Stabilisation in Distribution of Hybrid Stochastic Differential Equations by Feedback Control based on Discrete-Time State Observations ⋆

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Abstract

A new concept of stabilisation of hybrid stochastic systems in distribution by feedback controls based on discrete-time state observations is initialised. This is to design a controller to stabilise the unstable system such that the distribution of the solution process tends to a probability distribution. In addition, the discrete-time state observations are also taken into consideration to make the design of the controller more practical. Theorems on the stabilisation of hybrid stochastic systems in distribution are proved. The lower bound of the duration between two consecutive state observations is obtained. The implementation of theorems are demonstrated by designing the feedback controls in the structure cases and easy-to-rules are provided for the user. Numerical examples are discussed to illustrate the theoretical results.

Key words: Brownian motion, Markov chain, stability in distribution, feedback control, discrete-time state observation.

1 Introduction

One important class of hybrid systems, which has been used to model many practical systems where they may experience abrupt changes in their structure and parameters, is family of hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) [6,9,11,14,15,17–20,23,27,28,30,35,36]. When a given hybrid SDE is not stable, Mao [12] in 2013 discussed how to design a feedback control based on discrete-time state observations to stabilise the SDE in the sense of the mean-square exponential stability. Such a stabilisation problem has since then been studied by many authors (see, e.g., [2,3,8,16,22,24–26,31,32]). A common feature of these papers is that the stabilisation was achieved in the sense of asymptotic stability of the trivial solution, namely the solution of the controlled SDE will tend to zero (the steady state) asymptotically in mean-square or almost surely or in probability and so on.

However, many stochastic systems do not posses a deterministic steady state. Sometimes to achieve the stability of the deterministic steady state precisely is not a wise strategy of sustainable development. For population systems with environmental fluctuation such as tree, fish and other animal population, the stochastic permanence but not extinction is regarded as the control objective. In this situation it is useful to investigate whether or not the solution will converge in distribution (not necessary to converge to zero), which is known as the asymptotic stability in distribution. Especially, the limit invariant measure with supported set in positive cone implies the permanence of population system [4]. It should be pointed out that for some SDE models of the populations of species the solution to the equation does not have the second or even first moment but could be

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stable in distribution (see, e.g., [13]). In some cases, it is more practical to control the systems in the distribution sense rather than to force the solution to tend to zero. For example, for some viruses we can not make them extinct but only hope to control them in certain distributions with the probability of re-outbreak relatively low.

The aim of this paper is to discuss how to design a feedback control based on discrete-time state observations to stabilise a given unstable hybrid SDE in the sense of asymptotic stability in distribution [7]. As this is the first paper on such a problem of stabilisation in distribution, it is better to state it more precisely in terms of mathematics. Consider an unstable hybrid SDE

$$dX(t) = f(X(t), r(t))dt + g(X(t), r(t))dB(t), \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the state, $B(t) = (B_1(t), \cdots, B_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain with its state space S (please see Section 2 for the formal definitions) which represents the system mode. The aim here is to design a feedback control $u(X([t/\tau]\tau), r(t))$ in the drift part so that the controlled system

$$\begin{aligned} dX(t) = &\big(f(X(t), r(t)) + u(X([t/\tau]\tau), r(t))\big)dt \\ &+ g(X(t), r(t))dB(t) \end{aligned} \tag{2}$$

becomes asymptotically stable in distribution. By asymptotic stability in distribution we mean that the law (or probability distribution) of the solution $X(t)$ will converge to a unique probability measure μ_τ on $\mathbb{R}^n$ as t tends to infinity for any initial values $X(0) \in \mathbb{R}^n$ and $r(0) \in S$. Here $\tau > 0$ is a constant and $[t/\tau]$ is the integer part of t/τ.

It should be noted that we assume that we only need to observe $X(t)$ at time points $0, \tau, 2\tau, \cdots$, and the feedback control $u(X([t/\tau]\tau), r(t))$ will be designed only based on those discrete-time observations of the state $X(t)$, but not whole continuous-time state $X(t)$. This is one of the special features in this paper. It is due to this special feature, we need to cope with the mixture of the continuous-time state $X(t)$ and the discrete-time state $X([t/\tau]\tau)$ in the system. As we will explain later, the controlled SDE (2) can be regarded as a hybrid stochastic differential delay equation (SDDE). Although the asymptotic stability in distribution of hybrid SDDEs was studied before, for example, in [34], the existing theory is not applicable to our special controlled SDE (2). Instead, we will develop a new approach consisting of two stages: (i) Imaging if whole continuous-time state $X(t)$ is observable, we can then design the feedback control $u(X(t), r(t))$ to make the auxiliary SDE

$$\begin{aligned} dX(t) = &\big(f(X(t), r(t)) + u(X(t), r(t))\big)dt \\ &+ g(X(t), r(t))dB(t) \end{aligned} \tag{3}$$

to be asymptotically stable in distribution. (ii) Show that there is a positive number τ^* such that the feedback control $u(X([t/\tau]\tau), r(t))$ formed from the same control function $u(\cdot, \cdot)$ but based on the discrete-time state observations will make the controlled system (2) to be asymptotically stable in distribution provided $\tau < \tau^*$. We will also give a lower bound on τ^* which is computable numerically. We will see that stage (i) is relatively easier but stage (ii) is very technical. Let us begin to develop our new approach.

2 Mathematical Preliminaries

2.1 Notations

Throughout this paper, unless otherwise specified, we let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathcal{B}(\mathbb{R}^n)$ denote the family of all Borel measurable sets in $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, then $|x|$ is its Euclidean norm. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^TA)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If A is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalues, respectively. By $A > 0$ and $A \geq 0$, we mean A is positive and non-negative definite, respectively. If both a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $\mathbb{N}_+$ denote the set of nonnegative integers.

We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). For a subset $\bar{\Omega}$ of Ω, $I_{\bar{\Omega}}$ denotes its indicator function. Let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$, $t \geq 0$, be a right-continuous irreducible Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

2.2 Basic assumptions and discussions on the equations

Consider an n-dimensional hybrid SDE (1) on $t \geq 0$, where $f : \mathbb{R}^n \times S \to \mathbb{R}^n$ and $g : \mathbb{R}^n \times S \to \mathbb{R}^{n\times m}$ are Borel measurable functions satisfying the following assumption.

