# Conchoid surfaces of spheres 

Martin Peternell ${ }^{\text {a }}$, David Gruber ${ }^{\text {a }}$, Juana Sendra ${ }^{\text {b }}$<br>${ }^{a}$ Institute of Discrete Mathematics and Geometry,<br>Vienna University of Technology, Vienna, Austria<br>${ }^{b}$ Dpto. Matemática Aplicada a I. T. Telecomunicación, Univ. Politécnica de Madrid, Spain


#### Abstract

The conchoid of a surface $F$ with respect to given fixed point $O$ is roughly speaking the surface obtained by increasing the radius function with respect to $O$ by a constant. This paper studies conchoid surfaces of spheres and shows that these surfaces admit rational parameterizations. Explicit parameterizations of these surfaces are constructed using the relations to pencils of quadrics in $\mathbb{R}^{3}$ and $\mathbb{R}^{4}$. Moreover we point to remarkable geometric properties of these surfaces and their construction.


Keywords: sphere, pencil of quadrics, rational conchoid surface, polar representation, rational radius function.

## 1. Introduction

The conchoid is a classical geometric construction and dates back to the ancient Greeks. Given a planar curve $C$, a fixed point $O$ (focus point) and a constant distance $d$, the conchoid $D$ of $C$ with respect to $O$ at distance $d$ is the (Zariski closure of the) set of points $Q$ in the line $O P$ at distance $d$ of a moving point $P$ varying in the curve $C$,

$$
\begin{equation*}
D=\{Q \in O P \text { with } P \in C, \text { and } \overline{Q P}=d\}^{*}, \tag{1}
\end{equation*}
$$

where the asterisk denotes the Zariski closure. For a more formal definition of conchoids in terms of diagrams of incidence we refer to 12, 13.

The definition of the conchoid surface to a given surface $F$ in space with respect to a given point $O$ and distance $d$ follows analogous lines.

We aim at studying real rational surfaces in 3 -space whose conchoid surfaces are also rational and real. A surface $F \subset \mathbb{R}^{3}$ will be represented by a polar representation $\mathbf{f}(u, v)=\rho(u, v) \mathbf{k}(u, v)$, where $\mathbf{k}(u, v)$ is a parameterization of the unit sphere $S^{2}$. Without loss of generality we assume $O=(0,0,0)$. Consequently their conchoid surfaces $F_{d}$ for varying distance $d$ admit the polar representation $\mathbf{f}_{d}(u, v)=(\rho(u, v) \pm d) \mathbf{k}(u, v)$.

Since we want to determine classes of surfaces whose conchoid surfaces for varying distances are rational, we focus at rational polar surface representations. Then the 'base' surface $F$ and its conchoids $F_{d}$ correspond to the same rational parameterization $\mathbf{k}(u, v)$ of the unit sphere $S^{2}$. The following definition excludes possibly occurring cases where $F$ and $F_{d}$ are rational, but their rational parameterizations $\mathbf{f}$ and/or $\mathbf{f}_{d}$ are not corresponding to a rational representation $\mathbf{k}(u, v)$ of $S^{2}$.

Definition 1. A surface $F$ is called rational conchoid surface with respect to the focus point $O=(0,0,0)$ if $F$ admits a rational polar representation $\rho(u, v) \mathbf{k}(u, v)$, with a rational radius function $\rho(u, v)$ denoting the distance function from $O$ to $F$ and a rational parameterization $\mathbf{k}(u, v)$ of $S^{2}$.

Contribution: The main contribution of this article is the study of the conchoid surfaces of spheres. We prove that a sphere $F$ in $\mathbb{R}^{3}$ admits a rational polar representation $\mathbf{f}(u, v)=$ $\rho(u, v) \mathbf{k}(u, v)$ with a rational radius function $\rho(u, v)$ and a particular rational parameterization $\mathbf{k}(u, v)$ of the unit sphere $S^{2}$, independently of the relative position of the sphere $F$ and the focus point $O$. This implies that the conchoids $G$ of $F$ with respect to any focus in $\mathbb{R}^{3}$ admit rational parameterizations.

It is remarkable that an analogous result to this contribution for spheres does not exist for circles and conics in $\mathbb{R}^{2}$. The conchoid curves of conics $C$ are only rational if either $O \in C$ or $O$ coincides with one of $C$ 's focal points.

Two constructions to prove the main result are presented. The first one uses the cone model being introduced in Section 1.1 and studies a pencil of quadrics in $\mathbb{R}^{4}$. This construction is explicit and leads to a surprisingly simple solution and a rational polar representation of a sphere. The second approach investigates pencils of quadrics in $\mathbb{R}^{3}$ containing a sphere and a cone of revolution whose base locus is a rational quartic with rational distance from $O$.

### 1.1. The cone model

Let $F$ be a surface in $\mathbb{R}^{3}$ and let $G$ be its conchoid surface at distance $d$ with respect to the origin $O=(0,0,0)$ as focal point. The construction of the conchoid surfaces $G$ of the 'base' surface $F$ is performed as follows. Consider Euclidean 4 -space $\mathbb{R}^{4}$ with coordinate axis $x, y, z$ and $w$, where $\mathbb{R}^{3}$ is embedded in $\mathbb{R}^{4}$ as the hyperplane $w=0$. Consider the quadratic cone $D: x^{2}+y^{2}+z^{2}-w^{2}=0$ in $\mathbb{R}^{4}$. Further, let $A$ be the cylinder through $F$, whose generating lines are parallel to $w$. Note that $A$ as well as $D$ are three-dimensional manifolds in $\mathbb{R}^{4}$. The conchoid construction of the 'base' surface $F$ is based on the study of the intersection $\Phi=A \cap D$, which is typically a two-dimensional surface in $\mathbb{R}^{4}$.

For a given parameterization $\mathbf{f}(u, v)$ of $F$ in $\mathbb{R}^{3}$, the cylinder $A$ through $F$ admits the representation $\mathbf{a}(u, v, s)=\left(f_{1}, f_{2}, f_{3}, 0\right)+s(0,0,0,1)$. Let $F$ be a rational surface and $\mathbf{f}(u, v)$ be rational. If the intersection $\Phi=A \cap D$ is a rational surface in $\mathbb{R}^{4}$, then it is obvious that $F$ admits a rational polar representation. Let $\varphi(a, b)=\left(\varphi_{1}, \ldots, \varphi_{4}\right)(a, b)$ be a rational representation of $\Phi$ in $\mathbb{R}^{4}$, then $\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)(a, b)$ is obviously a rational polar representation of $F$. Since $\varphi_{4}^{2}=\varphi_{1}^{2}+\varphi_{2}^{2}+\varphi_{3}^{2}$ holds, $\mathbf{k}=1 / \varphi_{4}\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ is a rational parameterization of $S^{2}$ and $\rho(a, b)=\varphi_{4}(a, b)$ is a rational radius function of $F$. We summarize the construction.

