# Krall-type Orthogonal Polynomials in Several Variables ${ }^{1}$ 

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#### Abstract

For a bilinear form obtained by adding a Dirac mass to a positive definite moment functional in several variables, explicit formulas of orthogonal polynomials are derived from the orthogonal polynomials associated with the moment functional. Explicit formula for the reproducing kernel is also derived and used to establish certain inequalities for classical orthogonal polynomials.


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## 1 Introduction

Let $\Pi^{d}$ denote the space of polynomials in $d$-variables, and let $u$ be a moment functional, denoted by $\langle u, p\rangle$ for $p \in \Pi^{d}$, for which orthogonal polynomials exist. We define a new functional $v$ by adding a Dirac mass to $u$,

$$
\begin{equation*}
\langle v, p\rangle=\langle u, p\rangle+\lambda p(c), \quad c \in \mathbb{R}^{d}, \quad \lambda \in \mathbb{R}, \quad p \in \Pi^{d} \tag{1}
\end{equation*}
$$

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and study orthogonal polynomials with respect to the functional $v$.
In the case of one variable, this problem first arose from the work of A. M. Krall ([5]) when he studied the orthogonal polynomials that are eigenfunctions of a fourth order differential operator considered by H. L. Krall ([6]7]), and showed that the polynomials are orthogonal with respect to a measure that is obtained from a continuous measure on an interval by adding masses at the end points of the interval. In [3], Koornwinder studied the case that the measure is a Jacobi weight function together with additional mass points at 1 and -1 ; he constructed explicit orthogonal polynomials and studied their properties. Uvarov ([10]) considered the problem of orthogonal polynomials with respect to a measure obtained by adding a finite discrete part to another measure; his main result expresses the polynomials orthogonal with respect to the new measure in terms of the polynomials orthogonal with respect to the old one. More generally, one can consider perturbations of quasi definite linear functionals via the addition of Dirac delta functionals, orthogonal polynomials in such a general setting has been studied extensively in recent years (see, for instance [8], and the references therein).

The purpose of the present paper is to study this problem in several variables. After a brief section on notations and preliminaries in the next section, we state and prove our main results in Sections 3. The result gives a necessary and sufficient condition for the existence of orthogonal polynomials with respect to the linear functional $v$ defined by (1), and expresses orthogonal polynomials with respect to $v$ in terms of the orthogonal polynomials with respect to $u$. Furthermore, we can also express the reproducing kernel of polynomials with respect to $v$ in terms of the kernel with respect to $u$. Our formula on the reproducing kernel implies an inequality on the orthogonal polynomials with respect to $u$. Even in the case of one variable, it leads to new inequalities on classical orthogonal polynomials, which are stated in Section 4. Finally, in Section 5, we consider the example of orthogonal polynomials on the unit ball.

## 2 Orthogonal polynomials in several variables

In this section we recall necessary notations and definitions about orthogonal polynomials of several variables, following [1].

Throughout this paper, we will use the usual multi-index notation. For $\alpha=$ $\left(\alpha_{1}, \ldots, \alpha_{d}\right) \in \mathbb{N}_{0}^{d}$ and $x=\left(x_{1}, \ldots, x_{d}\right) \in \mathbb{R}^{d}$, we write $x^{\alpha}=x_{1}^{\alpha_{1}} \cdots x_{d}^{\alpha_{d}}$. The integer $|\alpha|=\alpha_{1}+\cdots+\alpha_{d}$ is called the total degree of $x^{\alpha}$. The linear combinations of $x^{\alpha},|\alpha|=n$, is a homogeneous polynomial of degree $n$. We denote by $\mathcal{P}_{n}^{d}$ the space of homogeneous polynomials of degree $n$ in $d$ variables, by $\Pi_{n}^{d}$ the space of polynomials of total degree not greater than $n$. It is well
known that

$$
\operatorname{dim} \Pi_{n}^{d}=\binom{n+d}{n} \quad \text { and } \quad \operatorname{dim} \mathcal{P}_{n}^{d}=\binom{n+d-1}{n}:=r_{n}^{d}
$$