Assumption 2.1 *There is a pair of positive constants α_1 and α_2 such that*

$$|f(x,i) - f(y,i)| \le \alpha_1 |x-y|$$

and

$$|g(x,i) - g(y,i)| \le \alpha_2 |x-y|$$

for all $x, y \in \mathbb{R}^n$ and $i \in S$.

It is easy to see from Assumption 2.1 that

$$|f(x,i)| \le \alpha_1|x| + d_1, \quad |g(x,i)| \le \alpha_2|x| + d_2, \tag{4}$$

for all $(x,i) \in \mathbb{R}^n \times S$, where $d_1 = \max_{i\in S}|f(0,i)|$ and $d_2 = \max_{i\in S}|g(0,i)|$.

It is well known (see, e.g., [19]) that under Assumption 2.1 the hybrid SDE (1) has a unique global solution $X(t)$ on $t \ge 0$ for any given initial values $X(0) \in \mathbb{R}^n$ and $r(0) \in S$. In addition, the pth moment of solution is bounded for any $p > 0$, which indicates the controllability of the system. Assume that this given SDE does not have the desired property of stability in distribution and we are required to design a feedback control $u(X(\delta_t), r(t))$, which is based on the the discrete-time observations of the state at times $0, \tau, 2\tau, \cdots$, to stabilise the system, where we have used the notation $\delta_t = [t/\tau]\tau$ for convenience. We see that the bigger τ could save more on the cost of state observations as the observations can be made less frequently. To make the design of the feedback control simpler, we will seek a linear form of feedback control, namely $u(X(\delta_t), r(t)) = A(r(t))X(\delta_t)$, where $A(i) \in \mathbb{R}^{n\times n}$ for all $i \in S$, and we will often write $A(i) = A_i$. The underlying controlled system (2) therefore becomes

$$\begin{aligned} dX(t) = &\big(f(X(t), r(t)) + A(r(t))X(\delta_t)\big)dt \\ &+ g(X(t), r(t))dB(t). \end{aligned} \tag{5}$$

Accordingly, our two-stage approach becomes to find N matrices A_i plus a positive number τ^*, as bigger as possible, so that this controlled system is asymptotically stable in distribution provided $\tau < \tau^*$. We should point out that in many practical situations, the matrices A_i have their structure form of $A_i = F_i G_i$ with $F_i \in \mathbb{R}^{n\times l}$ and $G_i \in \mathbb{R}^{l\times n}$ for some positive integer $l < n$. There are two cases which are known as: (i) State feedback: design F_i's when G_i's are given; (ii) Output injection: design G_i's when F_i's are given. We will treat two cases together by stating our new results in terms of matrices A_i but will explain how to deal with these two cases in Section 4.

We observe that the controlled system (5) is in fact a hybrid SDDE with a bounded variable delay. Indeed, if we define the bounded variable delay $\nu : [0,\infty) \to [0,\tau]$ by

$$\nu(t) = t - k\tau \quad \text{for } k\tau \le t < t(k+1)\tau \tag{6}$$

and $k = 0, 1, 2, \cdots$, then equation (5) can be written as

$$\begin{aligned} dX(t) =&[f(X(t,r(t)) + A(r(t))X(t-\nu(t))]dt \\ &+g(X(t), r(t)dB(t) \end{aligned} \tag{7}$$

It is therefore known (see, e.g., [19]) that under Assumption 2.1, for any given initial values $X(0) = x \in \mathbb{R}^n$ and $r(0) = i \in S$, equation (5) has a unique solution on $t \ge 0$. We will denote the solution by $X_{x,i}(t)$ while by $r_i(t)$ the Markov chain starting from i at time 0. Moreover, the solution has the property that $\mathbb{E}|X_{x,i}(t)|^p < \infty$ for all $t \ge 0$ and any $p > 0$.

However, the joint process $(X_{x,i}(t), r_i(t))$ does not have the Markov property on $t \ge 0$. Fortunately, the process does have a Markov property at discrete times $k\tau$. As a matter of fact, if we know $(X_{x,i}(k\tau), r_i(k\tau))$ at time $k\tau$, then $(X_{x,i}(t), r_i(t))$ on $t \ge k\tau$ is uniquely determined by solving equation (5) with initial vales $(X_{x,i}(k\tau), r_i(k\tau))$ but the information on how the process reaches $(X_{x,i}(k\tau), r_i(k\tau))$ starting from (x,i) is of no further use. In particular, $\{(X_{x,i}(k\tau), r_i(k\tau))\}_{k\ge 0}$ forms a discrete-time time-homogeneous Markov process and we define its k-step transition probability measure on $\mathbb{R}^n \times S$ by $p(k, x, i; dy \times \{j\})$. That is,

$$\begin{aligned} &\mathbb{P}((X_{x,i}(k\tau), r_i(k\tau)) \in U \times V) \\ =&\sum_{j\in V}\int_U p(k,x,i;dy\times\{j\}) \end{aligned} \tag{8}$$

for any $U \in \mathcal{B}(\mathbb{R}^n)$ and $V \subset S$.

2.3 Definition of stability in distribution

To state the definition of asymptotically stable in distribution, we still need a few more notations. Denote $\mathcal{C}_\tau$ the family of continuous functions ξ from $[0,\tau]$ to $\mathbb{R}^n$ with norm $\|\xi\|_\tau = \sup_{s\in[0,\tau]}|\xi(s)|$. Denote by $\mathcal{P}(\mathcal{C}_\tau)$ the family of probability measures on $\mathcal{C}_\tau$. For $P_1, P_2 \in \mathcal{P}(\mathcal{C}_\tau)$, define the Wasserstein metric $d_{\mathbb{L}}$ by

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{\phi\in\mathbb{L}}\Big|\int_{\mathcal{C}_\tau}\phi(\xi)P_1(d\xi) - \int_{\mathcal{C}_\tau}\phi(\zeta)P_2(d\zeta)\Big| \tag{9}$$

where

$$\begin{aligned} \mathbb{L} =&\{\phi : \mathcal{C}_\tau \to \mathbb{R} \text{ satisfying } |\phi(\xi) - \phi(\zeta)| \le \|\xi - \zeta\|_\tau \\ &\text{and } |\phi(\xi)| \le 1 \text{ for } \xi, \zeta \in \mathcal{C}_\tau\}. \end{aligned}$$

Moreover, for $t \ge 0$, define $\hat{X}_{x,i}(t) = \{X_{x,i}(t+s) : 0 \le s \le \tau\}$ which is $\mathcal{C}_\tau$-valued. Denote by $\mathcal{L}(\hat{X}_{x,i}(t))$ the probability measure on $\mathcal{C}_\tau$ generated by $\hat{X}_{x,i}(t)$. We refer readers to [5] for more details about probability measures generated by stochastic processes and the following definition.

