

A Variational Bayes Moving Horizon Estimation Adaptive Filter with Guaranteed Stability [★]

Xiangxiang Dong ^{a,b}, Giorgio Battistelli ^c, Luigi Chisci ^c, Yunze Cai ^{a,b}

^aDepartment of Automation, Shanghai Jiao Tong University, Shanghai, 200240, China

^bKey Laboratory of System Control and Information Processing, Ministry of Education of China, Shanghai 200240, China

^cDepartment of Information Engineering, University of Florence, Florence, 50139, Italy

Abstract

This paper addresses state estimation of linear systems with special attention on unknown process and measurement noise covariances, aiming to enhance estimation accuracy while preserving the stability guarantee of the Kalman filter. To this end, the full information estimation problem over a finite interval is firstly addressed. Then, a novel adaptive *variational Bayesian* (VB) *moving horizon estimation* (MHE) method is proposed, exploiting VB inference, MHE and Monte Carlo integration with importance sampling for joint estimation of the unknown process and measurement noise covariances, along with the state trajectory over a moving window of fixed length. Further, it is proved that the proposed adaptive VB MHE filter ensures mean-square boundedness of the estimation error with any number of importance samples and VB iterations, as well as for any window length. Finally, simulation results on a target tracking example demonstrate the effectiveness of the VB MHE filter with enhanced estimation accuracy and convergence properties compared to the conventional non-adaptive Kalman filter and other existing adaptive filters.

Key words: Variational Bayes; moving horizon estimation; Monte Carlo integration; importance sampling; stability.

1 Introduction

The Kalman filter (KF) is an optimal filter for state estimation of linear stochastic dynamical systems (Kalman, 1960). It has been widely exploited in a variety of applications (e.g., signal processing, target tracking, control systems, etc.) in view of its ease of implementation and strong exponential stability properties (Bar-Shalom et al., 2002; Joseph and Tou, 1961; Leung et al., 2000).

Even though the conventional KF performs state estimation with guaranteed stability, its performance is highly affected by the prior knowledge on process and measurement noises, which are typically assumed to have Gaussian distributions with known *process noise covariance matrix* (PNCM) and *measurement noise co-*

variance matrix (MNCM) (Dong et al., 2017; Mehra, 1972). Unfortunately, in practical situations the noise statistics are usually unknown or only partially known, due to partial and/or imprecise prior knowledge. To address this issue, many adaptive methods have been proposed (Mehra, 1972), including covariance matching, maximum likelihood and *variational Bayesian* (VB), where the VB approach (Dong et al., 2017; Tzikas et al., 2008; Zhang et al., 2019) is commonly used for joint estimation of state and unknown noise statistics in view of its high estimation accuracy.

An adaptive VB filter is presented by Huang et al. (2018) where the *predicted error covariance matrix* (PECM) and MNCM are jointly estimated together with the system state. While effective in many contexts, its filtering performance can be sensitive to the choice of the nominal process noise covariance, which is used to calculate the initial value of the PECM. The approach of Huang et al. (2018) has been extended and applied also in nonlinear, multi-sensor, and multimodal settings (Dong et al., 2021a,b; Youn et al., 2020). As an improvement, a *variational Bayes sliding window Kalman filter* (VB Sliding Window) is proposed in (Huang et al.,

[★] This paper was not presented at any IFAC meeting. Corresponding author: Yunze Cai.

Email addresses: js.danesir@sjtu.edu.cn (Xiangxiang Dong), giorgio.battistelli@unifi.it (Giorgio Battistelli), luigi.chisci@unifi.it (Luigi Chisci), yzcais@sjtu.edu.cn (Yunze Cai).

2020) by imposing an approximation on the smoothing posterior *probability density function* (PDF) of the sliding window states. However, the performance of this filter is affected by the window length and, more importantly, for both adaptive filters in (Huang et al., 2018) and (Huang et al., 2020) no theoretical guarantee of stability of the estimation error has been proved.

Besides the KF and its variants, another widely employed estimation technique is moving horizon estimation (MHE) (Alessandri et al., 2003; Rao et al., 2001). MHE is based on the idea of computing an estimate of the state trajectory over a moving window of fixed length by taking into account a limited amount of most recent information, after which the estimation results are propagated to the next time step and then the former stated estimation procedure is further repeated. The main positive feature of MHE is the possibility of defining a performance criterion that can be designed specifically for the problem under consideration. Thanks to its guaranteed stability and performance (Alessandri et al., 2008; Rao et al., 2003), MHE has been widely used in both linear and nonlinear contexts (Alessandri and Awawdeh, 2016; Alessandri and Gaggero, 2017; Battistelli et al., 2017; Gharbi et al., 2021; Zou et al., 2020a) for centralized, networked, and distributed estimation (Battistelli, 2019; Farina et al., 2010a,b; Lauricella et al., 2020; Liu et al., 2013; Schneider and Marquardt, 2016; Yin and Liu, 2017; Zou et al., 2020b). The interested reader is referred to the special issue (Alessandri and Battistelli, 2020) for recent advances on MHE. In a Bayesian framework, MHE can be conveniently exploited to approximate the full-information Bayesian estimation problem whenever the latter does not admit a closed-form recursive solution (Delgado and Goodwin, 2014; Fiedler et al., 2020).

In this work, in order to perform joint estimation of state, PNCM and, MNCM with bounded estimation error, a novel adaptive VB MHE filter is developed. To this end, inspired by the idea of MHE, the unknown noise covariances are regarded as nearly constant within the current window and estimated through the VB method while ensuring available bounds. The considered framework allows for imposing constraints on the unknown PNCM and MNCM in terms of a priori defined sets to which the respective estimates should belong. First, the full information estimation problem over a finite interval is addressed by modeling the PNCM and MNCM distributions as constrained inverse Wishart. In this context, we provide an algorithm based on the VB fixed-point method for computing the optimal factorized approximation of the joint posterior of state trajectory, PNCM and MNCM. Then, the MHE paradigm is employed to make the proposed approach recursive and ensure bounded memory and computational complexity. The resulting VB MHE filter exploits Monte Carlo integration with importance sampling for computing the estimates of the unknown covariances. Further, and most

importantly, it is proved that the proposed VB MHE filter ensures stability, in terms of mean-square boundedness of the estimation error, for any choice of the number of importance samples and VB iterations, as well as of the window length. Simulation results demonstrate the effectiveness of the proposed filter as compared to the state of the art, thus confirming the theoretical findings.

The remaining parts of this paper are organized as follows. Section 2 provides background and the problem formulation. In Section 3, the full information estimation problem over a finite interval is addressed via VB inference. Then, in Section 4 the MHE paradigm is applied to derive a recursive adaptive estimation algorithm. In Section 5, the stability of the proposed adaptive filter is analyzed. Performance assessment via simulation experiments concerning a target tracking example is provided in Section 6. Finally, conclusions and perspectives for future work are given in Section 7.

2 Problem formulation and preliminaries

Consider a linear discrete-time system

$$x_t = Ax_{t-1} + w_{t-1} \quad (1)$$

and linear measurements

$$y_t = Cx_t + v_t \quad (2)$$

where: t is the time index; x_t and y_t are the state and measurement vectors of dimensions n_x and n_y , respectively; A and C are the state transition and, respectively, measurement matrices; w_{t-1} and v_t denote white process and measurement noises, assumed to be Gaussian with zero mean but unknown covariances Q and R . It is also assumed that w_k and v_j are uncorrelated for any k and j . The unknown covariances Q and R are supposed to belong to known sets $\mathcal{Q} \subseteq \mathbb{S}_+^{n_x}$ and $\mathcal{R} \subseteq \mathbb{S}_+^{n_y}$, respectively, where \mathbb{S}_+^d denotes the set of real-valued positive definite $d \times d$ symmetric matrices.

Following a Bayesian approach, the process and measurement noise covariances are regarded as random matrices to be estimated together with the state trajectory. For the resulting adaptive estimation problem, although there have been some variational adaptive filters proposed in (Huang et al., 2018) and (Huang et al., 2020), for such filters there is no available proof of stability. Motivated by this, this paper aims to propose a novel adaptive filter for unknown PNCM and MNCM that ensures mean-square boundedness of the estimation error. The main contribution focuses on the derivation of the proposed filter as well as on the stability proof. The proposed adaptive filter will jointly exploit VB inference and MHE.

