Distributed Nash Equilibrium Seeking under Quantization Communication \star

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Abstract

This paper investigates Nash equilibrium (NE) seeking problems for noncooperative games over multi-players networks with finite bandwidth communication. A distributed quantized algorithm is presented, which consists of local gradient play, distributed decision estimating, and adaptive quantization. Exponential convergence of the algorithm is established, and a relationship between the convergence rate and the bandwidth is quantitatively analyzed. Finally, a simulation of an energy consumption game is presented to validate the proposed results.

Key words: Distributed Nash equilibrium seeking, quantization communication, exponential convergence.

1 Introduction

Game theory as a powerful tool for analyzing the interactions between rational decision-makers, has penetrated into various fields, including biology (Hammerstein & Selten, 1994), economics (Choi, Taleizadeh, & Yue, 2020) and computer sciences (Shoham, 2008). Nash equilibrium (NE), named after John Forbes Nash, Jr., is an important strategy profile of players in noncooperative games. Recently, advances in network optimization techniques have been applied to develop NE seeking algorithms (Salehisadaghiani & Pavel, 2016; Ye & Hu, 2017; Gadjov & Pavel, 2018; Lu, Jing, & Wang, 2018; De Persis & Grammatico, 2019; Zeng, Chen, Liang, & Hong, 2019; Zhu, Yu, Wen, & Chen, 2020).

Note that the above NE seeking algorithms mainly focused on infinite precision transmission. However, the communication bandwidth is limited in the actual network, such as underwater vehicles and low-cost unmanned aerial vehicles systems. Hence, each player should sample and quantize its real value into finite bits before transmitting it while receiving it from its neighbors. This quantized communication process overcomes the bandwidth constraints, significantly reduces storage consumption, and is suitable for solving the practical network problems (Rabbat & Nowak, 2005; Nedic, Olshevsky, Ozdaglar, & Tsitsiklis, 2008). In the existing quantization works (Yuan, Xu, Zhao, & Rong, 2012; Yi & Hong, 2014; Li, Liu, Soh, & Xie, 2017; Liu, Wu, Tian, & Ling, 2021; Kajiyama, Hayashi, & Takai, 2021), the following three problems were mainly concerned: i) How can it ensure convergence even with inexact iterations throughout the distributed quantized algorithm? ii) What is the required minimum bandwidth when convergence is obtained? iii) How does the bandwidth affect convergence rate? To answer these problems, a zooming-in quantization rule is used in (Yi & Hong, 2014), which proved that merely three bits could obtain the optimal solution. After that, only onebit transmission was required in (Li et al., 2017), which explicitly characterized the proposed algorithm's sublinear convergence rate. Further, the work (Kajiyama et al., 2021) guaranteed a linear convergence rate of the quantized gradient tracking algorithm.

Although the above three questions have been widely discussed in distributed quantized optimization problems, few answers for the distributed NE seeking problem. Primarily because the cost function of each player in distributed NE seeking problems depends on the actions of all players, while the cost function of each player in a distributed optimization only depends on the action of itself. Thus, the update of the action of each player is much more complex in distributed NE seeking problems, which further brings technical difficulties in the design of the adaptive quantization scheme that depends on

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the trajectories of the actions of the players. Hence, the quantization scheme in distributed optimization problems can not be directly extended to distributed NE seeking, which motivates our works. Notably, the literature (Nekouei, Nair, & Alpcan, 2016) tried to answer these problems for the distributed NE seeking, but in which each player was required to broadcast their quantized actions to all other players. It is still a centralized method in essence.

We take a step from our previous works on distributed NE seeking (Liang, Yi, & Hong, 2017) and distributed quantized cooperative problems (Ma, Ji, Sun, & Feng, 2018; Chen & Ji, 2020) toward distributed quantized NE seeking. The main contributions are as follows.

1) This is the first work to reveal that a distributed quantized NE seeking algorithm achieves exponential convergence under any positive bandwidth.

2) An affine inequality explicitly characterizes the relation between the convergence rate and bandwidth, which indicts linearly increased convergence rate would linearly increase the bandwidth requirement.

3) Our work is an extension to the distributed NE seeking with infinite precision transmission (Salehisadaghiani & Pavel, 2016; Ye & Hu, 2017; Gadjov & Pavel, 2018; Lu et al., 2018; De Persis & Grammatico, 2019). Further, the assumption on the Lipschitz condition of the augment game mapping is not required anymore.

4) Compared with the only distributed quantized NE seeking work (Nekouei et al., 2016), the communication graph must be fully connected. Our algorithm is distributed, and each player only interacts the quantized information with its neighbors, not all other players.

The rest of the paper is organized as follows. In Section 2, the problem is formulated. In Section 3, we propose the distributed quantized NE seeking algorithm based on the designed adaptive quantization scheme. In Section 4, the main results, including the convergence analysis and the quantitative analysis on bandwidth, are discussed. An energy consumption game example is presented in Section 5 and the conclusion is given in Section 6.

Notation: Denote \mathbb{R}^n as the *n*-dimensional Euclidean space. For $x \in \mathbb{R}^n$, denote the 2-norm by ||x||. $\operatorname{col}\{x_i\}_{i \in \mathcal{I}}$ stacks the vector x_i as a new column vector in the order of the index set \mathcal{I} . For matrices A and B, the Kronecker product is denoted as $A \otimes B$. Denote by 0_n , $1_n \in \mathbb{R}^n$, and $I_n \in \mathbb{R}^{n \times n}$ the vectors of all zero and ones, and the identical matrix. A function $J : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if, for all $x, y \in \mathbb{R}^n$ and $x \neq y$, J(tx + (1 - t)y) < tJ(x) + (1 - t)J(y) with $t \in (0, 1)$. A function $J(x) : \mathbb{R}^n \to \mathbb{R}$ is radially unbounded on \mathbb{R}^n if for every $x_n \in \mathbb{R}^n$ such that $||x_n|| \to \infty$, we also have $J(x_n) \to \infty$. For a differentiable function $J(x) : \mathbb{R}^n \to \mathbb{R}$, its gradient $\nabla_x J(x) = \operatorname{col}\{\frac{\partial J}{\partial x_i}\}_{i \in \{1, \dots, n\}} \in \mathbb{R}^n$. The minimum integer not smaller than $a \in \mathbb{R}$ is denoted as [a].

2 Problem statement

Consider the noncooperative game $G = \{\mathcal{V}, J_i, x_i\}$, where $\mathcal{V} = \{1, \dots, N\}$ is the set of players involved in the game. A variable $x_i \in \mathbb{R}^{n_i}$ is the action of player $i \in \mathcal{V}$. A differentiable function $J_i(x_i, x_{-i}) \in \mathbb{R}$ is the local cost function of each player $i \in \mathcal{V}$, where $x_i \in \mathbb{R}^{n_i}$ is its own action and $x_{-i} \in \mathbb{R}^{n-n_i}$ for $n = \sum_{i=1}^N n_i$ denotes all players' actions except player i.