Theorem 2. The rational conchoid surfaces $F \subset \mathbb{R}^{3}$ are in bijective correspondence to the rational 2-surfaces in the quadratic cone $D: x^{2}+y^{2}+z^{2}-w^{2}=0$ in $\mathbb{R}^{4}$.

Proof: We proved already that for a rational surface $\Phi \subset D$, its orthogonal projection $\left(\varphi_{1}, \varphi_{2}, \varphi_{3}\right)$ onto $\mathbb{R}^{3}$ is a rational conchoid surface with rational radius function $\varphi_{4}$. Conversely, any rational conchoid surface $F$ with respect to $O$ is defined by a rational polar parameterization $\rho(u, v) \mathbf{k}(u, v)$, with $\mathbf{k}=\left(k_{1}, k_{2}, k_{3}\right) \in \mathbb{R}(u, v)^{3}$ and $\|\mathbf{k}\|=1$. The corresponding surface $\Phi \subset D$ is represented by $\varphi(u, v)=\rho\left(k_{1}, k_{2}, k_{3}, 1\right)(u, v)$.

The quadratic cone $D$ possesses universal parameterizations and we may use them to specify all possible rational parameterizations of rational conchoid surfaces. The construction starts with rational universal parameterizations of the unit sphere $S^{2}$. Following [4] we choose four arbitrary polynomials $a(u, v), b(u, v), c(u, v)$ and $d(u, v)$ without common factor. Let

$$
\alpha=2(a c+b d), \beta=2(b c-a d), \gamma=a^{2}+b^{2}-c^{2}-d^{2}, \delta=a^{2}+b^{2}+c^{2}+d^{2}
$$

then $\mathbf{k}(u, v)=\frac{1}{\delta}(\alpha, \beta, \gamma)$ is a rational parameterization of the unit sphere $S^{2}$. Thus $\varphi(u, v)=$ $\rho(u, v)(\alpha, \beta, \gamma, \delta)$ with a non-zero rational function $\rho(u, v)$ is a rational parameterization of a twodimensional surface $\Phi \subset D$. Consequently

$$
\mathbf{f}(u, v)=\rho(u, v)\left(\frac{\alpha}{\delta}, \frac{\beta}{\delta}, \frac{\gamma}{\delta}\right)(u, v)=\rho(u, v) \mathbf{k}(u, v)
$$

is a rational polar representation of a rational conchoid surface $F$ in $\mathbb{R}^{3}$, with $\rho(u, v)$ as radius function and $\mathbf{k}(u, v)$ as rational parameterization of the unit sphere $S^{2}$. It is sufficient to consider polynomials and the construction reads as follows.

Corollary 3. Given six relatively prime polynomials $a(u, v), b(u, v), c(u, v), d(u, v)$, and $r(u, v)$ and $s(u, v)$, a universal parameterization of a rational 2-surface $\Phi \subset D$ in $\mathbb{R}^{4}$ is given by

$$
\begin{equation*}
\varphi(u, v)=\frac{r}{s}\left(2(a c+b d), 2(b c-a d), a^{2}+b^{2}-c^{2}-d^{2}, a^{2}+b^{2}+c^{2}+d^{2}\right)(u, v) \tag{2}
\end{equation*}
$$

Consequently, a universal rational parameterization of a rational conchoid surface reads

$$
\begin{equation*}
\mathbf{f}(u, v)=\frac{r}{s\left(a^{2}+b^{2}+c^{2}+d^{2}\right)}\left(2(a c+b d), 2(b c-a d), a^{2}+b^{2}-c^{2}-d^{2}\right)(u, v) \tag{3}
\end{equation*}
$$

This is a general result about all rational parameterizations of rational conchoid surfaces. For a particular given rational surface $F$ it is difficult to decide whether the intersection $\Phi=D \cap W$ admits rational parameterizations or not. Typically the surface $\Phi$ is not rational. Nevertheless, there are interesting non-trivial cases where $\Phi$ admits rational parameterizations.

In [7] it has been proved that conchoids of rational ruled surfaces $F$ are rational. We give a hint how this result can be proved with help of the cone model $D$ and Theorem 2 If $F$ is a ruled surface, the cylinder $A \subset \mathbb{R}^{4}$ carries a one-parameter family of planes parallel to the $w$-axis. These planes pass through the generating lines of $F$. This implies that typically the intersection $\Phi=A \cap D$ carries a one-parameter family of conics obtained as intersections of the mentioned planes with $D$. This family of conics is rational, and it is known ( $[6,9]$ ) that there exist rational parameterizations $\varphi(u, v)$ of $\Phi$. Thus the conchoids of real rational ruled surfaces are rational.

In this context we mention a trivial but useful statement which we prove for completeness.
Lemma 4. Given a rational curve $C$ with parameterization $\mathbf{c}(t)$ on a rotational cone $D$, then the distance $\|\mathbf{c}(t)-\mathbf{v}\|$ between the curve $C$ and the vertex $\mathbf{v}$ of $D$ is a rational function.

Proof: We use a special coordinate system with $\mathbf{v}$ at the origin, and $z$ as rotational axis of $D$. This implies that $D$ is the zero set of $x^{2}+y^{2}-\gamma^{2} z^{2}$. Without loss of generality we let $\gamma=1$. The given curve $C$ admits therefore a rational parameterization $\mathbf{c}(t)=\left(c_{1}, c_{2}, c_{3}\right)(t)$ satisfying $c_{1}^{2}+c_{2}^{2}=c_{3}^{2}$. Obviously one obtains $\|\mathbf{c}(t)\|=\sqrt{2} c_{3}(t)$ being rational.

## 2. Conchoids of spheres

Given a sphere $F$ in $\mathbb{R}^{3}$ and an arbitrary focus point $O$, the question arises if there exists a rational representation $\mathbf{f}(u, v)$ of $F$ with the property that $\|\mathbf{f}(u, v)\|$ is a rational function of the parameters $u$ and $v$. To give a constructive answer to this question we describe an approach using the cone-model presented in Section 1.1. Later on in Section 3 we study a different method working in $\mathbb{R}^{3}$ directly. There are several relations between these methods which will be discussed along their derivation.

