# On the Complexity of Smooth Spline Surfaces from Quad Meshes 

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October 23, 2018


#### Abstract

This paper derives strong relations that boundary curves of a smooth complex of patches have to obey when the patches are computed by local averaging. These relations restrict the choice of reparameterizations for geometric continuity.

In particular, when one bicubic tensor-product B -spline patch is associated with each facet of a quadrilateral mesh with $n$-valent vertices and we do not want segments of the boundary curves forced to be linear, then the relations dictate the minimal number and multiplicity of knots: For general data, the tensor-product spline patches must have at least two internal double knots per edge to be able to model a $G^{1}$-conneced complex of $C^{1}$ splines. This lower bound on the complexity of any construction is proven to be sharp by suitably interpreting an existing surface construction. That is, we have a tight bound on the complexity of smoothing quad meshes with bicubic tensor-product B-spline patches.


## 1 Introduction

Even though every newly proposed smooth surface construction seeks to be optimal in some aspect, the overall theory of smooth surface constructions offers few sharp lower bounds, i.e. proofs that no polynomial construction of lower degree is possible and that a construction of this least degree exists so that upper bound and lower bound match. One well-appreciated bound is the degree-6 bound for $C^{2}$ subdivision surfaces derived by Reif and Prautzsch Pra97 and shown to be sharp, for example by Rei98, PR99. Such sharp bounds allow us to

- understand the fundamental difficulty of the task, and to
- guide future research by showing where research is futile
- and what assumptions must be side-stepped to derive substantially new results.

We are motivated by a standard task of geometric design: to determine $G^{1}$ connected tensor-product B-spline patches approximating a quadrilateral mesh
whose vertices can have any fixed valence. While this challenge can be met by recursive subdivision CC78, representing the surface with a finite small number of patches defined by the quad and its neighbors is often preferable, for example to parallelize the construction (see e.g. [LS08, MYP08). This raises the question: (Q) what is the simplest structure (in distribution and number of knots) of degree bi-3 spline patches that allow a quad mesh to be converted by localized operations into a smooth surface with one spline patch per quad? Surprisingly, this basic question at the heart of a classical task of geometric design has not been settled to date.

To frame the question, Section 2 takes a more general view. We do not constrain the domain to be a collection of quadrilaterals or the functions to be polynomial splines. Also, the relations in Lemmas 1, 2 and 3 do not depend on locality of the construction but apply to any collection of sufficiently smooth patches coming together with a logically symmetric $G^{1}$ join: $\partial_{2} \mathbf{b}^{k}(u, 0)+\partial_{1} \mathbf{b}^{k-1}(0, u)=$ $\alpha^{k}(u) \partial_{1} \mathbf{b}^{k}(u, 0)$ (see Definition 1. page (4). Adding locality of operations as a requirement in Section 2.1 then rules out everywhere (piecewise) linear $\alpha^{k}$, still in the very general setting.

In Section 3, we specialize the setting to polynomial tensor-product splines of degree bi-3. For these, we obtain a lower bound on the number and multiplicity of knots. We prove that at least two internal double knots are required per edge to admit a local construction. This lower bound is tight, because the recentlypublished construction for smooth surfaces FP08 can be re-interpreted as a spline construction with exactly two internal double knots. Together, the lower and upper bound conclusively settle the question Q .

## 1.1 $\mathrm{Bi}-3$ constructions in the literature

Creating $C^{1}$ surfaces with a finite number of patches of degree bi-3, i.e. generalizing standard tensor-product B-splines to smooth surfaces from arbitrary manifold quad meshes, is a classic challenge of CAGD (see e.g. Bez77, vW86, Pet91). The assumption that a simple construction with a finite number of patches is not possible motivates the classic Catmull-Clark subdivision (Fig. 1 left). PCCM Pet00 is a finite construction that approximates Catmull-Clark limit surfaces with smoothly connected bi-3 patches. PCCM requires up to two steps of Catmull-Clark subdivision to separate non-4-valent vertices. This proves that a $4 \times 4$ arrangement of polynomial patches per quad suffices in principle, corresponding to two double interior knots and one single knot (Fig. [1 middle), However, PCCM can have poor shape for certain higher-order saddles (Fig. 5. URL, Pet01, LS08]). More recently, a number of papers appeared that are also predicated on the assumption that a simple construction with a finite number of patches is not possible. Shi et al. [SWWL04, SLW06] propose a subdivisionlike refinement approach with bi-3 tensor-product patches to obtain $C^{0}$ surfaces where ever more single knots are inserted. They correctly surmise that, in general, no finite $C^{1}$ construction with $C^{2}$ tensor-product splines of degree bi-3 is possible (see Theorem 1 of our paper). At the other extreme, using a single patch per quad, Loop and Schaefer LS08 propose a bi-3 $C^{0}$ surface construc-


Figure 1: Knot distribution. A quadrilateral piece generated by CatmullClark subdivision has (infinitely many) single knots, a piece of PCCM requires two double and at least one more single knot, and the construction FP08 has two double interior knots (which this paper shows to be the minimal number of knots).
tion with separate tangent patches to convey an impression of smoothness as in VPBM01, while Myles et al. MYP08 perturb a bi-3 base patch near non4 -valent vertices to obtain a $C^{1}$ surface of degree bi- 5 for CAD applications. Hahmann et al. HBC08 propose a $2 \times 2$ macro-patch per quad; and Fan and Peters [FP08 present an algorithm that constructs smoothly connected Bézier patches of degree bi-3 whose internal transitions allow re-interpretion as one tensor-product spline patch per quad with two internal double knots (Fig. [] right, Corollary (4). We will see that this is indeed the minimal number and multiplicity of knots for the standard Catmull-Clark layout of patches. The structurally different polar layout allows collapsed bi-3 spline patches with single internal knots to complete a $C^{1}$ surface MKP07.

