Cubature rule associated with a discrete blending sum of quadratic spline quasi-interpolants

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Cubature rule associated with a discrete blending sum of quadratic spline quasi-interpolants

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Abstract

A new cubature rule for a parallelepiped domain is defined by integrating a discrete blending sum of C^1 quadratic spline quasi-interpolants in one and two variables. We give the weights and the nodes of this cubature rule and we study the associated error estimates for smooth functions. We compare our method with cubature rules based on tensor products of spline quadratures and classical composite Simpson's rules.

Key words: Multivariate numerical integration, Spline quasi-interpolants

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1 Introduction

Let $\Omega := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]$ be a parallepiped endowed with the tensor product of arbitrary partitions on each subinterval $[\alpha_s, \beta_s], s = 1, 2, 3$. There are many possibilities of constructing quasi-interpolants (abbr. QIs) from univariate, bivariate or trivariate quadratic spline QIs. Such trivariate quadratic spline operators can be found for example in [22] and in references therein, but they need rather complex triangulations of the domain. On the other hand, one can use tensor products or discrete boolean sums of univariate QIs. The cubature formulas associated with tensor products are briefly studied in [14]. Trivariate blending sums are more complicated to define (see e.g. [13], [17], and chapter 8 of [6]). A third possibility is to combine univariate and bivariate quadratic QIs. Cubature rules associated with tensor products or blending sums of such QIs are also briefly studied in [14]. In the present paper, we study more completely the cubature rule associated with a trivariate spline QI obtained as a discrete blending sum of a bivariate and a univariate C^1 quadratic spline QIs. Generalities on spline QIs can be found e.g. in [1] - [5] [8] [15] [20] [31]. For cubature rules, see e.g. [9] [10] [16] [18].

Here is an outline of the paper. In Section 2, we recall the main properties of univariate quadratic spline QIs as they appear in [29] and [30]. In Section 3, we do the same for bivariate quadratic spline QIs on the so-called criss-cross triangulation of the domain $\Omega' := [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$, which are studied in [25] [26] [27]. In Section 4, we define and study the properties of the discrete blending sum of the two previous operators. In Section 5, we construct the cubature rule associated with this QI, using previous results on univariate quadrature

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and bivariate cubature rules given in [29] and [19], and we give error bounds for nonuniform partitions. In Section 6, we give more informations on cubature errors in the specific cases of symmetric nonuniform partitions and of uniform partitions. Finally, in Section 7, we give several examples where our cubature rule is compared with tensor product cubature rules based on univariate quadratic spline QIs and classical composite Simpson's rules. We also briefly consider the possibility of inserting multiple knots in the integration of nonsmooth functions by spline cubatures which improves the precision of the result by comparison with classical rules.

2 Univariate quadratic spline and discrete quasi-interpolants

Let $X_m = \{\alpha = x_0 < x_1 < \ldots < x_m = \beta\}$ be a partition of a bounded interval $I = [\alpha, \beta]$. For $1 \leq i \leq m$, let $h_i = x_i - x_{i-1}$ be the length of the subinterval $I_i = [x_{i-1}, x_i]$. Let $\mathcal{S}_2(X_m)$ be the $m+2$-dimensional space of C^1 quadratic splines on this partition and let $\Gamma'' = \{0, 1, \ldots, m+1\}$. A basis of this space is formed by quadratic B-splines $\{B_i, i \in \Gamma''\}$, with triple knots $x_0 = x_{-1} = x_{-2} = \alpha$, and $x_m = x_{m+1} = x_{m+2} = \beta$. We also use the set of $m+2$ data sites (or Greville abscissas):

$$S_m = \{s_i = \frac{1}{2}(x_{i-1} + x_i),\, i \in \Gamma''\}$$

(note that $s_0 = \alpha$, $s_{m+1} = \beta$, $h_{-1} = h_0 = h_{m+1} = h_{m+2} = 0$) and the following mesh ratios

$$\sigma_i = \frac{h_i}{h_{i-1} + h_i}, \quad \sigma_i' = 1 - \sigma_i = \frac{h_{i-1}}{h_{i-1} + h_i}, \quad 1 \leq i \leq m+1.$$

with the convention $\sigma_0 = \sigma'_{m+2} = 0$. Note also that $\sigma_1 = \sigma'_{m+1} = 1$. All these values lie in $[0, 1]$.

The simplest discrete quasi-interpolant (abbr. dQI) is the Schoenberg-Marsden operator:

$$Q_1 f = \sum_{i \in \Gamma''} f(s_i) B_i$$

which is exact on the space Π_1 of linear polynomials. In [25], another dQI has been studied

$$Q_2 f = \sum_{i \in \Gamma''} \mu_i(f) B_i$$

whose discrete coefficient functionals are respectively defined by

$\mu_0(f) = f(s_0), \mu_{m+1}(f) = f(s_{m+1})$ and, for $1 \leq i \leq m$,

$$\mu_i(f) = a_i f(s_{i-1}) + b_i f(s_i) + c_i f(s_{i+1}),$$

where

$$a_i = -\frac{\sigma_i^2 \sigma'_{i+1}}{\sigma_i + \sigma'_{i+1}}, \quad b_i = 1 + \sigma_i \sigma'_{i+1} \quad c_i = -\frac{\sigma_i (\sigma'_{i+1})^2}{\sigma_i + \sigma'_{i+1}}.$$

We implicitly assume that $a_0 = c_0 = 0, b_0 = 1$ and $a_{m+1} = c_{m+1} = 0, b_{m+1} = 1$. Let $e_s(x) = x^s, s \geq 0$, then using the following expansions

$$e_0 = \sum_{i \in \Gamma''} B_i, \quad e_1 = \sum_{i \in \Gamma''} s_i B_i, \quad e_2 = \sum_{i \in \Gamma''} x_{i-1} x_i B_i,$$

it is easy to verify that Q_2 is exact on the space Π_2 of quadratic polynomials.

We can now define the *fundamental functions* of $\mathcal{S}_2(X_m)$ associated with the dQI Q_2 as follows

$$\tilde{B}_0 = B_0 + a_1 B_1, \quad \tilde{B}_{m+1} = c_m B_m + B_{m+1},$$

and, for $1 \leq i \leq m$:

$$\tilde{B}_i = c_{i-1} B_{i-1} + b_i B_i + a_{i+1} B_{i+1} \; ,$$

They allow to express $Q_2 f$ in the following shorter form

$$Q_2 f = \sum_{i \in \Gamma''} f(s_i) \tilde{B}_i,$$

and to show that the infinity norm of Q_2 is equal to the Chebyshev norm of its Lebesgue function:

$$\Lambda_{Q_2}(x) = \sum_{i \in \Gamma''} |\tilde{B}_i(x)|.$$

The following result is proved in [29]

Theorem 1 *For any partition X_m of I, the infinity norm of Q_2 is uniformly bounded by* 2.5*. Moreover, if the partition is uniform, one has $\|Q_2\|_\infty = \frac{305}{207} \approx$* 1.4734.