Let $F$ be the sphere with center $\mathbf{m}=(m, 0,0)$ and radius $r$, and let $O=(0,0,0)$. Thus $F$ is given by

$$
\begin{equation*}
F:(x-m)^{2}+y^{2}+z^{2}-r^{2}=0 \tag{4}
\end{equation*}
$$

If $m=0$, the center of $F$ coincides with $O$. In this trivial situation the conchoid surface of $F$ is reducible and consists of two spheres, where one might degenerate to $F$ 's center if $d=r$. If $m^{2}-r^{2}=0$, the focal point $O$ is contained in $F$. To construct a rational polar representation, we make the ansatz $\mathbf{f}(u, v)=\rho(u, v) \mathbf{k}(u, v)$ with $\mathbf{k}(u, v)=\left(k_{1}, k_{2}, k_{3}\right)(u, v)$ and $\|\mathbf{k}(u, v)\|=1$ and an unknown radius function $\rho(u, v)$. Plugging this into (4), we obtain a rational polar representation with rational radius function $\rho(u, v)=2 m k_{1}(u, v)$. Note that in this case the conchoid is irreducible and rational.

### 2.1. Pencil of quadrics in $\mathbb{R}^{4}$

Consider the Euclidean space $\mathbb{R}^{4}$ with coordinate axes $x, y, z$ and $w$ and let $\mathbb{R}^{3}$ be embedded as the hyperplane $w=0$. Let a sphere $F \subset \mathbb{R}^{3}$ be defined by (4) and $O=(0,0,0)$. To study the general case we assume $m \neq 0$ and $m^{2} \neq r^{2}$. The equation of the cylinder $A \subset \mathbb{R}^{4}$ through $F$ with $w$-parallel lines agrees with the equation of $F$ in $\mathbb{R}^{3}$,

$$
\begin{equation*}
A:(x-m)^{2}+y^{2}+z^{2}-r^{2}=0 \tag{5}
\end{equation*}
$$

Consider the pencil $Q(t)=A+t D$ of quadrics in $\mathbb{R}^{4}$, spanned by $A$ and the quadratic cone $D: x^{2}+y^{2}+z^{2}=w^{2}$ from Section 1.1. Any point $\bar{X}=(x, y, z, w) \in D$ has the property that the distance from $X=(x, y, z)$ to $O$ in $\mathbb{R}^{3}$ equals $w$. We study the geometric properties of the del Pezzo surface $\Phi=A \cap D$ of degree four, the base locus of the pencil of quadrics $Q(t)$. According to Theorem 2, the sphere $F$ is a rational conchoid surface exactly if $\Phi$ admits rational parameterizations.

Besides $A$ and $D$ there exist two further singular quadrics in $Q(t)$. These quadrics are obtained for the zeros $t_{1}=-1$ and $t_{2}=r^{2} / \gamma^{2}$ of the characteristic polynomial

$$
\begin{equation*}
\operatorname{det}(A+t D)=-(1+t)^{2} t\left(\gamma^{2} t-r^{2}\right), \text { with } \gamma^{2}=m^{2}-r^{2} \neq 0 \tag{6}
\end{equation*}
$$

The quadric corresponding to the twofold zero $t_{1}=-1$ is a cylinder

$$
\begin{equation*}
R: w^{2}-2 m x+m^{2}-r^{2}=0 \tag{7}
\end{equation*}
$$

Its directrix is a parabola in the $x w$-plane and its two-dimensional generators are parallel to the $y z$-plane. The singular quadric $S$ corresponding to $t_{2}=r^{2} / \gamma^{2}$ is a quadratic cone and reads

$$
S:\left(x-\frac{m^{2}-r^{2}}{m}\right)^{2}+y^{2}+z^{2}=\frac{r^{2}}{m^{2}} w^{2} .
$$

Its vertex is the point $O^{\prime}=\left(\frac{m^{2}-r^{2}}{m}, 0,0,0\right)$. The intersections of $S$ with three-spaces $w=c$ are spheres $\sigma(c)$, whose top view projections in $w=0$ are centered at $O^{\prime}$ and their radii are $r c / \mathrm{m}$. The intersections of $D$ with three-spaces $w=c$ are spheres $d(c)$ whose top view projections in $w=0$ are centered at $O$ with radii $c$. The intersections $k(c)=s(c) \cap d(c)$ of these spheres $(w=c)$ are circles in planes $x=\left(c^{2}+m^{2}-r^{2}\right) /(2 m)$. Thus $\Phi$ contains a family of conics, whose top view projections are the circles $k(c)$. The conics in $\Phi$ are contained in the planes

$$
\varepsilon(c): x=\frac{c^{2}+m^{2}-r^{2}}{2 m}, w=c .
$$

The half opening angle $\delta$ of $D$ with respect to the $w$-axis is $\pi / 4$, thus $\tan \delta=1$. The half opening angle $\sigma$ of $S$ is given by $\tan \sigma=r / m$, see Figure 1(a). Applying the scaling

$$
\left(x^{\prime}, y^{\prime}, z^{\prime}, w^{\prime}\right)=(f x, f y, f z, w), \text { with } f=\frac{r}{m}
$$

in $\mathbb{R}^{4}$ maps $D$ to a congruent copy of $S$. Consider a point $\bar{X}=(x, y, z, w)$ in $\Phi=A \cap D$ and its projection $X=(x, y, z)$ in $F$. The distance $\operatorname{dist}(X, O)$ of $X$ to $O$ in $\mathbb{R}^{3}$ is $w$. For the distance $\operatorname{dist}\left(X, O^{\prime}\right)$ between $X$ and $O^{\prime}$ we consequently obtain

$$
\begin{equation*}
\operatorname{dist}\left(X, O^{\prime}\right)=\frac{r}{m} \operatorname{dist}(X, O), \text { for all } X \in F \tag{8}
\end{equation*}
$$

Remark on the circle of Apollonius. Note that $O^{\prime}$ is the inverse point of $O$ with respect to the sphere $F$. It is an old result by Apollonius Pergaeus (262-190 b.c.) that the set of points $X$ in the plane having constant ratio of distances $f=d / d^{\prime}$, with $d=\operatorname{dist}(O, X)$ and $d^{\prime}=\operatorname{dist}\left(O^{\prime}, X\right)$, from two given fixed points $O$ and $O^{\prime}$, respectively, is a circle $k$, see Fig. 1(b). Rotating $k$ around the line $O O^{\prime}$ gives the sphere $F$ and $O$ and $O^{\prime}$ are inverse points with respect to $F$ (and the circle $k$ ).