## 2 Unbiased $G^{1}$ constraints

We consider $n$ parameterically $C^{1}$ patches

$$
\begin{equation*}
\mathbf{b}^{k}: \square \subsetneq \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}, \quad k=1, \ldots, n \tag{1}
\end{equation*}
$$

meeting at a central point $\mathbf{b}^{k}(0,0)=\mathbf{p}$ such that $\mathbf{b}^{k}(u, 0)=\mathbf{b}^{k-1}(0, u)$ (see Fig. (2). We do not (yet) assume that $\square$ is the unit square but just that the origin is a corner of the domain $\square$ and that two edges emanate from it in independent directions. We also assume that the patches are not singular at the origin in the sense that $\partial_{2} \mathbf{b}^{k}(0,0) \times \partial_{1} \mathbf{b}^{k}(0,0) \neq 0$ where $\partial_{\ell}$ denotes differentiation with respect to the $\ell$ th argument.

To make the $n$ patches form a $C^{1}$ surface, we want to enforce logically symmetric (unbiased) $G^{1}$ constraints. (We will discuss the general case in Section (4)

Definition 1 (Unbiased $G^{1}$ constraints) With $\alpha^{k}: \mathbb{R} \rightarrow \mathbb{R}$ a sufficiently smooth, univariate scalar-valued function, the unbiased $G^{1}$ constraints between


Figure 2: Indexing and parameterization of adjacent patches at a vertex of valence $n$ (if $k=1$ then $\mathbf{b}^{k-1}=\mathbf{b}^{n}$ ), illustrating the $G^{1}$ constraints (2) .
consecutive patches are

$$
\begin{equation*}
\partial_{2} \mathbf{b}^{k}(u, 0)+\partial_{1} \mathbf{b}^{k-1}(0, u)=\alpha^{k}(u) \partial_{1} \mathbf{b}^{k}(u, 0) \tag{2}
\end{equation*}
$$

If $\alpha^{k} \equiv 0$, the constraints enforce parametric $C^{1}$ continuity.
We abbreviate

$$
\begin{equation*}
a_{\ell}^{k} \in \mathbb{R}, \quad \text { the } \ell \text { th derivative of } \alpha^{k} \text { evaluated at } 0 \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{t}^{k}:=\partial_{1} \mathbf{b}^{k}(0,0) \in \mathbb{R}^{3} \tag{4}
\end{equation*}
$$

so that relation (2) becomes at $(0,0)$

$$
\left.\mathbf{t}^{k+1}+\mathbf{t}^{k-1}=a_{0}^{k} \mathbf{t}^{k} . \quad \text { (2) }\right)_{u=0}
$$



That is, superscripts count sectors (modulo $n$ ) surrounding $(0,0)$ while subscripts indicate derivatives. Later, starting with (21), we will use a second subscript (and remove the superscript) to denote pieces of $\alpha^{k}$.

We now add the assumption that each $\mathbf{b}^{k}$ is twice continuously differentiable at $(0,0)$ (as are the polynomial pieces of a spline patch). In reference to the main application, we will call such smooth functions generalized splines.

Definition 2 (Knot lines and generalized splines) $A C^{s}$ generalized spline patch is a map $\mathbf{b}^{k}: \square \subsetneq \mathbb{R}^{2} \rightarrow \mathbb{R}^{3}$ that is s times continuously differentiable. The set of knot lines of $\mathbf{b}^{k}$ is a finite collection of lines in $\square$ such that at most
two distinct lines cross. Boundary edges of $\square$ are knot lines. An intersection of a boundary edge minus its end points with a non-parallel knot line is called an edge knot. At every edge knot, on either side of its knot line,

$$
\begin{equation*}
\partial_{1}^{i} \partial_{2}^{j} \mathbf{b}^{k} \text { is well-defined for } i+j \leq s+1 \quad \text { and } \quad \partial_{2} \mathbf{b}^{k} \times \partial_{1} \mathbf{b}^{k} \neq 0 \tag{5}
\end{equation*}
$$

The generalized spline definition is intentionally broader than its subclass of polynomial tensor-product splines that motivates it. It includes, for example, trigonometric splines or subdivision constructions.