Remark. The results of this section are also valid when X_m contains some knots of multiplicity 2 or 3. Assume first that $\xi = x_p = x_{p+1}$ is a *double knot*, then $Q_2 f$ is only continuous at that point. Moreover, as $h_{p+1} = 0$, we have $\mathrm{supp}(B_p) = [x_{p-2}, \xi]$, $\mathrm{supp}(B_{p+1}) = [\xi - h_p, \xi + h_{p+2}]$, $\mathrm{supp}(B_{p+2}) = [\xi, x_{p+3}]$. Similarly, as $\sigma_{p+1} = 0$, we have $a_{p+1} = c_{p+1} = 0$ and $b_{p+1} = 1$, hence:

$$Q_2 f = \sum_{i=0}^{p} \mu_i(f) B_i + f(\xi) B_{p+1} + \sum_{i=p+2}^{m+1} \mu_i(f) B_i.$$

Now, if $\eta = x_{q-1} = x_q = x_{q+1}$ is a *triple knot*, then $Q_2 f$ has a discontinuity at this point. Assume that f is itself discontinuous and admits left and right limits $f(\eta^-)$ and $f(\eta^+)$. Then as $h_q = h_{q+1} = 0$, we have $\mathrm{supp}(B_q) = [\eta - h_{q-1}, \eta]$, with $B_q(\eta^-) = 1$ and $B_q(\eta^+) = 0$, while $\mathrm{supp}(B_{q+1}) = [\eta, \eta + h_{q+2}]$, with $B_{q+1}(\eta^-) = 0$ and $B_{q+1}(\eta^+) = 1$. As $\sigma_q = \sigma_{q+1} = 0$, we get $a_q = a_{q+1} = c_q = c_{q+1} = 0$ and $b_q = b_{q+1} = 1$, hence:

$$Q_2 f = \sum_{i=0}^{q-1} \mu_i(f) B_i + f(\eta^-) B_q + f(\eta^+) B_{q+1} + \sum_{i=q+2}^{m+1} \mu_i(f) B_i.$$

Finally, from theorem 1 and standard arguments in approximation theory, we deduce (for a proof, see [26]).

Theorem 2 *There exists a constant* $0 < C_1 < 1$ *such that for all* $f \in W^{3,\infty}(I)$ *and for all partitions of* I*, with* $h = \max h_i$*,*

$$\|f - Q_2 f\|_\infty \leq C_1 h^3 \|f^{(3)}\|_\infty.$$

3 Bivariate quadratic splines and quasi-interpolants

In this Section, we recall the main results of [25] on bivariate C^1 quadratic splines and associated discrete quasi-interpolants defined on a nonuniform criss-cross triangulation of a rectangular domain.

3.1 Bivariate quadratic splines on a bounded rectangle

For $I = [\alpha_1, \beta_1]$ and $J = [\alpha_2, \beta_2]$, let Ω' be the rectangular domain $I \times J$ decomposed into mn subrectangles by the two partitions:

$$X_m = \{x_i, 0 \leq i \leq m\}, \quad Y_n = \{y_j, 0 \leq j \leq n\}$$

respectively of the segments I and J. We consider the associated extended partitions with triple knots $x_0 = x_{-1} = x_{-2} = \alpha_1$, $x_m = x_{m+1} = x_{m+2} = \beta_1$ and $y_0 = y_{-1} = y_{-2} = \alpha_2$, $y_n = y_{n+1} = y_{n+2} = \beta_2$. For $1 \leq i \leq m$ and $1 \leq j \leq n$, we set $h_i = x_i - x_{i-1}$, $k_j = y_j - y_{j-1}$, $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$, $s_i = \frac{1}{2}(x_{i-1} + x_i)$, $t_j = \frac{1}{2}(y_{j-1} + y_j)$. Moreover $h_{-1} = h_0 = h_{m+1} = h_{m+2} = k_{-1} = k_0 = k_{n+1} = k_{n+2} = 0$.

Let $\Gamma' = \Gamma'_{mn} = \{(i,j), 0 \leq i \leq m+1, 0 \leq j \leq n+1\}$, then the data sites are the mn intersection points of diagonals in subrectangles $\Omega'_{ij} = I_i \times J_j$, the

$2(m+n)$ midpoints of the subintervals on the four edges of Ω' and the four vertices of Ω', i.e. the $(m+2)(n+2)$ points of the following set:

$$\mathcal{D}_{mn} = \{M_{ij} = (s_i, t_j), \ \ (i,j) \in \Gamma'_{mn}\}$$

We denote by

$$\mathcal{B}_{mn} = \{B_{ij}, \ \ (i,j) \in \Gamma'_{mn}\}$$

the collection of $(m+2)(n+2)$ B-splines spanning the space $\mathcal{S}_2(\mathcal{T}_{mn})$ of all C^1 piecewise quadratic splines on the criss-cross triangulation $\mathcal{T}_{mn}$ associated with the partition $X_m \times Y_n$ of the rectangle Ω' (see e.g. [7] [8]), which is defined as follows.

The B-splines that we will use are completely defined by their Bernstein-Bézier (abbr. BB)-coefficients in each triangle of $\mathcal{T}_{mn}$. The latter can be found in [23] for inner B-splines (with full octagonal supports inside Ω') and more completely in the technical reports [24] (for uniform partitions) and [27] (for non-uniform partitions) for boundary B-splines. As $\dim \mathcal{S}_2(\mathcal{T}_{mn}) = (m+2)(n+2)-1$, the set $\mathcal{B}_{mn}$ is only a spanning system of $\mathcal{S}_2(\mathcal{T}_{mn})$. However, for the construction of QIs, we do not need that $\mathcal{B}_{mn}$ be a basis. A fundamental property is that B-splines are positive and form a partition of unity on Ω'. Moreover, monomials $e_{rs}(x,y) := x^r y^s, 0 \le r+s \le 2$, in $\Pi_2 = \Pi_2[x,y]$, the space of bivariate quadratic polynomials, have simple expansions in terms of B-splines

$$e_{10}(x,y) = x = \sum_{(i,j)\in\Gamma'} s_i B_{ij}(x,y), \ \ e_{01}(x,y) = y = \sum_{(i,j)\in\Gamma'} t_j B_{ij}(x,y),$$

$$e_{11}(x,y) = xy = \sum_{(i,j)\in\Gamma'} s_i t_j B_{ij}(x,y),$$

$$e_{20}(x,y) = x^2 = \sum_{(i,j)\in\Gamma'} (s_i^2 - \frac{h_i^2}{4}) B_{ij}(x,y),$$

$$e_{02}(x,y) = y^2 = \sum_{(i,j)\in\Gamma'} (t_j^2 - \frac{k_j^2}{4}) B_{ij}(x,y).$$