2.1 Idea of variational Bayes inference

Before deriving the proposed adaptive filter with guaranteed stability, this section briefly recalls the basic idea of VB inference.

The VB approach is based on the idea of approximating the true posterior $p(\theta)$ with a variational distribution $q(\theta)$ constrained to have a fixed form by minimizing the *Kullback-Leibler divergence* (KLD) from $p(\theta)$ (Tzikas et al., 2008), i.e.,

$$q = \arg \min_q \text{KLD}(q||p) \quad (3)$$

where KLD is defined as

$$\text{KLD}(q||p) = \int q(\theta) \log \frac{q(\theta)}{p(\theta)} d\theta. \quad (4)$$

When the variables to be estimated can be partitioned as $\theta = (\theta_1, \dots, \theta_M)$ and the variational distribution $q(\theta)$ is given the factorized form $q(\theta) = \prod_{i=1}^M q_i(\theta_i)$, then the optimal solution to (3) must satisfy

$$\log q_i(\theta_i) = E_{\theta_{j,j \neq i}} [\log p(\theta)] + \text{constant} \quad (5)$$

where E denotes expectation. Both VB and VB sliding-window filters of (Huang et al., 2018) and (Huang et al., 2020) have been derived via VB inference in factorized form.

3 Variational Bayes inference for full information estimation

In this section, the full information estimation problem over a finite interval is addressed and VB inference is used in order to compute a factorized approximation of the true joint posterior of the state trajectory, PNCM, and MNCM.

For the system (1)-(2), the initial state x_0 is assumed to be Gaussian-distributed with mean \bar{x}_0 and covariance \bar{P}_0 , i.e.,

$$p(x_0) = \mathcal{N}(x_0; \bar{x}_0, \bar{P}_0). \quad (6)$$

For the unknown covariance of a Gaussian distribution, its conjugate prior is the inverse Wishart distribution (Huang et al., 2018; O'Hagan and Forster, 2004), whose PDF is denoted as $\mathcal{W}^{-1}(G; \Delta, \gamma)$, indicating that the random matrix $G \in \mathbb{S}_+^d$ follows an inverse Wishart distribution with degree of freedom $\gamma > d + 1$ and scale matrix $\Delta \in \mathbb{S}_+^d$. Since the unknown covariances Q and R are supposed to belong to the sets \mathcal{Q} and \mathcal{R} , respectively, we take the priors for Q and R as constrained inverse Wishart distributions of parameters (\bar{M}, \bar{m}) and,

respectively, (\bar{S}, \bar{s}) , i.e.

$$p(Q) \propto \mathcal{W}^{-1}(Q; \bar{M}, \bar{m}) \mathbf{1}_{\mathcal{Q}}(Q) \quad (7)$$

$$p(R) \propto \mathcal{W}^{-1}(R; \bar{S}, \bar{s}) \mathbf{1}_{\mathcal{R}}(R) \quad (8)$$

where $\mathbf{1}_{\mathcal{Q}}(Q)$ is the indicator function taking value 1 if $Q \in \mathcal{Q}$ and 0 otherwise.

Let $y_{1:t}$ denote the sequence of measurements from time 1 to time t . Then, it is an easy matter to check that the joint posterior PDF of the state trajectory $x_{0:t}$ and the unknown noise covariances Q, R can be expressed as

$$\begin{aligned} & p(x_{0:t}, Q, R | y_{1:t}) \\ & \propto \mathcal{N}(x_0; \bar{x}_0, \bar{P}_0) \times \prod_{i=1}^t \mathcal{N}(x_i; Ax_{i-1}, Q) \mathcal{N}(y_i; Cx_i, R) \\ & \times \mathcal{W}^{-1}(Q; \bar{M}, \bar{m}) \mathcal{W}^{-1}(R; \bar{S}, \bar{s}) \mathbf{1}_{\mathcal{Q}}(Q) \mathbf{1}_{\mathcal{R}}(R). \end{aligned} \quad (9)$$

By exploiting the VB approach, a factorized approximation of the joint PDF in (9) is sought as

$$p(x_{0:t}, Q, R | y_{1:t}) \cong q_x(x_{0:t}) q_Q(Q) q_R(R) \quad (10)$$

where q_x, q_Q, q_R denote the factors of the approximated joint PDF. To this end, we notice preliminarily that, since the joint posterior (9) is null when $Q \notin \mathcal{Q}$ or $R \notin \mathcal{R}$, then in the VB approximation it must hold that

$$\begin{cases} q_Q(Q) = 0 & \text{for } Q \notin \mathcal{Q} \\ q_R(R) = 0 & \text{for } R \notin \mathcal{R}. \end{cases} \quad (11)$$

Otherwise the minimum in (3) could not be achieved since, by definition, the KLD is infinite whenever the support of q is not contained in the support of p .

Concerning the variational distribution of the state trajectory, the following result holds.

Proposition 1 *Given the joint posterior (9) and the factorized approximation (10), then the approximated PDF of the state trajectory according to the VB approach is of the form*

$$q_x(x_{0:t}) = \mathcal{N}(x_{0:t}; \hat{x}(\Psi, \Phi), P(\Psi, \Phi)) \quad (12)$$

where

$$\Phi = \int_{\mathcal{Q}} Q^{-1} q_Q(Q) dQ \quad (13)$$

$$\Psi = \int_{\mathcal{R}} R^{-1} q_R(R) dR \quad (14)$$

$$\hat{x}(\Psi, \Phi) = \Omega^{-1}(\Psi, \Phi) \omega(\Psi) \quad (15)$$

$$P(\Psi, \Phi) = \Omega^{-1}(\Psi, \Phi) \quad (16)$$

$$\omega(\Psi) = \begin{bmatrix} C'\Psi y_t \\ \vdots \\ C'\Psi y_1 \\ \bar{P}_0^{-1}\bar{x}_0 \end{bmatrix} \quad (17)$$

and the block matrix $\Omega(\Psi, \Phi)$ defined as in equation (18).

Proof. By taking logarithm on both sides of (10), we obtain

$$\begin{aligned} & \log p(x_{0:t}, Q, R | y_{1:t}) \\ &= -\frac{1}{2} \left[\|x_0 - \bar{x}_0\|_{\bar{P}_0^{-1}}^2 + \sum_{i=1}^t \left(\log|R| + \|y_i - Cx_i\|_{R^{-1}}^2 \right. \right. \\ & \quad \left. \left. + \log|Q| + \|x_i - Ax_{i-1}\|_{Q^{-1}}^2 \right) + (\bar{s} + n_y + 1) \log|R| \right. \\ & \quad \left. + (\bar{m} + n_x + 1) \log|Q| + \text{tr}(\bar{S}R^{-1}) + \text{tr}(\bar{M}Q^{-1}) \right] \\ & \quad + \text{constant} \\ &= -\frac{1}{2} \left\{ (x_0 - \bar{x}_0)' \bar{P}_0^{-1} (x_0 - \bar{x}_0) + \sum_{i=1}^t \left[(y_i - Cx_i)' \right. \right. \\ & \quad \left. \left. \times R^{-1} (y_i - Cx_i) + (x_i - Ax_{i-1})' Q^{-1} (x_i - Ax_{i-1}) \right] \right. \\ & \quad \left. + (\bar{s} + t + n_y + 1) \log|R| + (\bar{m} + t + n_x + 1) \log|Q| \right. \\ & \quad \left. + \text{tr}(\bar{S}R^{-1}) + \text{tr}(\bar{M}Q^{-1}) \right\} + \text{constant} \quad (19) \end{aligned}$$

for any $Q \in \mathcal{Q}$ and $R \in \mathcal{R}$. Hence, in view of (5) and (11), it can be obtained that

$$\begin{aligned} & q_x(x_{0:t}) \\ & \propto \exp \int_{\mathcal{R}} \int_{\mathcal{Q}} \log p(x_{0:t}, Q, R | y_{1:t}) q_Q(Q) q_R(R) dQ dR. \end{aligned} \quad (20)$$