For $C^{1}$ generalized splines, we can then differentiate relation (21) along (the respective domain edge of) the common boundary $\mathbf{b}^{k}(u, 0)=\mathbf{b}^{k-1}(0, u)$ :

$$
\begin{align*}
& \left(\partial_{1} \partial_{2} \mathbf{b}^{k}\right)(u, 0)+\left(\partial_{2} \partial_{1} \mathbf{b}^{k-1}\right)(0, u) \\
& =\alpha^{k}(u) \partial_{1}^{2} \mathbf{b}^{k}(u, 0)+\left(\alpha^{k}\right)^{\prime}(u) \partial_{1} \mathbf{b}^{k}(u, 0) \tag{6}
\end{align*}
$$

When we evaluate at $u=0$ then

$$
\begin{equation*}
\text { at }(0,0), \quad \partial_{1} \partial_{2} \mathbf{b}^{k}+\partial_{2} \partial_{1} \mathbf{b}^{k-1}=a_{0}^{k} \partial_{1}^{2} \mathbf{b}^{k}+a_{1}^{k} \partial_{1} \mathbf{b}^{k} \tag{7}
\end{equation*}
$$

If $n$ is even then the alternating sum of the left hand sides vanishes

$$
\begin{equation*}
\text { at }(0,0), \quad \sum_{k=1}^{n}(-1)^{k}\left(\partial_{1} \partial_{2} \mathbf{b}^{k}+\partial_{2} \partial_{1} \mathbf{b}^{k-1}\right)=0 \tag{8}
\end{equation*}
$$

and therefore so must the right hand side

$$
\begin{equation*}
\text { at }(0,0), \quad 0=\sum_{k=1}^{n}(-1)^{k} a_{0}^{k} \partial_{1}^{2} \mathbf{b}^{k}+\sum_{k=1}^{n}(-1)^{k} a_{1}^{k} \partial_{1} \mathbf{b}^{k} \tag{9}
\end{equation*}
$$

In particular, if the patches join smoothly and therefore have a unique normal $\mathbf{n} \in \mathbb{R}^{3}$ at $\mathbf{p}$ then, with $\cdot$ denoting the scalar product,

$$
\begin{equation*}
\text { if } n \text { is even, at }(0,0) \quad 0=\sum_{k=1}^{n}(-1)^{k} a_{0}^{k} \mathbf{n} \cdot \partial_{1}^{2} \mathbf{b}^{k} \tag{10}
\end{equation*}
$$

This is the vertex-enclosure constraint (see e.g. Pet02, p.205]).
We briefly focus on the important generic case where $n=4$ patches meet.
Definition 3 (tangent X) If $n=4, \partial_{1} \mathbf{b}^{1}(0,0)=-\partial_{1} \mathbf{b}^{3}(0,0)$ and $\partial_{1} \mathbf{b}^{2}(0,0)=$ $-\partial_{1} \mathbf{b}^{4}(0,0)$ then the tangents form an $X$.

Lemma 1 ( $\mathbf{X}$ tangent) If the tangents form an $X$, then $a_{1}^{1}=a_{1}^{3}$ and $a_{1}^{2}=a_{1}^{4}$.

Proof If the tangents form an X then $n=4$ and $a_{0}^{k}=0, k=1,2,3,4$ so that (9) simplifies to

$$
\begin{equation*}
\text { at }(0,0), \quad 0=\left(a_{1}^{1}-a_{1}^{3}\right) \partial_{1} \mathbf{b}^{1}-\left(a_{1}^{2}-a_{1}^{4}\right) \partial_{1} \mathbf{b}^{2} . \tag{11}
\end{equation*}
$$

Since the patches are regular at corners, both summands have to vanish, implying the claim.

We now consider the unbiased $G^{1}$ transition between two $C^{1}$ generalized spline patches. We focus on an edge vertex, the image of an edge knot on the common boundary. By definition, an edge vertex is not an end point of the boundary. That is, we consider a point where four polynomial pieces meet such that $\mathbf{b}^{1}$ and $\mathbf{b}^{2}$ belong to one generalized spline patch and $\mathbf{b}^{3}$ and $\mathbf{b}^{4}$ are adjacent pieces of the edge-adjacent generalized spline patch (Figure 3). Since each generalized spline patch is internally parameterically $C^{1}$, by Definition $\mathbb{}$

$$
\begin{equation*}
\alpha^{2} \equiv 0 \equiv \alpha^{4} \tag{12}
\end{equation*}
$$



Figure 3: Join across an edge knot on the boundary (solid) between two generalized splines. The first generalized spline has polynomial pieces $\mathbf{b}^{1}$ and $b^{2}$.

Lemma 2 ( $C^{1}$ generalized spline, edge knot) Let $(0,0)$ be the parameter associated with an edge knot on the boundary common to two $C^{1}$ generalized splines that are joined by unbiased $G^{1}$ constraints. Then

$$
\begin{align*}
a_{0}^{1} & =-a_{0}^{3},  \tag{13}\\
\text { at }(0,0): 0 & =a_{0}^{1}\left(\partial_{1}^{2} \mathbf{b}^{1}-\partial_{1}^{2} \mathbf{b}^{3}\right)+\left(a_{1}^{1}-a_{1}^{3}\right) \mathbf{t}^{1} . \tag{14}
\end{align*}
$$

Proof Since $n=4, a_{0}^{1} \mathbf{t}^{1}=\mathbf{t}^{2}+\mathbf{t}^{4}=a_{0}^{3} \mathbf{t}^{3}$ and the parametric $C^{1}$ constraints imply $\mathbf{t}^{1}:=-\mathbf{t}^{3}$ so that (13) follows. By (12), (9) specializes to

$$
\text { at }(0,0), \quad \begin{aligned}
0 & =a_{0}^{1} \partial_{1}^{2} \mathbf{b}^{1}+a_{0}^{3} \partial_{1}^{2} \mathbf{b}^{3}+a_{1}^{1} \partial_{1} \mathbf{b}^{1}+a_{1}^{3} \partial_{1} \mathbf{b}^{3} \\
& =a_{0}^{1}\left(\partial_{1}^{2} \mathbf{b}^{1}-\partial_{1}^{2} \mathbf{b}^{3}\right)+\left(a_{1}^{1}-a_{1}^{3}\right) \mathbf{t}^{1}
\end{aligned}
$$

as claimed.
So, remarkably, when two generalized spline patches meet along a common boundary, unbiased $G^{1}$ constraints across this boundary imply the constraint (14) exclusively in terms of derivatives along the boundary.