The aim of the NE seeking is that each selfish player i obtains x_{-i} through communication for minimizing its own cost function $J_i(x_i, x_{-i}) : \mathbb{R}^n \to \mathbb{R}$. The definition of the NE is given as follows.

Definition 1 (Nash Equilibrium) Given a game $G = \{\mathcal{V}, J_i, x_i\}$, a vector of actions $x^* = (x_1^*, \cdots, x_N^*) \in \mathbb{R}^n$ is a NE if $J_i(x_i^*, x_{-i}^*) \leq \inf_{x_i \in \mathbb{R}^{n_i}} J_i(x_i, x_{-i}^*), \forall i \in \mathcal{V}$ holds.

We describe the information sharing between players as an undirected and connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} as the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ as the edge set. Denote $\mathcal{N}_i \subseteq \mathcal{V}$ as the set of neighbors of player *i*. The adjacency matrix of the graph \mathcal{G} is denoted as $\mathcal{A} = [a_{ij}]_{N \times N}$, with $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$ otherwise. The corresponding Laplacian matrix is $L_{\mathcal{G}} = [l_{ij}]_{N \times N}$, with $l_{ij} = -a_{ij}$ if $i \neq j$, and $l_{ij} = \sum_{j\neq i}^{N} a_{ij}$ otherwise. For an undirected and connected graph \mathcal{G} , one has that $L_{\mathcal{G}} \mathbf{1}_N = \mathbf{0}_N$, $\mathbf{1}_N^T L_{\mathcal{G}} = \mathbf{0}_N^T$ and all eigenvalues of $L_{\mathcal{G}}$ are real numbers and could be arranged by an ascending order $0 = \lambda_1 < \cdots \leq \lambda_N$. To proceed, we further make the following technical assumptions.

Assumption 1 For every $i \in \mathcal{V}$, the local cost function $J_i(x_i, x_{-i})$ is continuously differentiable, strictly convex and radially unbounded in $x_i \in \mathbb{R}^{n_i}$ for any fixed x_{-i} .

Assumption 1 was widely used in the existing related works such as Assumption 2 in (Gadjov & Pavel, 2018) and Assumption 1 in (De Persis & Grammatico, 2019).

Definition 2 The game mapping $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is defined as $F(x) = col\{\nabla_i J_i(x_i, x_{-i})\}_{i \in \mathcal{V}}$.

The following assumptions formulate the restricted strongly monotone and the Lipschitz continuity of the elements of the game mapping F(x).

Assumption 2 The game mapping F(x) satisfies

 F(x) is μ-strongly monotone with the constant μ > 0, that is, for any x, y ∈ ℝⁿ

$$F(x) - F(y), x - y \ge \mu ||x - y||^2$$

• For every $i \in \mathcal{V}$, the gradient $\nabla_{x_i} J_i(x_i, x_{-i})$ is uniformly Lipschitz continuous in x_i , that is, there is some constants $\theta_i \geq 0$ such that for any fixed $x_{-i} \in$ \mathbb{R}^{n-n_i} .

$$\|\nabla_{x_i} J_i(x_i, x_{-i}) - \nabla_{x_i} J_i(y_i, x_{-i})\| \le \theta_i \|x_i - y_i\|.$$

Moreover, for every $i \in \mathcal{V}$ the gradient $\nabla_{x_i} J_i(x_i, x_{-i})$ is uniformly Lipschitz continuous in x_{-i} , that is, there is some constants $\theta_{-i} \geq 0$ such that for any fixed $x_i \in \mathbb{R}^{n-n_i},$

$$\|\nabla_{x_{-i}} J_i(x_i, x_{-i}) - \nabla_{x_{-i}} J_i(x_i, y_{-i})\| \le \theta_{-i} \|x_{-i} - y_{-i}\|.$$

Define $\theta = (\theta_i^2 + \theta_{-i}^2)^{1/2}$. It follows from Assumption 3 in (Gadjov & Pavel, 2018) and Assumption 2 in (De Persis & Grammatico, 2019) that Assumptions 1-2 ensure the existence and uniqueness of the NE for the game G.

Assumption 3 The initial states of all players satisfy $||x_i(0)||_{\infty} \leq M \text{ for } i \in \mathcal{V} \text{ and } ||x^*||_{\infty} \leq M'.$

Remark 1 It is worth pointing out that in existing works on distributed quantized consensus, the assumption on the initial state, that is, $||x_i(0)||_{\infty} \leq M$ was required to estimate the upper bound of tracking errors, see Assumption 2 in (You & Xie, 2011) and Assumption 3 in (Ma et al., 2018). These errors guide the design of the scaling function to avoid the saturation of quantizers at the initial time. However, in the game context, the upper bound of tracking errors is related to both $x_i(t)$ and x^* . Hence, $||x^*||_{\infty} \leq M'$ is also needed.

Algorithm design 3

In the distributed framework, each player has no access to the exact action of all other players, and it only receives a fixed number of bits from its neighbors. Frequently that means each player $i, i \in \mathcal{V}$ needs to estimate all other players' actions. We denote this estimated action as $\boldsymbol{x}^i = (x_1^i; \cdots; x_N^i) \in \mathbb{R}^n$, where x_i^i is actual actions of player *i* and x_j^i is an estimated action of player j.

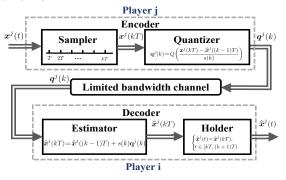


Fig. 1. Communication Process

Since the communication digital channels among players exist bandwidth constraints, each player i interacts the quantized version of x^j with its neighbors $j, j \in \mathcal{N}_i$. As shown in Fig. 1, quantized communication process for $(j,i) \in \mathcal{E}$ is summarized as the following two parts.

• Quantized communication process for $(j, i) \in \mathcal{E}$.

(i) Encoder: Player j samples its own estimation $x^{j}(t)$ at the fixed sampling time $kT, k \in \mathbb{N}$, then it encodes $x^{j}(kT)$ as the quantized message $q^{j}(k)$ with a uniform quantizer as follows,

$$\begin{aligned} \mathbf{q}^{j}(0) &= Q\left(\frac{\boldsymbol{x}^{j}(0)}{s(0)}\right), \\ \mathbf{q}^{j}(k) &= Q\left(\frac{\boldsymbol{x}^{j}(kT) - \widehat{\boldsymbol{x}}^{j}((k-1)T)}{s(k)}\right), \ k \in \mathbb{N}^{+}, \end{aligned}$$

where the multi-quantizer $Q(\cdot) = 1_n \otimes q(\cdot) \in \mathbb{R}^n$ with $n\log_2(2L+1)$ bits is designed as follows,

$$q(x) = \begin{cases} 0, & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ i, & \text{if } \frac{2i-1}{2} \le x < \frac{2i+1}{2}, i = 1, \cdots, L, \\ L, & \text{if } x \ge \frac{2L+1}{2}, \\ -q(-x), & \text{if } x \le -\frac{1}{2}. \end{cases}$$

Then, player *j* broadcasts the quantized message $\mathbf{q}^{j}(k)$ to its neighbor i at time kT.