Let $\left\{\mu_{\alpha}\right\}_{\alpha \in \mathbb{N}_{0}^{d}}$ be a multi-sequence of real numbers, and let $u$ be a real valued functional defined on $\mathcal{P}$ by means of

$$
\left\langle u, x^{\alpha}\right\rangle=u\left(x^{\alpha}\right)=\mu_{\alpha}
$$

and extended by linearity. Then, $u$ is called the moment functional determined by $\left\{\mu_{\alpha}\right\}_{\alpha \in \mathbb{N}_{0}^{d}}$. If $\left\langle u, p^{2}\right\rangle>0, \forall p \in \Pi_{n}^{d}$ and $p \neq 0$, then the moment $u$ is called positive definite and it induces an inner product accordingly by $(p, q):=$ $\langle u, p q\rangle, \forall p, q \in \Pi^{d}$. A typical example of a positive moment functional is an integral with respect to a positive measure $d \mu$ with all moments finite, $\langle u, p\rangle=\int_{\mathbb{R}^{d}} p(x) d \mu$.

A polynomial $P \in \Pi_{n}^{d}$ is called an orthogonal polynomial with respect to $u$ if $\langle u, P Q\rangle=0$ for all $Q \in \Pi_{n-1}^{d}$. Let $V_{n}^{d}$ denote the space of orthogonal polynomials with respect to the $u$. We are interested in the case when $u$ admits a basis of orthogonal polynomials; that is, $\operatorname{dim} V_{n}^{d}=r_{n}^{d}$ for all $n$. This happens whenever $u$ is positive definite. In general, we call a moment functional quasi definite if it admits a basis of orthogonal polynomials.

For the study of orthogonal polynomials it is often convenient to adopt a vector notation (4|11]). Let $\left\{P_{\alpha}^{n}\right\}_{|\alpha|=n}$ denote a basis of $V_{n}^{d}$. Let the elements of $\left\{\alpha \in \mathbb{N}^{d}:|\alpha|=n\right\}$ be ordered by $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{r_{n}^{d}}$ according to a fixed monomial order, say the lexicographical order. We then write the basis of $V_{n}^{d}$ as a column vector

$$
\mathbb{P}_{n}=\left(P_{\alpha}^{n}\right)_{|\alpha|=n}=\left(P_{\alpha_{1}}, P_{\alpha_{2}}, \ldots, P_{\alpha_{r_{n}^{d}}}\right)^{T}
$$

Using this notation, the orthogonality of $P_{\alpha}^{n}$ can be expressed as

$$
\left\langle u, \mathbb{P}_{n} \mathbb{P}_{m}^{T}\right\rangle=H_{n} \delta_{m, n},
$$

where $H_{n}$ is a matrix of size $r_{n}^{d} \times r_{n}^{d}$. That $u$ is quasi definite is equivalent to that $H_{n}$ is invertible for all $n \geq 0$. If $u$ is positive definite, then we can choose a basis so that $H_{n}$ is the identity matrix for all $n$; in other words, we can choose the basis to be orthonormal.

Let $u$ be a quasi definite moment functional and let $\left\{\mathbb{P}_{n}\right\}_{n \geq 0}$ be a sequence of orthogonal polynomials with respect to $u$. The reproducing kernel of $V_{n}^{d}$ and $\Pi_{n}^{d}$ are denoted by $\mathbf{P}_{n}(\cdot, \cdot)$ and $\mathbf{K}_{n}(\cdot, \cdot)$, respectively, and are given by

$$
\mathbf{P}_{k}(x, y)=\mathbb{P}_{k}^{T}(x) H_{k}^{-1} \mathbb{P}_{k}(y) \quad \text { and } \quad \mathbf{K}_{n}(x, y)=\sum_{k=0}^{n} \mathbf{P}_{k}(x, y)
$$

These kernels satisfy the usual reproducing property; for example,

$$
p(x)=\left\langle u, \mathbf{K}_{n}(x, \cdot) p(\cdot)\right\rangle=\left\langle u, p(\cdot) \mathbf{K}_{n}(\cdot, x)\right\rangle, \quad p \in \Pi_{n}^{d}
$$

which shows, in particular, that these functions are independent of a particular choice of the bases. The kernel $\mathbf{K}_{n}(x, y)$ satisfies an analog of the ChristoffelDarboux formula, and plays an important role in the study of orthogonal Fourier expansions. For this and further properties of orthogonal polynomials of several variables, see [1].