Lemma 3 ( $C^{2}$ generalized spline, edge knot) Let $(0,0)$ be the parameter associated with an edge knot of the boundary common to two $C^{2}$ generalized splines joined by unbiased $G^{1}$ constraints. Then, in addition to (13), at ( 0,0 ),

$$
\begin{align*}
a_{1}^{1} & =a_{1}^{3}  \tag{15}\\
0 & =a_{0}^{1}\left(\partial_{1}^{3} \mathbf{b}^{1}-\partial_{1}^{3} \mathbf{b}^{3}\right)+4 a_{1}^{1} \partial_{1}^{2} \mathbf{b}^{1}+\left(a_{2}^{1}-a_{2}^{3}\right) \mathbf{t}^{1} \tag{16}
\end{align*}
$$

Proof Since the generalized splines are $C^{2}, \partial_{1}^{2} \mathbf{b}^{1}(0,0)=\partial_{1}^{2} \mathbf{b}^{3}(0,0)$. Then (14) implies (15).

Parametric $C^{2}$ continuity across the spline-internal boundaries (see dashed lines in Fig. (3) implies

$$
\begin{equation*}
\text { for } k=2,4, \text { at }(0,0), \quad \partial_{2} \partial_{1} \partial_{2} \mathbf{b}^{k}+\partial_{1} \partial_{2} \partial_{1} \mathbf{b}^{k-1}=0 \tag{17}
\end{equation*}
$$

Differentiating (6) once more along the (direction corresponding to the) common boundary of the two generalized splines, we obtain for $k=1,3$, at $(0,0)$,

$$
\begin{equation*}
\partial_{1} \partial_{1} \partial_{2} \mathbf{b}^{k}+\partial_{2} \partial_{2} \partial_{1} \mathbf{b}^{k-1}=a_{0}^{k} \partial_{1}^{3} \mathbf{b}^{k}+2 a_{1}^{k} \partial_{1}^{2} \mathbf{b}^{k}+a_{2}^{k} \partial_{1} \mathbf{b}^{k} \tag{18}
\end{equation*}
$$

Summing the two instances of (18) and subtracting the two instances of (17) eliminates the mixed derivatives of the left hand side and yields at $(0,0)$

$$
\begin{align*}
0 & =a_{0}^{1} \partial_{1}^{3} \mathbf{b}^{1}+2 a_{1}^{1} \partial_{1}^{2} \mathbf{b}^{1}+a_{2}^{1} \partial_{1} \mathbf{b}^{1}  \tag{19}\\
& +a_{0}^{3} \partial_{1}^{3} \mathbf{b}^{3}+2 a_{1}^{3} \partial_{1}^{2} \mathbf{b}^{3}+a_{2}^{3} \partial_{1} \mathbf{b}^{3}
\end{align*}
$$

Parametric $C^{2}$ continuity then implies (16).

### 2.1 Linear $\alpha$ and vertex-localized constructions

The Taylor expansions up to order two of the patches joining at a point are strongly intermeshed by Equation (7). To avoid solving large, global systems, a vertex should not depend on the expansions at its neighbors.

Definition 4 (vertex-localized construction) A construction is $G^{1}$ vertexlocalized if we can solve at every vertex (with local parameters $(u, v)=(0,0)$ ) the unbiased $G^{1}$ constraints (2) $u=0$ and (6) on the second-order Taylor expansion $\partial_{1}^{i} \partial_{2}^{j} \mathbf{b}^{k}, 0 \leq i, j, i+j \leq 2$ independent of the expansions at its neighbors.

Note that a vertex-localized construction can use a priori known input, for example the local connectivity and the valence of the neighbors. Nevertheless,
the unbiased $G^{1}$ constraints imply a local, unbiased choice of the tangent directions, namely such that

$$
\begin{equation*}
\alpha^{k}(0):=2 \cos \frac{2 \pi}{n} \tag{20}
\end{equation*}
$$

(For a proof that logical symmetry implies (20) see e.g. Pet94, Prop 3].)
Corollary 1 (valence symmetry for $n=4$ and linear $\alpha$ ) Let $n=4$ and let $n^{k}$ denote the valence of the $k$ th neighbor vertex, $k=1, \ldots, n$. Then $a$ local, unbiased choice of the tangent directions and $\alpha^{k}$ linear are compatible with unbiased $G^{1}$ constraints only when the valences of opposite neighbors agree: $n^{k}=n^{k+2}$.

Proof The claim follows from Lemma 1 since by the unbiased choice $\alpha^{k}(0):=0$ and $\alpha^{k}(1):=2 \cos \frac{2 \pi}{n^{k}}$.