In turn, we have

$$\begin{aligned} & \int_{\mathcal{R}} \int_{\mathcal{Q}} \log p(x_{0:t}, Q, R | y_{1:t}) q_Q(Q) q_R(R) dQ dR \\ &= -\frac{1}{2} \left\{ (x_0 - \bar{x}_0)' \bar{P}_0^{-1} (x_0 - \bar{x}_0) \right. \\ & \quad \left. + \sum_{i=1}^t \left[(y_i - Cx_i)' \Psi (y_i - Cx_i) \right. \right. \\ & \quad \left. \left. + (x_i - Ax_{i-1})' \Phi (x_i - Ax_{i-1}) \right] \right\} + \text{constant} \quad (21) \end{aligned}$$

with Φ and Ψ defined as in (13)-(14). Then, with standard algebraic manipulations, it can be checked that the approximated PDF of the state trajectory has Gaussian distribution with mean and covariance given by (15) and

(16), respectively. \square

We remark that, by construction, the estimate and covariance of Proposition 1 are partitioned as follows

$$\hat{x} = \begin{bmatrix} \hat{x}_t \\ \hat{x}_{t-1} \\ \vdots \\ \hat{x}_0 \end{bmatrix}, P = \begin{bmatrix} P_t & P_{t,t-1} & \cdots & P_{t,0} \\ P_{t-1,t} & P_{t-1} & \cdots & P_{t-1,0} \\ \vdots & \vdots & \ddots & \vdots \\ P_{0,t} & P_{0,t-1} & \cdots & P_0 \end{bmatrix}. \quad (22)$$

Concerning the variational distributions of the PNCM and MNCM, the following result holds.

Proposition 2 *Given the joint posterior (9) and the factorized approximation (10), then the approximated PDFs of the PNCM and MNCM according to the VB approach are of the form*

$$q_Q(Q) \propto \mathcal{W}^{-1}(Q; M(\hat{x}, P), m) \mathbf{1}_{\mathcal{Q}}(Q) \quad (23)$$

$$q_R(R) \propto \mathcal{W}^{-1}(R; S(\hat{x}, P), s) \mathbf{1}_{\mathcal{R}}(R) \quad (24)$$

with

$$m = \bar{m} + t \quad (25)$$

$$s = \bar{s} + t \quad (26)$$

$$\begin{aligned} M(\hat{x}, P) &= \bar{M} + \sum_{i=1}^t \left[(\hat{x}_i - A\hat{x}_{i-1})(\hat{x}_i - A\hat{x}_{i-1})' \right. \\ & \quad \left. + P_i + AP_{i-1}A' - P_{i,i-1}A' - AP_{i-1,i} \right] \end{aligned} \quad (27)$$

$$\begin{aligned} S(\hat{x}, P) &= \bar{S} + \sum_{i=1}^t \left[(y_i - C\hat{x}_i)(y_i - C\hat{x}_i)' \right. \\ & \quad \left. + CP_iC' \right]. \end{aligned} \quad (28)$$

Proof. As already discussed, $q_Q(Q) = 0$ and $q_R(R) = 0$ for $Q \notin \mathcal{Q}$ and $R \notin \mathcal{R}$, respectively. For $Q \in \mathcal{Q}$, we have

$$\begin{aligned} & q_Q(Q) \propto \\ & \exp \int_{\mathcal{R}} \int \log p(x_{0:t}, Q, R | y_{1:t}) q_x(x_{0:t}) q_R(R) dx_{0:t} dR \end{aligned} \quad (29)$$

where

$$\begin{aligned} & \int_{\mathcal{R}} \int \log p(x_{0:t}, Q, R | y_{1:t}) q_x(x_{0:t}) q_R(R) dx_{0:t} dR \\ &= -\frac{1}{2} \left\{ (\bar{m} + n_x + t + 1) \log Q + \text{tr}(\bar{M}Q^{-1}) \right\} \end{aligned}$$

$$\Omega(\Psi, \Phi) = \begin{bmatrix} C'\Psi C + \Phi & -\Phi A & 0 & \dots & 0 \\ -A'\Phi & C'\Psi C + \Phi + A'\Phi A & -\Phi A & & \vdots \\ 0 & -A'\Phi & \ddots & \ddots & \vdots \\ \vdots & & & C'\Psi C + \Phi + A'\Phi A & -\Phi A \\ 0 & \dots & \dots & -A'\Phi & A'\Phi A + \bar{P}_0^{-1} \end{bmatrix} \quad (18)$$

$$+ \sum_{i=1}^t \text{tr} \left[\int (x_i - Ax_{i-1}) (x_i - Ax_{i-1})' \times q_x(x_{0:t}) dx_{0:t} Q^{-1} \right] \Big\} + \text{constant}. \quad (30)$$

Each integral in the summation turns out to be

$$\begin{aligned} & \int (x_i - Ax_{i-1}) (x_i - Ax_{i-1})' q_x(x_{0:t}) dx_{0:t} \\ &= (\hat{x}_i - A\hat{x}_{i-1}) (\hat{x}_i - A\hat{x}_{i-1})' + AP_{i-1}A' \\ & \quad + P_i - P_{i,i-1}A' - AP_{i-1,i} \end{aligned} \quad (31)$$

where P_i are the diagonal blocks of P in (22), and the cross-covariances $P_{i-1,i}$ are the lower diagonal blocks of P . Then the approximated PDF of the PNCM follows a bounded inverse Wishart distribution as in (23). Further, for $R \in \mathcal{R}$, we have

$$\begin{aligned} q_R(R) &\propto \\ \exp \int_{\mathcal{Q}} \int \log p(x_{0:t}, Q, R | y_{1:t}) q_x(x_{0:t}) q_Q(Q) dx_{0:t} dQ \end{aligned} \quad (32)$$

where

$$\begin{aligned} & \int_{\mathcal{Q}} \int \log p(x_{0:t}, Q, R | y_{1:t}) q_x(x_{0:t}) q_Q(Q) dx_{0:t} dQ \\ &= -\frac{1}{2} \left\{ (\bar{s} + n_y + t + 1) \log R + \text{tr}(\bar{S}R^{-1}) \right. \\ & \quad + \sum_{i=1}^t \text{tr} \left[\int (y_i - Cx_i) (y_i - Cx_i)' \right. \\ & \quad \left. \left. \times q_x(x_{0:t}) dx_{0:t} R^{-1} \right] \right\} + \text{constant}. \end{aligned} \quad (33)$$

Each integral in the summation turns out to be

$$\begin{aligned} & \int (y_i - Cx_i) (y_i - Cx_i)' q_x(x_{0:t}) dx_{0:t} \\ &= (y_i - C\hat{x}_i) (y_i - C\hat{x}_i)' + CP_iC'. \end{aligned} \quad (34)$$

Then the approximated PDF of the MNCM follows a

Table 1

Fixed point iteration for full-information VB estimation

Set: $m = \bar{m} + t$, $s = \bar{s} + t$;
for $k = 1 : N$
 (a) Compute $\hat{x}^{(k)} = \hat{x}(\Psi^{(k-1)}, \Phi^{(k-1)})$ via (15);
 (b) Compute $P^{(k)} = P(\Psi^{(k-1)}, \Phi^{(k-1)})$ via (16);
 (c) Compute $M^{(k)} = M(\hat{x}^{(k)}, P^{(k)})$ via (27);
 (d) Compute $S^{(k)} = S(\hat{x}^{(k)}, P^{(k)})$ via (28);
 (e) Compute:

$$\Phi^{(k)} = \frac{\int_{\mathcal{Q}} Q^{-1} \mathcal{W}^{-1}(Q; M^{(k)}, m) dQ}{\int_{\mathcal{Q}} \mathcal{W}^{-1}(Q; M^{(k)}, m) dQ} \quad (37)$$

$$\Psi^{(k)} = \frac{\int_{\mathcal{R}} R^{-1} \mathcal{W}^{-1}(R; S^{(k)}, s) dR}{\int_{\mathcal{R}} \mathcal{W}^{-1}(R; S^{(k)}, s) dR}; \quad (38)$$

end

bounded inverse Wishart distribution of the form (24). \square

By combining Propositions 1 and 2, we get a system of nonlinear equations in the unknowns \hat{x} , P , M and S which can be iteratively solved via the fixed-point method estimating one parameter at a time while fixing the others (Huang et al., 2018). See (Sato, 2001) for convergence results. First, the values of Φ and Ψ are initialized by using the prior PDFs of Q and R

$$\Phi^{(0)} = \frac{\int_{\mathcal{Q}} Q^{-1} \mathcal{W}^{-1}(Q; \bar{M}, \bar{m}) dQ}{\int_{\mathcal{Q}} \mathcal{W}^{-1}(Q; \bar{M}, \bar{m}) dQ} \quad (35)$$

$$\Psi^{(0)} = \frac{\int_{\mathcal{R}} R^{-1} \mathcal{W}^{-1}(R; \bar{S}, \bar{s}) dR}{\int_{\mathcal{R}} \mathcal{W}^{-1}(R; \bar{S}, \bar{s}) dR}. \quad (36)$$

Then, the algorithm of Table 1 is carried out.

