. Feng, Observability of Boolean networks via matrix equations, Automatica 111 (2020) 108621.
- [19] Y. Yu, B. Wang, J. Feng, Input observability of Boolean control networks, Neurocomputing 333 (2019) 22–28.
- [20] R. Zhou, Y. Guo, W. Gui, Set reachability and observability of probabilistic Boolean networks, Automatica 106 (2019) 230–241.
- [21] E. Fornasini, M. E. Valcher, Observability, reconstructibility and state observers of Boolean control networks, IEEE Transactions on Automatic Control 58(6) (2013) 1390–1401.
- [22] Y. Liu, B. Li, J. Lu, J. Cao, Pinning control for the disturbance decoupling problem of Boolean networks, IEEE Transactions on Automatic Control 62(12) (2017) 6595–6601.
- [23] M. E. Valcher, Input/output decoupling of Boolean control networks, IET Control Theory & Applications 11(13) (2017) 2081–2088.
- [24] Y. Yu, J. Feng, J. Pan, D. Cheng, Block decoupling of Boolean control networks, IEEE Transactions on Automatic Control 64(8) (2019) 3219–3140.
- [25] E. Fornasini, M. E. Valcher, Optimal control of Boolean control networks, IEEE Transactions on Automatic Control 59(5) (2014) 1258–1270.
- [26] Y. Wu, X.-M. Sun, X. Zhao, T. Shen, Optimal control of Boolean control networks with average cost: a policy iteration approach, Automatica 100 (2019) 378–387.
- [27] M. R. Rafimanzelat, F. Bahrami, Attractor stabilizability of Boolean networks with application to biomolecular regulatory networks, IEEE Transactions on Control of Network Systems 6(1) (2019) 72–81.
- [28] B. Wang, J. Feng, On detectability of probabilistic Boolean networks, Information Sciences 483 (2019) 383–395.
- [29] K. Zhang, L. Zhang, L. Xie, Invertibility and nonsingularity of Boolean control networks, Automatica 60 (2015) 155–164.
- [30] Z. Zhang, T. Leifeld, P. Zhang, Finite horizon tracking control of Boolean control networks, IEEE Transactions on Automatic Control 63(6) (2018) 1798–1805.
- [31] Y. Zou, J. Zhu, Kalman decomposition for Boolean

control networks, Automatica 54 (2015) 65-71.

- [32] Z. Gao, X. Chen, T. Başar, Controllability of conjunctive Boolean networks with application to gene regulation, IEEE Transactions on Control of Network Systems 5(2) (2018) 770–781.
- [33] E. Weiss, M. Margaliot, A polynomial-time algorithm for solving the minimal observability problem in conjunctive Boolean networks, IEEE Transactions on Automatic Control 64(7) (2019) 2727–2736.
- [34] T. Akutsu, M. Hayashida, W.-K. Ching, M. K. Ng, Control of Boolean networks: hardness results and algorithms for tree structured networks, Journal of Theoretical Biology 244(4) (2007) 670–679.
- [35] D. Laschov, M. Margaliot, On Boolean control networks with maximal topological entropy, Automatica 50(11) (2014) 2924–2928.
- [36] D. Laschov, M. Margaliot, G. Even, Observability of Boolean networks: a graph-theoretic approach, Automatica 49(8) (2013) 2351–2362.
- [37] K. Zhang, L. Zhang, R. Su, A weighted pair graph representation for reconstructibility of Boolean control networks, SIAM Journal on Control and Optimization 54(6) (2016) 3040–3060.
- [38] A. Chutinan, B. H. Krogh, Verification of infinite-state dynamic systems using approximate quotient transition systems, IEEE Transactions on Automatic Control 46(9) (2001) 1401–1410.
- [39] P. Tabuada, G. J. Pappas, Quotients of fully nonlinear control systems, SIAM Journal on Control and Optimization 43(5) (2005) 1844–1866.
- [40] P. Tabuada, Verification and Control of Hybrid Systems: A Symbolic Approach, Springer, 2009.
- [41] D. Cheng, H. Qi, Controllability and observability of Boolean control networks, Automatica 45(7) (2009) 1659–1667.
- [42] P. Tabuada, G. J. Pappas, Bisimilar control affine systems, Systems & Control Letters 52(1) (2004) 49–58.
- [43] R. Li, Q. Zhang, T. Chu, Reduction and analysis of Boolean control networks by bisimulation (submitted for publication), 2019.
- [44] K. H. Rosen, Discrete Mathematics and Its Applications, McGraw-Hill, 7th edition, 2012.
- [45] E. Fornasini, M. E. Valcher, On the periodic trajectories of Boolean control networks, Automatica 49(5) (2013) 1506–1509.
- [46] R. Li, M. Yang, T. Chu, State feedback stabilization for Boolean control networks, IEEE Transactions on Automatic Control 58(7) (2013) 1853–1857.
- [47] Y. Guo, P. Wang, W. Gui, C. Yang, Set stability and set stabilization of Boolean control networks based on invariant subsets, Automatica 61 (2015) 106–112.
- [48] A. Veliz-Cuba, B. Stigler, Boolean models can explain bistability in the *lac* operon, Journal of Computational Biology 18(6) (2011) 783–794.