It should be mentioned that there are other metrics to measure the distance between two probability measures. Our results in this paper hold for the Wasserstein metric defined above. It is also worth to investigate similar problems in other metrics.

Definition 1 *The controlled system (5) is said to be asymptotically stable in distribution if there exists a probability measure $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$ such that*

$$\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mu_\tau) = 0$$

for all $(x,i) \in \mathbb{R}^n \times S$.

It should be pointed out that in the literature (see, e.g., [33]), the asymptotic stability in distribution is in general defined on the joint process $(\hat{X}_{x,i}(k\tau), r_i(\kappa\tau))$. On the other hand, given that the law of the Markov chain $r_i(t)$ is already known to converge to its unique stationary distribution (see, e.g.,[1]), our definition here only on $\hat{X}_{x,i}(k\tau)$ is consistent with that in the literature.

3 Stabilisation in Distribution

Let us begin this section by stating a key assumption in this paper.

Assumption 3.1 *There exists a positive number β and N symmetric positive definite matrices Q_i $(1 \le i \le N)$ such that*

$$\begin{aligned}&2(x-y)^T Q_i[f(x,i) - f(y,i) + A_i(x-y)]\\&+ \operatorname{trace}[(g(x,i) - g(y,i))^T Q_i(g(x,i) - g(y,i))]\\&+ \sum_{j=1}^N \gamma_{ij}(x-y)^T Q_j(x-y) \le -\beta|x-y|^2 \qquad (10)\end{aligned}$$

for all $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$.

Remark 3.2 *Assumption 3.1 looks quite complicated at first glance. It is actually, to some extend, a version of the one-sided Lipschitz condition. The extra Q_i provides our more freedom to design the controllers when the theorem is applied.*

It is straightforward to show from Assumptions 2.1 and 3.1 that

$$\begin{aligned}&2x^T Q_i[f(x,i) + A_i x] + \operatorname{trace}[g(x,i)^T Q_i g(x,i)]\\&+ \sum_{j=1}^N \gamma_{ij} x^T Q_j x \le -\beta|x|^2 + \beta_1|x| + \beta_2 \qquad (11)\end{aligned}$$

for all $(x,i) \in \mathbb{R}^n \times S$, where β_1 and β_2 are positive numbers.

Throughout this paper, we will set

$$\alpha_3 = \max_{i\in S} \|A_i\| \text{ and } \alpha_4 = \max_{i\in S} \|Q_i A_i\| \qquad (12)$$

while define

$$H_1(\tau) = 6\tau[\tau(\alpha_1+\alpha_3)^2 + \alpha_2^2]e^{6\tau(\tau\alpha_1^2+\alpha_2^2)}, \qquad (13)$$

$$H_2(\tau) = [4\tau(2\tau\alpha_1^2+\alpha_2^2) + 4\tau^2\alpha_3^2]e^{4\tau(2\tau\alpha_1^2+\alpha_2^2)} \qquad (14)$$

for $\tau > 0$ as well as

$$H_3(\tau) = \frac{2H_1(\tau)}{1-2H_1(\tau)} \text{ and } H_4(\tau) = \frac{2H_2(\tau)}{1-2H_2(\tau)} \qquad (15)$$

for small $\tau > 0$ such that $2H_2(\tau) < 1$. We will not mention their definitions any more.

Remark 3.3 *The reason that we write down the forms of $H_1(\tau)$, $H_2(\tau)$, $H_3(\tau)$ and $H_4(\tau)$ explicitly is that we need them to calculate $\tau*$ in Theorem 3.7. Only when the value of $\tau*$ is known, we are able to design the feedback control based on the discrete-time state observations with any two observations having time grip less than $\tau*$. Such a process will be demonstrated in Example 5.1.*

3.1 Lemmas

To show our new results on the stabilisation in distribution, we first present a couple of lemmas.

Lemma 3.4 *Let Assumption 2.1 hold. Let τ be sufficiently small for $2H_1(\tau) < 1$. Then the solution $X_{x,i}(t)$ of equation (5) satisfies*

$$\mathbb{E}|X_{x,i}(t) - X_{x,i}(\delta_t)|^2 \le H_3(\tau)\mathbb{E}|X_{x,i}(t)|^2 + h(\tau) \quad (16)$$

for all $t \ge 0$, where $h(\tau) = \frac{2h_1(\tau)}{1-2H_1(\tau)}$ with $h_1(\tau)$ being defined by

$$h_1(\tau) = 6\tau(\tau d_1^2 + d_2^2)e^{6\tau(\tau\alpha_1^2+\alpha_2^2)}. \qquad (17)$$

Moreover, let τ be sufficiently small for $2H_2(\tau) < 1$, then for any $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$,

$$\mathbb{E}|Z(t) - Z(\delta_t)|^2 \le H_4(\tau)\mathbb{E}|Z(t)|^2 \qquad (18)$$

for all $t \ge 0$, where $Z(t) = X_{x,i}(t) - X_{y,i}(t)$.