Notice that in the unconstrained case, i.e. $\mathcal{Q} = \mathbb{S}_+^{n_x}$ and $\mathcal{R} = \mathbb{S}_+^{n_y}$, the integrals in (37)-(38) can be computed in closed form. In fact, in this case,

$$\Phi^{(k)} = \int Q^{-1} \mathcal{W}^{-1}(Q; M^{(k)}, m) dQ = m [M^{(k)}]^{-1} \quad (39)$$

$$\Psi^{(k)} = \int R^{-1} \mathcal{W}^{-1} \left(R; S^{(k)}, s \right) dR = s \left[S^{(k)} \right]^{-1}. \quad (40)$$

Conversely, when $\mathcal{Q} \subset \mathbb{S}_+^{n_x}$ and $\mathcal{R} \subset \mathbb{S}_+^{n_y}$, the integrals in (37)-(38) can no longer be computed in closed form but can, anyway, be easily approximated to any desired accuracy via Monte Carlo integration. Analogous considerations hold for the integrals in (35)-(36). The discussion on the application of Monte Carlo integration in the considered framework is deferred to the next section.

Notice that the above inference, relying on the whole measurement sequence $y_{1:t}$ up to time t , is characterized by memory and computational complexity growing with time. For the sake of implementation, a *moving horizon* approximation of finite fixed length $T \geq 1$ will be considered hereafter, by only exploiting at time t the measurement sub-sequence $y_{t-T+1:t}$ in order to estimate the state sub-trajectory $\hat{x}_{t-T:t}$.

4 Variational Bayes Moving-horizon estimation algorithm

The purpose of this section is to make the proposed approach recursive by means of the MHE approximation, where the estimation results at the current time index is used as the initial value for the next moving horizon estimation.

Specifically, suppose that the information collected up to time $t - T$ can be approximately summarized by the PDF

$$\begin{aligned} & p(x_{t-T}, Q, R | y_{1:t-T}) \\ & \propto \mathcal{N}(x_{t-T}; \bar{x}_{t-T}, \bar{P}_{t-T}) \mathcal{W}^{-1}(Q; \bar{M}_{t-T}, \bar{m}_{t-T}) \\ & \quad \times \mathcal{W}^{-1}(R; \bar{S}_{t-T}, \bar{s}_{t-T}) \mathbf{1}_{\mathcal{Q}}(Q) \mathbf{1}_{\mathcal{R}}(R). \end{aligned} \quad (41)$$

Then, we can apply the previously outlined VB approach to compute an approximation of the form

$$\begin{aligned} & p(x_{t-T:t}, Q, R | y_{t-T+1:t}) \\ & \propto \mathcal{N}(x_{t-T:t}; \hat{x}_{t-T:t|T}, P_{t-T:t|T}) \\ & \quad \times \mathcal{W}^{-1}(Q; M_t, m_t) \mathcal{W}^{-1}(R; S_t, s_t) \mathbf{1}_{\mathcal{Q}}(Q) \mathbf{1}_{\mathcal{R}}(R) \end{aligned} \quad (42)$$

given the measurement sequence $y_{t-T+1:t}$ and the prior knowledge summarized by \bar{x}_{t-T} , \bar{P}_{t-T} , \bar{m}_{t-T} , \bar{S}_{t-T} and \bar{s}_{t-T} , which are initialized at time $t = T$ from the prior distributions of x_0 , Q , R . Notice that, in order to account for the moving horizon, instead of the func-

tion $\omega(\Psi, \Phi)$ of Proposition 1 we consider

$$\omega_t(\Psi) = \begin{bmatrix} C' \Psi y_t \\ \vdots \\ C' \Psi y_{t-T+1} \\ \bar{P}_{t-T}^{-1} \bar{x}_{t-T} \end{bmatrix}. \quad (43)$$

Similarly, instead of the matrix $\Omega(\Psi, \Phi)$ in (18), we consider the matrix $\Omega_t(\Psi, \Phi)$ defined with respect to the sliding window $[t - T, t]$ with \bar{P}_0 replaced by \bar{P}_{t-T} . As to Proposition 2, instead of the functions $M(\hat{x}, P)$ and $S(\hat{x}, P)$, we consider

$$\begin{aligned} & M_t(\hat{x}_{t-T:t}, P_{t-T:t}) \\ & = \bar{M}_{t-T} + \sum_{i=t-T}^t \left[(\hat{x}_i - A\hat{x}_{i-1})(\hat{x}_i - A\hat{x}_{i-1})' \right. \\ & \quad \left. + P_i + AP_{i-1}A' - P_{i,i-1}A' - AP_{i-1,i} \right] \end{aligned} \quad (44)$$

$$\begin{aligned} & S_t(\hat{x}_{t-T:t}, P_{t-T:t}) = \bar{S}_{t-T} \\ & + \sum_{i=t-T}^t \left[(y_i - C\hat{x}_i)(y_i - C\hat{x}_i)' + CP_iC' \right]. \end{aligned} \quad (45)$$

In order to go from time t to time $t + 1$, one iteration of the KF can be performed starting from the most recent estimates \hat{Q}_t and \hat{R}_t of the PNCM and MNCM. The latter can be computed via the integrals

$$\hat{Q}_t = \frac{\int_{\mathcal{Q}} Q \mathcal{W}^{-1}(Q; M_t, m_t) dQ}{\int_{\mathcal{Q}} \mathcal{W}^{-1}(Q; M_t, m_t) dQ} \quad (46)$$

$$\hat{R}_t = \frac{\int_{\mathcal{R}} R \mathcal{W}^{-1}(R; S_t, s_t) dR}{\int_{\mathcal{R}} \mathcal{W}^{-1}(R; S_t, s_t) dR}. \quad (47)$$

Then, we can update \bar{x}_{t-T+1} and \bar{P}_{t-T+1} as follows

$$\bar{x}_{t-T+1} = A\bar{x}_{t-T} \quad (48)$$

$$\bar{P}_{t-T+1} = A\bar{P}_{t-T}A' + \hat{Q}_t \quad (49)$$

$$K_{t-T+1} = \bar{P}_{t-T+1}C' \left(C\bar{P}_{t-T+1}C' + \hat{R}_t \right)^{-1} \quad (50)$$

$$\bar{P}_{t-T+1} = (I - K_{t-T+1}C) \bar{P}_{t-T+1} \quad (51)$$

$$\bar{x}_{t-T+1} = \bar{x}_{t-T+1} + K_{t-T+1}(y_{t-T+1} - C\bar{x}_{t-T+1}). \quad (52)$$

For the time propagation of the parameters \bar{M}_{t-T} and \bar{m}_{t-T} , following (Huang et al., 2018), we can set

$$\bar{M}_{t-T+1} = \rho M_t \quad (53)$$

$$\bar{m}_{t-T+1} = \rho(m_t - n_x - 1) + n_x + 1 \quad (54)$$

Table 2
VB-MHE algorithm

Inputs: $y_{t-T+1:t}$, \bar{x}_{t-T} , \bar{P}_{t-T} , \bar{M}_{t-T} , \bar{m}_{t-T} , \bar{S}_{t-T} , \bar{s}_{t-T} , Φ_{t-1} , Ψ_{t-1}