## 3 Krall-type orthogonal polynomials in several variables

Let $u$ be a quasi definite moment functional defined on $\Pi^{d}$. We define a new moment functional $v$ as the perturbation of $u$ given by

$$
\langle v, p\rangle=\langle u, p\rangle+\lambda p(c), \quad \forall p(x) \in \Pi^{d}
$$

where $\lambda$ is a non zero real number and $c \in \mathbb{R}^{d}$ is a given point. Our first result gives a necessary and sufficient condition for $v$ to be quasi definite.

Theorem 1 The moment functional $v$ is quasi definite if and only if

$$
\lambda_{n}:=1+\lambda \mathbf{K}_{n}(c, c) \neq 0, \quad n \geq 0
$$

Furthermore, when $v$ is quasi definite, a sequence of orthogonal polynomials, $\left\{\mathbb{Q}_{n}\right\}_{n \geq 0}$, with respect to $v$ is given by

$$
\begin{equation*}
\mathbb{Q}_{n}(x)=\mathbb{P}_{n}(x)-\frac{\lambda}{\lambda_{n-1}} \mathbf{K}_{n-1}(c, x) \mathbb{P}_{n}(c) \quad n \geq 0 \tag{2}
\end{equation*}
$$

where $\left\{\mathbb{P}_{n}\right\}_{n \geq 0}$ denote orthogonal polynomials with respect to $u$ and $\mathbf{K}_{-1}(\cdot, \cdot):=$ 0 .

PROOF. First we assume that $v$ is quasi definite and $\left\{\mathbb{Q}_{n}\right\}_{n \geq 0}$ is a sequence of orthogonal polynomials with respect to $v$. Since $u$ is quasi definite, the leading coefficient of $\mathbb{P}_{n}$ is an invertible matrix. Hence, by multiplying $\mathbb{Q}_{n}$ by an invertible matrix, if necessary, we can assume that $\mathbb{Q}_{n}-\mathbb{P}_{n} \in \mathcal{P}_{n-1}$ for $n \geq 0$. This shows, in particular, that $\mathbb{Q}_{0}=\mathbb{P}_{0}$. Furthermore, since $\left\{\mathbb{P}_{n}\right\}_{n \geq 0}$ is a basis of $\Pi^{d}$, for each $n \geq 1$ we can express $\mathbb{Q}_{n}$ in terms of $\mathbb{P}_{n}$. Thus, there exist constant matrices $M_{i}^{n}$ of size $r_{n}^{d} \times r_{i}^{d}$ such that

$$
\mathbb{Q}_{n}(x)=\mathbb{P}_{n}(x)+\sum_{i=0}^{n-1} M_{i}^{n} \mathbb{P}_{i}(x)
$$

where, by the orthogonality of $\mathbb{P}_{n}$ and the definition of $v$,

$$
M_{i}^{n}=\left\langle u, \mathbb{Q}_{n} \mathbb{P}_{i}^{T}\right\rangle H_{i}^{-1}=\left[\left\langle v, \mathbb{Q}_{n} \mathbb{P}_{i}^{T}\right\rangle-\lambda \mathbb{Q}_{n}(c) \mathbb{P}_{i}^{T}(c)\right] H_{i}^{-1}
$$

for $0 \leq i \leq n-1$. Since $\left\langle v, \mathbb{Q}_{n} \mathbb{P}_{i}^{T}\right\rangle=0$ for $i \leq n-1$, we conclude then

$$
\mathbb{Q}_{n}(x)=\mathbb{P}_{n}(x)-\lambda \mathbb{Q}_{n}(c) \sum_{i=0}^{n-1} \mathbb{P}_{i}^{T}(c) H_{i}^{-1} \mathbb{P}_{i}(x)=\mathbb{P}_{n}(x)-\lambda \mathbb{Q}_{n}(c) \mathbf{K}_{n-1}(c, x)
$$

Evaluating the above expression at $x=c$, we obtain

$$
\begin{equation*}
\mathbb{Q}_{n}(c)\left[1+\lambda \mathbf{K}_{n-1}(c, c)\right]=\mathbb{P}_{n}(c) \tag{3}
\end{equation*}
$$