Corollary 1 is a remarkably strong restriction since vertices of valence $n=4$ are common. Choosing linear $\alpha$ can therefore be problematic. For example, the construction HBC08 can therefore not succeed in general.

In the most challenging case, the vertex enclosure constraint (10) applies at each vertex. While the vertex-enclosure constraint only restricts the normal component of the second derivatives along the curves at each vertex, independence of the normals at endpoints means that in general all three coordinates are constrained. So for vertex-localized constructions, we should assume that the second-order Taylor expansion has to be set independently at each vertex. It is this scenario with unrestricted choice of geometric instantiation of the secondorder Taylor expansion that we take into account when, in the following, we prefix a statement with 'in general'.


Figure 4: Propagation of $a_{0, j}=0$ in Lemma 4

Along a boundary curve, each scalar function $\alpha^{k}$ can consist of pieces that correspond to the knot segments of the two generalized splines meeting along the curve. Since, in this context, we only deal with one $k$ at a time, we drop this $k$ superscript and define

$$
\begin{equation*}
a_{\ell, j} \in \mathbb{R} \quad \text { to be the } \ell \text { th derivative of the } j \text { th piece } \alpha_{j} \text { at } 0 . \tag{21}
\end{equation*}
$$

For example, $\alpha^{3}$ and $\alpha^{1}$ in Fig. 3, can be relabeled $\alpha_{j}:=\alpha^{3}$ and $\alpha_{j+1}:=\alpha^{1}$.
Lemma 4 (everywhere piecewise linear $\alpha$ ruled out) In general, a vertexlocalized construction of unbiased $G^{1}$ transitions between $C^{1}$ generalized spline patches with everywhere at most linear $\alpha$ is not possible.

Proof Consider a vertex surrounded by vertices of valence $n=4$. Then vertexlocalized construction implies that $a_{0,0}=0$. Assume for now that edge knots exist. Then local construction implies also $a_{0,-1}=0$ (and $a_{0,1}=0$ ) for the immediate neighbor edge vertexes since all neighbor vertices are of valence 4 (by definition at most two knot lines intersect within a generalized spline). Shifting the focus to one such an edge vertex, say the one corresponding to $a_{0,-1}=0$, we observe that its tangents form an X since the two generalized splines, one at either side, are internally parametrically $C^{1}$ (across each dashed line in Fig. 3). So Lemma 1 and $a_{0,0}=0$ imply $a_{0,-2}=0$ and again this edge vertex's tangents form an X. In this manner, X configurations and $a_{0,-j}=0$ propagate, also across vertices of valence $n=4$ whose neighbors are not all of valence $n=4$ (see the arrows in Figure 4 for illustration). Once the propagation meets an original vertex with valence $n \neq 4$ (whether or not we had edge knots to start with), vertex-localized construction clashes with Lemma 1

Lastly, we characterize a known source of poor shape of smooth surface constructions due to restricted boundary curves Pet01. This limited flexibility is undesirable and constructions that cause it will later be excluded.


Figure 5: Shape defect (star shape) due to embedded straight line segments at a higher order saddle from URL.

Lemma 5 (Flatness at saddle points) Let $\mathbf{c}$ be a curve segment emanating from a higher-order saddle point $\mathbf{p}:=\mathbf{c}(0)$. If the derivative $\mathbf{c}^{\prime}$ of $\mathbf{c}$ factors into a linear vector-valued polynomial and a scalar factor:

$$
\begin{align*}
& \mathbf{c}^{\prime}:=\boldsymbol{\ell} \gamma  \tag{22}\\
& \boldsymbol{\ell}: \mathbb{R} \rightarrow \mathbb{R}^{3}, \operatorname{deg}(\boldsymbol{\ell}) \leq 1, \quad \gamma: \mathbb{R} \rightarrow \mathbb{R}, \operatorname{deg}(\gamma) \leq 1
\end{align*}
$$

then $\mathbf{c}$ is a planar curve segment. If the saddle is symmetric then $\mathbf{c}$ is a straight line segment.

Proof Let $\mathbf{n}$ be the normal at $\mathbf{p}$ and, without loss of generality, $\gamma(u):=1+\gamma_{1} u$ for some $\gamma_{1} \in \mathbb{R}$. Then $\mathbf{c}^{\prime}(0)=\boldsymbol{\ell}(0), \mathbf{c}^{\prime \prime}(0)=\boldsymbol{\ell}^{\prime}(0)+\boldsymbol{\ell}(0) \gamma_{1}$ and $\mathbf{c}^{\prime \prime \prime}(0)=$ $2 \ell^{\prime}(0) \gamma_{1}$. At a higher-order saddle point, the normal curvature is zero, and therefore $\mathbf{n} \cdot \mathbf{c}^{\prime \prime}(0)=0$. This implies $\mathbf{n} \cdot \boldsymbol{\ell}^{\prime}(0)=0$ and $\mathbf{n} \cdot \mathbf{c}^{\prime \prime \prime}(0)=0$ establishing planarity. If the saddle is symmetric then $\mathbf{c}^{\prime}(0)$ and $\mathbf{c}^{\prime \prime}(0)$ are collinear and so is $\mathbf{c}^{\prime \prime \prime}(0)=2 \boldsymbol{\ell}^{\prime}(0) \gamma_{1}$.