If we consider a varying constant ratio $f$, one obtains a family of spheres $F(f)$ with inverse points $O$ and $O^{\prime}$ which form an elliptic pencil of spheres. Their centers are on the line $O O^{\prime}$. Ratio $1\left(d=d^{\prime}\right)$ corresponds to the bisector plane of $O$ and $O^{\prime}$.


Figure 1: Pencil of quadrics in $\mathbb{R}^{4}$ and Apollonius circle

### 2.2. A rational quartic on the sphere

The pencil of quadrics $Q(t)$ in $\mathbb{R}^{4}$ spanned by the sphere $F$ and the cone $D$ contains the cylinder $R$. Expressing the variable $x$ from (7) one gets

$$
\begin{equation*}
x=\frac{w^{2}+m^{2}-r^{2}}{2 m} \tag{9}
\end{equation*}
$$

and inserting this into $D$ results in the polynomial

$$
\begin{equation*}
\alpha(w): 4 m^{2}\left(y^{2}+z^{2}\right)+p(w)=0, \text { with } p(w)=w^{4}-2 w^{2}\left(m^{2}+r^{2}\right)+\left(m^{2}-r^{2}\right)^{2} . \tag{10}
\end{equation*}
$$

Considering $y$ and $z$ as variables, $\alpha(w)$ is a one-parameter family of conics (circles) in the $y z$-plane, depending rationally on the parameter $w$. The circles $\alpha(w)$ do not possess real points for all $w$, but there exist intervals determining families of real circles $\alpha(w)$. To obtain real circles one has to perform a re-parameterization $w(u)$ within an appropriate interval. The factorization of $p(w)$ reads

$$
p(w)=(w+a)(w-a)(w+b)(w-b), \text { with } a=m+r, \text { and } b=m-r .
$$

If $O$ is outside of $F$, thus $m>r$, the polynomial $-p(w)$ is positive in the interval $[m-r, m+r]$. Thus a possible re-parameterization is

$$
\begin{equation*}
w(u)=\frac{a u^{2}+b}{1+u^{2}}=\frac{u^{2}(m+r)+m-r}{1+u^{2}} . \tag{11}
\end{equation*}
$$

Otherwise we could re-parameterize over another appropriate interval. Additionally we note that if $O$ is inside of $F$, the inverse point $O^{\prime}$ is outside of $F$. Since equation (8) holds for the distances of a point $X \in F$ to $O$ and $O^{\prime}$, we can exchange roles and perform the computation for the point $O^{\prime}$.

We return to the family of conics $\alpha(w)$. Substituting 111 into leads to a family of real conics

$$
\begin{equation*}
\alpha(u): y^{2}+z^{2}=\frac{4 r^{2} u^{2}}{m^{2}\left(1+u^{2}\right)^{4}}\left(a u^{2}+m\right)\left(m u^{2}+b\right) \tag{12}
\end{equation*}
$$

We are looking for rational functions $y(u)$ and $z(u)$ satisfying 12 identically. Therefore we introduce auxiliary variables $\tilde{y}$ and $\tilde{z}$ by the relations $y=2 \tilde{y} r u /\left(m\left(1+u^{2}\right)^{2}\right)$ and $z=2 \tilde{z} r u /(m(1+$ $\left.\left.u^{2}\right)^{2}\right)$. We obtain $\tilde{y}^{2}+\tilde{z}^{2}=\left(a u^{2}+m\right)\left(m u^{2}+b\right)$. Factorizing left and right hand side of this equation results in a linear system to determine $\tilde{y}$ and $\tilde{z}$,

$$
\begin{aligned}
& \tilde{y}+\mathrm{i} \tilde{z}=(\sqrt{a} u+\mathrm{i} \sqrt{m})(\sqrt{m} u+\mathrm{i} \sqrt{b}) \\
& \tilde{y}-\mathrm{i} \tilde{z}=(\sqrt{a} u-\mathrm{i} \sqrt{m})(\sqrt{m} u-\mathrm{i} \sqrt{b})
\end{aligned}
$$



Figure 2: Rational polar representation of a sphere and its conchoid surfaces

The solution $\tilde{y}=\sqrt{m}\left(\sqrt{a} u^{2}-\sqrt{b}\right), \tilde{z}=u(m+\sqrt{a b})$ finally leads to

$$
\begin{equation*}
y(u)=\frac{2 r \sqrt{m} u}{m\left(1+u^{2}\right)^{2}}\left(\sqrt{a} u^{2}-\sqrt{b}\right), \text { and } z(u)=\frac{2 r u^{2}}{m\left(1+u^{2}\right)^{2}}(m+\sqrt{a b}) \tag{13}
\end{equation*}
$$

which is a rational parameterization of a curve in the $y z$-plane, following the family of conics $\alpha(w)$.
We note that any real rational family of conics possesses real rational parameterizations, see for instance [6, 9]. The solution (13) together with (9) determines a curve $C \subset F$ which possesses the rational distance function

$$
\begin{equation*}
\|\mathbf{c}(u)\|=w(u)=\frac{u^{2}(m+r)+(m-r)}{1+u^{2}} \tag{14}
\end{equation*}
$$

with respect to $O$. Its parameterization is

$$
\mathbf{c}(u)=\frac{1}{m\left(1+u^{2}\right)^{2}}\left(\begin{array}{c}
u^{4} m(m+r)+2 u^{2}\left(m^{2}-r^{2}\right)+m(m-r)  \tag{15}\\
2 r \sqrt{m} u\left(u^{2} \sqrt{m+r}-\sqrt{m-r}\right) \\
2 r u^{2}\left(m+\sqrt{m^{2}-r^{2}}\right)
\end{array}\right)
$$

Theorem 5. Let $F$ be a sphere and let $O$ be an arbitrary point in $\mathbb{R}^{3}$. Then there exists a rational quartic curve $C \subset F$ and a rational parameterization $\mathbf{c}(u)$ of $C$ such that the distance of $C$ to $O$ is a rational function in the curve parameter $u$.

Rotating $C$ around the $x$-axis leads to a rational polar representation $r(u, v) \mathbf{k}(u, v)$ of $F$ with rational distance function $\rho(u, v)=w(u)$ from $O$. The quartic curve $C$ together with this parameterization is illustrated in Fig. 2(a) Fig. 2(b) displays a sphere $F$ together with both conchoid surfaces $G_{1}$ and $G_{2}$ for distances $d$ and $-d$ with respect to $O$. We summarize the presented construction.