(ii) Decoder: Player *i* receives $\mathbf{q}^{j}(k)$ from player *j*, then estimates $\boldsymbol{x}^{j}(t)$ as $\hat{\boldsymbol{x}}^{j}(t)$. The decoder is designed as follows,

$$\widehat{\boldsymbol{x}}^{j}(0) = s(0)\mathbf{q}^{j}(0), \tag{2}$$

$$\widehat{\boldsymbol{x}}^{j}(kT) = \widehat{\boldsymbol{x}}^{j}((k-1)T) + s(k)\mathbf{q}^{j}(k), \qquad (3)$$

$$\widehat{\boldsymbol{x}}^{j}(t) = \widehat{\boldsymbol{x}}^{j}(kT), \, kT \leq t < (k+1)T, \, k \in \mathbb{N}^{+}.$$

• Quantized parameters design:

(i) Select the sampling period T > 0 satisfying

$$(e^{\alpha\lambda_N T} - 1)(e^{\gamma T} - 1)\rho\varepsilon^{-1} \le a_1 < 1,$$
 (5)

where $\gamma = \varepsilon \nu \beta / 4$, $\alpha > \frac{\varepsilon}{\lambda_2} (\frac{\theta^2}{\mu} + \theta)$, $a_1, \beta \in (0, 1)$, $\nu = \langle \Gamma , \mu \rangle$

$$2\lambda_{\min}\left(\begin{bmatrix}\frac{\mu}{N} & -\frac{\sigma}{\sqrt{N}}\\ -\frac{\theta}{\sqrt{N}} & \frac{\alpha\lambda_2}{\varepsilon} - \theta\end{bmatrix}\right), \ \rho = \left(\frac{\theta_{\mathbf{F}}\varepsilon}{\alpha\lambda_N} + 1\right)\frac{8\alpha\lambda_N}{\nu^2\beta\varepsilon\sqrt{1-\beta}}.$$
(ii) Design the scaling function $s(k)$ as follows.

$$s(k) = s(0)e^{-\gamma kT},\tag{6}$$

where $s(0) = \frac{\varepsilon \nu M_0}{\alpha \lambda_N} \sqrt{1 - \beta} e^{-\gamma T - \alpha \lambda_N T}$ and $M_0 \triangleq M + M'$. (iii) Choose L as a positive integer satisfying

$$L \ge \max\left\{\frac{M_0}{s(0)}, \left\lceil \frac{\sqrt{Nn}e^{\varepsilon\nu\beta T/4 + \alpha\lambda_N T}}{2a_2} - \frac{1}{2} \right\rceil \right\},\$$

where $0 < a_2 < 1 - a_1$.

Remark 2 Notably, an exponentially decaying scaling function (6) is used here for the exponential convergence of the quantization errors, which is important to the exponential convergence of the NE seeking algorithm. On the other hand, s(k) should be large enough such that the quantizer keeps non-saturated. That is, the convergence rate of s(k) cannot be faster than that of the tracking error $\tilde{x}^{i}(t) = x^{i}(t) - x^{*}$. Hence, the designed convergence rate γ for s(k) matches that of $\tilde{x}^{i}(kT)$, whose convergence rate will be proved as γ in the following Theorem 1.

By using $\hat{x}^{i}(t)$ and $\hat{x}^{j}(t)$, player *i* updates its estimated action as follows,

$$\dot{\boldsymbol{x}}^{i}(t) = \alpha \sum_{j=1}^{N} a_{ij}(\widehat{\boldsymbol{x}}^{j}(t) - \widehat{\boldsymbol{x}}^{i}(t)) - R_{i} \nabla_{i} J_{i}(\boldsymbol{x}^{i}), \quad (7)$$

where $R_i = [\mathbf{0}_{n_1 \times n_i}, \cdots, \varepsilon I_{n_i \times n_i}, \cdots, \mathbf{0}_{n_N \times n_i}]^T \in \mathbb{R}^{n \times n_i}$.

The dynamic (7) is developed from (Gadjov & Pavel, 2018; Lu et al., 2018), which requires continuous communication and accurate message interaction. In (7), each player just exchanges the quantized information with its neighbors at the sampling instant. Thus, our approach significantly saves communication resources.

Remark 3 Compared with quantized NE seeking literature (Nekouei et al., 2016), in which the algorithm as $x_{k+1}^i = x_k^i + \mu_k \frac{\partial}{\partial x^i} U_i\left(x_k^i, \mathcal{D}_k\left(\boldsymbol{x}_k^{-i}\right)\right)$, where $\mathcal{D}_k\left(\boldsymbol{x}_k^{-i}\right)$ represents quantized actions received from all other players at time k. It means that the communication graph is assumed to be fully connected, in contrast, we need not this assumption anymore.

4 Main results

We prove that the dynamic (7) exponentially converges to a NE in Subsection 4.1 and then discuss quantitative properties on the required bandwidth in Subsection 4.2.

4.1 Convergence analysis

First, we present the following lemma to prove that the equilibrium of the dynamic (7) is a NE.

Lemma 1 The equilibrium x^* of the dynamic (7) is a NE of game G.

The proof is similar to the proof of Lemma 4 in (Gadjov & Pavel, 2018) and thus omitted here.

Definition 3 The augmented game mapping is defined as $\mathbf{F}(\mathbf{x}) = col\{\nabla_i J_i(x^i)\}_{i \in \mathcal{V}} : \mathbb{R}^{n^2} \to \mathbb{R}^n$.

The following lemma shows the Lipschitz continuity of the augmented mapping $\mathbf{F}(\boldsymbol{x})$.

Lemma 2 Under Assumptions 1 and 2, the augmented mapping $\mathbf{F}(\mathbf{x})$ is θ -Lipschitz continuous in $\mathbf{x} \in \mathbb{R}^{n^2}$.