Proof. Fix the initial values (x,i) arbitrarily and write $X_{x,i}(t) = X(t)$ simply. Let v be any non-negative integer. For $t \in [v\tau, (v+1)\tau)$, we have $\delta_t = v\tau$. It follows from (5) that

$$X(t) - X(\delta_t) = X(t) - X(v\tau) = \int_{v\tau}^{t} [f(X(s), r(s)) + A(r(s))X(v\tau)]ds + \int_{v\tau}^{t} g(X(s), r(s))dB(s).$$

This together with (4) implies easily that

$$\begin{aligned}
&\mathbb{E}|X(t) - X(\delta_t)|^2 \\
\le& 2\tau\mathbb{E}\int_{v\tau}^{t} (\alpha_1|X(s)| + d_1 + \alpha_3|X(v\tau)|)^2 ds \\
&+ 2\mathbb{E}\int_{v\tau}^{t} (\alpha_2|X(s)| + d_2)^2 ds \\
\le& 2\tau\mathbb{E}\int_{v\tau}^{t} (\alpha_1|X(s) - X(\delta_s)| + d_1 + (\alpha_1 + \alpha_3)|X(v\tau)|)^2 ds \\
&+ 2\mathbb{E}\int_{v\tau}^{t} (\alpha_2|X(s) - X(\delta_s)| + d_2 + \alpha_2|X(v\tau)|)^2 ds \\
\le& 6(\tau\alpha_1^2 + \alpha_2^2)\int_{v\tau}^{t} \mathbb{E}|X(s) - X(\delta_s)|^2 ds \\
&+ 6\tau[\tau(\alpha_1 + \alpha_3)^2 + \alpha_2^2]\mathbb{E}|X(v\tau)|^2 + 6\tau(\tau d_1^2 + d_2^2).
\end{aligned}$$

The Gronwall inequality shows

$$\begin{aligned}
&\mathbb{E}|X(t) - X(\delta_t)|^2 \\
\le& H_1(\tau)\mathbb{E}|X(v\tau)|^2 + h_1(\tau) \\
\le& 2H_1(\tau)\Big(\mathbb{E}|X(t) - X(\delta_t)|^2 + \mathbb{E}|X(t)|^2\Big) + h_1(\tau),
\end{aligned}$$

where $h_1(\tau)$ was defined by (17) already. Consequently

$$\mathbb{E}|X(t) - X(\delta_t)|^2 \le \frac{2H_1(\tau)}{1 - 2H_1(\tau)}\mathbb{E}|X(t)|^2 + \frac{h_1(\tau)}{1 - 2H_1(\tau)}.$$

This implies that (16) holds for $t \in [v\tau, (v+1)\tau)$. But $v \ge 0$ is arbitrary so the first assertion (16) must hold for all $t \ge 0$. The second assertion (18) can be proved in the similar fashion. □

Lemma 3.5 *Let Assumptions 2.1 and 3.1 hold. If τ is sufficiently small for*

$$2H_1(\tau) < 1 \text{ and } \beta > \beta_\tau := 2\alpha_4\sqrt{H_3(\tau)}, \tag{19}$$

then the solution of equation (5) satisfies

$$\mathbb{E}|X_{x,i}(t)|^2 \le \frac{c_2}{c_1}\Big(|x|^2 e^{-0.5t(\beta-\beta_\tau)/c_2} + \frac{2\beta_3}{\beta - \beta_\tau}\Big), \tag{20}$$

for all $t \ge 0$, where $c_1 = \min_{i\in S}\lambda_{\min}(Q_i)$, $c_2 = \max_{i\in S}\lambda_{\max}(Q_i)$ and $\beta_3 = \beta_2 + 2\alpha_4^2 h(\tau)/\beta_\tau + 0.5\beta_1^2/(\beta - \beta_\tau)$ with β_1 and β_2 being specified in (11) and $h(\tau)$ in Lemma 3.4.

Proof. Once again we fix (x, i) arbitrarily and write $X_{x,i}(t) = X(t)$. Let $\theta = 0.5(\beta - \beta_\tau)/c_2$ which is positive. Applying the generalized Itô formula (see, e.g., [19, Theorem 1.14 on page 48]) to $e^{\theta t}X^T(t)Q(r(t))X(t)$ and using (11), we can show easily that

$$\begin{aligned}
&c_1 e^{\theta t}\mathbb{E}|X(t)|^2 - c_2|x|^2 \\
&\le \int_0^t e^{\theta s}\mathbb{E}\Big(-(\beta - \theta c_2)|X(s)|^2 + \beta_1|X(s)| + \beta_2\Big)ds \\
&+ \mathbb{E}\int_0^t 2e^{\theta s}\alpha_4\mathbb{E}(|X(s)||X(s) - X(\delta_s)|)ds.
\end{aligned} \tag{21}$$

But, by Lemma 3.4, we have

$$\begin{aligned}
&2\alpha_4\mathbb{E}(|X(s)||X(s) - X(\delta_s)|) \\
\le& 0.5\beta_\tau\mathbb{E}|X(s)|^2 + \frac{2\alpha_4^2}{\beta_\tau}\mathbb{E}|X(s) - X(\delta_s)|^2 \\
\le& 0.5\beta_\tau\mathbb{E}|X(s)|^2 + \frac{2\alpha_4^2}{\beta_\tau}H_3(\tau)\mathbb{E}|X(s)|^2 + \frac{2\alpha_4^2}{\beta_\tau}h(\tau) \\
=& \beta_\tau\mathbb{E}|X(s)|^2 + \frac{2\alpha_4^2}{\beta_\tau}h(\tau).
\end{aligned} \tag{22}$$

Substituting this into (21) and recalling the definition of θ, we obtain

$$\begin{aligned}
&c_1 e^{\theta t}\mathbb{E}|X(t)|^2 - c_2|x|^2 \\
&\le \int_0^t e^{\theta s}\mathbb{E}\Big(-0.5(\beta - \beta_\tau)|X(s)|^2 \\
&\qquad + \beta_1|X(s)| + \beta_2 + \frac{2\alpha_4^2}{\beta_\tau}h(\tau)\Big)ds \\
&\le \int_0^t e^{\theta s}\beta_3 ds \le (\beta_3/\theta)e^{\theta t},
\end{aligned} \tag{23}$$

where β_3 was defined in the statement of Lemma 3.5. This implies the required assertion (20). □

Lemma 3.6 *Let Assumptions 2.1 and 3.1 hold. If τ is sufficiently small for*

$$2H_2(\tau) < 1 \text{ and } \beta > \beta'_\tau := 2\alpha_4\sqrt{H_4(\tau)}, \tag{24}$$

then for any $(x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$,

$$\mathbb{E}\|\hat{X}_{x,i}(k\tau) - \hat{X}_{y,i}(k\tau)\|_\tau^2 \le c_3|x - y|^2\varepsilon^{-\gamma k\tau} \tag{25}$$

for all $k \in \mathbb{N}_+$, where $c_3 = 4c_2[1 + \tau^2(\alpha_1^2 + \alpha_3^2) + \tau\alpha_2^2]/c_1$ and $\gamma = (\beta - \beta'_\tau)/c_2$.