Theorem 6. Spheres in $\mathbb{R}^{3}$ admit rational polar representations with respect to any focus point $O$. This implies that the conchoid surfaces of spheres admit rational parameterizations. The construction is based on rational quartic curves on $F$ with rational distance from $O$.

Rationality and Uni-Rationality. The construction performed in Section 2.2 yields a rational parameterization $\mathbf{f}(u, v)$ of the sphere $F$ with rational radius function $\rho(u, v)$, given by (14), such that $\mathbf{f}(u, v)=\rho(u, v) \mathbf{k}(u, v)$, where $\mathbf{k}(u, v)$ is an improper parameterization of the unit sphere $S^{2}$. This means that typically a point $X \in F$ corresponds to two points $\left(u, v_{1}\right)$ and $\left(u, v_{2}\right)$ in the parameter domain. Rotating the curve $C$ around the $x$-axis, the sphere $F$ is double covered.

The conchoid surface $G$ of $F$ at distance $d$ typically consists of two surfaces $G_{1}$ and $G_{2}$, which admit the rational parameterizations

$$
\begin{equation*}
\mathbf{g}_{1}=(\rho(u, v)+d) \mathbf{k}(u, v), \text { and } \mathbf{g}_{2}=(\rho(u, v)-d) \mathbf{k}(u, v) \tag{16}
\end{equation*}
$$

for positive and negative distance. The conchoid $G=G_{1} \cup G_{2}$ is an irreducible algebraic surface of order six. It is not bi-rational equivalent to the projective plane but each component $G_{1}$ as well as $G_{2}$ admits improper rational parameterizations. These components $G_{1}$ and $G_{2}$ are called uni-rational. This is not a contradiction to Castelnuovo's theorem since we are not working over an algebraically closed field but over the field of real numbers $\mathbb{R}$.

Let us consider an example to illustrate these properties. We consider the sphere $F$ with center $\mathbf{m}=(3 / 2,0,0)$ and radius $r=1$, and compute its conchoid $G$ for variable distance $d$. We obtain parameterizations $\mathbf{g}_{1}(u, v)$ and $\mathbf{g}_{2}(u, v)$ from equation for the real uni-rational varieties $G_{1}$ and $G_{2}$. The algebraic variety $G=G_{1} \cup G_{2}$ is given by the equation

$$
\begin{align*}
G: & \left(x^{2}+y^{2}+z^{2}\right)\left(4\left(x^{2}+y^{2}+z^{2}\right)-12 x+5\right)^{2} \\
& +d^{2}\left(40\left(x^{2}+y^{2}+z^{2}\right)-144 x^{2}+96 x\left(x^{2}+y^{2}+z^{2}\right)-32\left(x^{2}+y^{2}+z^{2}\right)^{2}\right)  \tag{17}\\
& +16 d^{4}\left(x^{2}+y^{2}+z^{2}\right)=0
\end{align*}
$$

Remarks on the parameterization. The rational quartic $C$ on $F$ is of course not unique but depends on the re-parameterization (11). An admissible rational re-parameterization of a real interval is of even degree. Let us consider a quadratic re-parameterization. Since $\alpha$ is of degree four in $w$, the re-parameterized family is typically of degree $\leq 8$ in $u$. This implies that the solutions $y(u)$ and $z(u)$ are of degree $\leq 4$, which holds also for $x(u)$ because of (9). The coefficient functions $\mathbf{c}(u)=(x, y, z)(u)$ determine a rational quartic $C$ on $F$, with rational norm $\|\mathbf{c}\|=w(u)$.

Different choices of the interval and a quadratic re-parameterization will typically result in different quartic curves on $F$. In (11) we have chosen the largest possible interval and a rational function satisfying $w(-u)=w(u)$ and obtained the curve $C$ through antipodal points of $F$. By rotating we obtain the full sphere, doubly covered.

For any quadratic re-parameterization, the quartic $C$ is the base locus of a pencil of quadrics $Q(t)=F+t K$, spanned by the sphere $F$ and, for instance, the quadratic projection cone $K$ with vertex at $C$ 's double point.

The particular choice (11) implies that the quartic $C$ is symmetric with respect to the $x z$-plane. This holds since $u$ appears only with even powers in $x$ and $z$, thus we have $x(-u)=x(u)$ and $z(-u)=z(u)$. The orthogonal projection of $C$ to the $x z$-plane is doubly covered, thus a conic. In this case $(x, z)(u)$ parameterizes a parabola, because of the factor $\left(1+u^{2}\right)^{2}$ in $\mathbf{c}(u)$ 's denominator. This implies that the pencil $Q(t)$ can also be spanned by the sphere $F$ and the parabolic cylinder $P$ passing through $C$, whose generating lines are parallel to $y$. It can be proved that all quadrics in $Q(t)$ except $P$ are rotational quadrics with parallel axes. This implies that $K$ is a rotational cone, and the remaining singular quadric $L$ is a rotational cone, too. For the particular choice (11) and for the generalized construction performed in Section 3, the rotational cone $L$ has the vertex $O$. We note that for any admissible re-parameterization $L$ 's vertex is typically different from $O$.

### 2.3. Pencil of quadrics in $\mathbb{R}^{3}$

The quartic curve $C$ from $\sqrt{15}$ on the sphere $F$ is the base locus of a pencil of quadrics $F+\lambda K$ in $\mathbb{R}^{3}$, spanned by $F$ and the projection cone $K$ of $C$ from its double point s, see Fig. 3. The double point $\mathbf{s}$ is located in the symmetry plane of $C$ and in the polar plane of the origin $O$ with respect to $F$. Its coordinates are

$$
\begin{equation*}
\mathbf{s}=\frac{1}{m}\left(\gamma^{2}, 0, r \gamma\right) \text { with } \gamma^{2}=m^{2}-r^{2} \tag{18}
\end{equation*}
$$

The pencil $F+\lambda K$ contains two further singular quadrics which are obtained for the zeros $\lambda_{1}=1 / \mathrm{m}$ and $\lambda_{2}=-1 / \gamma$ of the characteristic polynomial

$$
\operatorname{det}(F+\lambda K)=r^{2}(m \lambda-1)(\gamma \lambda+1)
$$

Corresponding to $\lambda_{1}$ there is a parabolic cylinder $P$ with $y$-parallel generating lines passing through $C$. Corresponding to $\lambda_{2}$ we find the rotational cone $L$ through $C$ with vertex $O$.