PROOF. Follows from Assumptions 1-2, for any $x, y \in \mathbb{R}^n$ such that $x_i, y_i \in \mathbb{R}^{n_i}$ and $x_{-i}, y_{-i} \in \mathbb{R}^{n-n_i}$, there is

$$\begin{aligned} \|\nabla_{i}J_{i}(x_{i}, x_{-i}) - \nabla_{i}J_{i}(y_{i}, y_{-i})\| \\ &= \|\nabla_{i}J_{i}(x_{i}, x_{-i}) - \nabla_{i}J_{i}(y_{i}, x_{-i}) + \nabla_{i}J_{i}(y_{i}, x_{-i}) \\ &- \nabla_{i}J_{i}(y_{i}, y_{-i})\| \\ &\leq \left(\beta\theta_{i}^{2}\|x_{i} - y_{i}\|^{2} + \beta/(\beta - 1)\theta_{-i}^{2}\|x_{-i} - y_{-i}\|_{2}^{2}\right)^{\frac{1}{2}}, \quad (8) \end{aligned}$$

where $\beta > 1$. Choose $\beta = 1 + \theta_{-i}^2 / \theta_i^2$ to rewrite (8) as

$$\|\nabla_i J_i(x) - \nabla_i J_i(y)\| \le (\theta_i^2 + \theta_{-i}^2)^{1/2} \|x - y\|, \tag{9}$$

Due to the arbitrary of $x, y \in \mathbb{R}^n$, Lemma 2 holds.

Note that the Lipschitz continuity of $\mathbf{F}(x)$ was assumed in most distributed NE seeking works, see Assumption 4 in (Gadjov & Pavel, 2018) and Assumption 5 in (De Persis & Grammatico, 2019). Lemma 2 indicts that using Assumptions 1-2, the Lipschitz continuity of $\mathbf{F}(x)$ can be yielded such that it need not be assumed in this work.

Define $\Phi_1 = \frac{1}{\sqrt{N}} \mathbf{1}_N$ and $\Phi_2 \in \mathbb{R}^{N \times (N-1)}$ to construct a unitary matrix $\Phi = [\Phi_1, \Phi_2]$ such that $\Phi^T L_{\mathcal{G}} \Phi =$ diag $(0, \lambda_2, \cdots, \lambda_N)$. Observe that $\Phi_2^T \Phi_2 = I_{N-1}$ and $\Phi_2 \Phi_2^T = I_N - \Phi_1 \Phi_1^T$. Further, define tracking errors as $\widetilde{x}^i = x^i - x^*$ and estimation errors as $e^i \triangleq x^i - \widehat{x}^i$. Stack the above vectors as $x^* = 1_N \otimes x^*$, $\widetilde{x} = \operatorname{col}{\{\widetilde{x}^i\}_{i \in \mathcal{V}}}$ and e = $\operatorname{col}{\{e^i\}_{i \in \mathcal{V}}}$, respectively. The coordinate transformation of \widetilde{x} and e is written as follows,

$$\overline{\boldsymbol{x}}_1(t) = (\boldsymbol{\Phi}_1^T \otimes \boldsymbol{I}_n) \widetilde{\boldsymbol{x}}(t) \in \mathbb{R}^n, \ \overline{\boldsymbol{x}}_2(t) = (\boldsymbol{\Phi}_2^T \otimes \boldsymbol{I}_n) \widetilde{\boldsymbol{x}}(t) \in \mathbb{R}^{(N-1)n}, \\ \overline{\boldsymbol{e}}_1(t) = (\boldsymbol{\Phi}_1^T \otimes \boldsymbol{I}_n) \boldsymbol{e}(t) \in \mathbb{R}^n, \ \overline{\boldsymbol{e}}_2(t) = (\boldsymbol{\Phi}_2^T \otimes \boldsymbol{I}_n) \boldsymbol{e}(t) \in \mathbb{R}^{(N-1)n}.$$

Next, we will prove the exponential convergence of the quantized NE seeking dynamic (7), to do so, we present the following lemma whose proof is given in Appendix.

Lemma 3 Construct the Lypaunov function as follows, $V(\overline{\boldsymbol{x}}) = \frac{1}{2}(\|\overline{\boldsymbol{x}}_1\|^2 + \|\overline{\boldsymbol{x}}_2\|^2).$

Under Assumptions 1-3, along with the dynamic (7), if the following three inequalities hold when $k = k_1, \forall k_1 \in \mathbb{N}$,

$$\|\boldsymbol{e}(kT)\| \leq \frac{a_2 \varepsilon \nu M_0 \sqrt{(1-\beta)Nn} e^{-\alpha \lambda_N T}}{2\alpha \lambda_N} e^{-\gamma (k+1)T}, (10)$$

$$\|\boldsymbol{e}(t)\| < \frac{\varepsilon \nu M_0 \sqrt{(1-\beta)Nn}}{2\alpha\lambda_N} e^{-\gamma(\lfloor t/T \rfloor + 1)T}, \ t \in [0, kT), (11)$$

$$V(\overline{\boldsymbol{x}}) \leq (NnM_0^2/2)e^{-2\lfloor t/T \rfloor T}, \ t \in [0, kT], \ k \in \mathbb{N},$$
(12)

then (10)-(12) hold when $k = k_1 + 1$.

Theorem 1 Given an undirected and connected graph \mathcal{G} , under Assumptions 1-3, the dynamic (7) exponentially converges to a NE of game G.

PROOF. We prove Theorem 1 via the principle of induction. When k = 0, it follows from $s(0) = \varepsilon \nu M_0/(\alpha \lambda_N) \sqrt{1-\beta} e^{-\gamma T - \alpha \lambda_N T}$ and Assumption 3 that (10)-(12) hold. Using the conclusion of Lemma 3, if (10)-(12) hold when $k = k_1$, it follows that (10)-(12) hold when $k = k_1 + 1$. We conclude that (10)-(12) hold for any $k \in \mathbb{N}$. By (12), \overline{x} exponentially converges to zero, which implies that $\tilde{x}(t)$ exponentially converges to zero. Then, based on Lemma 1, Theorem 1 holds.

4.2 Quantitative analysis on bandwidth

In this subsection, Theorem 2 gives the required minimum bandwidth to ensure the exponential convergence of the dynamic (7). Theorem 3 discusses the relation between the required communication bandwidth and the convergence rate. The bandwidth is defined as follows.

Definition 4 The bandwidth between the communication process $(j, i) \in \mathcal{E}$ is defined as

$$\mathcal{B} = \max_{(j,i)\in\mathcal{E}} \{\lim_{t\to\infty} \sup_{t'} \{\frac{1}{t'} \sum_{\substack{t_{\mu}^{ji} \leq t'}} r_{ji}(k), \ t' \geq t\} \} \ bits/sec$$

where t_k^{ji} , $k \in \mathbb{N}$ are the sampling instants and $r_{ji}(k)$ is the bits required to be transmitted at t_k^{ji} .

Theorem 2 Given an undirected and connected graph \mathcal{G} , under Assumptions 1-4, the dynamic (7) exponentially converges to a NE under any positive bandwidth.