Proof. Fix any $(x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$ and set $Z(t) = X_{x,i}(t) - X_{y,i}(t)$ and $\hat{Z}(t) = \{Z(t+s) : 0 \le s \le \tau\}$ for $t \ge 0$. So $Z(0) = x - y$. In a similar fashion as (23) was proved, we can apply the generalised Itô formula to

$e^{\gamma t}Z^T(t)Q(r(t))Z(t)$ and then use Assumption 3.1 and Lemma 3.4 to obtain

$$\begin{aligned} &c_1 e^{\gamma t}\mathbb{E}|Z(t)|^2 - c_2|Z(0)|^2 \\ &\le -(\beta-\gamma c_2-\beta'_\tau)\int_0^t e^{\theta s}\mathbb{E}|Z(s)|^2 ds = 0. \end{aligned} \tag{26}$$

This implies

$$\mathbb{E}|Z(t)|^2 \le \frac{c_2}{c_1}|x-y|^2\varepsilon^{-\gamma t}, \quad \forall t \ge 0. \tag{27}$$

Now, for any $k \in \mathbb{N}_+$, it follows easily from equation (5) and Assumption 2.1 that

$$\begin{aligned} &\mathbb{E}\|Z(k\tau)\|_\tau^2 \le 4\mathbb{E}|Z(k\tau)|^2 \\ &+4\int_{k\tau}^{(k+1)\tau}\big[(\tau\alpha_1^2+\alpha_2^2)\mathbb{E}|Z(t)|^2+\tau\alpha_3^2\mathbb{E}|Z(k\tau)|^2\big]dt. \end{aligned}$$

This, together with (27), implies assertion (25) immediately. □

3.2 Main theorem

Theorem 3.7 *Let Assumptions 2.1 and 3.1 hold. Let $\tau_1^*, \dots, \tau_4^*$ be the unique positive roots to the following equations*

$$2H_1(\tau_1^*) = 1 \text{ and } \beta = 2\alpha_4\sqrt{H_3(\tau_2^*)}, \tag{28}$$

$$2H_2(\tau_3^*) = 1 \text{ and } \beta = 2\alpha_4\sqrt{H_4(\tau_4^*)}, \tag{29}$$

respectively, and set $\tau^* = \tau_1^* \wedge \tau_2^* \wedge \tau_3^* \wedge \tau_4^*$. Then for each $\tau < \tau^*$, there exists a unique probability measure $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$ such that

$$\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mu_\tau) = 0 \tag{30}$$

for all $(x,i) \in \mathbb{R}^n \times S$. In other words, equation (5) is asymptotically stable in distribution provided $\tau < \tau^*$.

Proof. We divide the whole proof into three steps in order to make the technical proof more understandable. Fix $\tau < \tau^*$ arbitrarily.

Step 1. We first claim that for any compact subset K of $\mathbb{R}^n$,

$$\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mathcal{L}(\hat{X}_{y,j}(k\tau))) = 0 \tag{31}$$

uniformly in $x, y \in K$ and $i, j \in S$. Note that $\{r(k\tau)\}_{k\in\mathbb{N}_+}$ is a discrete-time ergodic Markov chain with its one-step transition probability matrix $e^{\tau\Gamma}$. Define the stopping time

$$\kappa_{ij} = \inf\{k\tau : r_i(k\tau) = r_j(k\tau),\ k \ge 0\}.$$

Then $\kappa_{ij} < \infty$ a.s. (see, e.g., [1]). Hence, for any $\varepsilon \in (0,1)$, there is a positive number $T_1 > 0$ such that

$$\mathbb{P}(\kappa_{ij} \le T_1) > 1 - \varepsilon/6 \quad \forall i, j \in S. \tag{32}$$

Recalling a known result ([19, p. 99, Theorem 3.24]) that

$$\sup_{(x,i)\in K\times S} \mathbb{E}\Big(\sup_{0\le t\le T_1} |X_{x,i}(t)|^2\Big) < \infty,$$

we see there is a sufficiently large $\rho > 0$ such that

$$\mathbb{P}(\Omega_{x,i}) > 1 - \varepsilon/12 \quad \forall (x,i) \in K \times S, \tag{33}$$

where $\Omega_{x,i} = \big\{\omega \in \Omega : \sup_{0\le t\le T_1} |X_{x,i}(t,\omega)| \le \rho\big\}$. We now fix $x, y \in K$ and $i, j \in S$ arbitrarily. For any $\phi \in \mathbb{L}$ and $k \in \mathbb{N}_+$ with $k\tau \ge T_1$, we have

$$|\mathbb{E}\phi(\hat{X}_{x,i}(k\tau)) - \mathbb{E}\phi(\hat{X}_{y,j}(k\tau))| \le \frac{\varepsilon}{3} + J_1(k\tau), \tag{34}$$

where

$$J_1(k\tau) := \mathbb{E}\Big(I_{\{\kappa_{ij}\le T_1\}}|\phi(\hat{X}_{x,i}(k\tau)) - \phi(\hat{X}_{y,j}(k\tau))|\Big).$$

Set $\Omega_1 = \Omega_{x,i} \cap \Omega_{y,j} \cap \{\kappa_{ij} \le T_1\}$. By the paragraph before (8), we derive

$$\begin{aligned} &J_1(k\tau) \\ =&\mathbb{E}\Big(I_{\{\kappa_{ij}\le T_1\}}\mathbb{E}\big(|\phi(\hat{X}_{x,i}(k\tau)) - \phi(\hat{X}_{y,j}(k\tau))|\big|\mathcal{F}_{\kappa_{ij}}\big)\Big) \\ =&\mathbb{E}\Big(I_{\{\kappa_{ij}\le T_1\}}\mathbb{E}|\phi(\hat{X}_{u,l}(k\tau-\kappa_{ij})) - \phi(\hat{X}_{v,l}(k\tau-\kappa_{ij})|\Big) \\ \le&\frac{\varepsilon}{3} + \mathbb{E}\Big(I_{\Omega_1}\mathbb{E}|\hat{X}_{u,l}(k\tau-\kappa_{ij}) - \hat{X}_{v,l}(k\tau-\kappa_{ij})|\Big), \end{aligned} \tag{35}$$

where $u = X_{x,i}(\kappa_{ij})$, $v = X_{y,j}(\kappa_{ij})$ and $l = r_i(\kappa_{ij}) = r_j(\kappa_{ij})$. Observing that for any given $\omega \in \Omega_1$, $|u| \vee |v| \le \rho$, we can apply Lemma 3.6 to see that there is another positive constant T_2 such that

$$\mathbb{E}|X_{u,l}(k\tau-\kappa_{ij}) - X_{v,l}(k\tau-\kappa_{ij})| \le \frac{\varepsilon}{3} \quad \forall k\tau \ge T_1 + T_2.$$