Recall that $\lambda_{k}=1+\lambda \mathbf{K}_{k}(c, c)$. If $\lambda_{n-1}=0$ for some value of $n \geq 1$, then $\mathbb{P}_{n}(c)=0$ by (3) and, furthermore, $\lambda_{n}=\lambda_{n-1}+\lambda \mathbb{P}_{n}^{T}(c) H_{n}^{-1} \mathbb{P}_{n}(c)=0$. Thus, we conclude that $\mathbb{P}_{n+1}(c)=0$ which, however, contradict to the fact that $\mathbb{P}_{n}$ and $\mathbb{P}_{n+1}$ cannot have a common zero ([1], p. 113). This shows that $\lambda_{n} \neq 0$, $\forall n \geq 0$, and also that (2) holds whenever $\lambda_{n} \neq 0$.

Conversely, if all $\lambda_{n}$ are non-zero, then we can define the polynomials $\mathbb{Q}_{n}$ by (21) and the above proof shows then that $\mathbb{Q}_{n}$ is orthogonal with respect to $v$. Since $\mathbb{Q}_{n}$ and $\mathbb{P}_{n}$ has the same leading coefficient matrix, it is evident that $\left\{\mathbb{Q}_{n}\right\}_{n \geq 0}$ contains a basis of $\Pi^{d}$.

Remark 2 If $u$ is a quasi definite moment functional, then $v$ is quasi definite except when $\lambda$ belongs to an infinite discrete set of values. If $u$ is positive definite and we choose $\lambda>0$, then it is easy to see that $v$ is also positive definite (see (4) below).

Let $\mathbb{Q}_{n}$ be as in the theorem, we define $\widetilde{H}_{k}=\left\langle v, \mathbb{Q}_{k} \mathbb{Q}_{k}^{T}\right\rangle$ and denote by

$$
\widetilde{\mathbf{P}}_{k}(x, y):=\mathbb{Q}_{k}^{T}(x) \widetilde{H}_{k}^{-1} \mathbb{Q}_{k}(y) \quad \text { and } \quad \widetilde{\mathbf{K}}_{n}(x, y):=\sum_{k=0}^{n} \widetilde{\mathbf{P}}_{k}(x, y)
$$

the reproducing kernels associated with the linear functional $v$. We will derive an explicit formula of these kernels when $u$ is positive definite.

Whenever $u$ is positive definite, we choose $\left\{\mathbb{P}_{n}\right\}_{n \geq 0}$ in the Theorem 11 as a sequence of orthonormal basis with respect to $u$, so that $H_{n}=\left\langle u, \mathbb{P}_{n} \mathbb{P}_{n}^{T}\right\rangle=I_{r_{n}}$, the identity matrix of size $r_{n}$.

Proposition 3 Let $u$ be a positive definite linear functional. Then, for $k \geq 0$,

$$
\begin{equation*}
\widetilde{H}_{k}=I_{r_{k}}+\frac{\lambda}{\lambda_{k-1}} \mathbb{P}_{k}(c) \mathbb{P}_{k}(c)^{T} \quad \text { and } \quad \widetilde{H}_{k}^{-1}=I_{r_{k}}-\frac{\lambda}{\lambda_{k}} \mathbb{P}_{k}(c) \mathbb{P}_{k}(c)^{T} \tag{4}
\end{equation*}
$$

PROOF. Since we choose $\mathbb{P}_{n}$ so that $H_{n}=I_{r_{n}}$, it follows from (3) that

$$
\begin{aligned}
\widetilde{H}_{k}=\left\langle v, \mathbb{Q}_{k} \mathbb{Q}_{k}^{T}\right\rangle=\left\langle v, \mathbb{Q}_{k} \mathbb{P}_{k}^{T}\right\rangle & =\left\langle u, \mathbb{Q}_{k} \mathbb{P}_{k}^{T}\right\rangle+\lambda \mathbb{Q}_{k}(c) \mathbb{P}_{k}(c)^{T} \\
& =I_{r_{k}}+\frac{\lambda}{\lambda_{k-1}} \mathbb{P}_{k}(c) \mathbb{P}_{k}(c)^{T}
\end{aligned}
$$