PROOF. Choose $\frac{\alpha}{\varepsilon} > \frac{1}{\lambda_2}(\frac{\theta^2}{\mu} + \theta)$. In this case, the parameters ν and ρ are two constants. For any T > 0,

$$\lim_{\varepsilon \to 0} (e^{\alpha \lambda_N T} - 1) (e^{\frac{\varepsilon \nu \beta}{4} T} - 1) \rho \varepsilon^{-1} = \lim_{\varepsilon \to 0} \alpha \lambda_N T^2 \varepsilon \nu \beta \rho / (4\varepsilon) = 0,$$

thus, (5) is satisfied. If ε is chosen properly, then

$$\lim_{\varepsilon \to 0} \sqrt{Nn} e^{\varepsilon \nu \beta T/4 + \alpha \lambda_N T} / (2a_2) = \sqrt{Nn} / (2a_2).$$

In this case, the transmitted quantized information $q^i(k)$ is represented by $n \log_2 \left(2 \left[\max\left\{ \frac{M_0}{s_1}, \frac{\sqrt{Nn}}{2a_2} \right\} \right] + 1 \right)$ bits at each T. Since T could be chosen by any positive constant, the bandwidth \mathcal{B} could be any positive constants.

Remark 4 By Shannon's rate-distortion theory, if there is a distributed algorithm achieving exponential convergence with the rate γ , then the communication bandwidth $\mathcal{B} > \gamma \log_2 e > 0$. Particularly, Theorem 2 establishes a sufficient and necessary condition on the required bandwidth for the exponential convergence of the dynamic (7).

Naturally, much bandwidth means relaxed communication constraints, which contributes to the fast convergence rate for the dynamic (7). We give an affine inequality in the following theorem to describe this fact. **Theorem 3** Given an undirected and connected graph \mathcal{G} , under Assumptions 1-3, the convergence rate and the minimum bandwidth required in the dynamic (7) satisfy

$$\mathcal{B} \le c_1 \gamma + c_2, \tag{13}$$

where $c_1, c_2 > 0$ are some constants independent of \mathcal{B} and γ .

PROOF. We prove (13) via computing the upper bound of the minimum bandwidth \mathcal{B}_0 for any given convergence rate γ_0 . Choose $\alpha/\varepsilon = c_0$ and β as two positive constants such that ν is a positive constant. Then, the convergence rate $\gamma_0 = \varepsilon_0 \nu \beta/4$ is determined by ε_0 . Let the sampling instants $T_0 = 1/(b_1\gamma_0 + b_2)$, where $b_1 = \frac{4c_0\lambda_N/(\nu\beta)+1}{\ln(\rho_0)}$, $b_2 = \frac{\rho_0\rho\nu\beta}{4a_1}$ and $\rho_0 > 1$. Using $T_0 \leq \min\{1/b_1\gamma_0, 1/b_2\}$, $e^{\alpha\lambda_N T_0} - 1 \leq e^{\alpha\lambda_N T_0}$ and $e^{\gamma_0 T_0} - 1 \leq \gamma_0 T_0 e^{\gamma_0 T_0}$, we observe the chosen T_0 satisfies $(e^{\alpha\lambda_N T_0} - 1)(e^{\gamma_0 T_0} - 1)\varepsilon^{-1} \leq e^{\alpha\lambda_N T_0 + \gamma_0 T_0}\nu\beta T_0/4 < a_1/\rho$,

which grantees that T_0 satisfies (5). Next, we estimate the number of quantization levels L_0 for computing \mathcal{B}_0 . Recalling from the definition of the bandwidth, it could be computed via

$$\mathcal{B} = \lim_{n \to \infty} \frac{\sum_{k=0}^{n-1} r_{ji}(k)}{nT} = \lim_{n \to \infty} \frac{\sum_{k=1}^{n-1} r_{ji}(k)}{(n-1)T}.$$
 (14)

It implies that the choice of the quantization levels at the initial time has no effect on the value of the communication bandwidth. Hence, we only consider the

case
$$L_0 = \left| \frac{\sqrt{Nne^{\gamma_0 T_0 + \alpha \lambda_N T_0}}}{2a_2} - \frac{1}{2} \right|$$
 and we obtain
 $\mathcal{B}_0 = \log_2(2L_0 + 1)/T_0 = \log_2(\sqrt{Nn}/a_2e^{\gamma_0 T_0 + \alpha \lambda_N T_0})/T_0$
 $= \log_2 e(1 + 4c_0\lambda_N/\nu\beta)\gamma_0 + \log_2(nN)/2 - \log_2 a_2/T_0$
 $\stackrel{\Delta}{=} c_1\gamma_0 + c_2,$

Since the chosen of γ_0 is arbitrary, Theorem 3 holds.

Remark 5 The problem of the minimum bandwidth for the fixed convergence rate γ_0 is complicated and still unsolved in the quantized control. In fact, for a given convergence rate γ_0 , Theorem 3 provides an upper bound of the minimum bandwidth $\mathcal{B}_0 = c_1\gamma_0 + c_2$, which partially deals with this problem.

5 An example

In this section, we utilize an energy consumption game for heating ventilation and air conditioning systems (Ye & Hu, 2017) to illustrate the effectiveness of our results. The cost function of player i is modeled as

$$f_i(x) = a_i ||x_i - b_i||^2 + x_i^T \left(c \sum_{i=1}^N x_i + d \right), \ i = 1, \cdots, 5,$$

where $x_i \in \mathbb{R}^3$, $a_i = 0.96 - 0.5i$, $b_i = [9; 11; 13] + 4(i-1)1_3$, c = 0.001, and d = [10; 12; 14]. Based

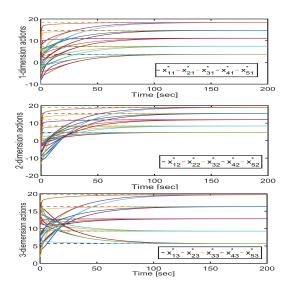


Fig. 3. Three dimensions of the theoretical NE x_i^* (dotted line), and dynamics x^i (solid line) for players $i, i = 1, \dots, 5$.

on theoretical analysis, the unique Nash equilibrium is computed as $x^* = [x_1^*; \cdots; x_5^*] =$ [3.7608; 4.7165; 5.6722; 7.4709; 8.3692; 9.2675; 11.1473; 11.9816; 12.8159; 14.7838; 15.5462; 16.3086; 18.3724; 19.0535; 19.7345] $\in \mathbb{R}^{15}$. The initial estimation is set as $x^i(0) = [x_1(0); \cdots; x_5(0)] + 2(i-1)\mathbf{1}_{15} \in \mathbb{R}^{15}$ with $x_i(0) = [-10+3i; -5-2i; 10-2i]$. The communication graph is given in Fig.2.

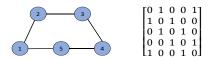


Fig. 2. Communication graph.

The parameters of quantization scheme are chosen as: (a) the sampling period T=0.1 sec; (b) the scaling function $s(k)=0.1e^{-0.1k}$; (c) the bandwidth $\mathcal{B}=270$ bit/sec.