Appendix

The matrix ${\cal F}$ for the biological model discussed in Section 5 is

$F = \delta_{432}[255]$	258	261	255	258	261	246	249	252	264	267	270	264	267	270	246	249	252	48	51	54
48	51	54	30	33	36	255	258	261	255	258	261	246	249	252	264	267	270	264	267	270
246	249	252	264	267	270	264	267	270	246	249	252	258	258	261	258	258	261	249	249	252
267	267	270	267	267	270	249	249	252	51	51	54	51	51	54	33	33	36	258	258	261
258	258	261	249	249	252	267	267	270	267	267	270	249	249	252	267	267	270	267	267	270
249	249	252	255	258	261	255	258	261	246	249	252	264	267	270	264	267	270	246	249	252
48	51	54	48	51	54	30	33	36	255	258	261	255	258	261	246	249	252	264	267	270
264	267	270	246	249	252	264	267	270	264	267	270	246	249	252	258	258	261	258	258	261
249	249	252	267	267	270	267	267	270	249	249	252	51	51	54	51	51	54	33	33	36
258	258	261	258	258	261	249	249	252	267	267	270	267	267	270	249	249	252	267	267	270
267	267	270	249	249	252	417	420	423	417	420	423	408	411	414	426	429	432	426	429	432
408	411	414	210	213	216	210	213	216	192	195	198	417	420	423	417	420	423	408	411	414
426	429	432	426	429	432	408	411	414	426	429	432	426	429	432	408	411	414	420	420	423
420	420	423	411	411	414	429	429	432	429	429	432	411	411	414	213	213	216	213	213	216
195	195	198	420	420	423	420	420	423	411	411	414	429	429	432	429	429	432	411	411	414
429	429	432	429	429	432	411	411	414	417	420	423	417	420	423	408	411	414	426	429	432
426	429	432	408	411	414	210	213	216	210	213	216	192	195	198	417	420	423	417	420	423
408	411	414	426	429	432	426	429	432	408	411	414	426	429	432	426	429	432	408	411	414
420	420	423	420	420	423	411	411	414	429	429	432	429	429	432	411	411	414	213	213	216
213	213	216	195	195	198	420	420	423	420	420	423	411	411	414	429	429	432	429	429	432
411	411	414	429	429	432	429	429	432	411	411	414	228	231	234	228	231	234	219	222	225
237	240	243	237	240	243	219	222	225	21	24	27	21	24	27	3	6	9	228	231	234
228	231	234	219	222	225	237	240	243	237	240	243	219	222	225	237	240	243	237	240	243
219	222	225	231	231	234	231	231	234	222	222	225	240	240	243	240	240	243	222	222	225
24	24	27	24	24	27	6	6	9	231	231	234	231	231	234	222	222	225	240	240	243
240	240	243	222	222	225	240	240	243	240	240	243	222	222	225	228	231	234	228	231	234
219	222	225	237	240	243	237	240	243	219	222	225	21	24	27	21	24	27	3	6	9
228	231	234	228	231	234	219	222	225	237	240	243	237	240	243	219	222	225	237	240	243
237	240	243	219	222	225	231	231	234	231	231	234	222	222	225	240	240	243	240	240	243
222	222	225	24	24	27	24	24	27	6	6	9	231	231	234	231	231	234	222	222	225
240	240	243	240	240	243	222	222	225	240	240	243	240	240	243	222	222	225	390	393	396
390	393	396	381	384	387	399	402	405	399	402	405	381	384	387	183	186	189	183	186	189
165	168	171	390	393	396	390	393	396	381	384	387	399	402	405	399	402	405	381	384	387
399	402	405	399	402	405	381	384	387	393	393	396	393	393	396	384	384	387	402	402	405
402	402	405	384	384	387	186	186	189	186	186	189	168	168	171	393	393	396	393	393	396
384	384	387	402	402	405	402	402	405	384	384	387	402	402	405	402	402	405	384	384	387
390	393	396	390	393	396	381	384	387	399	402	405	399	402	405	381	384	387	183	186	189
183	186	189	165	168	171	390	393	396	390	393	396	381	384	387	399	402	405	399	402	405
381	384	387	399	402	405	399	402	405	381	384	387	393	393	396	393	393	396	384	384	387
402	402	405	402	402	405	384	384	387	186	186	189	186	186	189	168	168	171	393	393	396
393	393	396	384	384	387	402	402	405	402	402	405	384	384	387	402	402	405	402	402	405
384	384	387]	•																	

The above notation means that the first column of F is δ_{432}^{255} , the second column is δ_{432}^{258} , and so on.