A higher-order saddle, such as the monkey saddle of Fig. 5. should have non-zero Gauss curvature apart from the central saddle point. Therefore, we will in the following disqualify constructions that force straight segments on the boundary for non-flat geometry.

To summarize, we showed that vertex-localized unbiased $G^{1}$ constructions with generalized splines are subject to strong restrictions on the reparametrization $\alpha$ (Lemma 1, 2 and 3) or the allowable valence of the vertices (Corollary (1). In the next section, we apply these general restrictions to polynomial splines.

## 3 Lower bounds for degree bi-3

We now argue that, in general, vertex-localized enforcement of unbiased $G^{1}$ constraints with polynomial tensor-product splines of degree bi-3 (bicubic) is possible only if the spline patches have at least two internal double knots per edge.

Since we specialize to polynomials $\mathbf{b}^{k}$ of degree bi-3, equality in the $G^{1}$ constraints implies that $\alpha$ is a rational function, $\alpha=: \frac{\beta}{\gamma}$. In fact, we have a low bound on the degrees of the numerator $\beta$ and the denominator $\gamma$.

Lemma 6 ( $\alpha$ degree restricted) If the two bi-3 patches $\mathbf{b}^{k}$ and $\mathbf{b}^{k-1}$ satisfy an unbiased $G^{1}$ constraint (2) then either

$$
\begin{align*}
& \alpha^{k}:=\frac{\beta}{\gamma} \text { is rational with }  \tag{23}\\
& (\operatorname{deg}(\beta), \operatorname{deg}(\gamma)) \in\{(2,1),(2,0),(1,1),(1,0),(0,1),(0,0)\} \\
& \text { and } \quad \partial_{1} \mathbf{b}^{k}(u, 0)=\boldsymbol{\ell}(u) \gamma(u), \operatorname{deg}(\boldsymbol{\ell}) \leq 2-\operatorname{deg}(\gamma) \tag{24}
\end{align*}
$$

or the boundary $\mathbf{b}^{k}(u, 0)$ is forced to have a straight segment.
Proof We may assume that $\beta$ and $\gamma$ are relatively coprime. Since the left hand side $\partial_{2} \mathbf{b}^{k}(u, 0)+\partial_{1} \mathbf{b}^{k-1}(0, u)$ of the $G^{1}$ constraint (2) is polynomial, $\gamma(u)$ must be a (scalar) factor of $\partial_{1} \mathbf{b}^{k}(u, 0) \in \mathbb{R}^{3}$, the (vector-valued) derivative of the boundary curve. Unless $\mathbf{b}^{k}(u, 0)$ is a line segment, $0<\operatorname{deg}\left(\partial_{1} \mathbf{b}^{k}(u, 0)\right) \leq 2$. Consequently $\operatorname{deg}(\gamma) \leq 2$ and since $\operatorname{deg}(\gamma)=2$ implies that $\partial_{1} \mathbf{b}^{k}(u, 0)=\mathbf{v} \gamma$ for a constant $\mathbf{v} \in \mathbb{R}^{3}, \operatorname{deg}(\gamma) \leq 1$ must hold to avoid that $\mathbf{b}^{k}(u, 0)$ is a straight segment. Since $\operatorname{deg}\left(\partial_{2} \mathbf{b}^{k}(u, 0)+\partial_{1} \mathbf{b}^{k-1}(0, u)\right) \leq 3$, also $\operatorname{deg}\left(\partial_{1} \mathbf{b}^{k}(u, 0) \beta\right) \leq 3$ and therefore $\operatorname{deg}(\beta) \leq 2$.

After scaling numerator and denominator, we may assume that $\gamma(u):=$ $1+\gamma_{1} u$. Not linear $\alpha$ then forces a particular boundary curve.

Corollary 2 ( $\alpha$ not linear restricts boundary curves) If $(\operatorname{deg}(\beta), \operatorname{deg}(\gamma)) \in$ $\{(2,1),(2,0),(1,1),(0,1)\}$ then the corresponding degree 3 boundary curve segment is of the form (22).

Proof The derivative of the curve segment either has a linear factor $\gamma$ or it is linear because $\operatorname{deg}(\beta)=2$.

Lemma 5 and Corollary 2 together imply that in general, at end points, $\alpha$ must be linear or constant if we require more flexibility than forced straight line segments.

Corollary 3 ( $\alpha$ not linear at higher-order saddle) If $\mathbf{b}^{k}(u, 0)$ emanates from a symmetric higher-order saddle point then $\alpha^{k}$ in the unbiased $G^{1}$ constraints (2) must be linear or constant for $\mathbf{b}^{k}(u, 0)$ not to be a straight segment.


Figure 6: (Figure 3 repeated) Join across an edge knot on the boundary (solid) between two splines. The first spline has polynomial pieces $\mathbf{b}^{1}$ and $\mathbf{b}^{2}$.

The next lemma shows that at edge knots, neighboring pieces of $\alpha$ constrain one another more than just by (14) and (16).