(1) Variational iterations:

Set $\Phi_t^{(0)} = \Phi_{t-1}$, $\Psi_t^{(0)} = \Psi_{t-1}$;
Set $m_t = \bar{m}_{t-T} + T$, $s_t = \bar{s}_{t-T} + T$;
for $k = 1 : N$
(a) Compute $P_{t-T:t}^{(k)} = \Omega_t^{-1} \left(\Psi_t^{(k-1)}, \Phi_t^{(k-1)} \right)$;
(b) Compute $\hat{x}_{t-T:t}^{(k)} = P_{t-T:t}^{(k)} \omega_t \left(\Psi_t^{(k-1)} \right)$;
(c) Compute $M_t^{(k)} = M_t \left(\hat{x}_{t-T:t}^{(k)}, P_{t-T:t}^{(k)} \right)$;
(d) Compute $S_t^{(k)} = S_t \left(\hat{x}_{t-T:t}^{(k)}, P_{t-T:t}^{(k)} \right)$;
(e) Compute

$$\Phi_t^{(k)} = \frac{\int_{\mathcal{Q}} Q^{-1} \mathcal{W}^{-1} \left(Q; M_t^{(k)}, m_t \right) dQ}{\int_{\mathcal{Q}} \mathcal{W}^{-1} \left(Q; M_t^{(k)}, m_t \right) dQ} \quad (57)$$

$$\Psi_t^{(k)} = \frac{\int_{\mathcal{R}} R^{-1} \mathcal{W}^{-1} \left(R; S_t^{(k)}, s_t \right) dR}{\int_{\mathcal{R}} \mathcal{W}^{-1} \left(R; S_t^{(k)}, s_t \right) dR}; \quad (58)$$

end for

Set $\hat{x}_{t-T:t|t} = \hat{x}_{t-T:t}^{(N)}$, $P_{t-T:t|t} = P_{t-T:t}^{(N)}$;
Set $M_t = M_t^{(N)}$, $S_t = S_t^{(N)}$;
Set $\Phi_t = \Phi_t^{(N)}$, $\Psi_t = \Psi_t^{(N)}$;
Compute \hat{Q}_t and \hat{R}_t via (46)-(47);

(2) Time update for moving horizon:

(a) Compute \bar{M}_{t-T+1} , \bar{m}_{t-T+1} , \bar{S}_{t-T+1} , \bar{s}_{t-T+1} for the next moving horizon filtering recursion via (53)-(56);
(b) Compute \bar{x}_{t-T+1} and \bar{P}_{t-T+1} via KF with estimated \hat{Q}_t and \hat{R}_t via (48)-(52);

Outputs: $\hat{x}_{t|t}$, $P_{t|t}$, \hat{Q}_t , \hat{R}_t , Φ_t , Ψ_t , \bar{x}_{t-T+1} , \bar{P}_{t-T+1} , \bar{M}_{t-T+1} , \bar{m}_{t-T+1} , \bar{S}_{t-T+1} , \bar{s}_{t-T+1}

where $\rho \in (0, 1)$ denotes a forgetting factor. Similarly, \bar{S}_{t-T+1} and \bar{s}_{t-T+1} can be obtained by

$$\bar{S}_{t-T+1} = \rho S_t \quad (55)$$

$$\bar{s}_{t-T+1} = \rho (s_t - n_y - 1) + n_y + 1. \quad (56)$$

To summarize, the implementation of the proposed VB MHE algorithm is outlined in Table 2.

The integrals in (46)-(47) can be approximated to any desired degree of accuracy by means of Monte Carlo integration with importance sampling. This amounts to drawing J samples Q_j and R_j , with $j = 1, 2, \dots, J$, from

suitable proposal distributions $\pi_Q(Q)$ and $\pi_R(R)$, respectively, and then setting

$$\hat{Q}_t = \frac{\sum_{j=1}^J Q_j \frac{\mathcal{W}^{-1}(Q_j; M_t, m_t)}{\pi_Q(Q_j)} \mathbf{1}_{\mathcal{Q}}(Q_j)}{\sum_{j=1}^J \frac{\mathcal{W}^{-1}(Q_j; M_t, m_t)}{\pi_Q(Q_j)} \mathbf{1}_{\mathcal{Q}}(Q_j)} \quad (59)$$

$$\hat{R}_t = \frac{\sum_{j=1}^J R_j \frac{\mathcal{W}^{-1}(R_j; S_t, s_t)}{\pi_R(R_j)} \mathbf{1}_{\mathcal{R}}(R_j)}{\sum_{j=1}^J \frac{\mathcal{W}^{-1}(R_j; S_t, s_t)}{\pi_R(R_j)} \mathbf{1}_{\mathcal{R}}(R_j)}. \quad (60)$$

A reasonable choice for the proposal distributions amounts to setting

$$\pi_Q(Q) = \mathcal{W}^{-1} \left(Q; (m_t - n_x - 1) \hat{Q}_{t-1}, m_t \right) \quad (61)$$

$$\pi_R(R) = \mathcal{W}^{-1} \left(R; (s_t - n_y - 1) \hat{R}_{t-1}, s_t \right). \quad (62)$$

In fact, since the previous estimates \hat{Q}_{t-1} and \hat{R}_{t-1} belong by construction to \mathcal{Q} and \mathcal{R} , respectively, then the above choice ensures that most of the samples are drawn inside those sets. The integrals in (57) and (58) can be computed a similar way.

As will be shown in the next section, the resulting recursive estimation algorithm, under the assumption of bounded \mathcal{Q} and \mathcal{R} , turns out to be mean-square stable for any J (number of samples), any N (number of VB iterations), and any T (length of the moving horizon window).

5 Stability analysis

The stability of the proposed VB MHE adaptive filter of Algorithm 2 is analyzed in this section in terms of boundedness of the estimation error $e_t = x_t - \hat{x}_{t|t}$. To this end, the following assumptions are needed.

A1. The pair (A, C) is detectable.

A2. There exist $\underline{\alpha}$ and $\bar{\alpha}$ with $0 < \underline{\alpha} \leq \bar{\alpha}$ such that $\underline{\alpha}I \leq Q \leq \bar{\alpha}I$, $\forall Q \in \mathcal{Q}$.

A3. There exist $\underline{\beta}$ and $\bar{\beta}$ with $0 < \underline{\beta} \leq \bar{\beta}$ such that $\underline{\beta}I \leq R \leq \bar{\beta}I$, $\forall R \in \mathcal{R}$.

Under the stated assumptions, the following result descends from classical results on KF (Jazwinski, 1970).

Lemma 1 *Let assumptions A1-A3 be satisfied. Then, there exist real numbers \underline{p} and \bar{p} with $0 < \underline{p} \leq \bar{p}$ such that*

$$\underline{p}I \leq \bar{P}_{t-T+1} \leq \bar{p}I \quad (63)$$

for any $t \geq T$.

Proof. Notice that \bar{P}_{t-T+1} is the covariance matrix of a KF constructed by using the estimated covariance matrices \hat{Q}_t and \hat{R}_t in place of the true ones. By construction, the estimates \hat{Q}_t and \hat{R}_t belong to the sets \mathcal{Q} and \mathcal{R} irrespectively of the number of samples J , the number of VB iterations N and the window length T . Hence, $\underline{\alpha}I \leq \hat{Q}_t \leq \bar{\alpha}I$ and $\underline{\beta}I \leq \hat{R}_t \leq \bar{\beta}I$ for any $t \geq T$. Then, the existence of uniform upper and lower bounds for \bar{P}_{t-T+1} can be proved as in (Jazwinski, 1970). \square

Let us now recall the following result.