We perform the proposed the dynamic (7) with $\alpha = 1$. Fig.3 compares theoretical NE and distributed estimated actions of all players for the three dimensions. It shows that the distributed estimates accurately track the theoretical NE [cf. Theorem 1].

Fig. 4 compares the tracking errors $\|\boldsymbol{x}(t)-1_5 \otimes x^*\|$ of our quantized algorithm with that of the existing distributed NE seeking algorithm presented in (Gadjov & Pavel, 2018; Lu et al., 2018) under an ideal communication channel. It shows that the quantized communication brings the difficulty to the NE seeking.

Fig. 5 shows simulation results for Theorem 3. It proves that for any given convergence rate γ_0 , the actually required bandwidth \mathcal{B} is less than the upper bound of the minimum bandwidth $\mathcal{B}_0 = c_0 \gamma_0 + c_1$ [cf. Theorem 3].

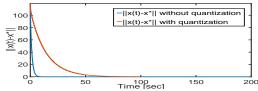


Fig. 4. The comparison of tracking errors between distributed NE seeking algorithm without quantization (blue line) and the dynamic (7) (red line).

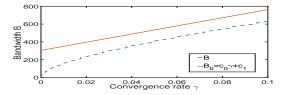


Fig. 5. The upper bound of minimum bandwidth \mathcal{B}_0 (red line) and practical bandwidth \mathcal{B} (blue line) for the given γ .

6 Conclusions

We investigated the distributed NE seeking with finite bandwidth constraints among each pair of players. To solve this problem, a distributed NE seeking algorithm with an adaptive quantization scheme was proposed. Theoretical and experimental results showed that for any bandwidth constraints, the proposed algorithm could achieve exponential convergence. In addition, an affine inequality was given to describe the relation between convergence rate and the required bandwidth.

7 Appendix

For notations simplicity, define

$$a(t) = (NnM_0^2/2)e^{-2\gamma \lfloor t/T \rfloor T}, \ \mathbf{R} = \operatorname{diag}\{R_1, \cdots, R_N\},$$

$$b(t) = \frac{\varepsilon \nu M_0 \sqrt{(1-\beta)Nn}}{2\alpha \lambda_N} e^{-\gamma (\lfloor t/T \rfloor + 1)T}, \ \Theta = \mathbf{F}(\boldsymbol{x}) - \mathbf{F}(\boldsymbol{x}^*).$$

Step 1. We prove that the following conclusion.

$$\|\boldsymbol{e}(t)\| \leq b(k_1T), \ \forall \ t \in (k_1T, t') \Rightarrow V(\overline{\boldsymbol{x}}) \leq a(t)$$

$$\forall \ t \in (k_1T, t'], \ \forall \ t' \in (k_1T, (k_1+1)T].$$
(15)

Follow the update dynamic (7) that

$$\begin{aligned} \dot{\overline{x}}_{1}(t) &= -(\varPhi_{1}^{T} \otimes I_{n}) \mathbf{R} \Theta, \\ \dot{\overline{x}}_{2}(t) &= -\alpha(\varPhi_{2}^{T} L_{\mathcal{G}} \varPhi_{2} \otimes I_{n}) \overline{x}_{2}(t) - (\varPhi_{2}^{T} \otimes I_{n}) \mathbf{R} \Theta \\ &+ \alpha(\varPhi_{2}^{T} L_{\mathcal{G}} \varPhi_{2} \otimes I_{n}) \overline{e}_{2}(t), \end{aligned}$$

Using
$$\Phi_2 \Phi_2^T = I_N - \Phi_1 \Phi_1^T$$
, compute $\dot{V}(\overline{\boldsymbol{x}})$ as
 $\dot{V} \leq -\widetilde{\boldsymbol{x}}^T \mathbf{R} \Theta - \alpha \lambda_2 \|\overline{\boldsymbol{x}}_2\|^2 + \alpha \overline{\boldsymbol{x}}_2^T [\Phi_2^T L_{\mathcal{G}} \Phi_2 \otimes I_n] \overline{\boldsymbol{e}}_2.$ (16)

Since
$$\widetilde{\boldsymbol{x}} = (\Phi_1 \otimes I_n) \overline{\boldsymbol{x}}_1 + (\Phi_2 \otimes I_n) \overline{\boldsymbol{x}}_2 = \widetilde{\boldsymbol{x}}_1 + \widetilde{\boldsymbol{x}}_2$$
, it yields

$$-\widetilde{\boldsymbol{x}}^{T}\mathbf{R}\Theta = -\widetilde{\boldsymbol{x}}_{1}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\widetilde{\boldsymbol{x}}_{2}+\boldsymbol{x}^{*})-\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})] -\widetilde{\boldsymbol{x}}_{2}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\widetilde{\boldsymbol{x}}_{2}+\boldsymbol{x}^{*})-\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})] -\widetilde{\boldsymbol{x}}_{1}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})-\mathbf{F}(\boldsymbol{x}^{*})] -\widetilde{\boldsymbol{x}}_{2}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})-\mathbf{F}(\boldsymbol{x}^{*})].$$
(17)

Since $\mathbf{F}(1_N \otimes x) = F(x)$ for any $x \in \mathbb{R}^n$ and F(x) is strong monotone, the third term of (17) is written as

$$\widetilde{\boldsymbol{x}}_{1}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})-\mathbf{F}(\boldsymbol{x}^{*})] = \varepsilon \overline{\boldsymbol{x}}_{1}^{T}/(N)^{1/2}[F(\overline{\boldsymbol{x}}_{1}/(N)^{1/2} + \boldsymbol{x}^{*})-F(\boldsymbol{x}^{*})] \ge \varepsilon \mu/N \|\overline{\boldsymbol{x}}_{1}\|^{2}, (18)$$

where $(1_N^T \otimes I_n)\mathbf{R} = \varepsilon I_n$ is utilized. Recalling $\|\mathbf{R}\| = \varepsilon \|\Phi_2\| = \varepsilon$, it follows from Lemma 2 that

$$\widetilde{\boldsymbol{x}}_{2}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})-\mathbf{F}(\boldsymbol{x}^{*})] \leq \varepsilon \theta / (N)^{1/2} \|\overline{\boldsymbol{x}}_{1}\| \|\overline{\boldsymbol{x}}_{2}\|.$$
(19)

Similarly, we further obtain

$$-\widetilde{\boldsymbol{x}}_{1}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\widetilde{\boldsymbol{x}}_{2}+\boldsymbol{x}^{*})-\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})] \leq \varepsilon \theta/(N)^{1/2} \|\overline{\boldsymbol{x}}_{1}\|\|\overline{\boldsymbol{x}}_{2}\|, \\ -\widetilde{\boldsymbol{x}}_{2}^{T}\mathbf{R}[\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\widetilde{\boldsymbol{x}}_{2}+\boldsymbol{x}^{*})-\mathbf{F}(\widetilde{\boldsymbol{x}}_{1}+\boldsymbol{x}^{*})] \leq \varepsilon \theta\|\overline{\boldsymbol{x}}_{2}\|^{2}.$$
(20)