Assuming that the inverse is of the form $\widetilde{H}_{k}^{-1}=I_{r_{k}}-\delta \mathbb{P}_{k}(c) \mathbb{P}_{k}(c)^{T}$, and using the fact that $\mathbb{P}_{k}(c)^{T} \mathbb{P}_{k}(c)=\mathbf{P}_{n}(c, c)$, a quick computation shows that $\widetilde{H}_{k} \widetilde{H}_{k}^{-1}=I_{r_{k}}$ is equivalent to $\lambda-\lambda_{k-1} \delta-\lambda \delta \mathbf{P}_{n}(c, c)=0$. Using the fact that $\mathbf{P}_{n}(c, c)=\mathbf{K}_{n}(c, c)-\mathbf{K}_{n-1}(c, c)$, it is easy to see that $\delta=\lambda / \lambda_{k}$.

Theorem 4 Let $u$ be a positive definite linear functional. Then, for $k \geq 0$,

$$
\begin{equation*}
\widetilde{\mathbf{P}}_{k}(x, y)=\mathbf{P}_{k}(x, y)-\frac{\lambda}{\lambda_{k}} \mathbf{K}_{k}(x, c) \mathbf{K}_{k}(c, y)+\frac{\lambda}{\lambda_{k-1}} \mathbf{K}_{k-1}(x, c) \mathbf{K}_{k-1}(c, y) \tag{5}
\end{equation*}
$$

Furthermore, for $n \geq 0$,

$$
\begin{equation*}
\widetilde{\mathbf{K}}_{n}(x, y)=\mathbf{K}_{n}(x, y)-\frac{\lambda}{\lambda_{n}} \mathbf{K}_{n}(x, c) \mathbf{K}_{n}(c, y) \tag{6}
\end{equation*}
$$

PROOF. Let $\gamma_{k}:=\lambda / \lambda_{k}$. By (2) and (4), it follows readily that

$$
\begin{aligned}
\widetilde{\mathbf{P}}_{k}(x, y)= & \mathbf{P}_{k}(x, y)-\gamma_{k}\left[\mathbf{P}_{k}(x, c) \mathbf{K}_{k-1}(x, c)+\mathbf{P}_{k}(y, c) \mathbf{K}_{k-1}(c, x)\right. \\
& \left.+\mathbf{P}_{k}(x, c) \mathbf{P}_{k}(y, c)\right]+\gamma_{k} \gamma_{k-1} \mathbf{P}_{k}(c, c) \mathbf{K}_{k-1}(x, c) \mathbf{K}_{k-1}(y, c) \\
= & \mathbf{P}_{k}(x, y)-\gamma_{k} \mathbf{K}_{k}(x, c) \mathbf{K}_{k}(y, c) \\
& +\gamma_{k}\left[1+\gamma_{k-1} \mathbf{P}_{k}(c, c)\right] \mathbf{K}_{k-1}(x, c) \mathbf{K}_{k-1}(y, c)
\end{aligned}
$$

as the first square bracket is equal to $\mathbf{K}_{k}(x, c) \mathbf{K}_{k}(y, c)-\mathbf{K}_{k-1}(x, c) \mathbf{K}_{k-1}(y, c)$. Since the definition of $\gamma_{k}$ leads readily to $1+\gamma_{k-1} \mathbf{P}_{k}(c, c)=\gamma_{k-1}$, this proves (5). Summing over (5) for $k=0,1, \ldots, n$ proves (6).

We note that, in the case of one variable, the formula (6) has appeared in [2], whereas the formula (5) appears to be new even in one variable.

## 4 An application of formula (5)

If $u$ is positive definite and $\lambda>0$, then $v$ is also positive definite. As a result, $\widetilde{H}_{n}$, hence $\widetilde{H}_{n}^{-1}$ are positive definite matrices by (4) for $n \geq 0$. In particular, $\widetilde{\mathbf{P}}_{n}(x, x)$ is nonnegative. In fact, if $d \geq 2$ and the linear functional $u$ is centrally symmetric (see [1] for definition), then $\widetilde{\mathbf{P}}_{n}(x, x)$ is strictly positive for all $x$, except for $n$ odd and $x=0$. As a consequence, we see that (5) implies that

$$
\mathbf{P}_{n}(x, x)-\frac{\lambda}{\lambda_{n}}\left[\mathbf{K}_{n}(x, c)\right]^{2}+\frac{\lambda}{\lambda_{n-1}}\left[\mathbf{K}_{n-1}(x, c)\right]^{2} \geq 0
$$

for all $x \in \mathbb{R}^{d}$ and for all $\lambda>0$. Recall that $\lambda_{n}=1+\lambda \mathbf{K}_{n}(c, c)$. Taking the limit $\lambda \rightarrow \infty$, we obtain an inequality which we state as a proposition.