3.2 Bivariate discrete quasi-interpolants

As in Section 2, we use the following notations, for $1 \leq i \leq m+1$ and $1 \leq j \leq n+1$:

$$\sigma_i = \frac{h_i}{h_{i-1}+h_i}, \quad \sigma_i' = 1-\sigma_i, \quad \tau_j = \frac{k_j}{k_{j-1}+k_j}, \quad \tau_j' = 1-\tau_j,$$

and we define the triplets of coefficients, for $1 \leq i \leq m, 1 \leq j \leq n$:

$$a_i = -\frac{\sigma_i^2 \sigma_{i+1}'}{\sigma_i + \sigma_{i+1}'}, \quad b_i = 1 + \sigma_i \sigma_{i+1}', \quad c_i = -\frac{\sigma_i (\sigma_{i+1}')^2}{\sigma_i + \sigma_{i+1}'},$$

$$\bar{a}_j = -\frac{\tau_j^2 \tau_{j+1}'}{\tau_j + \tau_{j+1}'}, \quad \bar{b}_j = 1 + \tau_j \tau_{j+1}', \quad \bar{c}_j = -\frac{\tau_j (\tau_{j+1}')^2}{\tau_j + \tau_{j+1}'}.$$

It is also convenient to set $a_0 = c_0 = c_{-1} = \bar{a}_0 = \bar{c}_0 = \bar{c}_{-1} = a_{m+1} = a_{m+2} = c_{m+1} = \bar{a}_{n+1} = \bar{a}_{n+2} = \bar{c}_{n+1} = 0$ and $b_0 = \bar{b}_0 = b_{m+1} = \bar{b}_{n+1} = 1$.

As in the univariate case, the simplest dQI is the analogue of the Schoenberg-Marsden operator:

$$P_1 f = \sum_{(i,j)\in\Gamma'} f(M_{ij}) B_{ij},$$

which is exact on the space Π_{11} of bilinear polynomials. Another quadratic spline dQI has been introduced in [25]

$$P_2 f = \sum_{(i,j)\in\Gamma'} \mu_{ij}(f) B_{ij},$$

whose discrete coefficient functionals are given by:

$$\mu_{ij}(f) = (b_i+\bar{b}_j-1) f(M_{ij}) + a_i f(M_{i-1,j}) + c_i f(M_{i+1,j}) + \bar{a}_j f(M_{i,j-1}) + \bar{c}_j f(M_{i,j+1}).$$

Using the expansions of monomials given at the end of the preceding section, it is easy to verify that P_2 is exact on Π_2, i.e. satisfies $P_2 e_{rs} = e_{rs}$ for $0 \leq r+s \leq 2$.

As in Section 2, we introduce the *fundamental functions* associated with this quasi-interpolant:

$$\bar{B}_{ij} = (b_i + \bar{b}_j - 1)B_{ij} + a_{i+1}B_{i+1,j} + c_{i-1}B_{i-1,j} + \bar{a}_{j+1}B_{i,j+1} + \bar{c}_{j-1}B_{i,j-1},$$

which allow to represent $P_2 f$ in the following simple form:

$$P_2 f = \sum_{(i,j)\in\Gamma'} f(M_{ij})\bar{B}_{ij},$$

and to define its Lebesgue function:

$$\Lambda_{P_2} = \sum_{(i,j)\in\Gamma'} |\bar{B}_{ij}|.$$

The Chebyshev norm of this function is equal to the infinity norm of P_2 and we get [25] [26] the following

Theorem 3 *The infinity norm of P_2 is uniformly bounded by 5 for any criss-cross partition $\mathcal{T}_{mn}$ of Ω. For uniform partitions, there holds the smaller bound $\|P_2\|_\infty \leq 2.4$.*

From theorem 3, the exactness of P_2 on Π_2 and standard arguments in approximation theory, we deduce the following result:

Theorem 4 *There exists a constant $C_2 > 0$ such that for all functions $f \in W^{3,\infty}(\Omega')$ and $h = \max\{diam(\tau) \,|\, \tau \in \mathcal{T}_{mn}\}$:*

$$\|f - P_2 f\|_\infty \leq C_2 h^3 \max\{\|D^{rs} f\|_\infty : r + s = 3\}.$$

4 Trivariate splines and quasi-interpolants

In this Section, we recall the properties of a trivariate dQI [14] [26] defined on a parallelepiped, which is the blending sum of trivariate extensions of univariate and bivariate dQIs already defined in Sections 2 and 3 above.

4.1 Trivariate splines

Let $\Omega = \Omega' \times \Omega''$ be a parallelepiped, with $\Omega' = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ and $\Omega'' = [\alpha_3, \beta_3]$. We consider the three partitions:

$$X_m = \{x_i, 0 \le i \le m\}, \;\; Y_n = \{y_j, 0 \le j \le n\}, \;\; Z_p = \{z_k, 0 \le k \le p\},$$

respectively of the segments $I = [\alpha_1, \beta_1] = [x_0, x_m]$, $J = [\alpha_2, \beta_2] = [y_0, y_n]$ and $\Omega'' = [\alpha_3, \beta_3] = [z_0, z_p]$. For Ω', we use the notations of Section 3. For Ω'', we use the following notations:

$$\ell_k = z_k - z_{k-1}, \;\; \Omega''_k = [z_{k-1}, z_k], \;\; u_k = \frac{1}{2}(z_{k-1} + z_k),$$

with $\ell_{-1} = \ell_0 = \ell_{p+1} = \ell_{p+2} = 0$, $u_0 = z_0$ and $u_{p+1} = z_p$. For mesh ratios, we define respectively

$$\upsilon_k = \frac{\ell_k}{\ell_{k-1} + \ell_k}, \quad \upsilon'_k = 1 - \upsilon_k = \frac{\ell_{k-1}}{\ell_{k-1} + \ell_k},$$

for $1 \le k \le p+1$, with $\upsilon_0 = \upsilon'_{p+2} = 0$, and, for $k \in \Gamma''_p = \{0, 1, \ldots, p+1\}$:

$$\tilde{a}_k = -\frac{\upsilon_k^2 \upsilon'_{k+1}}{\upsilon_k + \upsilon'_{k+1}}, \quad \tilde{b}_k = 1 + \upsilon_k \upsilon'_{k+1}, \quad \tilde{c}_k = \frac{\upsilon_k (\upsilon'_{k+1})^2}{\upsilon_k + \upsilon'_{k+1}},$$

with the convention $\tilde{a}_0 = \tilde{c}_0 = \tilde{a}_{p+1} = \tilde{c}_{p+1} = 0$. We also need the fundamental functions:

$$\tilde{B}_k = \tilde{c}_{k-1} B_{k-1} + \tilde{b}_k B_k + \tilde{a}_{k+1} B_{k+1},$$

with $\tilde{c}_{-1} = \tilde{a}_{p+2} = 0$.