Lemma 7 ( $\alpha$ not linear at single knot) Let the segments be arranged as in Figure [6] (the same as Figure [3), the edge knot single and the left boundary segment ( $\mathbf{b}^{3}(u, 0)$ shared by the two bi-3 splines) fixed but general (in the sense that the control points cannot be assumed to be in a particular relation). Then $\alpha^{1}$ can only be not linear if

$$
\begin{equation*}
a_{0}^{3}=0, a_{1}^{3}=0, \text { and } a_{2}^{3}=a_{2}^{1} \neq 0 . \tag{25}
\end{equation*}
$$

In particular, $\alpha^{3}$ must also be not linear.
Proof If $\alpha:=\alpha^{1}$ is not linear then Lemma 6 implies $(\operatorname{deg}(\beta), \operatorname{deg}(\gamma)) \in$ $\{(2,1),(2,0),(1,1),(0,1)\}$ and therefore $\partial_{1} \mathbf{b}^{1}(u, 0):=\ell(u) \gamma(u)$, a linear vectorvalued polynomial times the scalar (possibly constant) factor $\gamma(u):=1+\gamma_{1} u$. By (13) and (15) and the $C^{2}$ constraints for the boundary curve, constraint (16) becomes

$$
\begin{equation*}
\text { at }(0,0), \quad 0=a_{0}^{3}(\underbrace{\partial_{3}^{3} \mathbf{b}^{3}-2 \gamma_{1} \ell^{\prime}(0)}_{=: \mathbf{v}})+4 a_{1}^{3} \partial_{1}^{2} \mathbf{b}^{3}+\left(a_{2}^{3}-a_{2}^{1}\right) \mathbf{t}^{3} . \tag{26}
\end{equation*}
$$

By $C^{1}$ continuity $\boldsymbol{\ell}(0) \gamma(0)=\boldsymbol{\ell}(0)=-\mathbf{t}^{3}$ and hence the $C^{2}$ constraint $\partial_{1}^{2} \mathbf{b}^{3}=$ $\boldsymbol{\ell}(0) \gamma_{1}+\boldsymbol{\ell}^{\prime}(0)=-\mathbf{t}^{3} \gamma_{1}+\boldsymbol{\ell}^{\prime}(0)$ implies

$$
\begin{equation*}
\boldsymbol{\ell}^{\prime}(0)=\mathbf{t}^{3} \gamma_{1}+\partial_{1}^{2} \mathbf{b}^{3}(0,0) \tag{27}
\end{equation*}
$$

Therefore, at $(0,0), \mathbf{v}=\partial_{1}^{3} \mathbf{b}^{3}-2 \gamma_{1}\left(\mathbf{t}^{3} \gamma_{1}+\partial_{1}^{2} \mathbf{b}^{3}\right)$. Since, in general, $\partial_{1}^{3} \mathbf{b}^{3}(0,0)$, $\partial_{1}^{2} \mathbf{b}^{3}(0,0)$ and $\mathbf{t}^{3}$ are linearly independent, the scalar $\gamma_{1}$ can not force $\mathbf{v}=0$ (recall that $\mathbf{b}^{3}$ is fixed), and since $\mathbf{v}, \partial_{1}^{2} \mathbf{b}^{3}(0,0)$ and $\mathbf{t}^{3}$ are linearly independent, we must have $a_{0}^{3}=0$ and $a_{1}^{3}=0$ and $a_{2}^{1}=a_{2}^{3}$ in order for (26) to hold.

If $\alpha^{3}$ is linear then $a_{2}^{3}=0$ and since $\alpha^{\prime \prime}(0)=\left(\frac{\beta}{\gamma}\right)^{\prime \prime}(0)=\beta^{\prime \prime}(0)$ when $\alpha(0)=\alpha^{\prime}(0)=0$ (note that $\gamma(0)=1$ and hence $\beta(0)=\beta^{\prime}(0)=0$ ), we have $\alpha^{1} \equiv 0$ contradicting the assumption that $\alpha^{1}$ is not linear.

We now have all the pieces in place to prove the main theorem of smooth surface construction with bi-3 splines.

Theorem 1 (two double edge knots needed) In general, using splines of degree bi-3 for a vertex-localized unbiased $G^{1}$ construction without forced linear boundary segments requires the splines to have at least two internal double knots.

Proof In general, if the boundary curve has only a single 1-fold knot (hence two $C^{2}$-connected segments) there are not enough degrees of freedom to enforce $C^{2}$ continuity of the piecewise curve. If there are two 1-fold knots (three $C^{2}$-connected segments), $C^{2}$ continuity uniquely determines all boundary coefficients. If there is one 2 -fold knot (two $C^{1}$-connected segments), $C^{1}$ continuity uniquely determines all boundary coefficients. However, in these last two cases, (16) is unresolved at the (two, respectively one) edge knots $\left\{\tau_{i}\right\}$ and therefore, in general, these base cases allow for constructing a $C^{2}$ boundary curve but not for enforcing (2).

Inserting one additional edge knot that is 1-fold creates one additional boundary curve segment $j$ of degree 3 constrained by four vector-valued constraints: the parametric $C^{0}, C^{1}$ and $C^{2}$ constraints plus (16) or, equivalently, one free spline control point subject to (16). If $\alpha_{j}$ is linear, its two coefficients are determined via (13) and (15) by those of the neighbor segment, and therefore the free (B-spline) control point must be used to resolve (16). That is, if $\alpha_{j}$ is linear, we do not gain degrees of freedom that would enable enforcing (16) at the edge knots $\left\{\tau_{i}\right\}$ of the base case.