Lemma 2 *If a stochastic process $V_t(\varepsilon_t)$ satisfies the following conditions (where $\underline{\gamma}, \bar{\gamma}, \lambda$ and μ are real numbers satisfying $0 < \underline{\gamma} \leq \bar{\gamma}$, $0 \leq \lambda < 1$ and $\mu > 0$):*

$$\underline{\gamma} \|\varepsilon_t\|^2 \leq V_t(\varepsilon_t) \leq \bar{\gamma} \|\varepsilon_t\|^2 \quad (64)$$

$$\{E[V_t(\varepsilon_t)]\}^{1/2} \leq \lambda \{E[V_{t-1}(\varepsilon_{t-1})]\}^{1/2} + \mu \quad (65)$$

then the stochastic process is exponentially bounded in mean square, i.e

$$E[\|\varepsilon_t\|^2] \leq \frac{2\bar{\gamma}}{\underline{\gamma}} E[\|\varepsilon_0\|^2] \lambda^{2t} + \frac{2}{\underline{\gamma}} \left(\mu \sum_{i=0}^{t-1} \lambda^i \right)^2. \quad (66)$$

Based on Lemma 1 and Lemma 2, the following stability result can be proven.

Theorem 3 *Let assumptions A1-A3 be satisfied. Then the state estimation error sequence $e_t = x_t - \hat{x}_{t|t}$ is uniformly bounded in mean square.*

Proof. We first prove that $\bar{e}_t = x_t - \bar{x}_t$ is uniformly bounded and then prove that e_t is uniformly bounded as well.

Let us first define a candidate Lyapunov function

$$V_{t+1}(\tilde{e}_{t+1}) = \tilde{e}_{t+1}' \bar{P}_{t+1}^{-1} \tilde{e}_{t+1} \quad (67)$$

where $\tilde{e}_{t+1} = x_{t+1} - \tilde{x}_{t+1}$. In view of (49) and Lemma 1, we have $\underline{\alpha}I \leq \bar{P}_{t+1} \leq \bar{p}\|A\|^2 I + \bar{\alpha}I$. Hence, the Lyapunov candidate function satisfies (64) with $\bar{\gamma} = 1/\underline{\alpha}$ and $\underline{\gamma} = 1/(\bar{p}\|A\|^2 + \bar{\alpha})$.

Next, in view of (48)-(52), we can write

$$\tilde{e}_{t+1} = A(I - K_t C) \tilde{e}_t + A K_t v_t + w_t. \quad (68)$$

Consider now the square root expected value $\tilde{V}_{t+1}(e) = \{E[V_{t+1}(e)]\}^{1/2}$ of the candidate Lyapunov function.

Since $\tilde{V}_{t+1}(e)$ is a norm, we can apply the triangular inequality and write

$$\begin{aligned} \tilde{V}_{t+1}(e_{t+1}) &\leq \tilde{V}_{t+1}(A(I - K_t C) \tilde{e}_t) \\ &\quad + \tilde{V}_{t+1}(A K_t v_t) + \tilde{V}_{t+1}(w_t). \end{aligned} \quad (69)$$

Notice that

$$\tilde{V}_{t+1}(w_t) \leq (\bar{\gamma})^{1/2} \{E[\|w_t\|^2]\}^{1/2} \leq [\bar{\gamma} \text{tr}(Q)]^{1/2}. \quad (70)$$

Further, under the stated assumptions, the Kalman gain is bounded in that $\|K_{t-T}\| \leq \|\bar{P}_{t-T}\| \|C\| \underline{\beta}^{-1} \leq (\bar{p}\|A\|^2 + \bar{\alpha})\|C\| \underline{\beta}^{-1}$. Then, we have

$$\begin{aligned} \tilde{V}_{t+1}(A K_t v_t) &\leq (\bar{\gamma})^{1/2} \{E[\|A K_t v_t\|^2]\}^{1/2} \\ &\leq [\bar{\gamma} \text{tr}(R)]^{1/2} \mathcal{K} \|A\| \end{aligned} \quad (71)$$

where $\mathcal{K} = (\bar{p}\|A\|^2 + \bar{\alpha})\|C\| \underline{\beta}^{-1}$. Finally, with standard manipulations (see (Wanasinghe et al., 2015)), under the stated assumptions it can be shown that

$$V_{t+1}(A(I - K_t C) \tilde{e}_t) \leq \lambda^2 V_t(\tilde{e}_t) \quad (72)$$

for some λ with $0 \leq \lambda < 1$. Hence, from (69), we can derive (65) by setting $\mu = [\bar{\gamma} \text{tr}(Q)]^{1/2} + [\bar{\gamma} \text{tr}(R)]^{1/2} \mathcal{K} \|A\|$. Then, according to Lemma 2, \tilde{e}_t is mean-square bounded under the given assumptions.

Further, since

$$\bar{e}_t = (I - K_t C) \tilde{e}_t + K_t v_t, \quad (73)$$

we have that

$$E[\|\bar{e}_t\|^2] \leq 2(1 + \kappa\|C\|)^2 E[\|\tilde{e}_t\|^2] + 2\mathcal{K}^2 \text{tr}(R). \quad (74)$$

Therefore, the estimation error \bar{e}_t is also bounded in the mean-square sense.

Next, we will prove that $e_{t-T:t}$ is bounded. To this end, it is convenient to decompose $\hat{\Omega}_t = \Omega_t(\Psi_t^{(N-1)}, \Phi_t^{(N-1)})$ as follows

$$\hat{\Omega}_t = \Omega_{t,1} + \Omega_{t,2} + \Omega_{t,3} \quad (75)$$

where

$$\Omega_{t,1} = \begin{bmatrix} C' \Psi_t^{(N-1)} C & 0 & \cdots & 0 \\ 0 & C' \Psi_t^{(N-1)} C & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & C' \Psi_t^{(N-1)} C & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\Omega_{t,2} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ 0 & 0 & 0 \\ 0 & \cdots & 0 & \bar{P}_{t-T}^{-1} \end{bmatrix}$$

$$\Omega_{t,3} = \begin{bmatrix} \Omega_3^- & \Omega_3^\circ & 0 & \cdots & 0 \\ \Omega_3^\times & \Omega_3^- + \Omega_3^+ & \Omega_3^\circ & & \vdots \\ 0 & \Omega_3^\times & \ddots & \ddots & \\ \vdots & & \ddots & \Omega_3^- + \Omega_3^+ & \Omega_3^\circ \\ 0 & \cdots & & \Omega_3^\times & \Omega_3^+ \end{bmatrix} \quad (76)$$

with

$$\begin{cases} \Omega_3^- &= \Phi^{(N-1)} \\ \Omega_3^+ &= A^T \Phi^{(N-1)} A \\ \Omega_3^\times &= -A' \Phi^{(N-1)} \\ \Omega_3^\circ &= -\Phi^{(N-1)} A. \end{cases} \quad (77)$$

Further, it is an easy matter to check that $\hat{\omega}_t = \omega_t(\Psi_t^{(N-1)})$ can be decomposed as follows

$$\hat{\omega}_t = \omega_{t,1} + \omega_{t,2} + \omega_{t,3} \quad (78)$$

where

$$\omega_{t,1} = \Omega_{t,1} x_{t-T:t}$$

$$\omega_{t,2} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \quad \omega_{t,3} = \begin{bmatrix} C' \Psi_t^{(N-1)} v_t \\ \vdots \\ C' \Psi_t^{(N-1)} v_{t-T+1} \\ 0 \end{bmatrix}. \quad (79)$$

Clearly, from (75), the true state trajectory satisfies the identity

$$x_{t-T:t} = \hat{\Omega}_t^{-1} (\Omega_{1,t} + \Omega_{2,t} + \Omega_{3,t}) x_{t-T:t}. \quad (80)$$

Further, for the estimated state trajectory we have

$$\begin{aligned} \hat{x}_{t-T:t|T} &= \hat{\Omega}_t^{-1} \hat{\omega}_t = \hat{\Omega}_t^{-1} (\omega_{t,1} + \omega_{t,2} + \omega_{t,3}) \\ &= \hat{\Omega}_t^{-1} \Omega_{t,1} x_{t-T:t} + \hat{\Omega}_t^{-1} (\omega_{t,2} + \omega_{t,3}). \end{aligned} \quad (81)$$