Let $\Gamma = \Gamma_{mnp} = \{\gamma = (i, j, k), 0 \le i \le m+1, 0 \le j \le n+1, 0 \le k \le p+1\}$, then the set of data sites is:

$$\mathcal{D} = \mathcal{D}_{mnp} = \{N_\gamma = (s_i, t_j, u_k),\ \gamma = (i, j, k) \in \Gamma\}.$$

The partition $\mathcal{P} = \mathcal{P}_{mnp}$ of Ω considered here is the tensor product of partitions of Ω' and Ω''. As the partition of Ω' is the criss-cross triangulation $\mathcal{T}' = \mathcal{T}'_{mn}$ defined in Section 3, we see that $\mathcal{P}$ is a partition of Ω into *vertical prisms with triangular horizontal sections*.

We also consider the following families of bivariate B-splines and fundamental functions on Ω' introduced in Section 3

$$\mathcal{B}' = \{B_{ij}, (i, j) \in \Gamma'\}, \quad \bar{\mathcal{B}}' = \{\bar{B}_{ij}, (i, j) \in \Gamma'\},$$

and the univariate B-splines and fundamental functions on $\Omega'' = [\alpha_3, \beta_3]$ defined in Section 2:

$$\mathcal{B}'' = \{B_k, k \in \Gamma''\}, \quad \tilde{\mathcal{B}}'' = \{\tilde{B}_k, k \in \Gamma''\}.$$

Therefore the spline space $\mathcal{S}_2(\mathcal{P})$ is generated by the $(m + 2)(n + 2)(p + 2)$ tensor-product B-splines:

$$B_\gamma(x, y, z) = B_{ij}(x, y)B_k(z), \quad \gamma \in \Gamma.$$

Their properties are immediate consequences of properties of bivariate and univariate B-splines. In particular, they are positive and form a partition of unity on Ω. As the spline space $\mathcal{S}_2(\mathcal{P})$ contains the space of polynomials $\bar{\Pi}_2 = \bar{\Pi}_2[x, y, z] = \Pi_2[x, y] \otimes \Pi_2[z]$, we can expand the monomials of this space in terms of B-splines. Using the notation $e_{pqr} = x^p y^q z^r$ for monomials, we have:

$$e_{100} = \sum_{\gamma\in\Gamma} s_i B_\gamma, \quad e_{010} = \sum_{\gamma\in\Gamma} t_j B_\gamma, \quad e_{001} = \sum_{\gamma\in\Gamma} u_k B_\gamma,$$

$$e_{110} = \sum_{\gamma\in\Gamma} s_i t_j B_\gamma, \quad e_{101} = \sum_{\gamma\in\Gamma} s_i u_k B_\gamma, \quad e_{011} = \sum_{\gamma\in\Gamma} t_j u_k B_\gamma,$$

$$e_{200} = \sum_{\gamma\in\Gamma}(s_i^2 - \frac{h_i^2}{4})\, B_\gamma, \;\; e_{020} = \sum_{\gamma\in\Gamma}(t_j^2 - \frac{k_j^2}{4})\, B_\gamma, \;\; e_{002} = \sum_{\gamma\in\Gamma}(u_k^2 - \frac{\ell_k^2}{4})\, B_\gamma.$$

4.2 Trivariate discrete quasi-interpolants

For the construction of our trivariate dQI, we need the following trivariate extensions of bivariate and univariate dQIs defined in the previous sections, for which we use the same notations:

$$P_1 f(x,y,z) = \sum_{(i,j)\in\Gamma'} f(s_i, t_j, z)\, B_{ij}(x,y),$$

$$P_2 f(x,y,z) = \sum_{(i,j)\in\Gamma'} f(s_i, t_j, z)\, \bar{B}_{ij}(x,y),$$

$$Q_1 f(x,y,z) = \sum_{k\in\Gamma''} f(x, y, u_k)\, B_k(z),$$

$$Q_2 f(x,y,z) = \sum_{k\in\Gamma''} f(x, y, u_k)\, \tilde{B}_k(z),$$

We now define the trivariate blending sum (see e.g. [13] and [25] for these notions)

$$R = P_1 Q_2 + P_2 Q_1 - P_1 Q_1.$$

Setting, for $\gamma = (i,j,k) \in \Gamma$:

$$B^*_\gamma(x,y,z) = B_{ij}(x,y)\tilde{B}_k(z) + \bar{B}_{ij}(x,y) B_k(z) - B_{ij}(x,y) B_k(z),$$

we can write

$$Rf = \sum_{\gamma\in\Gamma} f(N_\gamma)\, B^*_\gamma.$$

In terms of tensor product B-splines $B_\gamma = B_{ij} B_k$, we get the following expression:

$$Rf = \sum_{\gamma\in\Gamma} \nu_\gamma(f) B_\gamma,$$

where the discrete coefficient functional $\nu_\gamma(f)$ is a linear combination of values of f at the seven neighbours of N_γ in $\mathbb{R}^3$ (we use the notations $\varepsilon_1 = (1,0,0), \varepsilon_2 = (0,1,0), \varepsilon_3 = (0,0,1)$):

$$\nu_\gamma(f) = a_i f(N_{\gamma-\varepsilon_1}) + c_i f(N_{\gamma+\varepsilon_1}) + \bar{a}_j f(N_{\gamma-\varepsilon_2}) + \bar{c}_j f(N_{\gamma+\varepsilon_2})$$

$$+\tilde{a}_k f(N_{\gamma-\varepsilon_3}) + \tilde{c}_k f(N_{\gamma+\varepsilon_3}) + (b_i + \bar{b}_j + \tilde{b}_k - 1) f(N_\gamma)$$

The following important property of this dQI can be proved:

Theorem 5 *The operator R is exact on the 16-dimensional subspace $\Pi_R = (\Pi_{11}[x,y] \otimes \Pi_2[z]) \oplus (\Pi_2[x,y] \otimes \Pi_1[z])$ of the 18-dimensional tensor-product space $\Pi_2[x,y] \otimes \Pi_2[z]$. Moreover, for any nonuniform partition $\mathcal{P}$ of the domain Ω, its infinity norm satisfies $\|R\|_\infty \leq 8$. When the partition $\mathcal{P}$ is uniform, there holds the smaller upper bound $\|R\|_\infty \leq 5$.*