Summing up both side of (18)-(20), we have

$$-\widetilde{\boldsymbol{x}}^T \mathbf{R} \Theta \leq 2\varepsilon \theta / (N)^{1/2} \|\overline{\boldsymbol{x}}_1\| \|\overline{\boldsymbol{x}}_2\| + \varepsilon \theta \|\overline{\boldsymbol{x}}_2\|^2 - \varepsilon \mu / N \|\overline{\boldsymbol{x}}_1\|^2.$$

Then the derivative of V in (16) is rewritten as follows,

$$\dot{V} \leq 2\varepsilon\theta/(N)^{1/2} \|\overline{\boldsymbol{x}}_1\| \|\overline{\boldsymbol{x}}_2\| + \varepsilon\theta \|\overline{\boldsymbol{x}}_2\|^2 - \varepsilon\mu/N \|\overline{\boldsymbol{x}}_1\|^2
-\alpha\lambda_2 \|\overline{\boldsymbol{x}}_2\|^2 + \alpha\overline{\boldsymbol{x}}_2^T (\Phi_2^T L_{\mathcal{G}} \Phi_2 \otimes I_n)\overline{\boldsymbol{e}}_2
\leq -\varepsilon\nu V + \alpha\overline{\boldsymbol{x}}_2^T (\Phi_2^T L_{\mathcal{G}} \Phi_2 \otimes I_n)\overline{\boldsymbol{e}}_2,$$
(21)

where the second inequality holds using the fact $\frac{\alpha}{\varepsilon} > \frac{1}{\lambda_2} (\frac{\theta^2}{\mu} + \theta)$. Since $\Phi_2 \Phi_2^T = I_N - \Phi_1 \Phi_1^T$ and $\mathbf{1}_N^T L = \mathbf{0}_N^T$, the second term of (21) is expressed as follows,

$$lpha \overline{oldsymbol{x}}_2^T (\Phi_2^T L_\mathcal{G} \Phi_2 \otimes I_n) \overline{oldsymbol{e}}_2 \!=\! lpha \widetilde{oldsymbol{x}}^T (L_\mathcal{G} \otimes I_n) oldsymbol{e}.$$

Thus, (21) is equivalent to $\dot{V} \leq -\varepsilon \nu V + \alpha \tilde{\boldsymbol{x}}^T (L_{\mathcal{G}} \otimes I_n) \boldsymbol{e}.$ (22)

It implies that

$$\dot{V} \le -\varepsilon \nu V/2 - \left(\varepsilon \nu \|\widetilde{\boldsymbol{x}}\|^2/4 - \alpha \|\widetilde{\boldsymbol{x}}\| \|L_{\mathcal{G}}\| \|\boldsymbol{e}\|\right).$$
(23)

Use $\alpha \|\widetilde{\boldsymbol{x}}\| \|L_{\mathcal{G}}\| \|\boldsymbol{e}\| \leq \varepsilon \nu \|\widetilde{\boldsymbol{x}}\|^2 / 4 + \alpha^2 \|L_{\mathcal{G}}\|^2 \|\boldsymbol{e}\|^2 / (\varepsilon \nu)$ such that $\dot{V} \leq -\varepsilon \nu V / 2 + \alpha^2 \|L_{\mathcal{G}}\|^2 \|\boldsymbol{e}\|^2 / (\varepsilon \nu)$. Select $\beta \in (0,1)$ such that $\dot{V} \leq -\varepsilon \beta \nu V / 2 - \varepsilon (1-\beta) \nu V / 2 + \alpha^2 \|L_{\mathcal{G}}\|^2 \|\boldsymbol{e}\|^2 / (\varepsilon \nu)$, and for $t \in (k_1 T, (k_1+1)T)$, $V(\overline{\boldsymbol{x}}) \geq a((k_1+1)T) > a(t), \|\boldsymbol{e}(t)\| < b(t) \Rightarrow \dot{V} \leq \frac{-\varepsilon \nu V}{2}.$ (24)

Since $a(t) \equiv a(k_1T), t \in [k_1T, (k_1 + 1)T)$, it follows from (24) that (15) holds for $t' \in (kT, (k_1 + 1)T)$. We then consider the situation on $t' = (k_1 + 1)T$. Assume that $V(\overline{x}) > a((k_1 + 1)T)$ for $t \in (k_1T, (k_1 + 1)T]$. Based on (23), there is $\dot{V}(\overline{x}) \leq -\varepsilon\nu V/2$. Hence, $\mathcal{A} = \{\overline{x}|V(\overline{x}) \leq a((k_1 + 1)T)\}$ is an invariant set. Note that $V(\overline{x}) \leq a(k_1T)$ when $t = k_1T$, then from (24), $\overline{x}(t)$ enters into the set \mathcal{A} not later than the time $t = (k_1+1)T$. Thus, (15) holds for $t = (k_1 + 1)T$.

Step 2. We prove the following conclusion

$$V(\overline{\boldsymbol{x}}) \leq a(k_1T), \ \forall \ t \in (k_1T, t'] \Rightarrow \|\boldsymbol{e}(t)\| < b(k_1T), \\ \forall \ t \in (k_1T, t'], \ \forall t' \in [k_1T, (k_1+1)T).$$

$$(25)$$

From (7), $\dot{\boldsymbol{e}}(t) = -\alpha \mathbf{L} \widetilde{\boldsymbol{x}}(t) + \alpha \mathbf{L} \boldsymbol{e}(t) - \mathbf{R} \Theta$. It implies that $\boldsymbol{e}(t) = e^{\alpha \mathbf{L}(t-k_1T)} \boldsymbol{e}(k_1T) - \int_{k_1T}^t e^{\alpha \mathbf{L}(t-\tau)} (\alpha \mathbf{L} \widetilde{\boldsymbol{x}}(\tau) + \mathbf{R} \Theta) \, \mathrm{d}\tau.$ (26)

Taking the Euclidean norm of both side of (26), it yields

$$\|\boldsymbol{e}(t)\| \leq \| e^{\alpha \mathbf{L}(t-k_1T)} \| \| \boldsymbol{e}(k_1T) \| + \int_{k_1T}^t \| e^{\alpha \mathbf{L}(t-\tau)} \| (\|\alpha \mathbf{L}\widetilde{\boldsymbol{x}}(\tau)\| + \|\mathbf{R}\Theta\|) \, \mathrm{d}\tau, \ k_1T < t \leq t'.$$
(27)

For any $t \in (k_1T, t']$, the first term of (27) satisfies $\| e^{\alpha \mathbf{L}(t-k_1T)} \| \| \boldsymbol{e}(k_1T) \| \leq e^{\alpha \lambda_N T} \| \boldsymbol{e}(k_1T) \|.$ (28)