Proposition 5 For $n \geq 1$, and $x, c \in \mathbb{R}^{d}$,

$$
\begin{equation*}
\mathbf{P}_{n}(x, x)+\frac{\left[\mathbf{K}_{n-1}(x, c)\right]^{2}}{\mathbf{K}_{n-1}(c, c)} \geq \frac{\left[\mathbf{K}_{n}(x, c)\right]^{2}}{\mathbf{K}_{n}(c, c)} . \tag{7}
\end{equation*}
$$

If $x=c$, then the two sides of (7) are equal. This inequality tends out to be non-trivial even in the case of one variable. Notice that in one variable, $\mathbf{P}_{n}(x, x)=\left[p_{n}(x)\right]^{2}$, where $p_{n}$ is the orthonormal polynomial. Let us specify the inequality (7) in the cases of classical orthogonal polynomials of Jacobi and Laguerre. We use the standard notation $P_{n}^{(\alpha, \beta)}, \alpha, \beta>-1$, for Jacobi polynomials, which are orthogonal with respect to $(1-x)^{\alpha}(1+x)^{\beta}$ on $[-1,1]$, and $L_{n}^{(\alpha)}, \alpha>-1$, for the Laguerre polynomials, which are orthogonal with respect to $x^{\alpha} e^{-x}$ on $[0, \infty)$. Using the formulas in [9], especially (4.3.3), (4.5.3) and (5.1.1), (5.1.13), the inequality (7) for the Jacobi polynomials with $c=$ 1 , and for the Laguerre polynomials with $c=0$, respectively, becomes the following:

Proposition 6 For $n \geq 1$ and $x \in[-1,1]$,

$$
\frac{\left[P_{n}^{(\alpha, \beta)}(x)\right]^{2}}{P_{n}^{(\alpha, \beta)}(1)}+\frac{n+\beta}{2 n+\alpha+\beta+1} \frac{\left[P_{n-1}^{(\alpha+1, \beta)}(x)\right]^{2}}{P_{n-1}^{(\alpha+1, \beta)}(1)} \geq \frac{n+\alpha+\beta+1}{2 n+\alpha+\beta+1} \frac{\left[P_{n}^{(\alpha+1, \beta)}(x)\right]^{2}}{P_{n}^{(\alpha+1, \beta)}(1)} .
$$

For $n \geq 1$ and $x \in[0, \infty)$,

$$
\frac{\left[L_{n}^{(\alpha)}(x)\right]^{2}}{L_{n}^{(\alpha)}(0)}+\frac{\left[L_{n-1}^{(\alpha+1)}(x)\right]^{2}}{L_{n-1}^{(\alpha+1)}(0)} \geq \frac{\left[L_{n}^{(\alpha+1)}(x)\right]^{2}}{L_{n}^{(\alpha+1)}(0)} .
$$

As far as we are aware, these inequalities are new. For example, in the case of Chebyshev polynomials or $\alpha=\beta=-1 / 2$ in the Jacobi polynomials, the inequality becomes

$$
2 \cos ^{2} n \theta+\frac{1}{2 n-1}\left(\frac{\sin \left(n-\frac{1}{2}\right) \theta}{\sin \frac{\theta}{2}}\right)^{2} \geq \frac{1}{2 n+1}\left(\frac{\sin \left(n+\frac{1}{2}\right) \theta}{\sin \frac{\theta}{2}}\right)^{2}, \quad 0 \leq \theta \leq \pi
$$