Proof: The monomial basis of Π_R being

$$\{1, x, y, z, x^2, y^2, z^2, xy, xz, yz, x^2z, y^2z, xyz, xz^2, yz^2, xyz^2\}$$

and R being the discrete boolean sum

$$R = P_2Q_1 + P_1Q_2 - P_1Q_1,$$

it is easy to verify that $Rm_{rst} = m_{rst} := x^r y^s z^t$ for all monomials in this basis.

$$Rm_{rst} = P_2 m_{rs} Q_1 m_t + P_1 m_{rs} Q_2 m_t - P_1 m_{rs} Q_1 m_t.$$

For $t = 0, 1$, we have $Q_1 m_t = Q_2 m_t = m_t$ and, for $0 \leq r+s \leq 2$, $P_2 m_{rs} = m_{r,s}$, thus we obtain

$$Rm_{rst} = P_2 m_{rs} m_t + P_1 m_{rs} m_t - P_1 m_{rs} m_t = P_2 m_{rs} m_t = m_{rs} m_t = m_{rst}.$$

For $t = 2$, we have $P_1 m_{rs} = m_{r,s}$ for $0 \leq r + s \leq 1$, thus we obtain

$$Rm_{rs2} = m_{rs}Q_1 m_2 + m_{rs}m_2 - m_{rs}Q_1 m_2 = m_{rs}m_t = m_{rst}.$$

Using the representation $Rf = \sum_{\gamma\in\Gamma} \nu_\gamma(f)B_\gamma$, we deduce that

$$\|Rf\|_\infty \leq \max \mid \nu_\gamma(f) \mid, \quad \text{for} \quad \|f\|_\infty \leq 1,$$

with

$$\nu_\gamma(f) \leq\mid a_i \mid + \mid c_i \mid + \mid \bar{a}_j \mid + \mid \bar{c}_j \mid + \mid \tilde{a}_k \mid + \mid \tilde{c}_k \mid +(b_i + \bar{b}_j + \tilde{b}_k - 1).$$

As the ($\mid a \mid + \mid c \mid$)'s are uniformly bounded by 1 and the b's by 2, we obtain

$$\nu_\gamma(f) \leq 3 + 5 = 8.$$

For uniform partitions, due to boundary coefficient functionals, the ($\mid a \mid + \mid c \mid$)'s are uniformly bounded by $\frac{1}{2}$, the $(b_i + \bar{b}_j)$'s by 3 and the $\tilde{b}_k$'s by $\frac{3}{2}$, so we obtain

$$\nu_\gamma(f) \leq \frac{3}{2} + 2 + \frac{3}{2} = 5.$$

$\square$

From Theorem 5, we deduce that $\|f - Rf\|_{\infty,\pi} \leq 9d(f, \Pi_2)_{\infty,\pi}$ where $d(f, \Pi_2)_{\infty,\pi} = \inf\{\|f - p\|_{\infty,\pi} \mid p \in \Pi_2\}$, in each prism $\pi \in \mathcal{P}$. Now we observe that Π_R contains the 10-dimensional subspace $\Pi_2[x, y, z]$ of trivariate quadratic polynomials, therefore standard arguments in approximation theory allows us to deduce the following result:

Theorem 6 *There exists a constant $C_3 > 0$ such that for all functions $f \in W^{3,\infty}(\Omega)$ and $h = \max\{diam(\pi) \mid \pi \in \mathcal{P}\}$:*

$$\|f - Rf\|_\infty \leq C_3 h^3 \max\{\|D^{pqr} f\|_\infty : p + q + r = 3\}.$$

Remark. We might partly explain our choice of the QI R by the following observation. All QIs briefly described in the introduction are of the form

$$Qf := \sum_{\gamma\in\Gamma} \mu_\gamma(f) C_\gamma,$$

where the C_γ's are compactly supported splines, with support centered at M_γ, and the coefficients $\mu_\gamma(f)$ are discrete linear functionals based on points lying in some neighbourhood of M_γ

$$\mu_\gamma(f) := \sum_{\alpha\in A} a_\alpha f(M_{\gamma+\alpha}),$$

(here A denotes a finite set of triplets of indices). When Q is the trivariate tensor product of univariate QIs, then $\#A = 27$. When Q is the tensor product of a bivariate discrete blending sum of univariate QIs with a third univariate QI, then $\#A = 15$. The same result holds when the bivariate discrete blending sum is substituted with the QI described in Section 3. Finally, the choice $Q = R$ leads to $\#A = 7$, which is the lowest possible cardinality of A in $\mathbb{R}^3$ for operators reproducing $\mathbb{P}_2[x, y, z]$. Therefore, the computation of fundamental functions, of operator norms and cubature weights is easier.

5 Cubature rule for non-uniform partitions

From the preceding Section, we deduce that

$$\mathcal{I}(f) = \int_\Omega f \approx \int_\Omega Rf = \mathcal{I}(Rf) = \sum_{\gamma\in\Gamma} w_\gamma f(N_\gamma)$$

where, for each $\gamma = (i, j, k) \in \Gamma$:

$$w_\gamma = \int_\Omega B^*_\gamma = \int_{\Omega'} B_{ij} \int_{\Omega''} \tilde{B}_k + \int_{\Omega'} \bar{B}_{ij} \int_{\Omega''} B_k - \int_{\Omega'} B_{ij} \int_{\Omega''} B_k$$

Therefore we need the following values:

$$w_k = \int_{\Omega''} B_k, \quad \tilde{w}_k = \int_{\Omega''} \tilde{B}_k, \quad w_{ij} = \int_{\Omega'} B_{ij}, \quad \bar{w}_{ij} = \int_{\Omega'} \bar{B}_{ij}.$$

5.1 Univariate quadrature rule

It is well known that

$$w_k = \int_{x_{k-2}}^{x_{k+1}} B_k = \frac{1}{3}(\ell_{k-1} + \ell_k + \ell_{k+1}),$$

with $\ell_{-1} = \ell_0 = \ell_{p+1} = \ell_{p+2} = 0$. From Sections 2 and 3, we have, for $1 \leq k \leq p$

$$\tilde{B}_k = \tilde{c}_{k-1} B_{k-1} + \tilde{b}_k B_k + \tilde{a}_{k+1} B_{k+1},$$

and, for boundary B-splines

$$\tilde{B}_0 = B_0 + \tilde{a}_1 B_1, \quad \tilde{B}_{p+1} = \tilde{c}_p B_p + B_{p+1}.$$

(with the convention $\tilde{c}_{-1} = \tilde{a}_{p+1} = 0$). Therefore it is easy to compute, for $0 \leq k \leq p+1$:

$$\tilde{w}_k = \tilde{c}_{k-1} w_{k-1} + \tilde{b}_k w_k + \tilde{a}_{k+1} w_{k+1}.$$

It is proved in [29] that those weights are positive for any nonuniform partition of the given interval.