Hence, by subtracting the two latter equations, we get

$$x_{t-T:t} - \hat{x}_{t-T:t|T}$$

$$= \hat{\Omega}_t^{-1} (\Omega_{2,t} x_{t-T:t} - \omega_{2,t} + \Omega_{3,t} x_{t-T:t} - \omega_{3,t}). \quad (82)$$

Notice that

$$\Omega_{2,t} x_{t-T:t} - \omega_{2,t} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{P}_{t-T}^{-1} \bar{e}_{t-T} \end{bmatrix} \quad (83)$$

which turns out to be uniformly bounded in mean square as previously shown. Notice also that

$$\Omega_{3,t} x_{t-T:t} = \begin{bmatrix} \Phi_t^{(N-1)} w_{t-1} \\ -A' \Phi_t^{(N-1)} w_{t-1} + \Phi_t^{(N-1)} w_{t-2} \\ \vdots \\ -A' \Phi_t^{(N-1)} w_{t-T+1} + \Phi_t^{(N-1)} w_{t-T} \\ -A' \Phi_t^{(N-1)} w_{t-T} \end{bmatrix} \quad (84)$$

Notice finally that, irrespectively of the number of samples J , the number of VB iterations N and the window length T , by construction the matrices $\Phi_t^{(k)}$ and $\Psi_t^{(k)}$ can be bounded as $\bar{\alpha}^{-1} I \leq \hat{\Phi}_t^{(k)} \leq \underline{\alpha}^{-1} I$ and $\bar{\beta}^{-1} I \leq \hat{\Psi}_t^{(k)} \leq \underline{\beta}^{-1} I$ for any $t \geq T$. As a consequence, all the matrices involved in (82) are uniformly bounded. Then, in view of (79) (82), (83), (84), we can conclude that there exist suitable constants c_1, c_2 and c_3 such that

$$E [\|e_t\|^2] \leq c_1 E [\|\bar{e}_{t-T}\|^2] + c_2 \text{tr}(Q) + c_3 \text{tr}(R). \quad (85)$$

By combining the latter inequality with (74), the uniform mean square boundedness of e_t follows. \square

Remark 1 The proposed algorithm has been developed under the assumption that the unknown PNCM and MNCM are nearly constant within the sliding window $[t-T, t]$. This condition is satisfied whenever the unknown covariance matrices are constant or their variations are slow compared to the size T of the sliding window. The forgetting factor ρ in the time propagation (53)-(56) can be tuned so as to make the filter more able to promptly detect variations in the unknown covariance matrices (by choosing ρ close to 0) or to improve the estimation accuracy for nearly constant matrices (by choosing ρ close to 1). Nevertheless, the stability of the estimation error is guaranteed for any choice of ρ and for any time-varying Q_t and R_t provided that they remain uniformly bounded.

6 Simulations

To assess the performance of the proposed adaptive VB MHE filter, a 2-dimensional target tracking example is

considered in this section. The target moves according to (1) with state $x = [\xi_t, \eta_t, \dot{\xi}_t, \dot{\eta}_t]'$, where (ξ_t, η_t) and $(\dot{\xi}_t, \dot{\eta}_t)$ denote target position and velocity in Cartesian coordinates, respectively. The state transition matrix is $A = \begin{bmatrix} I_2 & \mathcal{T} I_2 \\ 0 & I_2 \end{bmatrix}$ where I_n is the $n \times n$ identity and $\mathcal{T} = 1[s]$ the sampling interval. The target position coordinates are measured according to the measurement model (2) with $C = [I_2 \ 0]$. The unknown process and measurement noise covariances are supposed to belong to the bounded sets

$$\mathcal{Q} = \{Q \in \mathbb{S}_+^4 : 0.001 Q_0 \leq Q \leq 1000 Q_0\} \quad (86)$$

$$\mathcal{R} = \{R \in \mathbb{S}_+^2 : 0.1 R_0 \leq R \leq 10 R_0\} \quad (87)$$

where the nominal PNCM and MNCM are given by

$$Q_0 = \begin{bmatrix} 1/3 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 1/2 \\ 1/2 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1 \end{bmatrix}, \quad R_0 = 100 \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}.$$

Monte Carlo simulations with 50 independent trials of duration $t = 500[s]$ have been carried out to compare the nominal Kalman filter (NKF), the conventional VB filter of Huang et al. (2018) and the sliding window variational Kalman filter (VB Sliding Window, VB SW) of Huang et al. (2020) with the proposed adaptive VB MHE filter. Initial state and covariance for all filters are set to: $x_0 = [0[m], 10[m], 0[m/s], 10[m/s]]'$ and $P_0 = \text{diag}\{100[m^2], 100[m^2], 100[m^2/s^2], 100[m^2/s^2]\}$. For the nominal KF, the PNCM and MNCM are set to Q_0 and R_0 ; for the conventional VB, the nominal PNCM is set to Q_0 while the PECM and MNCM are estimated adaptively; the VB parameters for the conventional VB, VB Sliding Window and proposed VB MHE are set as in (Huang et al., 2018), i.e., $\rho = 0.9$, $\hat{S}_{0|0}^i = \kappa R_0$, $\hat{s}_{0|0}^i = \kappa + n_y + 1$, $\kappa = 3$, $\tau = 3$, $N = 1$. Further, for the VB Sliding Window and VB MHE, different values of the window length T are considered, i.e. $T \in \{4, 5, 10, 20\}$. The number of importance samples of the proposed bounded VB MHE is set to $J = 100$. The unknown true PNCM and MNCM are set to $Q = 50 Q_0$ and $R = 3 R_0$, respectively.

For the filtering performance assessment, the *root mean square error* (RMSE) versus time and the *time-averaged RMSE* (ARMSE) for position and velocity over the whole simulation are provided in Fig. 1, and respectively Tables 3-4, demonstrating the outperformance of the proposed filter with respect to the others. It can be seen from Tables 3-4 that, when $T = 20$, the proposed VB MHE filter provides improvement with respect to the conventional VB, nominal KF, VB Sliding Window

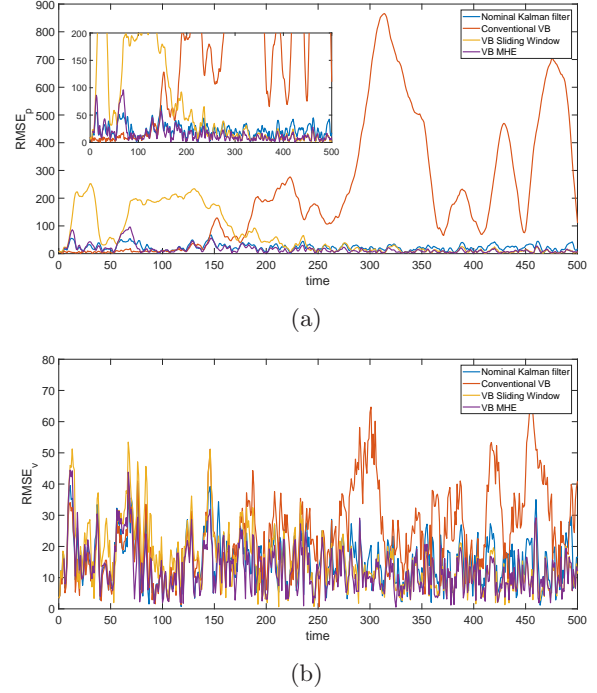


Fig. 1. Position (a) and velocity (b) RMSEs ($T = 5$)

of 96%, 59%, 42% in position ARMSE and 56%, 29%, 11% in velocity ARMSE. Conversely, when $T = 4$, the corresponding improvement with respect to the conventional VB, nominal KF, VB Sliding Window is 91%, 13%, 88% in position ARMSE and 45%, 12%, 36% in velocity, respectively. Although the results show performance degradation of both the VB Sliding Window and the proposed VB MHE when the window length decreases, the VB MHE degrades gracefully by providing smaller position and velocity ARMSEs as well as quicker convergence for all values of the window length, especially for low values of T for which the VB Sliding Window may exhibit much worse performance.