## 5 An example: Krall-type orthogonal polynomials in the unit ball

Let $B^{d}$ denote the unit ball of $\mathbb{R}^{d}$. We consider the inner product

$$
\langle f, g\rangle_{\mu}=c_{\mu} \int_{B^{d}} f(x) g(x)\left(1-\|x\|^{2}\right)^{\mu-1 / 2} d x
$$

where $\mu>-1 / 2$, and $c_{\mu}=\Gamma\left(\mu+\frac{d+1}{2}\right) /\left(\pi^{d / 2} \Gamma\left(\mu+\frac{1}{2}\right)\right)$ is the normalization constant so that $\langle 1,1\rangle_{\mu}=1$. As an example for our general results, we add the mass point at the origin, and consider the inner product

$$
\begin{equation*}
\langle f, g\rangle=\langle f, g\rangle_{\mu}+\lambda f(0) g(0), \quad \lambda>0 \tag{8}
\end{equation*}
$$

We now use (2) in Theorem 1 to find an orthogonal basis for $\langle\cdot, \cdot\rangle$.
Let $\mathcal{H}_{n}^{d}$ denote the space of spherical harmonic polynomials of degree $n$ in $d$ variables. Let $Y_{\nu}^{n}, 1 \leq \nu \leq \operatorname{dim} \mathcal{H}_{n}^{d}$, be an orthonormal basis for $\mathcal{H}_{n}^{d}$ in the following. An orthonormal basis for $\langle f, g\rangle_{\mu}$ is given explicitly by ([1, p. 39])

$$
\begin{equation*}
P_{j, \nu}^{n}(x)=\left[h_{j, \nu}^{n}\right]^{-1} p_{j}^{\left(\mu-\frac{1}{2}, n-2 j+\frac{d-2}{2}\right)}\left(2\|x\|^{2}-1\right) Y_{\nu}^{n-2 j}(x), \quad 0 \leq j \leq n / 2 \tag{9}
\end{equation*}
$$

where $p_{j}^{\left(\mu, n-2 j+\frac{d-2}{2}\right)}$ denote the orthonormal Jacobi polynomials, and $h_{j, \nu}^{n}$ is the normalizing constant given by $\left[h_{j, \nu}^{n}\right]^{2}=(d / 2)_{n-2 j} /\left(\mu+\frac{d+1}{2}\right)_{n-2 j}$, in which $(a)_{k}:=a(a+1) \ldots(a+k-1)$ denotes the shifted factorial. Since $Y_{\nu}^{n-2 j}$ is a homogeneous polynomial, $Y_{\nu}^{n-2 j}(0)=0$ unless its degree is zero, that is, unless $n=2 j$. Consequently, $P_{j, \nu}^{n}(0)=0$ unless $j=n / 2$ and $n$ is even. Notice that when $n=2 j, h_{j, \mu}^{n}=1$. Hence, it follows that $P_{\frac{n}{2}, \nu}^{n}(0)=p_{\frac{n}{2}}^{\left(\mu-\frac{1}{2}, \frac{d-2}{2}\right)}(-1)$ if $n$ is even and $P_{j, \nu}^{n}(0)=0$ in all other cases. Consequently, if we define polynomials $Q_{j, \mu}^{n}$ by

$$
Q_{j, \nu}^{n}(x)= \begin{cases}P_{n}^{n}, \nu  \tag{10}\\ P_{j, \nu}^{n}(x)-\rho_{n} \mathbf{K}_{n-1}(x, 0), & \text { if } n \text { is even } \\ \text { otherwise }\end{cases}
$$

where $\rho_{n}:=\lambda p_{n / 2}^{\left(\mu-\frac{1}{2}, \frac{d-2}{2}\right)}(-1) /\left(1+\lambda \mathbf{K}_{n-1}(0,0)\right)$, then according to (22) in Theorem 11, $\left\{Q_{j, \mu}^{n}: 1 \leq \nu \leq \operatorname{dim} \mathcal{H}_{n-2 j}^{d}, 0 \leq 2 j \leq n\right\}$ constitutes an orthogonal basis with respect to the inner product (8).