To give explicit representations for the quadrics we use homogeneous coordinates $\mathbf{y}=(1, x, y, z)^{T}$. Since there should not be any confusion, we use same notations for the quadric $F$ and its coordinate matrix appearing in the homogeneous quadratic equation $\mathbf{y}^{T} \cdot F \cdot \mathbf{y}=0$. The coefficient matrices $F$ and $K$ read

$$
F=\left(\begin{array}{cccc}
m^{2}-r^{2} & -m & 0 & 0  \tag{19}\\
-m & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), K=\left(\begin{array}{cccc}
\gamma^{3} & -\gamma m & 0 & 0 \\
-\gamma m & \gamma & 0 & r \\
0 & 0 & -m & 0 \\
0 & r & 0 & -\gamma
\end{array}\right)
$$

An elementary computation shows that $K$ is a cone of revolution with opening angle $\pi / 2$ and $\mathbf{a}=(m+\gamma, 0, r)$ denotes a direction vector of its axis.

The cone $L$ through $C$ with vertex at $O$ is again a cone of revolution, whose axis is parallel to a. The parabolic cylinder $P$ through the quartic $C$ has $y$-parallel generating lines. The axis of the cross section parabola in the $x z$-plane is orthogonal to a, see Fig. 3(a). The coefficient matrices $L$ and $P$ are

$$
L=\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{20}\\
0 & 0 & 0 & -r \\
0 & 0 & m+\gamma & 0 \\
0 & -r & 0 & 2 \gamma
\end{array}\right), P=\left(\begin{array}{cccc}
\gamma^{2}(m+\gamma) & -m(m+\gamma) & 0 & 0 \\
-m(m+\gamma) & m+\gamma & 0 & r \\
0 & 0 & 0 & 0 \\
0 & r & 0 & m-\gamma
\end{array}\right)
$$

A trigonometric parameterization of the quartic $C$ is obtained by intersecting the cone $K$ with one quadric of the pencil $F+\lambda K$, for instance $F$. Let a be a unit vector in direction of $K$ 's axis, and $\mathbf{b}$ and $\mathbf{c}$ complete it to an orthonormal basis in $\mathbb{R}^{3}$. A trigonometric parameterization of $K$ is given by

$$
\begin{aligned}
& \mathbf{k}(t, v):=\mathbf{s}+v(\mathbf{a}+(\mathbf{b} \cos t+\mathbf{c} \sin t)), \text { with } \\
& \mathbf{a}=\frac{1}{\sqrt{2 m(m+\gamma)}}(m+\gamma, 0, r), \mathbf{b}=(0,-1,0), \text { and } \mathbf{c}=\frac{1}{\sqrt{2 m(m+\gamma)}}(r, 0,-(m+\gamma))
\end{aligned}
$$

Thus $K$ admits the explicit parameterization

$$
\mathbf{k}(t, v)=\frac{1}{2 m \sqrt{m(m+\gamma)}}\left(\begin{array}{c}
2 \gamma^{2} \sqrt{m(m+\gamma)}+v \sqrt{2} m(m+\gamma+r \sin t) \\
-2 v m \sqrt{m(m+\gamma)} \cos t \\
2 r \gamma \sqrt{m(m+\gamma)}+v \sqrt{2} m(r-(m+\gamma) \sin t)
\end{array}\right)
$$

Finally, a trigonometric parameterization of the quartic $C$ follows by

$$
\mathbf{c}(t)=\frac{1}{2 m}\left(\begin{array}{c}
(m+r \sin t)^{2}+\gamma^{2}  \tag{21}\\
\sqrt{2} \sqrt{m(m+\gamma)} \cos t(\gamma-m-r \sin t) \\
r(m+\gamma) \cos ^{2} t
\end{array}\right), \text { with }\|\mathbf{c}(t)\|=m+r \sin t
$$

The correspondence of the trigonometric parameterization and its norm with the expressions 15 and (14) in terms of rational functions is realized by the substitutions $\sin t=\left(u^{2}-1\right) /\left(u^{2}+1\right)$ and $\cos t=2 u /\left(u^{2}+1\right)$ and some rearrangement of the equations. Section 2.4 discusses relations to Viviani's curve (or Viviani's window). This particular quartic has a similar shape and its pencil of quadrics has similar properties. Viviani's curve has an additional symmetry.


Figure 3: Geometric properties of the conchoid construction

Remark. The inversion with center $O$ at the sphere which intersects the given sphere $F$ perpendicularly, maps the sphere $F$ onto itself. Analogously this inversion fixes the rotational cone $L$. Thus the quartic intersection curve $C=F \cap L$ remains fixed as a whole, but of course not point-wise. The product of the distances $\operatorname{dist}(O, P)$ and $\operatorname{dist}\left(O, P^{\prime}\right)$ of two inverse points $P \in F$ and $P^{\prime} \in F$ equals $\sqrt{m^{2}-r^{2}}$. This property follows from the elementary tangent-secant-theorem of a circle.

### 2.4. Relations to Viviani's curve

The quartic curve $C$, the base locus of the pencil of quadrics $F+t K$, can be considered as generalization of Viviani's curve $V$. This particularly well known curve $V$ is the base locus of a pencil of quadrics, spanned by a sphere $F$ and a cylinder of revolution $L$ touching $F$ and passing through the center of $F$. The pencil of quadrics of Viviani's curve also contains a right circular cone $K$ with vertex in $V$ 's double point and opening angle $\pi / 2$, and further a parabolic cylinder $P$. Viviani's curve $V$ is obtained from $C$ by letting $O \rightarrow \infty$. Consequently, the inverse point $O^{\prime}$ becomes the center of the sphere $F$.

Choosing the inverse point $O^{\prime}=\left(\frac{m^{2}-r^{2}}{m}, 0,0\right)$ as origin, the parameterization 21) of $C$ becomes

$$
\mathbf{c}(t)=\frac{1}{2 m}\left(\begin{array}{c}
r^{2}\left(1+\sin ^{2} t\right)+2 m r \sin t  \tag{22}\\
\sqrt{2} \sqrt{m(m+\gamma)} \cos t(\gamma-m-r \sin t) \\
r(m+\gamma) \cos ^{2} t
\end{array}\right)
$$

By letting $m \rightarrow \infty$ one obtains $V$ as limit curve

$$
\begin{equation*}
\mathbf{v}(t)=\left(r \sin t,-r \sin t \cos t, r \cos ^{2} t\right) \tag{23}
\end{equation*}
$$

Fig. 4(a) illustrates Viviani's curve $V$, together with the sphere and the singular quadrics belonging to the pencil. The generalized Viviani curve $C$ being the base locus of the pencil appearing in the conchoid construction of the sphere is illustrated in Fig. 4(b). In contrast to the classical Viviani curve $V$ whose single parameter $r$ is the radius of the sphere $F$, the quartic curve $C$ has two parameters $r$ and $m$.