7 Conclusions

An adaptive variational Bayes moving horizon estimation method for state estimation under unknown process and measurement noise covariances has been proposed. Stability analysis has shown that the proposed filter ensures mean-square boundedness of the state estimation error for any number of VB iterations and any length of the moving window. Simulation results on a target tracking example have demonstrated the effectiveness of the proposed filter. Future work will focus on consensus adaptive state estimation for networked filtering with unknown noise covariances as well as the related stability analysis of the distributed filter.

Table 3
Position ARMSE vs. window length T

T	Conventional VB	NKF	VB SW	VB MHE
20	-	-	16.2	9.4
10	-	-	19.4	10.5
5	-	-	65.6	15.9
4	-	-	172.2	20.1
-	236.7	23.2	-	-

Table 4
Velocity ARMSE vs. window length T

T	Conventional VB	NKF	VB SW	VB MHE
20	-	-	12.0	10.7
10	-	-	12.4	11.2
5	-	-	15.3	12.5
4	-	-	20.9	13.4
-	24.6	15.2	-	-

Acknowledgements

This work was partly funded by National Natural Science Foundation of China (61627810), National Science and Technology Major Program of China (2018YFB1305003), and China Scholarship Council.

References

- Alessandri, A., Baglietto, M., and Battistelli, G. (2003). Receding-horizon estimation for discrete-time linear systems. *IEEE Transactions on Automatic Control*, 48(3), 473–478.
- Alessandri, A. and Awawdeh, M. (2016). Moving-horizon estimation with guaranteed robustness for discrete-time linear systems and measurements subject to outliers. *Automatica*, 67, 85–93.
- Alessandri, A., Baglietto, M., and Battistelli, G. (2008). Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes. *Automatica*, 44(7), 1753–1765.
- Alessandri, A. and Battistelli, G. (2020). Moving horizon estimation: Open problems, theoretical progress, and new application perspectives. *International Journal of Adaptive Control and Signal Processing*, 34(6), 703–705.
- Alessandri, A. and Gaggero, M. (2017). Fast moving horizon state estimation for discrete-time systems using single and multi iteration descent methods. *IEEE Transactions on Automatic Control*, 62(9), 4499–4511.
- Bar-Shalom, Y., Kirubarajan, T., and Li, X.R. (2002). *Estimation with Applications to Tracking and Navigation*. John Wiley & Sons, Inc., USA.
- Battistelli, G. (2019). Distributed moving-horizon estimation with arrival-cost consensus. *IEEE Transactions on Automatic Control*, 64(8), 3316–3323.
- Battistelli, G., Chisci, L., and Gherardini, S. (2017). Moving horizon estimation for discrete-time linear systems with binary sensors: Algorithms and stability results. *Automatica*, 85, 374–385.
- Delgado, R.A. and Goodwin, G.C. (2014). A combined MAP and Bayesian scheme for finite data and/or moving horizon estimation. *Automatica*, 50(4), 1116–1121.
- Dong, P., Jing, Z., Leung, H., and Shen, K. (2017). Variational Bayesian adaptive cubature information filter based on Wishart distribution. *IEEE Transactions on Automatic Control*, 62(11), 6051–6057.
- Dong, X., Battistelli, G., Chisci, L., and Cai, Y. (2021a). An adaptive consensus filter for distributed state estimation with unknown noise statistics. *IEEE Signal Processing Letters*.
- Dong, X., Chisci, L., and Cai, Y. (2021b). An adaptive variational Bayesian filter for nonlinear multi-sensor systems with unknown noise statistics. *Signal Processing*, 179, 107837.
- Farina, M., Ferrari-Trecate, G., and Scattolini, R. (2010a). Distributed moving horizon estimation for linear constrained systems. *IEEE Transactions on Automatic Control*, 55(11), 2462–2475.
- Farina, M., Ferrari-Trecate, G., and Scattolini, R. (2010b). Moving-horizon partition-based state estimation of large-scale systems. *Automatica*, 46(5), 910–918.
- Fiedler, F., Baumbach, D., Börner, A., and Lucia, S. (2020). A probabilistic moving horizon estimation framework applied to the visual-inertial sensor fusion problem. In *2020 European Control Conference (ECC)*, 1009–1016.
- Gharbi, M., Bayer, F., and Ebenbauer, C. (2021). Proximity moving horizon estimation for discrete-time nonlinear systems. *IEEE Control Systems Letters*, 5(6), 2090–2095.
- Huang, Y., Zhang, Y., Wu, Z., Li, N., and Chambers, J. (2018). A novel adaptive Kalman filter with inaccurate process and measurement noise covariance matrices. *IEEE Transactions on Automatic Control*, 63(2), 594–601.
- Huang, Y., Zhu, F., Jia, G., and Zhang, Y. (2020). A slide window variational adaptive Kalman filter. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(12), 3552–3556.
- Jazwinski, A.H. (1970). *Stochastic processes and filtering theory*. Academic Press, Inc., New York.
- Joseph, P.D. and Tou, J.T. (1961). On linear control theory. *Transactions of the American Institute of Electrical Engineers, Part II: Applications and Industry*, 80(4), 193–196.
- Kalman, R.E. (1960). A new approach to linear filtering and prediction problems. *Journal of Basic Engineering*, 82(1), 35–45.
- Lauricella, M., Farina, M., Schneider, R., and Scattolini, R. (2020). Iterative distributed fault detection and

- isolation for linear systems based on moving horizon estimation. *International Journal of Adaptive Control and Signal Processing*, 34(6), 743–756.
- Leung, H., Zhu, Z., and Ding, Z. (2000). An aperiodic phenomenon of the extended Kalman filter in filtering noisy chaotic signals. *IEEE Transactions on Signal Processing*, 48(6), 1807–1810.
- Liu, A., Yu, L., Zhang, W.A., and Chen, M.Z.Q. (2013). Moving horizon estimation for networked systems with quantized measurements and packet dropouts. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 60(7), 1823–1834.
- Mehra, R.K. (1972). Approaches to adaptive filtering. *IEEE Transactions on Automatic Control*, 17(5), 693–698.
- O’Hagan, A. and Forster, J.J. (2004). *Kendall’s Advanced Theory of Statistics, volume 2B: Bayesian Inference, second edition*, volume 2B. Arnold.
- Rao, C.V., Rawlings, J.B., and Lee, J.H. (2001). Constrained linear state estimation—a moving horizon approach. *Automatica*, 37(10), 1619–1628.
- Rao, C., Rawlings, J., and Mayne, D. (2003). Constrained state estimation for nonlinear discrete-time systems: stability and moving horizon approximations. *IEEE Transactions on Automatic Control*, 48(2), 246–258.
- Sato, M.a. (2001). Online model selection based on the variational bayes. *Neural Computation*, 13(7), 1649–1681.
- Schneider, R. and Marquardt, W. (2016). Convergence and stability of a constrained partition-based moving horizon estimator. *IEEE Transactions on Automatic Control*, 61(5), 1316–1321.
- Tzikas, D.G., Likas, A.C., and Galatsanos, N.P. (2008). The variational approximation for Bayesian inference. *IEEE Signal Processing Magazine*, 25(6), 131–146.
- Wanasinghe, T.R., Mann, G.K.I., and Gosine, R.G. (2015). Stability analysis of the discrete-time cubature kalman filter. In *2015 54th IEEE Conference on Decision and Control (CDC)*, 5031–5036.
- Yin, X. and Liu, J. (2017). Distributed moving horizon state estimation of two-time-scale nonlinear systems. *Automatica*, 79, 152–161.
- Youn, W., Huang, Y., and Myung, H. (2020). Outlier-robust Student’s-t-based IMM-VB localization for manned aircraft using TDOA measurements. *IEEE/ASME Transactions on Mechatronics*, 25(3), 1646–1658.
- Zhang, C., Büttepage, J., Kjellström, H., and Mandt, S. (2019). Advances in variational inference. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 41(8), 2008–2026.
- Zou, L., Wang, Z., Hu, J., and Zhou, D. (2020a). Moving horizon estimation with unknown inputs under dynamic quantization effects. *IEEE Transactions on Automatic Control*, 65(12), 5368–5375.
- Zou, L., Wang, Z., and Zhou, D. (2020b). Moving horizon estimation with non-uniform sampling under component-based dynamic event-triggered transmission. *Automatica*, 120, 109154.