5.2 Bivariate cubature rule

Now, it remains to compute the weights $w_{ij} = \int_{\Omega'} B_{ij}$ since we have

$$\bar{w}_{ij} = \int_{\Omega'} \bar{B}_{ij} = (b_i + \bar{b}_j - 1) w_{ij} + a_{i+1} w_{i+1,j} + c_{i-1} w_{i-1,j} + \bar{a}_{j+1} w_{i,j+1} + \bar{c}_{j-1} w_{i,j-1}.$$

Theorem 7 *The integral of the B-spline B_{ij} on the rectangular domain $\Omega' = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ is given by the formula:*

$$\int_{\Omega'} B_{ij} = \frac{1}{24}[(h_{i-1}+h_{i+1})(k_{j-1}+4k_j+k_{j+1})+(h_{i-1}+4h_i+h_{i+1})(k_{j-1}+k_{j+1})].$$

Proof: It is mainly a technical calculation using the BB representations of all pieces of quadratic polynomials composing the B-spline B_{ij} (more details are given in [19]). One can find these representations in the technical report [27]. On the other hand, a quadratic polynomial $p \in \Pi_2[x, y]$ on a triangle $\tau = A_1, A_2, A_3$ of the partition T has an expansion in the local Bernstein basis with respect to the barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$ of that triangle:

$$p(\lambda) = \sum_{|\alpha|=2} c(\alpha) b_\alpha(\lambda), \quad \text{with} \quad b_\alpha(\lambda) = \frac{2}{\alpha!}\lambda^\alpha.$$

When the triangle τ is included in the rectangle with horizontal (resp. vertical) edges of lengths h_i (resp. k_j), its area is equal to $\frac{1}{4}h_i k_j$. On the other hand, it is well known that all quadratic Bernstein polynomials have the same integral

$$\int_\tau b_\alpha = \frac{1}{24} h_i k_j,$$

therefore we obtain

$$\int_\tau p = \frac{1}{24} h_i k_j \sum_{|\alpha|=2} c(\alpha).$$

As the support of B_{ij} is composed of 28 triangles, we have just to sum up the BB-coefficients in each of them and to multiply this sum by $\frac{1}{24}$ times the area of the rectangle containing the triangle.

$\square$

Remark 1. The general expression of the integral of B_{ij} is still valid for boundary B-splines since in that case we have just to take some of the mesh-lengths $\{h_r, r = i-1, i, i+1\}$ or $\{k_s, s = j-1, j, j+1\}$ equal to zero.

Theorem 8 *The sum of bivariate cubature weights is uniformly bounded as follows*

$$\sum_{i,j} |\bar{w}_{ij}| \leq \int_{\Omega'} \sum_{i,j} |\bar{B}_{ij}| \leq mes(\Omega') \|P_2\|_\infty \leq 5\, mes(\Omega'),$$

5.3 Trivariate cubature rule

From sections 5.1 and 5.2, for each $\gamma = (i, j, k) \in \Gamma$ we can deduce that:

$$w_\gamma = w_{i,j} \tilde{w}_k + \bar{w}_{i,j} w_k - w_{i,j} w_k.$$

By inserting w_γ in $\mathcal{I}(Rf)$ defined in section 5 we obtain the desired trivariate cubature rule.

Theorem 9 *The sum of trivariate cubature weights is uniformly bounded as follows*

$$\sum_{\gamma} |w_\gamma| \leq \int_{\Omega} \sum_{\gamma} |B_\gamma| \leq vol(\Omega) \|R\|_\infty \leq 8\, vol(\Omega).$$

5.4 Error estimates for non-uniform partitions

A detailed study of the cubature error

$$E_R(f) = \int_{\Omega} f - \int_{\Omega} Rf = \mathcal{I}(f) - \mathcal{I}(Rf),$$

on arbitrary partitions would need a corresponding deep study of Sard kernels (see e.g. [16]) and we only give a rough result giving the approximation order $O(h^3)$ of this error:

Theorem 10 *There exists a constant $C_3^* > 0$ such that for any function $f \in W^{3,\infty}(\Omega)$ and for any partition $\mathcal{P}$ of Ω into prisms, with $h = \max\{diam(\pi),\ |$*

$\pi \in \mathcal{P}\}$:

$$\|E_R(f)\| \le C_3^* h^3 \max\{\|D^{pqr} f\|_\infty : p+q+r=3\}.$$

Proof: From theorem 6, we deduce immediately

$$|E_R(f)| \le \int_\Omega |f - Rf| \le \text{vol}(\Omega)\|f - Rf\|_\infty$$

$$\le C_3 \text{ vol}(\Omega) h^3 \max\{\|D^{pqr} f\|_\infty : p+q+r=3\},$$

therefore we can take $C_3^* = C_3 \text{ vol}(\Omega)$.

□

6 Cubature rule for a uniform partition

6.1 Uniform partition on an interval

We assume that the partition of $\Omega'' = [z_0, z_p] = [\alpha_3, \beta_3]$ is uniform with mesh-length ℓ and we denote $f_s = f(u_s)$ for $0 \le s \le p+1$. In that case, it is easy to verify the following result [28] [29]

Theorem 11 *The quadrature rule associated with the univariate dQI $Q_2 f$ can be written in the following form*

$$\int_{\alpha_3}^{\beta_3} Q_2 f = \ell\{\frac{1}{9}(f_0 + f_{p+1}) + \frac{7}{8}(f_1 + f_p) + \frac{73}{72}(f_2 + f_{p-1}) + \sum_{s=3}^{p-2} f_s\}.$$

Moreover, it is exact on Π_3 and not only on Π_2, as in the case of a non-uniform partition. This is due to the symmetry of nodes and weights w.r.t. the midpoint. Details on this method, with error estimates and comparison with composite Simpson's rule, are given in [29]. In fact, for f sufficiently smooth, *it is proved that*

$$\int_{\alpha_3}^{\beta_3} (f - Q_2 f) = C_2 \ell^4 f^{(4)}(c_2) + O(\ell^5), \quad c_2 \in [\alpha_3, \beta_3].$$

where $C_2 = \frac{23}{5760} \approx 0.004$.