Substituting this into (35) yields that $J_1(t) \le 2\varepsilon/3$ for all $k\tau \ge T_1 + T_2$. This, together with (34), implies that

$$|\mathbb{E}\phi(\hat{X}_{x,i}(k\tau) - \mathbb{E}\phi(\hat{X}_{y,j}(k\tau))| \le \varepsilon \quad \forall k\tau \ge T_1 + T_2. \tag{36}$$

Since ϕ is arbitrary, we must have

$$d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mathcal{L}(\hat{X}_{y,j}(k\tau))) \le \varepsilon \quad \forall k\tau \ge T_1 + T_2$$

for all $x, y \in K$ and $i, j \in S$. This proves our claim.

Step 2. We next claim that for any $(x,i) \in \mathbb{R}^n \times S$, $\{\mathcal{L}(\hat{X}_{x,i}(k\tau))\}_{k\in\mathbb{N}_+}$ is a Cauchy sequence in $\mathcal{P}(\mathcal{C}_\tau)$ with metric $d_{\mathbb{L}}$. In other words, we need to show that for any $\varepsilon > 0$, there is an integer $k_0 > 0$ such that

$$d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}((v+u)\tau)), \mathcal{L}(\hat{X}_{x,i}(u\tau))) \le \varepsilon \tag{37}$$

for all integers $u \ge k_0$ and $v \ge 1$. Let $\varepsilon \in (0,1)$ be arbitrarily. By Lemma 3.5, there is a $\rho > 0$ such that

$$\mathbb{P}\{\omega \in \Omega : |X_{x,i}(v\tau,\omega)| \le \rho\} > 1 - \varepsilon/8 \quad \forall v \ge 1. \tag{38}$$

For any $\phi \in \mathbb{L}$ and $u \ge 1$, we can then derive, using (8) and (38), that

$$\begin{aligned}
&|\mathbb{E}\phi(\hat{X}_{x,i}((v+u)\tau)) - \mathbb{E}\phi(\hat{X}_{x,i}(u\tau))| \\
=&|\mathbb{E}(\mathbb{E}[\phi(\hat{X}_{x,i}((v+u)\tau))|\mathcal{F}_{v\tau}]) - \mathbb{E}\phi(\hat{X}_{x,i}(u\tau))| \\
=&\Big|\sum_{j\in S}\int_{\mathbb{R}^n} \mathbb{E}\phi(\hat{X}_{y,j}(u\tau))p(v,x,i;dy\times\{j\}) - \mathbb{E}\phi(\hat{X}_{x,i}(u\tau))\Big| \\
\le&\sum_{j\in S}\int_{\mathbb{R}^n} |\mathbb{E}\phi(\hat{X}_{y,j}(u\tau)) - \mathbb{E}\phi(\hat{X}_{x,i}(u\tau))|p(v,x,i;dy\times\{j\}) \\
\le&\frac{\varepsilon}{2} + \sum_{j\in S}\int_{B_\rho} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{y,j}(u\tau)), \mathcal{L}(\hat{X}_{x,i}(u\tau)))p(v,x,i;dy\times\{j\})
\end{aligned}$$

where $B_\rho = \{y \in \mathbb{R}^n : |y| \le \rho\}$. But, by (31), there is a positive integer k_0 such that

$$d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{y,j}(u\tau)), \mathcal{L}(\hat{X}_{x,i}(u\tau))) \le \frac{\varepsilon}{2} \quad \forall u \ge k_0$$

whenever $(y,j) \in B_\rho \times S$. We therefore obtain

$$|\mathbb{E}\phi(\hat{X}_{x,i}((v+u)\tau)) - \mathbb{E}\phi(\hat{X}_{x,i}(u\tau))| \le \varepsilon$$

for $u \ge k_0$ and $v \ge 1$. As this holds for any $\phi \in \mathbb{L}$, we must have (37) as claimed.

Step 3. It follows from Step 2 that there is a unique $\mu_\tau \in \mathcal{P}(\mathcal{C}_\tau)$ such that

$$\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{0,1}(k\tau)), \mu_\tau) = 0.$$

This, together with (31), implies that

$$\begin{aligned}
&\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mu_\tau) \\
\le&\lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{x,i}(k\tau)), \mathcal{L}(\hat{X}_{0,1}(k\tau))) \\
&+ \lim_{k\to\infty} d_{\mathbb{L}}(\mathcal{L}(\hat{X}_{0,1}(k\tau)), \mu_\tau) = 0
\end{aligned}$$

for all $(x,i) \in \mathbb{R}^n \times S$, which is assertion (30). □

4 Implementation: Structure Feedback Controls

The application of our main result, Theorem 3.7, depends on the design of N matrices A_i's. In this section we will explain how to design the matrices in the situation of structure feedback controls. That is, we will look for the matrices in the structure form of $A_i = F_i G_i$ with $F_i \in \mathbb{R}^{n\times l}$ and $G_i \in R^{l\times n}$ for some positive integer l. Two cases can be discussed: (i) State feedback: design F_i's when G_i's are given; (ii) Output injection: design G_i's when F_i's are given (see, e.g., [14]). But, due to the limit of the length of paper, we only give the details of the state feedback and leave the case of output injection to the readers.