Figure 4: Quadric pencils of Viviani's curve and its generalization

## 3. Rotational quadrics with parallel axes

We consider the mentioned pencil of quadrics $Q(t)=A+t D$ from Section 2.1, and a hyperplane $E: a x+b y+c z-d w=0$ passing through $O=(0,0,0,0)$. The intersection $D \cap E$ is a quadratic cone whose projection onto $\mathbb{R}^{3}$ is a cone of revolution $L$ with axis in direction of $\mathbf{a}=(a, b, c)$. Assuming $\|\mathbf{a}\|=1$, the opening angle $2 \tau$ of $L$ is determined by $d=\cos \tau$.

Consider the quartic intersection curve $C=F \cap L$ of a sphere $F$ and the cone of revolution $L$. It is rational exactly if the cone $L$ is touching $F$ at a single point. Since this touching point has to be contained in the polar plane of $O=(0,0,0)$ with respect to $F$, we choose $\mathbf{s}=\left(\gamma^{2} / m, 0, r \gamma / m\right)$ (compare 18) and prescribe an arbitrary opening angle $2 \tau$ for $L$. Thus the unit direction vector of $L$ 's axis is

$$
\mathbf{a}=\frac{1}{m}(\gamma \cos \tau-r \sin \tau, 0, \gamma \sin \tau+r \cos \tau)=(a, b, c)
$$

The quartic $C$ is real if the axis is contained in the wedge formed by $\mathbf{s}$ and the $x$-axis, see Figure 3(b). Thus $-r / \gamma \leq \tan \tau \leq 0$, because the rotation from $\mathbf{s}$ to $\mathbf{a}$ by $\tau \leq 0$ is counterclockwise. In the following we use the abbreviations $c t:=\cos \tau$ and $s t:=\sin \tau$. The quadrics of the pencil with base locus $C$ are denoted similarly to Section 2.3 . The coefficient matrix of the projection cone $L$ reads

$$
L(\tau)=\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & r^{2}\left(c t^{2}-s t^{2}\right)+2 \gamma r s t c t & 0 & -\gamma r\left(c t^{2}-s t^{2}\right)+\left(r^{2}-\gamma^{2}\right) s t c t \\
0 & 0 & m^{2} c t^{2} & 0 \\
0 & -\gamma r\left(c t^{2}-s t^{2}\right)+\left(r^{2}-\gamma^{2}\right) s t c t & 0 & \gamma^{2}\left(c t^{2}-s t^{2}\right)-2 \gamma r s t c t
\end{array}\right)
$$

Rewriting $L(\tau)$ in terms of the double angle $2 \tau$ and substituting

$$
\begin{equation*}
\cos 2 \tau=\gamma / m, \text { and } \sin 2 \tau=-r / m \tag{24}
\end{equation*}
$$

we obtain $L$ from equation 20 . This holds for all equations and parameterizations in this section in an analogous way.

The pencil of quadrics $F+t L(\tau)$ contains two further singular quadrics. The first is a parabolic cylinder $P(\tau)$ passing through $C$. It corresponds to the eigenvalue $\frac{-1}{m^{2} c t^{2}}$ and its generating lines
are parallel to the $y$-axis. Its coefficient matrix of cylinder reads

$$
P(\tau)=\left(\begin{array}{cccc}
\gamma^{2} m^{2} c t^{2} & -m^{3} c t^{2} & 0 & 0 \\
-m^{3} c t^{2} & \gamma^{2}\left(c t^{2}-s t^{2}\right)+m^{2} s t^{2}-2 r \gamma s t c t & 0 & \left(\gamma^{2}-r^{2}\right) s t c t+r \gamma\left(c t^{2}-s t^{2}\right) \\
0 & 0 & 0 & 0 \\
0 & \left(\gamma^{2}-r^{2}\right) s t c t+r \gamma\left(c t^{2}-s t^{2}\right) & 0 & m^{2} c t^{2}-\gamma^{2}\left(c t^{2}-s t^{2}\right)+2 r \gamma s t c t
\end{array}\right)
$$

Our goal is not only to characterize the pencil of quadrics but to provide an explicit parameterization of the quartic curve $C$ on $F$ whose distance from $O$ is rational. This is performed by using a parameterization of the second singular quadric $K$ which corresponds to the zero $\frac{r}{\gamma m^{2} c t s t}$ of the characteristic polynomial $\operatorname{det}(F+t L(\tau)) . K$ is a cone of revolution with axis parallel to a, and its coefficient matrix reads

$$
K(\tau)=\left(\begin{array}{cccc}
\gamma^{2} & -m & 0 & 0 \\
-m & \frac{\gamma\left(m^{2}+2 r^{2}\right) s t c t+r^{3}\left(c t^{2}-s t^{2}\right)}{\gamma m^{2} s t c t} & 0 & \frac{-r\left(\left(\gamma^{2}-r^{2}\right) s t c t+\gamma r\left(c t^{2}-s t^{2}\right)\right)}{\gamma m^{2} s t c t} \\
0 & 0 & \frac{\gamma s t+r c t}{\gamma s t} & 0 \\
0 & \frac{-r\left(\left(\gamma^{2}-r^{2}\right) s t c t+\gamma r\left(c t^{2}-s t^{2}\right)\right)}{\gamma m^{2} s t c t} & 0 & \frac{\gamma\left(\gamma^{2}-r^{2}\right) s t c t+r \gamma^{2}\left(c t^{2}-s t^{2}\right)}{\gamma m^{2} s t c t}
\end{array}\right) .
$$

A parameterization of the cone of revolution $K$ with respect to its vertex $\mathbf{s}$ is

$$
\mathbf{k}(u, v)=\mathbf{s}+v(\mathbf{a}+R(\mathbf{b} \cos u+\mathbf{c} \sin u)),
$$

where $\mathbf{a}$ is a unit vector in direction of its axis, and $\mathbf{b}$ and $\mathbf{c}$ complete $\mathbf{a}$ to an orthonormal basis in $\mathbb{R}^{3}$, and $R$ denotes the radius of the cross section circle at distance 1 from $\mathbf{s}$ which has still to be determined. In detail this reads

$$
\mathbf{k}(u, v)=\left(\begin{array}{c}
\frac{\gamma^{2}}{m}+v\left(\frac{\gamma c t-r s t}{m}+R \frac{\sin u(\gamma s t+r c t)}{m}\right) \\
-v R \cos u \\
\frac{\gamma r}{m}+v\left(\frac{\gamma s t+r c t}{m}+R \frac{\sin u(-\gamma c t+r s t)}{m}\right)
\end{array}\right) .
$$