6.2 Uniform partition on a rectangular domain

We assume that the two partitions on $[\alpha_s, \beta_s], s = 1, 2$ are uniform. Let $h = h_i$ for $1 \leq i \leq m$ and $k = k_j$ for $1 \leq j \leq n$. In that case, we have the following result

Theorem 12 *The bivariate quadrature rule associated with the dQI* $P_2 f$ *on* Ω' *is given by*

$$\begin{gathered}
\int_{\Omega'} P_2 f = \frac{hk}{12} [\sum_{i=2}^{m-1} \sum_{j=2}^{n-1} f(M_{ij}) + \\
4 \left(\sum_{i=2}^{m-1} (f(M_{i,0}) + f(M_{i,n+1})) + \sum_{j=2}^{n-1} (f(M_{0,j}) + f(M_{m+1,j})) \right) + \\
8 \left(\sum_{i=2}^{m-1} (f(M_{i,1}) + f(M_{i,n})) + \sum_{j=2}^{n-1} (f(M_{1,j}) + f(M_{m,j})) \right) + \\
3 \left(f(M_{1,0}) + f(M_{m,0}) + f(M_{0,1}) + f(M_{m+1,1}) \right) + \\
3 \left(f(M_{0,n}) + f(M_{m+1,n}) + f(M_{1,n+1}) + f(M_{m,n+1}) \right) + \\
\left(f(M_{0,0}) + f(M_{m+1,0}) + f(M_{0,n+1}) + f(M_{m+1,n+1}) \right) + \\
5 \left(f(M_{1,1}) + f(M_{m,1}) + f(M_{1,n}) + f(M_{m,n}) \right)].
\end{gathered}$$

Moreover, this rule is exact on the space Π_3 *of cubic polynomials. When* $k = h$, *there holds, for any sufficiently smooth function* f,

$$\int_{\Omega'} (f - P_2 f) = O(h^4).$$

Details on this cubature formula are given in [19].

6.3 Symmetric or uniform partitions on the trivariate domain

Finally, we come back to the tridimensional domain $\Omega = \Omega' \times \Omega''$, equipped with symmetric or uniform partitions on $\Omega' = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]$ and $\Omega'' = [\alpha_3, \beta_3]$. Due to the symmetry of weights it is easy to prove the following

Theorem 13 *Assume that the partitions are symmetrical w.r.t. the midpoints of the three intervals. Then the cubature rule $\mathcal{I}(Rf)$ is exact on trivariate cubic polynomials. Therefore, when these partitions have the same meshlength h, there holds, for any sufficiently smooth function f,*

$$\int_{\Omega} (f - Rf) = O(h^4).$$

7 Numerical results and comparison

We compare the cubature rule $\mathcal{I}(Rf)$ defined in Section 5 with the tensor product cubature rules $\mathcal{I}_S(f)$ and $\mathcal{I}(Pf)$ based on univariate composite Simpson's rules and univariate quadratic spline QIs [14] respectively.

We set $[\alpha_1, \beta_1] = [\alpha_2, \beta_2] = [\alpha_3, \beta_3] = [0, 1]$. In this case, the integration domain becomes $\Omega = [0, 1]^3$.

We consider the uniform partitions X_m, Y_n and Z_p of $[\alpha_1, \beta_1], [\alpha_2, \beta_2]$ and $[\alpha_3, \beta_3]$ respectively. We need that m, n and p are even numbers, since we construct the composite Simpson's rule on the $m + 1, n + 1$ and $p + 1$ points of the partitions X_m, Y_n and Z_p.

We apply the cubature rules $\mathcal{I}(Rf)$, $\mathcal{I}_S(f)$ and $\mathcal{I}(Pf)$ to several integrands f. We choose $f = f_j, j = 1, \ldots, 4$ and $f = f_6$ from the testing package of Genz [21] which provides test families with pertinent attributes, whereas $f = f_5$ and f_8 are presented in [14]. We consider both smooth integrands, as $f_1, \ldots, f_5$, and only continous ones, as f_6, f_7 and f_8.

We denote by $E_R(f_j)$ the error of cubature $\mathcal{I}(Rf_j)$ and define

$$E_P(f_j) = \mathcal{I}(f_j) - \mathcal{I}(Pf_j), \quad E_S(f_j) = \mathcal{I}(f_j) - \mathcal{I}_S(f_j), \quad j = 1, \ldots, 8,$$

where the cubatures $\mathcal{I}(Rf_j)$, $\mathcal{I}(Pf_j)$ and $\mathcal{I}_S(f_j)$ are based on X_m, Y_n and Z_p. Moreover, $E_R^d(f_j)$ and $E_P^d(f_j)$ denote the errors of $\mathcal{I}(Rf_j)$ and $\mathcal{I}(Pf_j)$, respectively, based on knot sequences obtained by inserting knots of multiplicity two in X_m, Y_n and Z_p, as the case may be.

Assuming $m = n = p$, we give the cubature errors for the considered integrands in terms of the number n of subintervals.

Example 1 $f_1(x, y, z) = \cos(\frac{9\pi x}{2} + \frac{9\pi y}{2} + \frac{9\pi z}{2})$ – Oscillatory

$$\mathcal{I}(f_1) = -\frac{16}{729\pi^3}$$

n	$E_S(f_1)$	$E_R(f_1)$	$E_P(f_1)$
8	1.94(-4)	-1.70(-5)	3.98(-5)
16	7.95(-6)	-1.27(-5)	-1.52(-6)
32	4.60(-7)	-1.28(-6)	-2.18(-7)
64	2.82(-8)	-9.57(-8)	-1.71(-8)
128	1.76(-9)	-6.46(-9)	-1.17(-9)
256	1.10(-10)	-4.19(-10)	-7.59(-11)

Example 2 $f_2(x,y,z) = \frac{1}{[1+(x-0.5)^2][1+(y-0.5)^2][1+(z-0.5)^2]}$ – Product peak

$$\mathcal{I}(f_2) = 0.7973592937$$

n	$E_S(f_2)$	$E_R(f_2)$	$E_P(f_2)$
8	-2.60(-5)	4.09(-5)	2.18(-5)
16	-1.62(-6)	2.50(-6)	1.27(-6)
32	-1.01(-7)	1.53(-7)	7.62(-8)
64	-6.17(-9)	9.63(-9)	4.78(-9)
128	-2.64(-10)	7.20(-10)	4.16(-10)
256	1.04(-10)	1.67(-10)	1.48(-10)

Example 3 $f_3(x,y,z) = \frac{1}{(1+x+y+z)^4}$ – Corner peak

$$\mathcal{I}(f_3) = \frac{1}{24}$$

n	$E_S(f_3)$	$E_R(f_3)$	$E_P(f_3)$
8	-1.41(-5)	4.16(-5)	7.01(-6)
16	-9.20(-7)	3.06(-6)	5.41(-7)
32	-5.81(-8)	2.09(-7)	3.76(-8)
64	-3.64(-9)	1.37(-8)	2.48(-9)
128	-2.28(-10)	8.78(-10)	1.59(-10)
256	-1.35(-11)	5.58(-11)	1.03(-11)