4.1 State feedback: general results

We will use the technique of linear matrix inequalities (LMIs, see, e.g., [35]) to design F_i's. Under Assumption 2.1, we will introduce some rules which lead to the design of F_i's. Our first rule is:

Rule 4.1 *Find N pairs of symmetric matrices Q_i and $\hat{Q}_i$ $(1 \le i \le N)$ with $Q_i > 0$ such that*

$$\begin{aligned}
&2(x-y)^T Q_i[f(x,i) - f(y,i)] \\
&+\text{trace}[(g(x,i) - g(y,i))^T Q_i(g(x,i) - g(y,i))] \\
\le&(x-y)^T \hat{Q}_i(x-y)
\end{aligned} \tag{39}$$

for all $(x,y,i) \in \mathbb{R}^n \times \mathbb{R}^n \times S$.

It is very easy to meet this rule under Assumption 2.1. The simplest one is to let all $Q_i = I_n$, the $n\times n$ identity matrix, and $\hat{Q}_i = (2\alpha_1 + \alpha_2^2)I_n$. However, it is wise to find alternative matrices in order to make use of the given structures of f and g so that the following rule can be met more easily.

Rule 4.2 *Find a solution of matrices F_i of the LMIs*

$$\hat{Q}_i + F_i G_i + G_i^T F_i^T + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in S. \tag{40}$$

This rule guarantees that Assumption 3.1 is satisfied with

$$\beta = -\max_{i\in S} \lambda_{\max}\Big(\hat{Q}_i + F_i G_i + G_i^T F_i^T + \sum_{j=1}^N \gamma_{ij} Q_j\Big). \tag{41}$$

We hence have the following corollary.

Corollary 4.3 *Under Assumption 2.1, find matrices F_i ($i \in S$) by Rules 4.1 and 4.2. Then Theorem 3.7 holds with β defined by (41) and $A_i = F_i G_i$.*

4.2 State feedback: linear case

In the previous subsection, the matrices F_i or G_i are determined in two steps based on Rules 4.1 and 4.2. In particular, Rule 4.1 may have to be performed manually. However, in the linear case, we can set up a set of LMIs so that Matlab can be used to search for matrices Q_i, F_i or G_i together automatically. Assume that f and g have the linear forms

$$\begin{cases} f(x,i) = u_i + U_i x, \\ g(x,i) = (v_{1i} + V_{1i}x, \cdots, v_{mi} + V_{mi}x) \end{cases} \tag{42}$$

for $(x,i) \in \mathbb{R}^n \times S$, where $u_i, v_{1i}, \cdots, v_{mi} \in \mathbb{R}^n$ and $U_i, V_{1i}, \cdots, V_{mi} \in \mathbb{R}^{n\times n}$.

Assumption 3.1 means that we need to find F_i and $Q_i = Q_i^T > 0$ ($i \in S$) in order for

$$\begin{aligned} &Q_i(U_i + F_iG_i) + (U_i + F_iG_i)^T Q_i \\ &+ \sum_{k=1}^m V_{ki}^T Q_i V_{ki} + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in S. \end{aligned} \tag{43}$$

These matrix inequalities are not linear in Q_i and F_i's. However, if we set $J_i = Q_iF_i$, then they become the following LMIs

$$\begin{aligned} &Q_iU_i + J_iG_i + U_i^T Q_i + G_i^T J_i^T \\ &+ \sum_{k=1}^m V_{ki}^T Q_i V_{ki} + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in S. \end{aligned} \tag{44}$$

The following corollary is therefore immediate.

Corollary 4.4 *Consider the controlled system (5) with f and g having the linear forms (42). If the LMIs (44) have their solutions J_i and $Q_i = Q_i^T > 0$, then Theorem 3.7 holds with $A_i = Q_i^{-1} J_i G_i$ and*

$$\begin{aligned} \beta = -\max_{i\in S} \lambda_{\max}\Big(&Q_iU_i + J_iG_i + U_i^T Q_i + G_i^T J_i^T \\ &+ \sum_{k=1}^m V_{ki}^T Q_i V_{ki} + \sum_{j=1}^N \gamma_{ij} Q_j \Big). \end{aligned} \tag{45}$$

5 Example

Let us now discuss an example to illustrate our theory.

Example 5.1 Consider a linear hybrid SDE

$$\begin{aligned} dx(t) = &[u(r(t)) + U(r(t))x(t)]dt \\ &+ [v(r(t)) + V(r(t))x(t)]dB(t) \end{aligned} \tag{46}$$

on $t \geq 0$. Here $x(t) = (x_1(t), x_2(t))^T$; $B(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix};$$

and the system matrices are $u_1 = (10, 5)^T$, $u_2 = (15, 10)^T$, $v_1 = (0.5, 0.3)^T$, $v_2 = (0.4, 0.6)^T$,

$$U_1 = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -1 & 2 \\ -2 & 1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & -0.1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -0.1 & -0.1 \\ -0.1 & 0.1 \end{bmatrix}.$$

By Theorem 10.13 in Page 383 of [19], it is not hard to verify that both $x_1(t)$ and $x_2(t)$ will tend to infinity almost surely as $t \to \infty$, namely, every sample path will get large as the time advances, which indicates the instability in distribution. In addition, the K-S test [21] for 2-dimensional data is used to test the difference between probability distribution functions (PDFs) at time points with 0.2 difference, namely difference between PDF at $t = 90 + 0.2k$ and PDF at $t = 90 + 0.2(k+1)$ for $k = 0, 1, 2, ..., 99$. The simulation is conducted by using the EM method with 5000 paths. The result is drawn in the upper plot of Fig. 1. Logarithm of the corresponding p values is drawn in the lower plot [1]. It is clear that the difference between PDFs does not vanish for large t and the p values also indicate the significant difference between PDFs at quite close time points with much more than 99% confidence. All those observations indicate the instability in distribution.