By Corollary 3, the starting segment's $\alpha_{0}$ can be assumed to be linear. Let $\alpha_{j}$ be not linear while $\alpha_{l}, l=0, \ldots, j-1, j \geq 1$, are linear. By the reasoning of the previous paragraph all $\mathbf{b}^{l}(u, 0), l=0, \ldots, j-1$ are determined so that Lemma 7 applies: that is, $\alpha_{j}$ can only be not linear if there is at least by one double knot between some segment $\mathbf{b}^{l-1}(u, 0)$ and $\mathbf{b}^{l}(u, 0)$.

The symmetric argument at the other end implies the claim.
The proof of Theorem 1 reveals slightly more than its claim: the interior segment with $\alpha_{j}$ not linear must be separated by double knots from either end segment. The simplest such construction is then based on three segments with
the middle segment bracketed by two double knots, and such that $\alpha_{0}$ and $\alpha_{2}$ are linear and $\alpha_{1}$ quadratic (see Fig. 3, right).

Corollary 4 (lower bound is sharp) The construction in [FP08] uses the fewest knots when creating a smooth surface without forced linear segments with one bi-3 spline associated with each quad of a general quad mesh.

Proof By covering each quad with a $3 \times 3$ arrangement of parametrically $C^{1}$ connected bi-3 patches in Bernstein-Bézier-form, the construction in FP08] uses exactly two edge knots, both 2-fold. By its choice of quadratic $\alpha_{1}$ just for the $G^{1}$ constraints across the middle segment and linear $\alpha_{0}$ and $\alpha_{2}$ for the end segments, it does not have the shape problem characterized by Lemma 5. |||


Figure 7: No Shape defect (no forced straight line segments) in a higher order saddle (cf. Figure 5).

## 4 Discussion and Conclusion

Remarkably, the results in Section 2 do not depend on the degree or even the polynomial nature of splines, but assume only sufficiently smooth functions that are piecewise with smooth transitions between the pieces. In particular, the results apply to finite refinement by subdivision which creates parametrically smooth transitions within each generalized spline. The extension to generalized splines mapping to $\mathbb{R}^{d}, d>3$ is straightforward.

For bi- 3 splines these general constraints imply a lower bound on the number and distribution of knots. The construction in [FP08 shows the lower bound to be tight.

The results extend to constructions based on $G^{1}$ transitions of the form $\beta^{k}(u) \partial_{2} \mathbf{b}^{k}(u, 0)+\gamma^{k}(u) \partial_{1} \mathbf{b}^{k-1}(0, u)=\alpha^{k}(u) \partial_{1} \mathbf{b}^{k}(u, 0)$ for which there is a sufficiently rich set of input data that imply $\beta=\gamma$. For example, if $\left(\alpha^{k}, \beta^{k}, \gamma^{k}\right)$ reflect the local geometric distribution of the input data, any locally symmetric input yields $\beta=\gamma$ and the results of the paper hold.

The bounds provide a checklist for constructions. Theorem 1 implies for example that there is a subtle error in the proof of the non-trivial construction [HBC08] which uses one double edge knot only: the construction falls foul of Corollary 1. Such a $2 \times 2$ split construction can only succeed in special cases. Choosing generic input data and $n^{1}=n^{2}=n^{3}=4$ but $n^{4}=3$ shows the problem. As a second example, Lemma 4 prevents a vertex-localized solution with all $\alpha_{j}$ linear. When this lemma is specialized by fixing the degree to be 3 , by increasing the patch continuity to $C^{2}$ and by choosing $\alpha_{0 j}:=\frac{q-j}{q} \alpha_{00}+\frac{j}{q} \alpha_{0 q}$ then it yields a proof of the claim SWWL04, Thm 3.1]. (In light of (16), we might adjust the titles of [SWWL04] and SLW06] since we cannot have $G^{1}$ surfaces when adding single knots.)

When we restrict connectivity, i.e. drop the assumption made at the outset that the construction applies to general input and uses one tensor-product spline per quad, then constructions with fewer edge knots are possible. For restricted connectivity, it is well known that if all valences are odd or tangents are in an X configuration, then vertex-enclosure does not impose constraints and simple Bézier constructions are possible (e.g. vW86, Pet91, GZ94). If $n^{0}=n^{1}$ always holds, say when smoothing a cube, then we can choose linear $\alpha^{1}$ and $\alpha^{3}$ with $a_{1}^{1}=a_{1}^{3}$ and $a_{0}^{1}=0$ to enforce (14). That is, a construction with one double edge knot is possible. Such a construction, covering a quad by $2 \times 2$ bi- 3 patches, is proposed in HBC08. A similar but dual, spline-like construction appears in [ZT95. Global constructions, singular parameterization, or control of the valence, for example by splitting patches, can allow for structurally or degreewise simpler constructions, e.g. Rei95, PBP02, 9.11], Pet91, Pet95b.

If we allow higher degree, then general constructions of smooth surfaces with one patch per quad are shown possible for degree bi-5, for example MYP08. For degree bi-4, a single knot (a $2 x 2$-split) must be introduced (see e.g. Pet95a).

The case of several $G^{1}$-connected patches per quad still awaits full investigation, as does the case of rational bi-3 patches and the generalization of the problem to unbiased $G^{k}$ transitions for $k>1$.

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