Example 4 $f_4(x, y, z) = e^{-[(x-0.5)^2+(y-0.5)^2+(z-0.5)^2]}$

$$\mathcal{I}(f_4) = 0.7852115962$$

n	$E_S(f_4)$	$E_R(f_4)$	$E_P(f_4)$
8	-2.74(-5)	4.55(-5)	1.85(-5)
16	-1.69(-6)	2.96(-6)	1.19(-6)
32	-1.05(-7)	1.88(-7)	7.53(-8)
64	-6.56(-9)	1.19(-8)	4.75(-9)
128	-3.86(-10)	7.68(-10)	3.21(-10)
256	8.99(-13)	7.02(-11)	4.22(-11)

Example 5 $f_5(x,y,z) = \frac{\pi}{2(e-2)} e^{xy} \sin(\pi z)$

$\mathcal{I}(f_5) = 1$

n	$E_S(f_5)$	$E_R(f_5)$	$E_P(f_5)$
8	-1.38(-4)	3.47(-5)	9.40(-5)
16	-8.53(-6)	2.20(-6)	6.04(-6)
32	-5.31(-7)	1.37(-7)	3.80(-7)
64	-3.32(-8)	8.52(-9)	2.38(-8)
128	-2.07(-9)	5.31(-10)	1.49(-9)
256	1.29(-10)	3.30(-11)	9.30(-11)

Example 6.1 $f_6 = e^{-(|x-0.5|+5|y-0.5|+0.1|z-0.5|)}$

$\mathcal{I}(f_6) = 0.2818326003$

n	$E_S(f_6)$	$E_R(f_6)$	$E_R^d(f_6)$	$E_P(f_6)$	$E_P^d(f_6)$
8	-2.29(-4)	5.42(-3)	1.31(-4)	5.41(-3)	1.02(-4)
16	-1.48(-5)	1.37(-3)	1.05(-5)	1.37(-3)	8.38(-6)
32	-9.32(-7)	3.43(-4)	7.37(-7)	3.43(-4)	5.94(-7)
64	-5.84(-8)	8.59(-5)	4.87(-8)	8.59(-5)	3.95(-8)
128	-3.69(-9)	2.15(-5)	3.09(-9)	2.15(-5)	2.59(-9)
256	-2.66(-10)	5.37(-6)	1.61(-10)	5.37(-6)	1.24(-10)

Example 6.2 $f_6 = e^{-(|x-0.5|+|y-0.5|+|z-0.5|)}$

$\mathcal{I}(f_6) = 0.4873294738$

n	$E_S(f_6)$	$E_R(f_6)$	$E_R^d(f_6)$	$E_P(f_6)$	$E_P^d(f_6)$
8	-1.98(-6)	2.43(-3)	5.63(-6)	2.41(-3)	9.55(-7)
16	-1.24(-7)	6.05(-4)	4.27(-7)	6.04(-4)	7.43(-8)
32	-7.77(-9)	1.51(-4)	3.75(-8)	1.51(-4)	5.08(-9)
64	-5.08(-10)	3.78(-5)	1.10(-8)	3.78(-5)	3.10(-10)
128	-5.45(-11)	9.46(-6)	9.32(-9)	9.45(-6)	-2.63(-12)
256	-3.60(-11)	2.37(-6)	9.21(-9)	2.36(-6)	-2.68(-11)

Example 7 $f_7 = \frac{27}{8}\sqrt{1-|2x-1|}\sqrt{1-|2y-1|}\sqrt{1-|2z-1|}$

$$\mathcal{I}(f_7) = 1$$

n	$E_S(f_7)$	$E_R(f_7)$	$E_R^d(f_7)$	$E_P(f_7)$	$E_P^d(f_7)$
8	4.49(-2)	1.52(-2)	8.22(-3)	3.92(-3)	1.93(-3)
16	1.60(-2)	3.98(-3)	2.42(-3)	1.18(-3)	6.96(-4)
32	5.70(-3)	1.16(-3)	7.83(-4)	3.68(-4)	2.46(-4)
64	2.02(-3)	3.59(-4)	2.67(-4)	1.17(-4)	8.69(-5)
128	7.13(-4)	1.16(-4)	9.30(-5)	3.84(-5)	3.07(-5)
256	2.52(-4)	3.84(-5)	3.27(-5)	1.28(-5)	1.09(-5)

Example 8 $f_8 = \frac{27}{2}\sqrt{1-|2x-1|}y^2z^2$

$\mathcal{I}(f_8) = 1$

n	$E_S(f_8)$	$E_R(f_8)$	$E_R^d(f_8)$	$E_P(f_8)$	$E_P^d(f_8)$
8	1.52(-2)	3.15(-3)	1.38(-3)	3.92(-3)	1.98(-3)
16	5.38(-3)	1.11(-3)	6.34(-4)	1.18(-3)	6.96(-4)
32	1.90(-3)	3.61(-4)	2.40(-4)	3.68(-4)	2.46(-4)
64	6.73(-4)	1.17(-4)	8.63(-5)	1.17(-4)	8.69(-5)
128	2.38(-4)	3.83(-5)	3.07(-5)	3.84(-5)	3.07(-5)
256	8.41(-5)	1.28(-5)	1.09(-5)	1.28(-5)	1.09(-5)

The numerical results in Examples $1,\ldots,5$, with smooth integrands $f = f_j, j = 1,\ldots,5$, confirm the convergence properties of section 6.3. Moreover, the proposed cubature rule $\mathcal{I}(Rf)$ is comparable and in some case better than tensor product cubature rules $\mathcal{I}_S(f)$ and $\mathcal{I}(Pf)$ (see for instance Example 5).

For only continuous integrands, the spline cubature rules $\mathcal{I}(Rf)$ and $\mathcal{I}(Pf)$ include the possibility of inserting multiple knots where it is suspected that the integrand has singularities. The cubature errors in Examples $6, 7$ and 8 show that $\mathcal{I}(Rf)$ and $\mathcal{I}(Pf)$, based on spline QIs with double knots at $x_{n/2}, y_{n/2}$ and $z_{n/2}$, perform better than Simpson's cubature rule. The accuracy of $\mathcal{I}(Rf)$ and $\mathcal{I}(Pf)$ are comparable, even if $\mathcal{I}(Pf)$ can perform better than $\mathcal{I}(Rf)$ when the integrand is the product of same functions along x, y and z (see for instance Examples 6.1 and 6.2).

Remark. It is proved in [29] that the signs of the quadrature errors $\int_{\Omega''}(f(x)-Q_2f(x))dx$ (for the same order $O(h^4)$) are opposite to those of Simpson's rule: the numerical results in Examples $1,\ldots,6$ seem to confirm that the same property holds for $E_R(f)$.

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