Inserting $\mathbf{k}(u, v)$ into the equation $\mathbf{y}^{T} \cdot K(\tau) \cdot \mathbf{y}=0$ defines the radius

$$
R=\frac{\sqrt{-c t s t(\gamma s t+r c t)(\gamma c t-r s t)}}{c t(\gamma s t+r c t)}=\sqrt{\frac{-s t(\gamma c t-r s t)}{c t(\gamma s t+r c t)}} .
$$

The final parameterization of the quartic curve $C$ is obtained for $v=\frac{2 r(R \sin u c t-s t)}{1+R^{2}}$ and is a bit lengthy. It reads

$$
\mathbf{c}(u)=\left(\begin{array}{c}
\frac{\left(4 R r \sin u c t(\gamma c t-r s t)+2 r c t(\gamma s t+r c t)\left(R^{2} \sin ^{2} u-1\right)+m^{2}+r^{2}+R^{2}\left(m^{2}-r^{2}\right)-2 R r \gamma \sin u\right)}{m\left(1+R^{2}\right)}  \tag{25}\\
\frac{-2 R r \cos u(R c t \sin u-s t)}{1+R^{2}} \\
\frac{r\left(2 R^{2} c t \sin ^{2} u(r s t-\gamma c t)+4 R \gamma \sin u c t s t-\gamma\left(1-R^{2}\right)-2 r c t s t+2 \gamma c t^{2}+2 R r \sin u\left(c t^{2}-s t^{2}\right)\right)}{m\left(1+R^{2}\right)}
\end{array}\right),
$$

and its norm is

$$
\|\mathbf{c}(u)\|=\frac{\gamma c t\left(1+R^{2}\right)-2 r s t+2 r R c t \sin (u)}{c t\left(1+R^{2}\right)}
$$

This is proved by using the incidence $\mathbf{c} \subset E$, thus $a \mathbf{c}_{1}+b \mathbf{c}_{2}+c \mathbf{c}_{3}=c t w$, with $w=\|\mathbf{c}\|$. Note that $R$ is not rational in any rational substitution for the trigonometric functions $\cos \tau$ and $\sin \tau$. Rotating $C$ around the $x$-axis gives a rational polar representation $\mathbf{f}(u, v)$ of the sphere $F$. The resulting parameterization $\mathbf{f}$ of $F$ is not proper, but almost all points of $F$ are traced twice, therefore belonging to two parameter values $\left(u_{1}, v\right)$ and $\left(u_{2}, v\right)$. We summarize the construction.
Corollary 7. There exists a one-parameter family of quartic curves $C(\tau) \subset F$ with double point at $\mathbf{s}$ and symmetry plane $y=0$. The corresponding pencils of quadrics $Q(t)=F+\lambda L(\tau)$ contain rotational cones $K(\tau)$ and $L(\tau)$, where the vertex of the latter is at $O$, and a parabolic cylinder $P(\tau)$. Besides $P(\tau)$ all quadrics have rotational symmetry with parallel axes $\mathbf{a}(\tau)$. The distance function $\operatorname{dist}(O C)=\|\mathbf{c}(u)\|$ is rational in the curve parameter, but not rational in the angleparameter $\tau$.

## 4. Conclusion

We have discussed the conchoid construction for spheres and have shown that a sphere in $\mathbb{R}^{3}$ admits a rational polar representation with respect to an arbitrary chosen focus point, which implies that the conchoid surfaces of spheres possess rational parameterizations. Additionally we have given a geometric construction for these parameterizations which are based on a rational curve of degree four being the base locus of a pencil of quadrics in $\mathbb{R}^{3}$. Relations to the classical Viviani curve have been addressed. The construction of the rational parameterization of the conchoids is also based on a pencil of quadrics in $\mathbb{R}^{4}$.

## Acknowledgments

This work was developed, and partially supported, under the research project MTM2008-04699-C03-01 Variedades paramtricas: algoritmos y aplicaciones, Ministerio de Ciencia e Innovacin, Spain and by "Fondos Europeos de Desarrollo Regional" of the European Union.

## References

[1] M. Aigner, Bert Jüttler, Laureano Gonzalez-Vega, Josef Schicho, Parameterizing surfaces with certain special support functions, including offsets of quadrics and rationally supported surfaces, Journal of Symbolic Computation 44, 2009, 180-191.
[2] A. Albano, M. Roggero: Conchoidal transform of two plane curves, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, 2010, pp. 309-328.
[3] D. Cox D., Little J. and O'Shea D., 2010. Ideals, Varieties, and Algorithms. Springer-Verlag, New York.
[4] Dietz, R., Hoschek, J., and Jüttler, B., An algebraic approach to curves and surfaces on the sphere and other quadrics, Comp. Aided Geom. Design 10, pp. 211229, 1993.
[5] Peternell, M. and Pottmann, H., 1998. A Laguerre geometric approach to rational offsets, Comp. Aided Geom. Design 15, 223-249.
[6] Peternell, M., 1997. Rational Parametrizations for Envelopes of Quadric Families, Thesis, University of Technology, Vienna.
[7] Peternell, M., Gruber, D. and Sendra, J., 2011: Conchoid surfaces of rational ruled surfaces, Comp. Aided Geom. Design 28, 427-435.
[8] Pottmann, H., and Peternell, M. 1998. Applications of Laguerre geometry in CAGD, offsets, Comp. Aided Geom. Design 15, 165-186.
[9] Schicho, J.,1997. Rational Parametrization of Algebraic Surfaces. Technical report no. 97-10 in RISC Report Series, University of Linz, Austria, March 1997, PhD Thesis.
[10] Schicho, J., 2000. Proper Parametrization of Real Tubular Surfaces, J. Symbolic Computation 30, 583-593.
[11] Schicho, J., 1998. Rational Parametrization of Surfaces, J. Symbolic Computation 26, 1-29.
[12] J.R. Sendra and J. Sendra, 2008. An algebraic analysis of conchoids to algebraic curves, Applicable Algebra in Engineering, Communication and Computing, Vol.19, No.5, pp. 285-308.
[13] J. Sendra and J.R. Sendra, 2010. Rational parametrization of conchoids to algebraic curves, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, pp. 413-428.
[14] Shafarevich, R.I., 1994. Basic Algebraic Geometry, Vol.I, Springer, Heidelberg.