For any $t \in (k_1T, t']$, due to $\|\widetilde{\boldsymbol{x}}(t)\| \leq NnM_0^2 e^{-2\gamma t}$, then

$$\int_{k_{1}T}^{t} \| e^{\alpha \mathbf{L}(t-\tau)} \| \| \alpha \mathbf{L} \widetilde{\mathbf{x}}(\tau) \| d\tau$$

$$\leq \alpha \lambda_{N} \sqrt{Nn} M_{0} e^{\alpha \lambda_{N} t} \int_{k_{1}T}^{t} e^{-\alpha \lambda_{N} \tau} d\tau \int_{k_{1}T}^{(k+1)T} e^{-\frac{\varepsilon \nu \beta}{4} \tau} d\tau$$

$$\leq \frac{4\sqrt{Nn} M_{0}}{\varepsilon \nu \beta} (e^{\alpha \lambda_{N} T} - 1) (e^{\frac{\varepsilon \nu \beta}{4} T} - 1) e^{-\frac{\varepsilon \nu \beta}{4} (k+1)T}. \quad (29)$$

For any $t \in (k_1T, t']$, the third term of (27) satisfies

$$\int_{k_{1}T}^{t} \| e^{\alpha \mathbf{L}(t-\tau)} \| \| \mathbf{R} \Theta \| d\tau$$

$$\leq (e^{\alpha \lambda_{N}T} - 1) \frac{4\sqrt{Nn}M_{0}\theta_{F}}{\alpha \lambda_{N}\beta \nu} (e^{\frac{\varepsilon\nu\beta}{4}T} - 1)e^{-\frac{\varepsilon\nu\beta}{4}(k_{1}+1)T}.(30)$$

Denote $\Delta_{\boldsymbol{e}}^{k} = \sup_{t \in (k_{1}T, t']} \|\boldsymbol{e}(t)\|$. By (28)-(30),

$$\Delta_{\boldsymbol{e}}^{k} \leq \mathrm{e}^{\alpha\lambda_{N}T} \|\boldsymbol{e}(k_{1}T)\| + \left(\frac{\theta_{\mathbf{F}}\varepsilon}{\alpha\lambda_{N}} + 1\right) \frac{4\sqrt{Nn}M_{0}}{\varepsilon\nu\beta} \\ (e^{\alpha\lambda_{N}T} - 1)(e^{\frac{\varepsilon\nu\beta}{4}T} - 1)e^{-\frac{\varepsilon\nu\beta}{4}(k_{1}+1)T}.$$
(31)

From $\|\boldsymbol{e}(k_1T)\| \leq a_2 e^{-\alpha\lambda_N T} b(k_1T)$, we have

$$||e(t)|| \le e^{\alpha \lambda_N T} ||e(k_1 T)|| \le a_2 b(k_1 T), \ t \in (k_1 T, t'].$$
 (32)

It follows from (5) and (31) that

$$\left(\frac{\theta_{\mathbf{F}}\varepsilon}{\alpha\lambda_{N}}+1\right)\frac{4\sqrt{NnM_{0}}}{\varepsilon\nu\beta}\left(e^{\alpha\lambda_{N}T}-1\right)\left(e^{\frac{\varepsilon\nu}{4}T}-1\right)e^{-\frac{\varepsilon\nu}{4}(k+1)T}\right) \leq a_{1}\frac{\varepsilon\nu M_{0}}{2\alpha\lambda_{N}}\sqrt{Nn(1-\beta)}e^{-\frac{\varepsilon\nu}{4}\beta(k+1)T}.$$
(33)

Since $a_1 + a_2 < 1$, (31)-(33) yields (25).

Step 3. We prove the following conclusion

$$\|\boldsymbol{e}(t)\| < b(t), \ t \in [k_1 T, (k_1 + 1)T),$$

$$\Rightarrow \|\boldsymbol{e}((k_1 + 1)T)\| \le a_2 e^{-\alpha \lambda_N T} b((k_1 + 1)T).$$
(34)

Denote the left limit of $\boldsymbol{e}(t)$ as $\boldsymbol{e}^{-}(t)$. Since $\hat{\boldsymbol{x}}(t) \equiv \hat{\boldsymbol{x}}(k_1T), t \in [k_1T, (k_1+1)T),$

$$\boldsymbol{x}((k_1+1)T) - \widehat{\boldsymbol{x}}(k_1T) = \lim_{t \to ((k+1)T)^-} \boldsymbol{x}(t) - \widehat{\boldsymbol{x}}(t) = \boldsymbol{e}^-((k+1)T).$$

Using
$$s(k) = \frac{\alpha_2 \varepsilon \nu M_0}{\alpha \lambda_N} \sqrt{1 - \beta} e^{-\alpha \lambda_N T - \frac{\varepsilon \beta \nu (k+1)T}{4}}$$
, we have
 $\left\| \frac{\boldsymbol{x}((k_1+1)T) - \hat{\boldsymbol{x}}(k_1T)}{s(k_1+1)} \right\| \le \left\| \frac{b(k_1T)}{s(k_1+1)} \right\| \le L$, (35)

that is, the quantizer is unsaturated at $t = (k_1+1)T$. Then $\|\boldsymbol{e}((k_1+1)T)\| \leq \frac{\sqrt{Nn}}{2}s(k_1+1) = a_2e^{-\alpha\lambda_N T}b((k_1+1)T).$

Step 4. Based on Steps 1-3, we conclude the proof of Lemma 3. First, denote $\Omega = \{t \in (k_1T, (k_1 + 1)T) | || \boldsymbol{e}(t) || < b(k_1T) \}$, which is nonempty because of $|| \boldsymbol{e}(k_1T) || < b(k_1T)$.

Then, we show that $\sup_{t\in\Omega} t = (k_1 + 1)T$ via a contradiction argument. Assume that there exists $t' \in [k_1T, (k_1 + 1)T)$ such that $t' = \sup_{t\in\Omega} t$, then $||e(t')|| = b(k_1T)$, drawing on the fact that e(t) is continuous on any $t \in [k_1T, (k_1 + 1)T)$. For any $t \in [k_1T, t']$, since $||e(t)|| < b(k_1T)$, it follows from (15) in Step 1 that $V(\overline{\boldsymbol{x}}) \leq a(t), t \in [k_1T, t']$. With (25) in Step 2, we further obtain $||e(t)|| < b(k_1T)$, $\forall t \in [k_1T, t']$, which contradicts to $||e(t')|| = b(k_1T)$. Hence, $\sup_{t\in\Omega} t = (k_1 + 1)T$.

To sum up, we can conclude that $||e(t)|| < b(k_1T)$ for any $t \in [k_1T, (k_1+1)T)$ and (11) holds for $k = k_1 + 1$. Combining with (15) and (34), we further conclude that (12) and (10) hold for $k = k_1 + 1$. Lemma 3 is verified.

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