We therefore need to design a feedback control to stabilize the system in distribution sense. We consider the state feedback control and assume that we could only observe x_1-component in mode 1 and x_2-component in mode 2. In terms of mathematics, we look for a controller function of the form $u(x,i) = F_iG_ix$ with $G_1 = (1, 0)$, $G_2 = (0, 1)$ and $F_1,\ F_2 \in R^{2\times 1}$. Our aim is to find F_1 and F_2 as well as a positive number τ^* so that the controlled system

$$\begin{aligned} dx(t) = &[u(r(t)) + U(r(t))x(t) + F_{r(t)}G_{r(t)}x(\delta_t)]dt \\ &+ [v(r(t)) + V(r(t))x(t)]dB(t) \end{aligned} \tag{47}$$

[1] Since the p values are extremely small, we take logarithm of them to make them visible

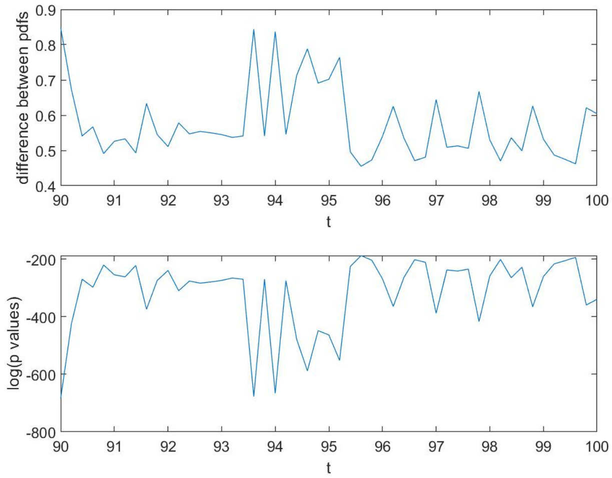

Fig. 1. Upper: difference between PDFs. Lower: logarithm of the corresponding p values

becomes stable in distribution as long as $\tau < \tau^*$.

To apply Corollary 4.4, it is easy to verify that both Q_1 and Q_2 are the 2×2 identity matrix. $J_1 = (-3, 0)^T$ and $J_2 = (0, -2)^T$ form a set of solutions to the LMI (44). In fact, the left-hand-side terms of (44) become

$$U_i + J_i G_i + U_i^T + G_i^T J_i^T + V_i^T V_i = \begin{bmatrix} -1.98 & 0 \\ 0 & -1.98 \end{bmatrix} \tag{48}$$

for both $i = 1$ and 2. We therefore see that, by setting $F_i = J_i$ $(i = 1, 2)$, Theorem 3.7 can be applied to the controlled system (47) with $\beta = 1.98$ and

$$A_1 = \begin{bmatrix} -3 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}.$$

To determine τ^*, we compute

$$\alpha_1 = 3, \ \alpha_2 = 0.1414214, \ \alpha_3 = 3, \ \alpha_4 = 3.$$

Consequently

$$H_1(\tau) = 6\tau(36\tau + 0.02)e^{6\tau(9\tau+0.02)},$$
$$H_2(\tau) = [4\tau(18\tau + 0.02) + 36\tau^2]e^{4\tau(9\tau+0.02)}.$$

We can then compute

$$\tau_1^* = 0.04513, \ \tau_2^* = 0.01476, \ \tau_3^* = 0.06284, \ \tau_4^* = 0.02086$$

and hence $\tau^* = 0.01476$. By Corollary 4.4 and Theorem 3.7, we can finally conclude that the controlled system (47) with $F_1 = (-3, 0)^T$ and $F = (0, -2)^T$ is stable in distribution as long as $\tau < 0.01476$.

Now, we choose $\tau = 0.01$ and use the EM method with 5000 paths to simulate samples. The K-S test for 2-dimensional data is used to test the difference between PDFs at time points with 1 difference, namely difference between PDF at $t = k$ and PDF at $t = (k + 1)$ for $i = 1, 2, ..., 100$. The result is drawn in the upper plot of Fig. 2. It is clear that the difference between PDFs vanishes as t advances. The p values drawn in the lower plot of Fig. 2, which are larger than 0.2 for large t, also indicate that we can not reject that PDFs at quite distant time points follow the same distribution.

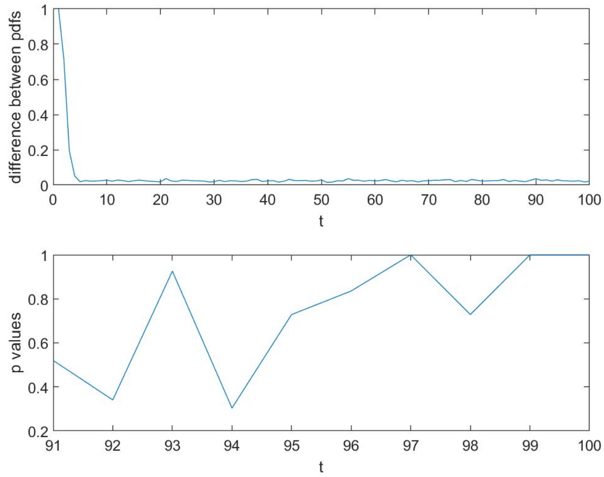

Fig. 2. Upper: difference between PDFs. Lower: the p values

6 Conclusion and future research

In this paper, we initiated the new problem of stabilisation in distribution for a class of hybrid SDEs. We showed that the stabilisation in distribution can be achieved by linear feedback controls based on the discrete-time state observations. We also gave a lower bound on τ^* so that the feedback control works as long as $\tau \le \tau^*$, where τ stands for the duration between two consecutive state observations (namely one state observation per τ unit of time). The lower bound can be determined numerically so that our theory can be applied more easily in practice. It should be pointed out that the lower bound derived in this paper is not optimal. It is worth to search a larger τ^*, which can help to save the cost as the frequency of the observations can be reduced when τ is allowed to be larger. To demonstrate how to implement our new theory, we discuss how to design the feedback control in the case of the state feedback. In the nonlinear systems, we proposed a couple of simple rules so that the feedback control can be designed by following these simple rules. In the linear systems, we showed the design of the feedback control can be done by solving LMIs and this can be achieved by using Matlab. An example were discussed to illustrate our theory.

It is still an open problem that whether a general theorem employing the general set-up of Lyapunov functions can be built up and proved, which may help to further release the constraint on τ^*. We will investigate this problem in future. Since we only used the discrete-time observations for the state to design the control in this paper, it is also worth to investigate designing the controllers based on discrete-time observations for both the state, $X(t)$ and the mode, $r(t)$.

At last, it should be noted that we proved the existence and uniqueness of such a μ_τ in Theorem 3.7 for those controlled hybrid stochastic systems. But, the explicit form the μ_τ can rarely be found. Therefore, to find a reliable numerical method that can approximate μ_τ efficiently is another very interesting open question.

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