To make the expression for transparent, we assume $\mu \geq 0$ and make use of the following explicit formula for the reproducing kernel $\mathbf{K}_{n}(\cdot, \cdot)$ in [12],

$$
\mathbf{K}_{n}(x, y)=A_{n}^{\mu} \int_{-1}^{1} P_{n}^{\left(\mu+\frac{d}{2}, \mu+\frac{d}{2}-1\right)}\left(x \cdot y+\sqrt{1-\|x\|^{2}} \sqrt{1-\|y\|^{2}} t\right)\left(1-t^{2}\right)^{\mu-1} d t
$$

where $\mu>0$ (see [12] for the case $\mu=0$ ) and

$$
A_{n}^{\mu}:=\frac{2 \Gamma\left(\mu+\frac{1}{2}\right) \Gamma\left(\mu+\frac{d+2}{2}\right) \Gamma(n+2 \mu+d)}{\pi^{1 / 2} \Gamma(\mu) \Gamma\left(n+\mu+\frac{d}{2}\right) \Gamma(2 \mu+d+1)}
$$

We set $y=0$ in this formula and follow through a sequence of manipulations of formulas. First we use [9, (4.5.3)] to write $P_{n}^{\left(\mu+\frac{d}{2}, \mu+\frac{d}{2}-1\right)}(z)$ as a sum of Gegenbauer polynomials, so that we can apply [1, Theorem 1.5.6] to get ride of the integral (for the terms of odd degree Gegenbauer polynomials, the integrals
are automatically zero), the result is a sum of $\left\lfloor\frac{n}{2}\right\rfloor$ terms of Jacobi polynomials upon using the first formula on [1, p. 27], which we can use [9, (4.5.3)] again to sum up. The final result is the following identity,

$$
\begin{equation*}
\mathbf{K}_{n}(x, 0)=\frac{\left(\mu+\frac{d-1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}{\left(\mu+\frac{1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}} P_{\left\lfloor\frac{n}{2}\right\rfloor}^{\left(\frac{d}{2}, \mu-\frac{1}{2}\right)}\left(1-2\|x\|^{2}\right), \tag{11}
\end{equation*}
$$

where $\lfloor x\rfloor$ denotes the integer part of $x$. In particular, setting $x=0$ gives

$$
\begin{equation*}
\mathbf{K}_{n}(0,0)=\frac{\left(\mu+\frac{d-1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}{\left(\mu+\frac{1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}} P_{\left\lfloor\frac{n}{2}\right\rfloor}^{\left(\frac{d}{2}, \mu-\frac{1}{2}\right)}(1)=\frac{\left(\mu+\frac{d-1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}{\left(\mu+\frac{1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}\binom{\left\lfloor\frac{n}{2}\right\rfloor+\frac{d}{2}}{\left\lfloor\frac{n}{2}\right\rfloor} \tag{12}
\end{equation*}
$$

Substituting these formulas into (10) gives a basis of explicit orthogonal polynomials with respect to the inner product in (8). Furthermore, using (11) and (12) in the formula (6), we obtain a compact formula for the reproducing kernel $\widetilde{\mathbf{K}}_{n}(\cdot, \cdot)$ associated with $\langle\cdot, \cdot\rangle$ in (8). We sum up these results as a proposition.

Proposition 7 For the inner product $\langle\cdot, \cdot\rangle$ in (8), the polynomials $Q_{j, \nu}^{n}, 1 \leq$ $\nu \leq \operatorname{dim} \mathcal{H}_{n-2 j}^{d}, 0 \leq j \leq n$, in (10) form an orthogonal basis of degree $n$. Furthermore, the reproducing kernel of $\Pi_{n}^{d}$ with respect to $\langle\cdot, \cdot\rangle$ is given by

$$
\widetilde{\mathbf{K}}_{n}(x, y)=\mathbf{K}_{n}(x, y)-d_{n} P_{\left\lfloor\frac{n}{2}\right\rfloor}^{\left(\frac{d}{2}, \mu-\frac{1}{2}\right)}\left(1-2\|x\|^{2}\right) P_{\left\lfloor\frac{n}{2}\right\rfloor}^{\left(\frac{d}{2}, \mu-\frac{1}{2}\right)}\left(1-2\|y\|^{2}\right),
$$

where the constant $d_{n}$ is given by

$$
d_{n}=\frac{\lambda}{1+\lambda \mathbf{K}_{n}(0,0)}\left[\frac{\left(\mu+\frac{d-1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}{\left(\mu+\frac{1}{2}\right)_{\left\lfloor\frac{n}{2}\right\rfloor}}\right]^{2}
$$

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